國 立 交 通 大 學

應 用 數 學 系

碩 士 論 文

在 C 型代數結構下之 N 相黎曼空間的單擺運動之確切理論與數值計算

The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surface of Genus N with Cut-Structure of Type C

研 究 生：范名宏

指導教授：李榮耀　教授

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摘要

$$u'' + \sin u = 0$$

在數學的歸類中是一個二階常微分方程，同時也是單擺運動的數學模型。

應用大學課程所學到的微積分技術，可以得到

$$\int \frac{1}{\sqrt{2(E+\cos u)}} du = \int dt,$$

其中 E 是積分常數，並且 u 是時間 t 的函數。

以大學課程的能力，上述的積分是難以處理的，因為 $\sqrt{2(E+\cos u)}$ 不是單值函數。因此我們先探討此常微分方程的解 $u(t)$ 所處的空間，並且討論 $\sin u$ 的非線性逼近在此空間上的運算情形，此空間就是 N 相黎曼空間。

除此之外，我們研究橢圓函數，並應用雅可比橢圓函數來分析理想單擺運動的數學模型，也就是我們在摘要開始時所提到的微分方程 $u'' + \sin u = 0$ ，並且確實的求解，以及討論解的週期性及相關性質。

中華民國 一Ｏ二 年 六 月

The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surface of Genus N with Cut-Structure of Type C

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Abstract

$$u'' + \sin u = 0$$

is a second order differential equation, which is a pendulum motion.

In the process of solving the O.D.E., we have the integral form

$$\int \frac{1}{\sqrt{2(E + \cos u)}} du = \int dt$$

where E is the integration constant (a parameter), and u is a function of time t.

The integration is noway to solve due to that $\sqrt{2(E + \cos u)}$ is not a single-valued function. So we study the space where the solution $u(t)$ resides, which is a Riemann Surface of genus N when $\sin u$ is replaced by the N-th partial sum of its Taylor series (which is a polynomial). And we study the corresponding O.D.E. under this Riemann Surface with the help of Mathematica computations.

Next, we study the classical elliptic function to solve the exact O.D.E. $u'' + \sin u = 0$ and analyze the associated properties.

June 2013

誌謝

碩士生涯以及論文能夠順利完成，最感謝的人就是指導教授李榮耀 老師。跟隨老師的這段時間以來，老師從不強硬指示我們該做什麼，而是告訴我們研究的方向之後，便讓同一團隊的大家各自發揮；大家在過程中各自對於不同的問題有所困惑，之後互相討論、互相幫助，其中的心路歷程就是最珍貴的；除了學業，也感謝老師這些時間裡在生活中的協助。感謝李榮耀 老師給我們這麼好的經歷，能夠成為老師的指導學生真的是萬分有幸。

感謝王夏聲 老師的教導，引領我重新認識實變數函數；感謝郎正廉 老師、余啓哲 老師，老師的指導與建議使論文得以更臻完備。由衷感謝各位老師。

生活中好多事情都和身邊的朋友們一起走過，在此也要向你們說聲感謝。謝謝智龍、奕倫，我們日夜相處，一起說笑、一起度過碩士生活的兩年；謝謝卓時、建偉、仁益，好多時刻，沒有你們的陪伴我都不知道該怎麼渡過；謝謝竣富 學長，在一群晚輩當中，總能分享我們所沒有的經驗，並且給我們更成熟的建議，沒有學長，畢業的道路肯定更加艱辛；謝謝團隊的夥伴，我們一同困惑、一同成長、一同享受最後的成果。謝謝分析、組合組的所有同學，你們每一個人都是我在交大的生活點滴裡重要的人。短暫的相處雖不如大學四年來的長久，但因為碩班的生活型態讓彼此緊密依靠；人生之中能有這樣的摯友以及回憶，就是在交大除了知識以外，我所擁有最大的寶藏了。

預料之外的三年，是我人生中的禮物。感謝教育所嘉凌 姐、師資培育中心雅怡 姐、佩萱 姐，能在這一年與你們共事，讓我對於碩士生涯的第三年有了完全不一樣的體驗，這將成為我永難忘懷的經歷與回憶。

最後要感謝我的父母，當我在外地生活的時候，你們總會不時地打電話關心我，也讓我在身心俱疲地回到台北的時候，能有個完全放鬆的地方。時而嚴厲、時而溫柔的管教方式，我相信這都是你們愛的表現；在你們庇護之下茁壯的我，除了說不完的感謝，以後的日子我也會盡心盡力地孝順你們，並努力做好每一件事，將榮耀歸於你們，我的父母，范祥茂、鄭心瑜。

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Chapter 1

Introduction

In this paper, we want to study a pendulum motion. An ideal pendulum motion is energy-conservative, and it can be shown in mathematical model as

$$u'' + \sin u = 0.$$

Since the ideal pendulum motion is a second order differential equation, [1] we reduce it to a first order differential equation and try to solve it naturally. There is

$$u'u'' + u' \sin u = 0,$$

and

$$\frac{1}{2}(u')^2 - \cos u = E$$

where E is the integration constant, then

$$u' = \frac{du}{dt} = \pm\sqrt{2(E + \cos u)}$$

where u is a function of time t,

$$\int \frac{1}{\sqrt{2(E + \cos u)}} du = \int dt. \tag{1.1}$$

Although we together all skills of integration in calculus, we still has no ability to deal with the equation (1.1). So we change the point to discuss where the solution $u(t)$ resides in short term.

Let $f(z) = \sqrt{2(E+\cos u)}$. Since the $f(z)$ is in the type of radical expression, the output of $f(z)$ needs to be confirmed in some domain. For any $E+\cos u \in \mathbb{C}$, it can be shown in the polar form, $E+\cos u = z = |z|e^{i\theta}$, and, $f(z)$ is a two-valued function of z on complex plane $\mathbb{C}$ since $|z|e^{i\theta} = |z|e^{i\theta+2n\pi}$. Indeed, $f(z)$ works on Riemann Surface, [2] [3] and the computations of the integral on Riemann Surface rely on the theorems of complex analysis, mainly the Cauchy integral formula. [4]

After studying the Riemann Surface, we use the tools of computing by Mathematica [5] on Riemann Surface applying to the nonlinear approximation of $\sin u$ into $P_N(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!}$ by the Taylor expansion, $\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

Since we are trapped into the equation (1.1), i.e., we are not able to integrate the fraction $\frac{1}{\sqrt{2(E+\cos u)}}$, trying a brand-new theory is the idea to know the exact theorem of the pendulum motion.

By the equation (1.1), we transferred the nonlinear O.D.E. problem into the so-called inverse problem (in an integral form), and it can be replaced by elliptic functions. [6] Furthermore, solve and express the solutions $u(t)$ in terms of classical elliptic functions. [7]

To construct the theorem enough to solve the original question, the pendulum motion, we will introduce several cases of elliptic functions, including the definitions, properties, and relations among each other.

Consider the condition in ideal environment, the pendulum motion works without friction, we can derive the mathematical model of the simple pendulum motion.

In addition, discuss the reasonable E as the total energy in the sense of physics and take the place of equation (1.1) by Jacobian elliptic functions so that it yields the exact solutions and the periods case by case.

Chapter 2

Riemann Surface

2.1 Introduction

$u'' + P_N(u) = 0$ is also a second order different equation where the degree of polynomial $P_N(u)$ is N.

By the derivation, we have

$$\int \frac{1}{\sqrt{2(E - P_{N+1}(u))}} du = \int dt$$

where E is the integration constant.

According to the fundamental theorem of Algebra, there is

$$\sqrt{2(E - P_{N+1}(u))} = \sqrt{c \prod_{k=1}^{N+1} (u - u_k)}, \qquad c \in \mathbb{C},$$

hence we must investigate the space where u resides.

Actually, $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$ is a two-valued function of z on complex plane $\mathbb{C}$. We use algebra and analysis to develop a new surface such that f becomes a single-valued and analytic function on this surface, namely, a Riemann Surface. [2]

2.1.1 Construct the corresponding Riemann Surface

First, take $f(z) = \sqrt{z}$ for example, $f : \mathbb{C} \to \mathbb{C}$. Using polar form, let $z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}, n \in \mathbb{Z}$, then

$$\begin{aligned} f(z) = \sqrt{z} = |z|^{\frac{1}{2}}e^{\frac{\theta+2n\pi}{2}i} \\ = \begin{cases} |z|^{\frac{1}{2}}e^{\frac{\theta}{2}i} & \text{if } n \text{ is even} \\ -|z|^{\frac{1}{2}}e^{\frac{\theta}{2}i} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

is a two-valued function. Now we want to let $f(z)$ becomes a single-valued function, so we modify its domain $\mathbb{C}$ to develop the corresponding Riemann Surface such that f becomes a single-valued and analytic function on this surface.

Starting at $z = re^{i\theta}$, we have $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{\theta}{2}i}, r \neq 0$. Fixing r and continuing along a closed path once around the origin so that θ increase by 2π, $f(z)$ comes to the value $\sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}i}$ which is just the negative of its original value. Continuing above way then θ increase by 2π and $f(z)$ comes to original value. First, image two sheets lying over the complex plane and cut the plane along negative real axis (i.e. from zero to infinite) and restrict ourselves so as never to continue $f(z)$ over this cuts, we get single-valued branches of $f(z)$.

Define that

$$
\begin{aligned}
f(z) &= |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \qquad -\pi \le \theta < \pi, \\
f(z) &= |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \qquad \pi \le \theta < 3\pi,
\end{aligned}
$$

called sheet-I and sheet-II respectively. The cut in each sheet has two edges, label the edge of starting edge with $+$ and the edge of terminal edge with $-$ (Show in Figure 2.1). Moreover, we cross the cut and pass from one sheet to another. Second we extend the plane of complex numbers with one additional point at infinity constitute a number system known as the extended complex numbers. Use stereographic projection, we can consider the two sheets to be a sphere.

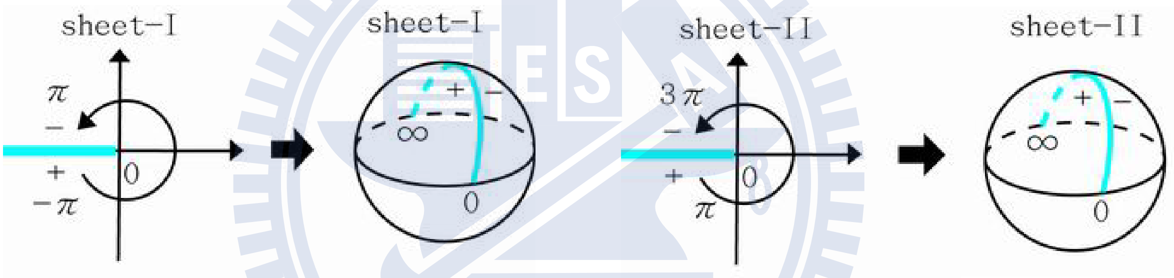

Figure 2.1: Complex plane and extended complex plane

Next, image that the spheres are made of rubber and stretch each cut into circular holes.

Rotate the spheres until the holes face each other, and paste two cuts together (+)edge of sheet-I with (−)edge of sheet-II and (−)edge of sheet-I with (+)edge of sheet-II. We can derive a sphere. We called this sphere, Riemann Surface of genus 0, denoted R_0. Show in Figure 2.3.

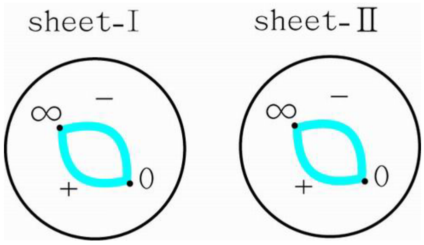

Figure 2.2: Place the cuts open

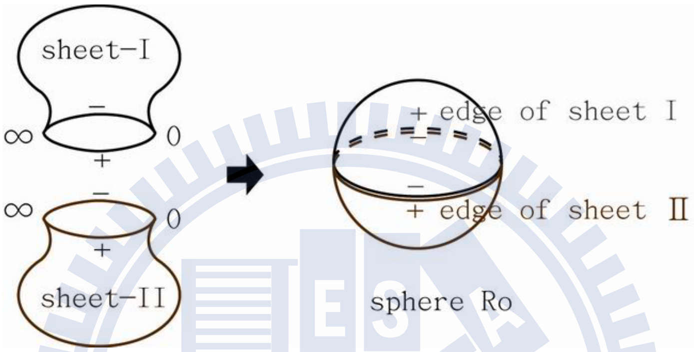

Figure 2.3: Construct R_0

Notice that in Riemann Surface (+)edge of sheet-I is equivalent to (−)edge of sheet-II and (−)edge of sheet-I is equivalent to (+)edge of sheet-II.

We could using similar way to develop the corresponding Riemann Surface for $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$.

In general situation, using same idea to construct Riemann Surface of $f(z)$ where $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)} = \prod_{k=1}^{n}\sqrt{(z - z_k)}, z_k \in \mathbb{R}, z_1 > z_2 > \ldots > z_n$ for horizontal cuts. First, we cut plane starts from z_k to $-\infty$. If the curve cross even cuts, it will not change that is becomes no cut. If the curve cross odd cuts, it will has a branch cut.

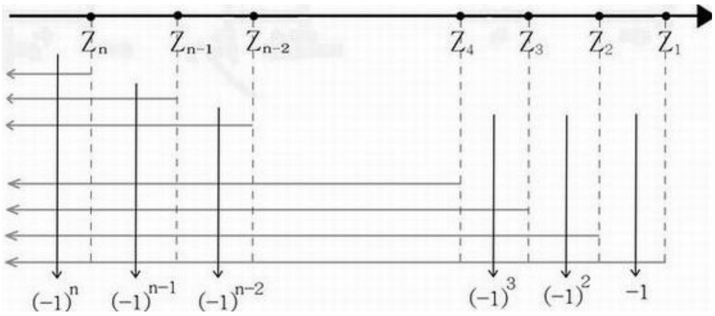

Figure 2.4: $n = 2N - 1$ or $2N$

Case 1: If $n = 2N-1$. There are cuts along $(-\infty, z_{2N-1}], \dots, [z_{2j}, z_{2j-1}], \dots, [z_4, z_3], [z_2, z_1]$.

Figure 2.5: $n = 2N - 1$

Case 2: If $n = 2N$. There are cuts along $[z_{2N}, z_{2N-1}], \dots, [z_{2j}, z_{2j-1}], \dots, [z_4, z_3], [z_2, z_1]$.

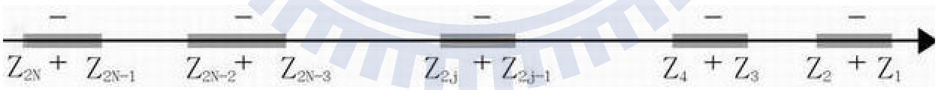

Figure 2.6: $n = 2N$

We use same idea to construct the corresponding Riemann Surface:

Case 1: $n = 2N - 1$

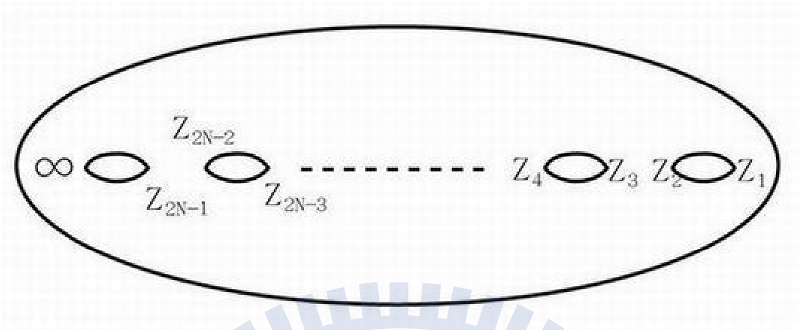

Figure 2.7: Placing cuts open in both sheets

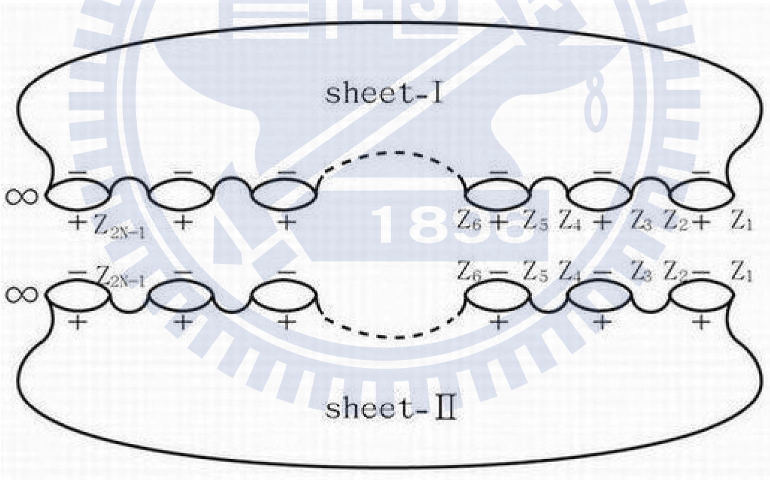

Figure 2.8: Together two sheets

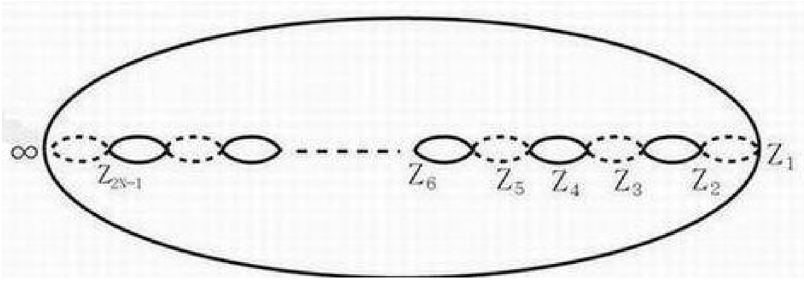

Figure 2.9: $N-1$ holes for $n = 2N - 1$

It becomes Riemann Surface with $N - 1$ holes, that is R_{N-1}.

Case 2: $n = 2N$

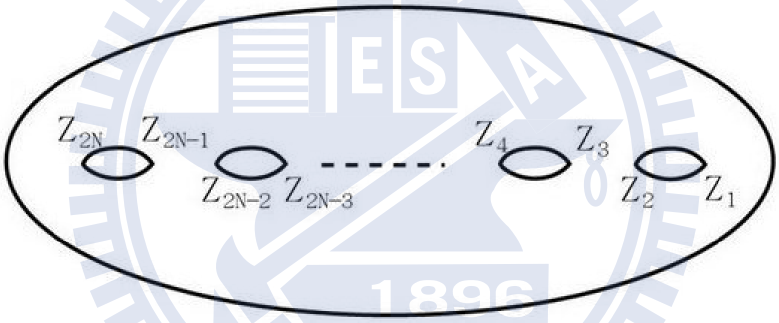

Figure 2.10: Placing cuts open in both sheets

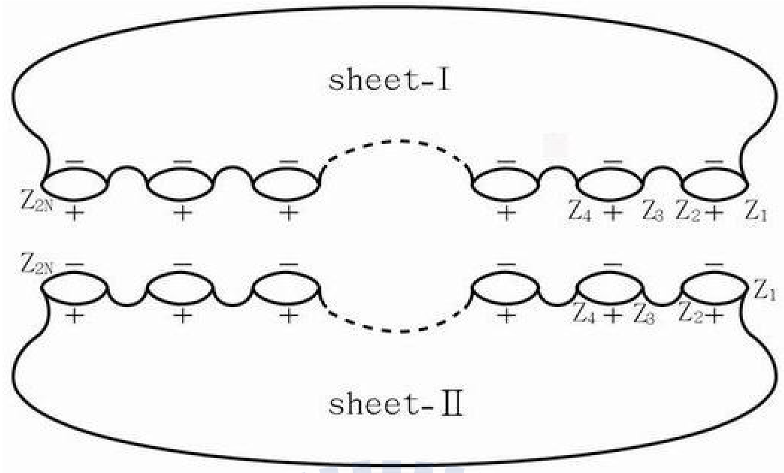

Figure 2.11: Together two sheets

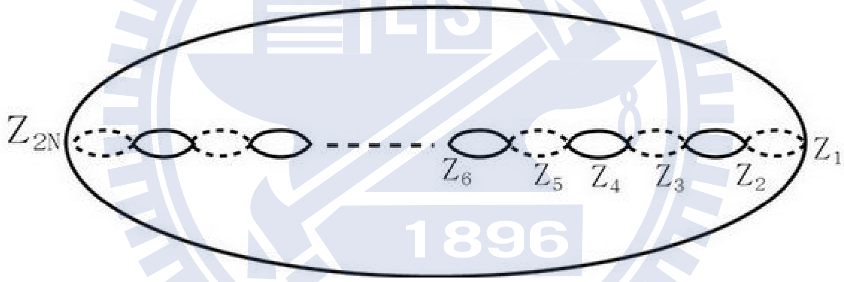

Figure 2.12: $N-1$ holes for $n=2N$

It also becomes Riemann Surface with $N-1$ holes, that is R_{N-1}.

So

$$f(z)=\sqrt{\prod_{k=1}^{2N-1}(z-z_k)}\overset{or}{=}\sqrt{\prod_{k=1}^{2N}(z-z_k)}$$

will make N cuts and construct Riemann Surface of genus $N-1$, i.e., $N-1$ holes in its geometric graph.

2.1.2 The curve in algebraic and geometric structure

For convenience, we use algebraic to discuss and compute the integrals later. We already know the relation of algebraic and geometric structure with $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$ and how to create the Riemann Surface.

We defined something as follow:

1. The curve in sheet-I is solid line and the curve in sheet-II is dash line in algebraic structure.

2. The curve in overhead Riemann Surface is solid line and the curve in ventral Riemann Surface is dash line in geometric structure.

2.1.3 The a, b-cycles and its equivalent paths

We know every closed curve on Riemann Surface R_N can be deformed into an integral combination of the loop-cut a_i and b_i, $i = 1, 2, \ldots, N$. So in this paper, we will consider the integrals of $f(z)$ over a, b-cycles help us to obtain the integrals easier.

If $f(z)$ has $2N-1$ or $2N$ roots, there are loop-cuts a_i, b_i, $i = 1, 2, \ldots, N-1$.

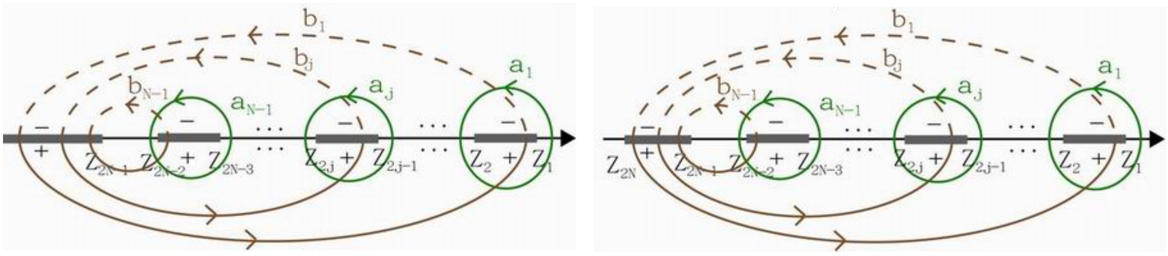

Figure 2.13: a, b-cycles on complex plane

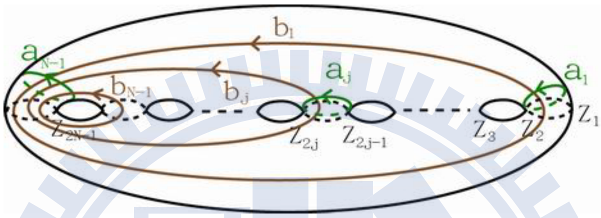

Figure 2.14: a, b-cycles on Riemann Surface

Each a-cycles are non-overlapping and each b-cycles are non-overlapping. Also a, b-cycles have the same number.

Sometimes the curves are difficult to write out their parameters, but always easy to straight lines. It could help us quicker and easier to obtain the integrals over the curves. So now using homotopic of curves to find the equivalent paths of curves. Take an example to explain.

From C is homotopic to C_1, denotes $C \approx C_1$. We have

$$\int_C \frac{1}{f(z)} dz = \int_{C_1} \frac{1}{f(z)} dz$$

in Figure 2.15, $C \approx C_1 \approx C_2 \approx C_3$, and finally we compression the curve C until

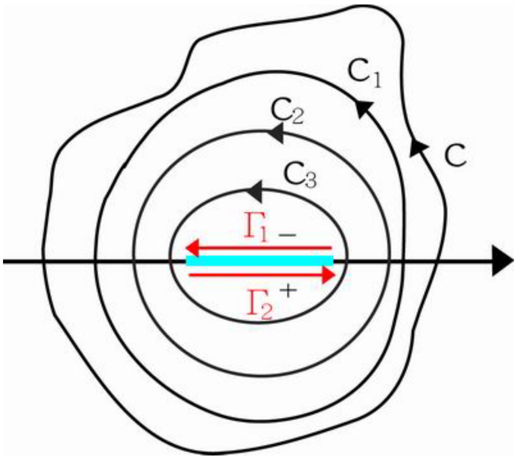

Figure 2.15: Homotopic

we find the equivalent paths of curves $C \approx \Gamma_1 \cup \Gamma_2$. So

$$\int_C \frac{1}{f(z)} dz = \int_{\Gamma_1 \cup \Gamma_2} \frac{1}{f(z)} dz \tag{2.1}$$

$$= \int_{\Gamma_1} \frac{1}{f(z)} dz + \int_{\Gamma_2} \frac{1}{f(z)} dz. \tag{2.2}$$

Here we give a simple example to confirm the theorem above.

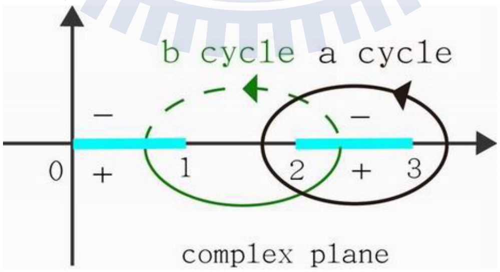

Figure 2.16: Cut plane and a, b-cycle of $f(z) = \sqrt{z(z-1)(z-2)(z-3)}$

Let $f(z) = \sqrt{z(z-1)(z-2)(z-3)} = \sqrt{z}\sqrt{z-1}\sqrt{z-2}\sqrt{z-3}$ and a, b-cycle show as Figure 2.16, then there are two circular paths display the a, b-cycle, $x = \frac{5}{2}, r = 1$, and $x = \frac{3}{2}, r = 1$ respectively.

So we have the parametric forms for the integral of a-cycle, $z = \frac{5}{2} + e^{i\theta}$,

$$\begin{aligned}\int_a \frac{1}{f(z)}dz &\overset{Math.}{=} \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{f(\frac{5}{2}+e^{i\theta})}d\theta \\ &= 0. + 3.3715i,\end{aligned}$$

and b-cycle, $z = \frac{3}{2} + e^{i\theta}$. Since the path in sheet-II are equivalent to the minus of path in sheet-I, which will introduce later, there is

$$\begin{aligned}\int_b \frac{1}{f(z)}dz &\overset{Math.}{=} \int_{-\pi}^{0} \frac{ie^{i\theta}}{f(\frac{3}{2}+e^{i\theta})}d\theta - \int_{0}^{\pi} \frac{ie^{i\theta}}{f(\frac{3}{2}+e^{i\theta})}d\theta \\ &= -4.31303 + 0.\ i.\end{aligned}$$

Next, we use the homotopic paths to get the integral of a, b-cycle.

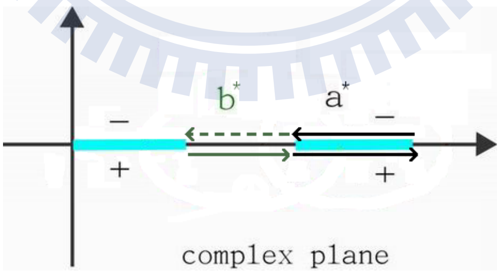

Figure 2.17: Equivalent paths of a, b-cycle

The equivalent path of a-cycle, from 2 to 3 on the (+)edge in sheet-I and from 3 to 2 on the (−)edge in sheet-I, is called a^*-cycle.

$$\begin{aligned}\int_{a^*} \frac{1}{f(z)} dz &\overset{Math.}{=} 2\int_3^2 \frac{1}{f(z)} dz \\ &= 0. + 3.3715i\end{aligned}$$

The equivalent path of b-cycle, from 1 to 2 in sheet-I and from 2 to 1 in sheet-II, is called b^*-cycle. Similarly, since the path in sheet-II are equivalent to the minus of path in sheet-I, there is

$$\begin{aligned}\int_{b^*} \frac{1}{f(z)} dz &\overset{Math.}{=} 2\int_1^2 \frac{1}{f(z)} dz \\ &= -4.31303 + 0.\ i\end{aligned}$$

By the calculations above, we verify the theorem for the equivalent path, i.e., homotopic. It means that we can choose the simplest path for the close contour, and get the numerical result in a efficient way.

In the next pages, we will use the skill (2.1) and (2.2) to compute all cases.

2.1.4 Conclusion of structure of Riemann Surface

For arbitrary cut, if $f(z)$ has $2N-1$ or $2N$ roots, then

1. There are N cuts in complex plane.

2. Its geometric graph has $N-1$ holes, and construct corresponding Riemann Surface of genus $N-1$, i.e., R_{N-1}.

3. There are $N-1$ a-cycles and $N-1$ b-cycles.

2.2 The integrals of $\dfrac{1}{f(z)}$ over a, b-cycles for horizontal cuts

We will use Mathematica helping us to obtain the values of integrals of $\dfrac{1}{f(z)}$ over a, b-cycles. First, discuss the values in sheet-I, sheet-II and Mathematica for horizontal cuts. $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$, using polar form $\prod_{k=1}^{n}(z - z_k)$. Let θ_1 denotes θ in sheet-I and θ_2 denotes in sheet-II. So

$$\theta_2 = \theta_1 + 2\pi.$$

We have

$$\begin{aligned} f(z)|_{(II)} &= \sqrt{r}e^{\frac{\theta_2}{2}i} \\ &= \sqrt{r}e^{\frac{\theta_1+2\pi}{2}i} \\ &= \sqrt{r}e^{\frac{\theta_1}{2}i}e^{\pi i} \\ &= -\sqrt{r}e^{\frac{\theta_1}{2}i} = -f(z)|_{(I)} \end{aligned}$$

where $f(z)|_{(I)}$ denote the value of $f(z)$ with z in sheet-I and $f(z)|_{(II)}$ means z in sheet-II. Because the difference of argument between z in sheet-I and sheet-II is 2π, there is the difference between $f(z)|_{(I)}$ and $f(z)|_{(II)}$ is π. So, there is $f(z)|_{(II)} = -f(z)|_{(I)}$.

Now discuss the difference in sheet-I of theory and Mathematica. First, $\sqrt{-1} = -i$, but we compute $\sqrt{-1}$ in Mathematica obtain $\sqrt{-1} \overset{Math.}{=} i$. Why? We found that $\theta \in (-\pi, \pi]$ of $re^{i\theta}$ in Mathematica, actually. For any other θ of $re^{i\theta}$ which does not belong to $(-\pi, \pi]$, Mathematica will conversion $re^{i\theta}$ into $re^{i\theta^*}$, $\theta^* \in (-\pi, \pi]$ where $re^{i\theta} = re^{i\theta^*}$.

Compare the value of $f(z)$ with z in sheet-I and in Mathematica, we discover that

Lemma 2.1. *If* $\prod_{k=1}^{n}(z-z_k) = re^{i\theta}$ *in sheet-I for horizontal cut,*

$$f(z)|_{(I)} = \begin{cases} f(z)|_{Mathematica} & if \;\; \theta \in (-\pi,\pi), \\ -f(z)|_{Mathematica} & if \;\; \theta = -\pi. \end{cases}$$

Proof.
Since $-\pi$ does not in $(-\pi,\pi]$, Mathematica will conversion $re^{-i\pi}$ into $re^{i\pi}$, but $f(re^{-i\pi})$ and $f(re^{i\pi})$ are different.

In theory:	$-1 = e^{-i\pi}$	$\Rightarrow \sqrt{-1} = e^{-\frac{i\pi}{2}} = -i$
In Mathematica:	$-1 = e^{-i\pi} \overset{Math.}{=} e^{i\pi}$	$\Rightarrow \sqrt{-1} = e^{\frac{i\pi}{2}} = i$

So $f(z) \overset{Math.}{=} -f(z)$ if $\theta = -\pi$ in Mathematica. ∎

In whole paper, $f(z) \overset{Math.}{=} -f(z)$ denotes the polynomial $f(z)$ in front of $\overset{Math.}{=}$ is the value of $f(z)$ in theory and the polynomial $f(z)$ behind the $\overset{Math.}{=}$ is the value of $f(z)$ in Mathematica.

Clearly, there is a mistake when $\theta = -\pi$. When we use Mathematica to get the value of integration we want, we need modify some range where the value will wrong. Determine the difference of $sign(f)$ (same or negative) and then modify the computation of Mathematica to get right value. Because sometimes the form of integration is complex, if we could simplify the way about modify the difference of $sign(f)$, it will help us to get right value easier.

Discuss in general situation:

Compute $\int \frac{1}{f(z)} dz$ over a, b-cycles for horizontal cuts where $f(z) = \sqrt{\prod_{k=1}^{m} (z - z_k)} = \prod_{k=1}^{m} \sqrt{z - z_k}, z_k \in \mathbb{R}, \forall k = 1 \sim m$ and $z_1 > z_2 > \ldots > z_m$.

1. a-cycles:

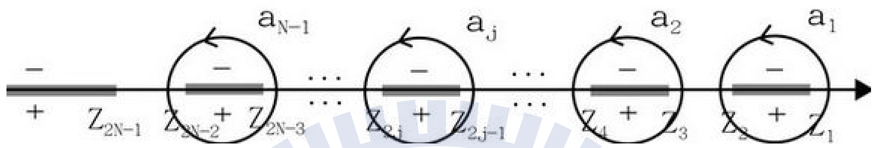

Figure 2.18: a-cycles for $2N - 1$ points

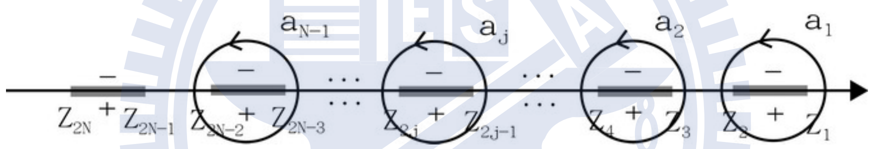

Figure 2.19: a-cycles for $2N$ points

There are N cuts ($N - 1$ holes), we give that a_j is a cycle, center at x with radius r, enclosed $[z_{2j}, z_{2j-1}]$ and doesn't intersect with other cuts. (parametric form)

If $z \in a_j$, let $z = x + re^{i\theta}$ where $\theta \in [-\pi, \pi)$,

$$\begin{aligned}\int_{a_j} \frac{1}{f(z)} dz &= \int_{a_j} \frac{1}{\sqrt{\prod_{k=1}^{m} (z - z_k)}} dz \\ &= \int_{-\pi}^{\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta.\end{aligned}$$

2. Consider $\int_{a_j^*} \frac{1}{f(z)} dz$ where a_j^* is an equivalent path for a_j and it's from z_{2j} to z_{2j-1} on (+)edge and then from z_{2j-1} to z_{2j} on (−)edge:

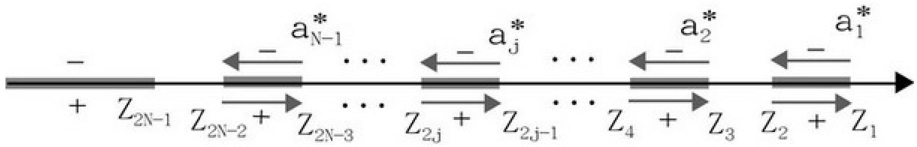

Figure 2.20: a^*-cycles for $2N-1$ points

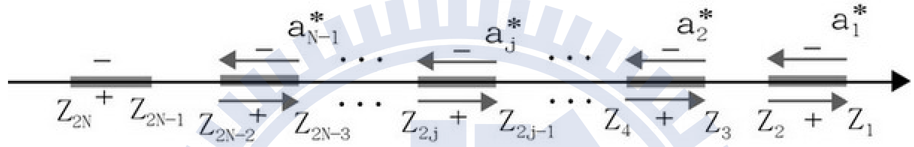

Figure 2.21: a^*-cycles for $2N$ points

By Cauchy integral formula, [4] we can get that

$$\int_{a_j} \frac{1}{f(z)} dz = \int_{a_j^*} \frac{1}{f(z)} dz.$$

Using Lemma 2.1 to compute:

(i) $z_{2j} \overset{+}{\to} z_{2j-1}$:

$$arg(z - z_k) = 0 \qquad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \qquad k = 2j, 2j+1, \ldots, m$$
$$arg(z - z_k) = -\pi \qquad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \qquad k = 1, 2, \ldots, 2j-1$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2j-1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2j} \overset{+}{\to} z_{2j-1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz. \tag{2.3}$$

(ii) $z_{2j} \overset{-}{\leftarrow} z_{2j-1}$:

$$\begin{aligned} &arg(z - z_k) = 0 &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, && k = 2j, 2j+1, \ldots, m \\ &arg(z - z_k) = \pi &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, && k = 1, 2, \ldots, 2j-1 \end{aligned}$$

So

$$f(z) \overset{Math.}{=} f(z),$$

$$\int_{z_{2j} \overset{-}{\leftarrow} z_{2j-1}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)} dz. \tag{2.4}$$

Conclusion of a-cycles: By (2.3) and (2.4),

$$\int_{a_j^*} \frac{1}{f(z)} dz \overset{Math.}{=} 2 \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)} dz.$$

3. b-cycles:

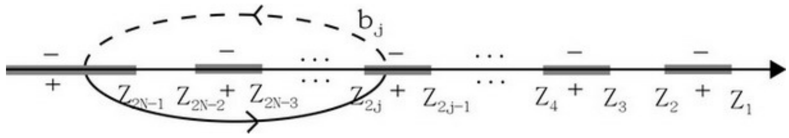

Figure 2.22: b-cycles for $2N-1$ points

Give b_j is a circle, center at x with radius r, enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $(-\infty, z_{2N-1}]$.

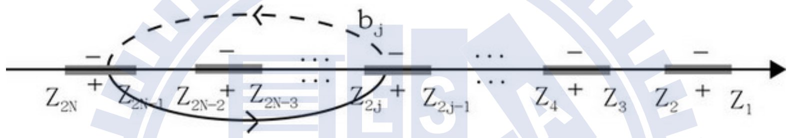

Figure 2.23: b-cycles for $2N$ points

Give b_j is a circle, center at x with radius r, enclosed the $[z_{2N-1}, z_{2j}]$ and intersect at the points on $[z_{2j}, z_{2j-1}]$ and $(-\infty, z_{2N-1}]$. If $z \in b_j, z = x + re^{i\theta}$ where $\theta \in [-\pi, 0) \cup [2\pi, 3\pi)$. From

$$f(z)|_{(II)} = -f(z)|_{(I)},$$

there is

$$\begin{aligned}\int_{b_j} \frac{1}{f(z)} dz &= \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta + \int_{2\pi}^{3\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta \\ &= \int_{-\pi}^{0} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta - \int_{0}^{\pi} \frac{rie^{i\theta}}{\prod_{k=1}^{m} \sqrt{x + re^{i\theta} - z_k}} d\theta.\end{aligned}$$

4. The equivalent path b_j^*:

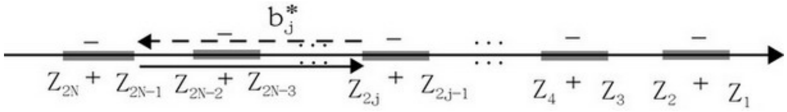

Figure 2.24: b^*-cycles for $2N-1$ points

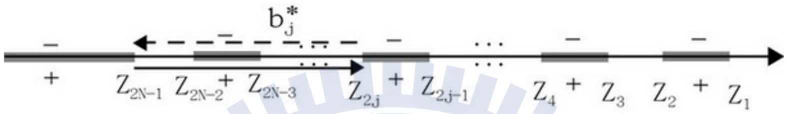

Figure 2.25: b^*-cycles for $2N$ points

From Cauchy integral formula, we have

$$\int_{b_j} \frac{1}{f(z)} = \int_{b_j^*} \frac{1}{f(z)}$$

where b_j^* is a path from z_m to z_{2j} in sheet-I and then from z_{2j} to z_m in sheet-II. Similarly, using Lemma 2.1 to compute:

(1) The path on cut, i.e., the path from z_{2s+2} to z_{2s+1} on (+)edge of sheet-I and the path from z_{2s+1} to z_{2s+2} on (−)edge of sheet-II, $s = j, j+1, \ldots, N-2$.

(i) $z_{2s+2} \xrightarrow{+} z_{2s+1}$:

$$\begin{aligned}
&arg(z-z_k) = 0 && \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, && k = 2s+2, 2s+3, \ldots, m \\
&arg(z-z_k) = -\pi && \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, && k = 1, 2, \ldots, 2s+1
\end{aligned}$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s+1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2s+2} \overset{+}{\rightarrow} z_{2s+1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2s+2}}^{z_{2s+1}} \frac{1}{f(z)} dz. \tag{2.5}$$

(ii) $z_{2s+2} \overset{-}{\leftarrow} z_{2s+1}$ on $(-)$edge of sheet-II is same as on $(+)$edge of sheet-I, so consider $z_{2s+2} \overset{+}{\leftarrow} z_{2s+1}$:

$$\begin{aligned} &arg(z - z_k) = 0 &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, && k = 2s+2, 2s+3, \ldots, m \\ &arg(z - z_k) = -\pi &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, && k = 1, 2, \ldots, 2s+1 \end{aligned}$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s+1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2s+2} \overset{-}{\leftarrow} z_{2s+1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2s+1}}^{z_{2s+2}} \frac{1}{f(z)} dz. \tag{2.6}$$

(2) Without cuts, i.e., the path from z_{2s+1} to z_{2s} in sheet-I and the path from z_{2s} to z_{2s+1} in sheet-II, $s = j, j+1, \ldots, N-2$.

(i) $z_{2s+1} \to z_{2s}$:

$$\begin{aligned} arg(z-z_k)=0 \quad &\Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k=2s+1,2s+2,\ldots,m \\ arg(z-z_k)=-\pi \quad &\Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k=1,2,\ldots,2s \end{aligned}$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s} f(z) \\ &= f(z), \end{aligned}$$

$$\int_{z_{2s+1}\to z_{2s}} \frac{1}{f(z)}dz \overset{Math.}{=} \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)}dz. \tag{2.7}$$

(ii) $z_{2s+1} \dashleftarrow z_{2s}$, by $f(z)|_{(II)} = -f(z)|_{(I)}$, we consider $z_{2s+1} \leftarrow z_{2s}$ first:

$$\begin{aligned} arg(z-z_k)=0 \quad &\Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k=2s+1,2s+2,\ldots,m \\ arg(z-z_k)=-\pi \quad &\Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k=1,2,\ldots,2s \end{aligned}$$

Hence

$$\begin{aligned} f(z)|_{z_{2s+1} \dashleftarrow z_{2s}} &= -f(z)|_{z_{2s+1}\leftarrow z_{2s}} \\ &\overset{Math.}{=} -(-1)^{2s} f(z)|_{z_{2s+1}\leftarrow z_{2s}} \\ &= -f(z)|_{z_{2s+1}\leftarrow z_{2s}}, \end{aligned}$$

$$\int_{z_{2s+1} \dashleftarrow z_{2s}} \frac{1}{f(z)}dz \overset{Math.}{=} -\int_{z_{2s}}^{z_{2s+1}} \frac{1}{f(z)}dz. \tag{2.8}$$

Conclusion of b-cycles: By (2.5), (2.6), (2.7) and (2.8),

$$\int_{b_j^*} \frac{1}{f(z)} dz \overset{Math.}{=} \sum_{s=j}^{N-1} \left(2 \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz \right).$$

2.3 The integrals of $\frac{1}{f(z)}$ over a, b-cycles for vertical cuts

After knowing the integrals in horizontal cuts introducing in the previous section, we will discuss the integrals for vertical cuts. In this case, we define that

$$z - z_k = re^{i\theta}, \qquad \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) \qquad \text{if } z \text{ in sheet-I}$$
$$z - z_k = re^{i\theta}, \qquad \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) \qquad \text{if } z \text{ in sheet-II},$$

the cut in each sheet has two edges, label the starting edge with $+$ and the terming edge with $-$ and z_k is the end point of vertical cut.

As the previous section, we need to modify the computation in Mathematica such that the numerical result of Mathematica is identical to the numerical result of theory when $\theta \in [-\frac{3\pi}{2}, -\pi]$.

Lemma 2.2. *When z in sheet-I for vertical cut whose one of the end points is z_k,*

$$\sqrt{z - z_k} \overset{Math.}{=} \begin{cases} -\sqrt{z - z_k} & if \;\; arg(z - z_k) \in [-\frac{3\pi}{2}, -\pi], \\ \sqrt{z - z_k} & if \;\; arg(z - z_k) \in (-\pi, \frac{\pi}{2}). \end{cases}$$

Proof.
Let z in sheet-I and using polar form $z - z_k = re^{i\theta}$. When $\theta \in (-\pi, \frac{\pi}{2})$, the argument in theory or Mathematica is the same. When $\theta \in [-\frac{3\pi}{2}, -\pi]$, Mathematica will conversion θ into $\theta + 2\pi$ where $\theta + 2\pi \in [\frac{\pi}{2}, \pi]$ and $re^{i\theta} = re^{(\theta+2\pi)i}$, but

In theory: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta}{2}i}$

In Mathematica: $\sqrt{z - z_k} = \sqrt{r}e^{\frac{\theta+2\pi}{2}i} = -\sqrt{r}e^{\frac{\theta}{2}i}$

Thus, if $\theta \in [-\frac{3\pi}{2}, -\pi]$,

$$\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}.$$ ∎

As same as horizontal cut. We first discuss the difference between the value in theory and the value in Mathematica. Compare their $sign(f)$ is different or not? Using statement before about modify and get value, the result will be the same or not?

Discuss in general situation of case 1:
Compute $\int \frac{1}{f(z)}dz$ over a, b-cycles for vertical cuts where $f(z) = \sqrt{\prod_{k=1}^{m}(z - z_k)} = \prod_{k=1}^{m}\sqrt{z - z_k}$, $z_k = r_k i, r_k \in \mathbb{R}, \forall k = 1 \sim m$ and $r_1 < r_2 < \ldots < r_m$.

1. a-cycles:

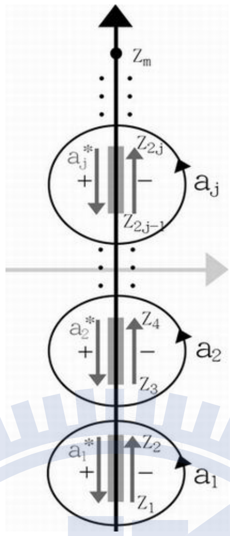

Figure 2.26: a-cycles and their equivalent paths a^*

a_j is a cycle, center at x with radius r, enclosed $[z_{2j}, z_{2j-1}]$ and doesn't intersect with other cuts. $\displaystyle\int_{a_j} \frac{1}{f(z)}dz = \int_{a_j^*} \frac{1}{f(z)}dz$ in sheet-I. The equivalent path a_j^* is the path on a vertical cut from z_{2j} to z_{2j-1} on $(+)$edge and then from z_{2j-1} to z_{2j} on $(-)$edge.

Using Lemma 2.2 to compute:

(i) $z_{2j} \overset{+}{\to} z_{2j-1}$:

$$arg(z - z_k) = -\frac{\pi}{2} \qquad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \qquad k = 2j, 2j+1, \ldots, m$$
$$arg(z - z_k) = -\frac{3\pi}{2} \qquad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \qquad k = 1, 2, \ldots, 2j-1$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2j-1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2j} \overset{+}{\rightarrow} z_{2j-1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz. \tag{2.9}$$

(ii) $z_{2j} \overset{-}{\leftarrow} z_{2j-1}$:

$$arg(z - z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2j, 2j+1, \ldots, m$$
$$arg(z - z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2j-1$$

So

$$f(z) \overset{Math.}{=} f(z),$$

$$\int_{z_{2j} \overset{-}{\leftarrow} z_{2j-1}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)} dz. \tag{2.10}$$

Conclusion of a-cycles of case 1: By (2.9) and (2.10),

$$\int_{a_j^*} \frac{1}{f(z)} dz \overset{Math.}{=} 2 \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)} dz.$$

2. b-cycles:

b_j is a cycle, center at x with radius r, enclosed $[z_{2N-1}, z_{2j}]$ and intersect the points on $[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, z_{2N}]$ in the case of $2N$ points, or b_j is a cycle, center at x with radius r, enclosed $[z_{2N-1}, z_{2j}]$ and intersect the points on

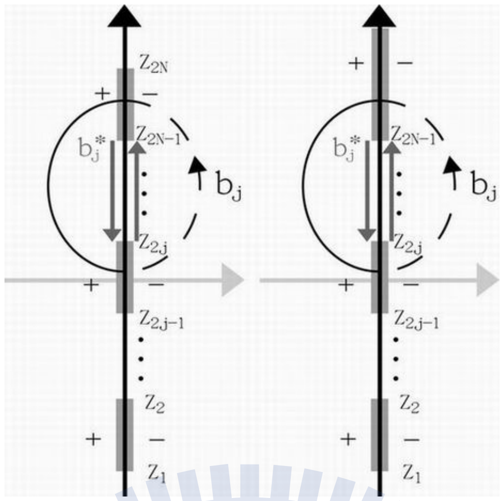

Figure 2.27: b_j and b_j^* of $2N-1$ and $2N$ points

$[z_{2j}, z_{2j-1}]$ and $[z_{2N-1}, \infty)$ in the case of $2N-1$ points.

By Cauchy integral formula, we know that

$$\int_{b_j} \frac{1}{f(z)} dz = \int_{b_j^*} \frac{1}{f(z)} dz$$

where b_j^* is a path from z_m to z_{2j} in sheet-I and then from z_{2j} to z_m in sheet-II. Similarly, using Lemma 2.2 to compute:

(1) The path on cut, i.e., the path from z_{2s+2} to z_{2s+1} on (+)edge of sheet-I and the path from z_{2s+1} to z_{2s+2} on (−)edge of sheet-II, $s = j, j+1, \ldots, N-2$.

(i) $z_{2s+2} \xrightarrow{+} z_{2s+1}$:

$$arg(z - z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \qquad k = 2s+2, 2s+3, \ldots, m$$
$$arg(z - z_k) = -\frac{3\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2s+1$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s+1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2s+2} \overset{+}{\rightarrow} z_{2s+1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2s+2}}^{z_{2s+1}} \frac{1}{f(z)} dz. \tag{2.11}$$

(ii) $z_{2s+2} \overset{-}{\dashleftarrow} z_{2s+1}$ on $(-)$edge of sheet-II is same as on $(+)$edge of sheet-I, so consider $z_{2s+2} \overset{+}{\leftarrow} z_{2s+1}$:

$$\begin{aligned} arg(z - z_k) = -\frac{\pi}{2} &\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2s+2, 2s+3, \ldots, m \\ arg(z - z_k) = -\frac{3\pi}{2} &\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2s+1 \end{aligned}$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s+1} f(z) \\ &= -f(z), \end{aligned}$$

$$\int_{z_{2s+2} \overset{-}{\dashleftarrow} z_{2s+1}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2s+1}}^{z_{2s+2}} \frac{1}{f(z)} dz. \tag{2.12}$$

(2) Without cuts, i.e., the path from z_{2s+1} to z_{2s} in sheet-I and the path from z_{2s} to z_{2s+1} in sheet-II, $s = j, j+1, \ldots, N-2$.

(i) $z_{2s+1} \to z_{2s}$:

$$arg(z-z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k = 2s+1, 2s+2, \ldots, m$$

$$arg(z-z_k) = -\frac{3\pi}{2} \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k = 1, 2, \ldots, 2s$$

So

$$\begin{aligned} f(z) &\overset{Math.}{=} (-1)^{2s} f(z) \\ &= f(z), \end{aligned}$$

$$\int_{z_{2s+1} \to z_{2s}} \frac{1}{f(z)} dz \overset{Math.}{=} \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz. \tag{2.13}$$

(ii) $z_{2s+1} \dashleftarrow z_{2s}$, by $f(z)|_{(II)} = -f(z)|_{(I)}$, we consider $z_{2s+1} \leftarrow z_{2s}$ first:

$$arg(z-z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k = 2s+1, 2s+2, \ldots, m$$

$$arg(z-z_k) = -\frac{3\pi}{2} \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k = 1, 2, \ldots, 2s$$

Hence

$$\begin{aligned} f(z)|_{z_{2s+1} \dashleftarrow z_{2s}} &= -f(z)|_{z_{2s+1} \leftarrow z_{2s}} \\ &\overset{Math.}{=} -(-1)^{2s} f(z)|_{z_{2s+1} \leftarrow z_{2s}} \\ &= -f(z)|_{z_{2s+1} \leftarrow z_{2s}}, \end{aligned}$$

$$\int_{z_{2s+1} \dashleftarrow z_{2s}} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_{z_{2s}}^{z_{2s+1}} \frac{1}{f(z)} dz. \tag{2.14}$$

Conclusion of b-cycles of case 1: By (2.11), (2.12), (2.13) and (2.14),

$$\int_{b_j^*} \frac{1}{f(z)} dz \overset{Math.}{=} \sum_{s=j}^{N-1} (2 \int_{z_{2s+1}}^{z_{2s}} \frac{1}{f(z)} dz).$$

When we want to modify the computation of $f(z)$ which has m roots, we needs to consider $\sqrt{z - z_k}, k = 1, 2, \ldots, m$. There are m steps of modifying the computation, and if m is large, it will become troublesome. Here provides a way to reduce the step. We can divided domain R into many areas to discuss the way to modify on vertical cuts.

If $f(z) = \sqrt{\prod_{k=1}^{m} (z - z_k)} = \prod_{k=1}^{m} \sqrt{z - z_k}$ for vertical cut in general situation:

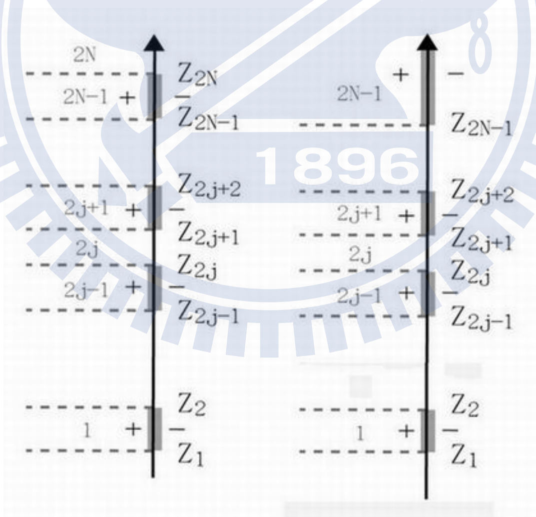

Figure 2.28: The areas with $2N-1$ and $2N$ points in vertical cuts

In each case we can determine that where $f(z) \overset{Math.}{=} f(z)$ or $f(z) \overset{Math.}{=} -f(z)$.

Case 1: $z_k = a_k i, a_k \in \mathbb{R}, k = 1, 2, \ldots, 2N$

Case 2: $z_k = a_k i, a_k \in \mathbb{R}, k = 1, 2, \ldots, 2N-1$

1. $z \in (+)$edge of the cut $[z_{2j-1}, z_{2j})$:

$$arg(z - z_k) = -\frac{3\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2j-1$$
$$arg(z - z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2j, 2j+1, \ldots, 2N-1 \text{ or } 2N$$

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z) = -f(z)$$

2. $z \in (-)$edge of the cut $[z_{2j-1}, z_{2j})$:

$$arg(z - z_k) = \frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2j-1$$
$$arg(z - z_k) = -\frac{\pi}{2} \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2j, 2j+1, \ldots, 2N-1 \text{ or } 2N$$

$$f(z) \overset{Math.}{=} f(z)$$

3. $z \in \{(x, y) : x < 0, a_{2j-1} \leq y < a_{2j}\} =$ region-$(2j-1)$:

$$arg(z - z_k) \in (-\frac{3\pi}{2}, -\pi] \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2j-1$$
$$arg(z - z_k) \in (-\pi, -\frac{\pi}{2}) \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2j, 2j+1, \ldots, 2N-1 \text{ or } 2N$$

$$f(z) \overset{Math.}{=} (-1)^{2j-1} f(z) = -f(z)$$

4. $z \in \{(x, y) : x \leq 0, a_{2j} \leq y < a_{2j+1}\} =$ region-$(2j)$:

$$arg(z - z_k) \in [-\frac{3\pi}{2}, -\pi] \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2j$$
$$arg(z - z_k) \in (-\pi, -\frac{\pi}{2}] \quad \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2j+1, 2j+2, \ldots, 2N-1 \text{ or } 2N$$

$$f(z) \overset{Math.}{=} f(z)$$

5. $z \in \{(x,y) : x \leq 0, y < a_1\} \cup \{(x,y) : 0 < x\}$:

$$arg(z - z_k) \in (-\pi, \frac{\pi}{2}) \ \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \ \forall k$$

$$f(z) \overset{Math.}{=} f(z)$$

Conclusion:

$$f(z) \overset{Math.}{=} \begin{cases} -f(z) & \text{if } z \in \text{region-}(2j-1) \cup (+)\text{edge of the cut } [z_{2j-1}, z_{2j}) \\ f(z) & \text{otherwise} \end{cases}$$

After studying the above skill, now let us discuss the other general case of vertical cuts whose figure showed below by two ways.

Consider $f(z) = \sqrt{\prod_{k=1}^{m}(z - z_k)} = \prod_{k=1}^{m}\sqrt{z - z_k}$, where $n = 2N, z_{2k-1} = \overline{z_{2k}}, k = 1, 2, \ldots, N$ and $Re(z_1) > Re(z_3) > \ldots > Re(z_{2N-1})$, also $Im(z_1) = Im(z_3) = \ldots = Im(z_{2N-1})$ and $Im(z_2) = Im(z_4) = \ldots = Im(z_{2N})$.

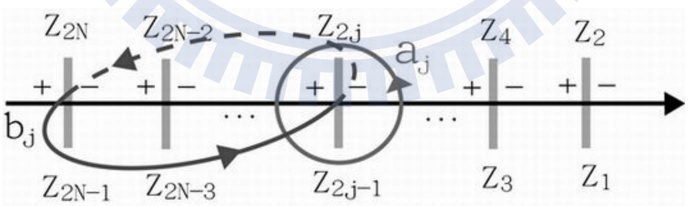

Figure 2.29: a, b-cycles in other kind of vertical cuts

1. Compute $\displaystyle\int_{a_j^*} \frac{1}{f(z)}dz$ where a_j^* is an equivalent path for a_j:

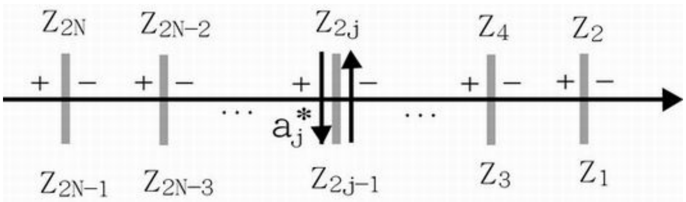

Figure 2.30: a_j^*-cycles in other kind of vertical cuts

(1) Compute by using argument of complex number to modify:

(i) The path from z_{2j} to z_{2j-1} on $(+)$edge of sheet-I:

$$arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi] \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k = 1, 3, \ldots, 2j-1$$
$$arg(z-z_k) \in (-\pi, \frac{\pi}{2}) \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k = \text{otherwise}$$

$$f(z) \overset{Math.}{=} (-1)^j f(z)$$

(ii) The path from z_{2j-1} to z_{2j} on $(-)$edge of sheet-I:

$$arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi] \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} -\sqrt{z-z_k}, \quad k = 1, 3, \ldots, 2j-3$$
$$arg(z-z_k) \in (-\pi, \frac{\pi}{2}) \quad \Rightarrow \sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, \quad k = \text{otherwise}$$

$$f(z) \overset{Math.}{=} (-1)^{j-1} f(z)$$

By (i), (ii), and Cauchy integral formula,

$$\begin{aligned}\int_{a_j} \frac{1}{f(z)}dz &= \int_{a_j^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} (-1)^j \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)}dz + (-1)^{j-1} \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)}dz \\ &= (-1)^j 2 \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)}dz.\end{aligned}$$

(2) Using the result about modify of blocks and then we can compute $\int_{a_j^*} \frac{1}{f(z)}dz$:

(i) The path from z_{2j} to z_{2j-1} on $(+)$edge of sheet-I:

$$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \overset{Math.}{=} -\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}, \qquad k = 1, 2, \ldots, j$$

$$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \overset{Math.}{=} \sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}, \qquad k = j+1, j+2, \ldots, N$$

$$f(z) \overset{Math.}{=} (-1)^j \prod_{k=1}^{m} \sqrt{z - z_k} = (-1)^j f(z)$$

(ii) The path from z_{2j-1} to z_{2j} on $(-)$edge of sheet-I:

$$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \overset{Math.}{=} -\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}, \qquad k = 1, 2, \ldots, j-1$$

$$\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} \overset{Math.}{=} \sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}}, \qquad k = j, j+1, \ldots, N$$

$$f(z) \overset{Math.}{=} (-1)^{j-1} \prod_{k=1}^{m} \sqrt{z - z_k} = (-1)^{j-1} f(z)$$

By (i),(ii), we have

$$\begin{aligned}\int_{a_j} \frac{1}{f(z)} dz &= \int_{a_j^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} (-1)^j \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz + (-1)^{j-1} \int_{z_{2j-1}}^{z_{2j}} \frac{1}{f(z)} dz \\ &= (-1)^j 2 \int_{z_{2j}}^{z_{2j-1}} \frac{1}{f(z)} dz.\end{aligned}$$

2. Compute $\int_{b_j^*} \frac{1}{f(z)} dz$ where b_j^* is an equivalent path for b_j, and $b_j^* = \cup_{k=j+1}^{N} a_k^* \cup \{z_{2N} \to z_{2j}\} \cup \{z_{2N} \leftarrow\!-\,- z_{2j}\}$:

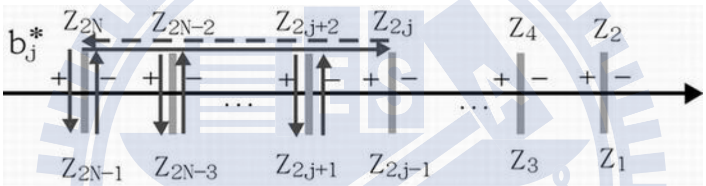

Figure 2.31: b_j^*-cycles in other kind of vertical cuts

(1) $z \in a_k^*$: done above.

(2) $\{z_{2N} \to z_{2j}\} \cup \{z_{2N} \leftarrow\!-\,- z_{2j}\}$:

(a) Compute by using argument of complex number to modify:

(i) $z_{2s+2} \to z_{2s}$ (namely b_{s1}^*), $s = j, j+1, \ldots, N-1$:

$$\begin{aligned}&arg(z - z_k) \in [-\frac{3\pi}{2}, -\pi] &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, \quad k = 1, 2, \ldots, 2s \\ &arg(z - z_k) \in (-\pi, \frac{\pi}{2}) &&\Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, \quad k = 2s, 2s+1, \ldots, 2N\end{aligned}$$

$$f(z) \overset{Math.}{=} (-1)^{2s} f(z) = f(z)$$

(ii) $z_{2s+2} \leftarrow z_{2s}$ (namely b_{s2}^*), $s = j, j+1, \ldots, N-1$. Since $f(z)|_{(II)} = -f(z)|_{(I)}$, so we consider $z_{2s+2} \leftarrow z_{2s}$ first:

$$\begin{aligned} &arg(z - z_k) \in [-\frac{3\pi}{2}, -\pi] && \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, && k = 1, 2, \ldots, 2s \\ &arg(z - z_k) \in (-\pi, \frac{\pi}{2}) && \Rightarrow \sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, && k = 2s, 2s+1, \ldots, 2N \end{aligned}$$

$$f(z) \overset{Math.}{=} -(-1)^{2s} f(z) = -f(z)$$

By (i),(ii), and letting $b_{s1}^* \cup b_{s2}^* = b_s^*$, we have

$$\begin{aligned} \int_{b_s^*} \frac{1}{f(z)} dz &= \int_{b_{s1}^*} \frac{1}{f(z)} dz + \int_{b_{s2}^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{z_{2s+2}}^{z_{2s}} \frac{1}{f(z)} dz + (-1) \int_{z_{2s}}^{z_{2s+2}} \frac{1}{f(z)} dz \\ &= 2 \int_{z_{2s+2}}^{z_{2s}} \frac{1}{f(z)} dz. \end{aligned}$$

(b) Using the result about modify of blocks to compute in Mathematica:

(i) $z_{2s+2} \rightarrow z_{2s}$ (b_{s1}^*), $s = j, j+1, \ldots, N-1$:

$$\begin{aligned} \sqrt{z - z_{2k-1}} \sqrt{z - z_{2k}} &\overset{Math.}{=} \sqrt{z - z_{2k-1}} \sqrt{z - z_{2k}}, \ \forall k \\ f(z) &\overset{Math.}{=} f(z) \end{aligned}$$

(ii) $z_{2s+2} \leftarrow\text{-}\text{-} z_{2s}$ (b^*_{s2}), $s = j, j+1, \ldots, N-1$. Since $f(z)|_{(II)} = -f(z)|_{(I)}$, so we consider $z_{2s+2} \leftarrow z_{2s}$ first:

$$\begin{aligned}\sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}} &\overset{Math.}{=} \sqrt{z - z_{2k-1}}\sqrt{z - z_{2k}},\ \forall k \\ f(z) &\overset{Math.}{=} -f(z)\end{aligned}$$

By the above compute and Cauchy integral formula,

$$\begin{aligned}\int_{b_j} \frac{1}{f(z)}dz &= \int_{b^*_j} \frac{1}{f(z)}dz \\ &= \sum_{k=j+1}^{N} \int_{a^*_k} \frac{1}{f(z)}dz + \sum_{s=j}^{N-1} \int_{b^*_s} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \sum_{k=j+1}^{N} \left((-1)^j 2 \int_{z_{2k}}^{z_{2k-1}} \frac{1}{f(z)}dz \right) + \sum_{s=j}^{N-1} \left(2 \int_{z_{2s+2}}^{z_{2s}} \frac{1}{f(z)}dz \right).\end{aligned}$$

No matter what methods we use, way of areas or the arguments of complex number, the modifying is the same. This means that we could choose the way letting the computation easier for different situation.

Together with the consequences, the next chapter shows how we apply the all conclusions above to a polynomial, surely, we can deal with the computation of known function which we are interesting about its construction on Riemann Surface.

Chapter 3

Apply Riemann Surface to Nonlinear Approximation of Sine

After the study of previous chapter, we apply the conclusion of Riemann Surface to the approximation of sine, and compute the integration on the Riemann Surface. Here we replace $\sin u$ by $P_N(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!}$ since $\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, the Taylor expansion shows.

So that, there is a new differential equation

$$\begin{aligned} u'' + P_N(u) &= 0, \\ u'u'' + u'P_N(u) &= 0. \end{aligned}$$

Integrating both sides, we obtain

$$\frac{1}{2}(u')^2 + P_{N+1}(u) = E$$

where E is the integration constant, $P_{N+1}(u) = \frac{u^2}{2!} - \frac{u^4}{4!} + \frac{u^6}{6!} - \frac{u^8}{8!} + \frac{u^{10}}{10!}$, then

$$u' = \frac{du}{dt} = \sqrt{2(E - P_{N+1}(u))}$$

where u is a function of time t,

$$\int \frac{1}{\sqrt{2(E - P_{N+1}(u))}} du = \int dt.$$

According to the fundamental theorem of Algebra, there is

$$\sqrt{2(E - P_{N+1}(u))} = \sqrt{c \prod_{k=1}^{N+1} (u - u_k)}, \qquad c \in \mathbb{C}.$$

Let $f(u, E) = \sqrt{2(E - P_{N+1}(u))}$, here we take E be 1.5, so that we can use Mathematica and obtain

$$f(u, 1.5) = \sqrt{\prod_{k=1}^{10} (u - u_k)} = \prod_{k=1}^{10} \sqrt{u - u_k}$$

where

$$u_1 = -5.7957 - 3.81789i \qquad u_2 = -5.7957 + 3.81789i$$
$$u_3 = -5.42043 \qquad u_4 = -4.26672$$
$$u_5 = -2.09438 \qquad u_6 = 2.09438$$
$$u_7 = 4.26672 \qquad u_8 = 5.42043$$
$$u_9 = 5.7957 - 3.81789i \qquad u_{10} = 5.7957 + 3.81789i.$$

For convenience, we will replace $f(u, 1.5)$ by $f(u)$ in the following pages.

Hence, we have ten branch points and obtain five branch cuts as the figure below.

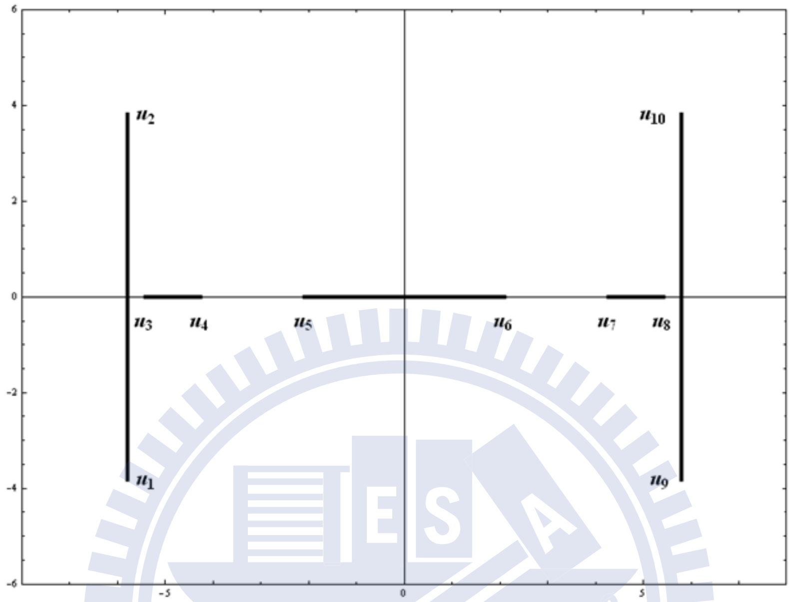

Figure 3.1: Branch points and branch cuts

As we learned in Chapter 2, all integration of closed contour on Riemann Surface is homotopic to the linear combination of a, b-cycles since the Cauchy integral formula. Hence, it is going to show the a, b-cycles in the integral

$$\int \frac{1}{f(u)} du = \int \frac{1}{\prod_{k=1}^{10} \sqrt{u - u_k}} du.$$

Since there exist five branch cuts, there are four a-cycles and four b-cycles we need to discuss, and the following computation will be completed by parametric form. The all closed contour cycles are counterclockwise.

The a-cycles is showed below.

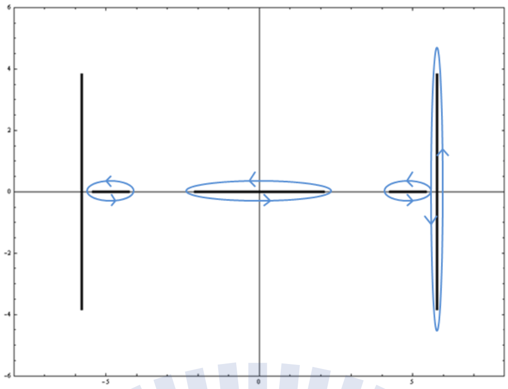

Figure 3.2: a-cycles

The equivalent a_1-cycle is showed below.

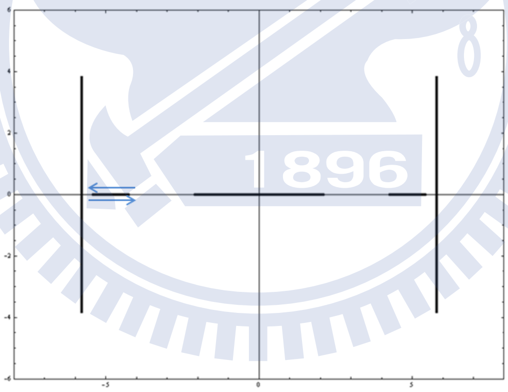

Figure 3.3: Equivalent a_1-cycle

$$
\begin{aligned}
\int_{a_1} \frac{1}{f(u)} du &\overset{Math.}{=} 2\int_{u_3}^{u_4} \frac{1}{f(r)} dr \\
&= -1.03837 \times 10^{-20} - 0.00339657i
\end{aligned}
$$

The equivalent a_2-cycle is showed below.

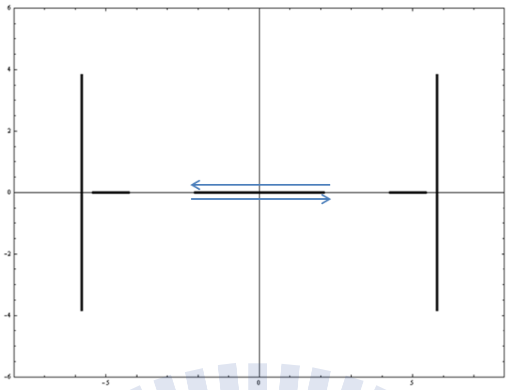

Figure 3.4: Equivalent a_2-cycle

$$\begin{aligned}\int_{a_2} \frac{1}{f(u)} du &\overset{Math.}{=} 2\int_{u_5}^{u_6} \frac{1}{f(r)} dr \\ &= -2.29246 \times 10^{-20} + 0.00640376i\end{aligned}$$

The equivalent a_3-cycle is showed below.

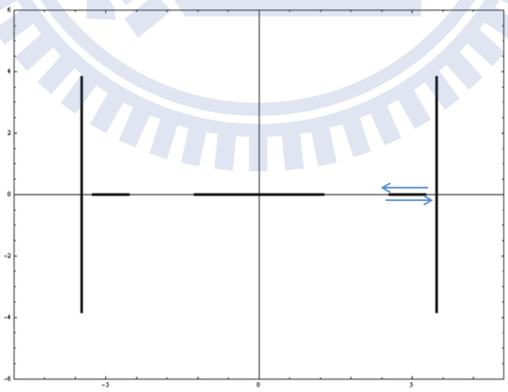

Figure 3.5: Equivalent a_3-cycle

$$\int_{a_3} \frac{1}{f(u)} du \overset{Math.}{=} 2\int_{u_7}^{u_8} \frac{1}{f(r)} dr$$
$$= 2.30733 \times 10^{-21} - 0.00339657i$$

The equivalent a_4-cycle is showed below.

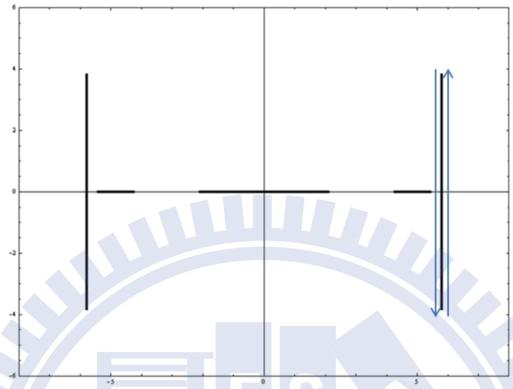

Figure 3.6: Equivalent a_4-cycle

$$\int_{a_4} \frac{1}{f(u)} du \overset{Math.}{=} 2\int_{Im(u_9)}^{Im(u_{10})} \frac{1}{f(Re(u_9) + ri)} idr$$
$$= -3.52366 \times 10^{-19} + 0.000194696i$$

Hence, we have all numerical results of a-cycles, and the next step is computing the b-cycles.

The b-cycles is showed below.

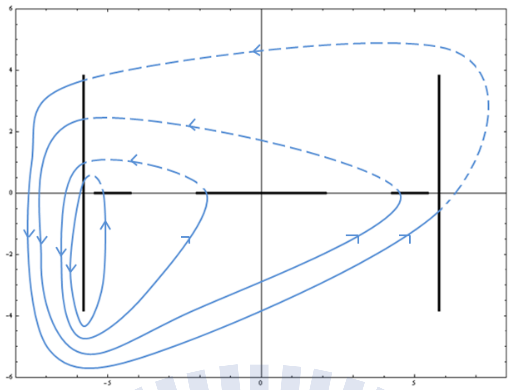

Figure 3.7: b-cycles

The equivalent b_1-cycle is showed below.

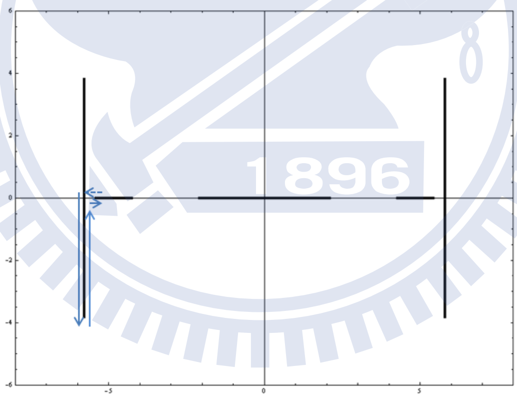

Figure 3.8: Equivalent b_1-cycle

$$\begin{aligned}\int_{b_1} \frac{1}{f(u)} du \quad &\overset{Math.}{=} \quad 2\int_0^{Im(u_1)} \frac{1}{f(Re(u_1)+ri)} idr - 2\int_{Re(u_1)}^{u_3} \frac{1}{f(r)} dr \\ &= \quad 0.00226652 + 0.0000973482i\end{aligned}$$

The equivalent b_2-cycle is showed below.

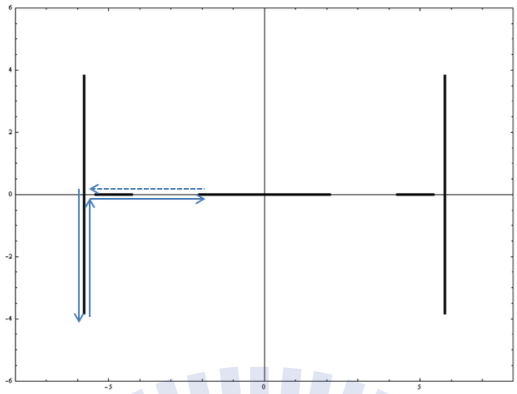

Figure 3.9: Equivalent b_2-cycle

$$\begin{aligned}\int_{b_2}\frac{1}{f(u)}du &\overset{Math.}{=} \int_{b_1}\frac{1}{f(u)}du - 2\int_{u_4}^{u_5}\frac{1}{f(r)}dr \\ &= -0.00300153 + 0.0000973482i\end{aligned}$$

The equivalent b_3-cycle is showed below.

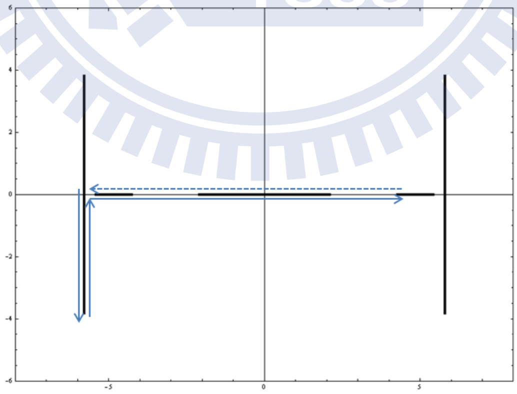

Figure 3.10: Equivalent b_3-cycle

$$\int_{b_3} \frac{1}{f(u)} du \overset{Math.}{=} \int_{b_2} \frac{1}{f(u)} du - 2 \int_{u_6}^{u_7} \frac{1}{f(r)} dr$$
$$= 0.00226652 + 0.0000973482i$$

The equivalent b_4-cycle is showed below.

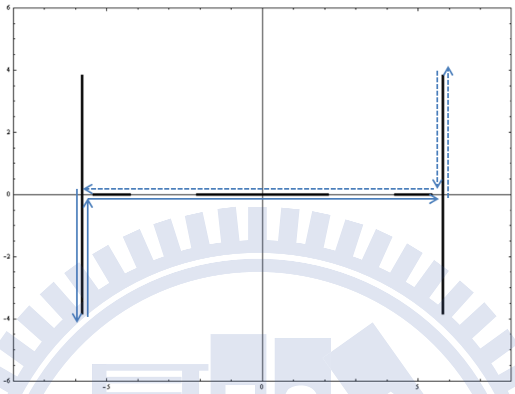

Figure 3.11: Equivalent b_4-cycle

$$\int_{b_4} \frac{1}{f(u)} du \overset{Math.}{=} \int_{b_3} \frac{1}{f(u)} du - 2 \int_{u_8}^{Re(u_{10})} \frac{1}{f(r)} dr + 2 \int_{Im(u_{10})}^{0} \frac{1}{f(Re(u_{10}) + ri)} i dr$$
$$= 8.67362 \times 10^{-19} - 4.06576 \times 10^{-20} i$$

So far, we have all numerical results of a, b-cycles, and, hence, we can obtain any integration of closed contour in this specific case.

Chapter 4

Elliptic Functions

In our original question, we want to solve the different equation

$$u'' + \sin u = 0,$$

and since the difficulty in integration, the ideal of solving the equation changes to the theory of elliptic functions. [6] There are several typical cases of elliptic functions, and the following pages will introduce the definitions, properties, and relations among these cases.

4.1 General definitions and properties of elliptic functions

4.1.1 Introduction

The first mathematician who studied the theory of elliptic integrals systematically is Legendre, and the ideal of inverting an elliptic integral to obtain an elliptic function is due to Abel, Jacobi and Gauss. The elliptic function is originated from the problem of finding the circumference of the ellipse, and in the view of differential equations, the elliptic function can solve kinds of problem with complex integrations.

4.1.2 Doubly-periodic functions and elliptic functions

A function f is called periodic with period 2ω if

$$f(z+2\omega)=f(z).$$

A function f is called a doubly-periodic function with $2\omega_1$ and $2\omega_2$ if

$$f(z+2\omega_1)=f(z+2\omega_2)=f(z)$$

where $\dfrac{2\omega_2}{2\omega_1}$ is not purely real.

Moreover, a doubly-periodic function f is called an elliptic function if it is analytic except poles and has no singularities other than poles in the finite part of the plane.

4.1.3 Period-parallelograms

Suppose that in the plane of the variable z we mark the points $0, 2\omega_1, 2\omega_2$ and $2\omega_1+2\omega_2$, generally, all the points whose complex coordinates are of the form $2m\omega_1+2n\omega_2$, where m and n are integers. Consider the points of set $0, 2\omega_1, 2\omega_2$ and $2\omega_1+2\omega_2$, and we obtain a parallelogram as the vertices. If there is no point

ω inside or on the boundary of this parallelogram such that

$$f(z+\omega) = f(z)$$

for all values of z, this parallelogram is called a fundamental period-parallelogram for an elliptic function with periods $2\omega_1, 2\omega_2$.

Such a translated parallelogram, without zeros or poles on its boundary, is called a cell.

4.1.4 Simple properties of elliptic functions

1. The number of poles of an elliptic function in any cell is finite.
2. The number of zeros of an elliptic function in any cell is finite.
3. The sum of the residues of an elliptic function at its poles in any cell is zero.
4. Liouville's theorem:
An elliptic function with no poles in a cell is merely a constant.

4.2 Weierstrass elliptic function

4.2.1 Definition

The Weierstrass elliptic function $\wp(z)$ is one of the famous elliptic function, which is defined by the equation

$$\wp(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left\{ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\} \tag{4.1}$$

where $\sum{}'$ denotes that the sum excludes the term when $m = n = 0$ and ω_1, ω_2 satisfy the condition that the ratio is not purely real. For brevity, we write $\Omega_{m,n}$ in place of $2m\omega_1 + 2n\omega_2$. So that the equation (4.1) will be

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ (z - \Omega_{m,n})^{-2} - \Omega_{m,n}^{-2} \right\}.$$

4.2.2 Properties of $\wp(z)$

1. $\wp(z)$ is an even function with single double pole at $\Omega_{m,n}$ for integers m, n.

2. $\wp(z)$ satisfies the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$$

where g_2 and g_3 (called the invariants) are given by the equations

$$g_2 = 60 \sum_{m,n}{}' \Omega_{m,n}^{-4}, \quad g_3 = 140 \sum_{m,n}{}' \Omega_{m,n}^{-6}.$$

3. (Properties of homogeneity)

$$\wp(\lambda z; \lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z; \omega_1, \omega_2), \quad \lambda \neq 0$$

$$\wp(\lambda z; \lambda^{-4} g_2, \lambda^{-6} g_3) = \lambda^{-2}\wp(z; g_2, g_3), \quad \lambda \neq 0$$

where $\wp(z; \omega_1, \omega_2)$ denote the function formed with periods $2\omega_1, 2\omega_2$ and $\wp(z; g_2, g_3)$ denote the function formed with invariants g_2, g_3.

4. (Addition-theorem)

a. If $u + v + w = 0$, then

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0.$$

b.

$$\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\right\}^2 - \wp(z) - \wp(y).$$

c.

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)$$

unless $2z$ is a period. The result is called the duplication formula.

4.2.3 The constants e_1, e_2, e_3

Let $\wp(z)$ be the Weierstrass elliptic function with periods $2\omega_1, 2\omega_2$. The value $\wp(\omega_1), \wp(\omega_2), \wp(\omega_3)$ (where $\omega_3 = -\omega_1 - \omega_2$) are all unequal; and, if their value be e_1, e_2, e_3, respectively, then the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$ and $e_1 \neq e_2 \neq e_3$. We have

$$e_1 + e_2 + e_3 = 0,$$

$$e_2e_3 + e_3e_1 + e_1e_2 = -\frac{1}{4}g_2,$$

$$e_1e_2e_3 = \frac{1}{4}g_3.$$

4.2.4 The Weierstrass-zeta function

First of all, the Weierstrass-zeta function should not be confused with the Zeta-function of Riemann discussed in Chapter XIII in [6].

The Weierstrass-zeta function $\zeta(z)$ is defined by the equation

$$\frac{d\zeta(z)}{dz} = -\wp(z),$$

coupled with the condition $\lim_{z\to 0}\left\{\zeta(z) - \frac{1}{z}\right\} = 0.$

Since the series for $\wp(z) - \frac{1}{z^2}$ converges uniformly throughout any domain from which the neighbourhoods of the points $\Omega'_{m,n}$ are excluded, we can integrate term-by-term and get

$$\begin{aligned}\zeta(z) - \frac{1}{z} &= -\int_0^z \left\{\wp(z) - \frac{1}{z^2}\right\} dz \\ &= -\sum_{m,n}{}' \int_0^z \left\{(z-\Omega_{m,n})^{-2} - \Omega_{m,n}^{-2}\right\} dz,\end{aligned}$$

and so

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{\frac{1}{z-\Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2}\right\}.$$

4.2.5 Properties of $\zeta(z)$

1. $\zeta(z)$ is an odd function. It is not a doubly-periodic function, and the residue of $\zeta(z)$ at every pole is 1.

2. If we integrate the equations

$$\wp(z+2\omega_1) = \wp(z) \quad \text{and} \quad \wp(z+2\omega_2) = \wp(z),$$

we get

$$\zeta(z+2\omega_1) = \zeta(z) + 2\eta_1$$

$$\zeta(z+2\omega_2) = \zeta(z) + 2\eta_2$$

where $2\eta_1$ and $2\eta_2$ are the constants introduced by integration; putting $z = -\omega_1, z = -\omega_2$, respectively, and taking account of the fact that $\zeta(z)$ is an odd function, we have

$$\eta_1 = \zeta(\omega_1),$$

$$\eta_2 = \zeta(\omega_2).$$

3. (Properties of homogeneity)

$$\zeta(\lambda z; \lambda\omega_1, \lambda\omega_2) = \lambda^{-1}\zeta(z; \omega_1, \omega_2), \quad \lambda \neq 0$$

4. (Legendre's relation)

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{1}{2}\pi i$$

4.2.6 The Weierstrass-sigma function

The Weierstrass-sigma function $\sigma(z)$ is defined by the equation

$$\frac{d}{dz}\log\sigma(z) = \zeta(z)$$

coupled with the condition $\lim_{z\to 0}\left\{\frac{\sigma(z)}{z}\right\} = 1.$

On account of the uniformity of convergence of the series for $\zeta(z)$, except near the poles of $\zeta(z)$, we may integrate the series term-by-term. Doing so, and taking the exponential of each side of the resulting equation, we get

$$\sigma(z) = z \prod_{m,n}{}' \left\{ (1 - \frac{z}{\Omega_{m,n}}) \exp(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2}) \right\}.$$

4.2.7 Properties of $\sigma(z)$

1. The product for $\sigma(z)$ converges absolutely and uniformly in any bounded domain of values of z.

2. The function $\sigma(z)$ is an odd integral function of z with simple zeros at all the points $\Omega_{m,n}$.

3. If we integrate the equations

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1 \quad \text{and} \quad \zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2,$$

we get

$$\sigma(z + 2\omega_1) = c_1 e^{2\eta_1 z} \sigma(z)$$

$$\sigma(z + 2\omega_2) = c_1 e^{2\eta_2 z} \sigma(z)$$

where c_1 and c_2 are the constants of integration; to determine c_1, c_2, we put $z = -\omega_1, z = -\omega_2$, respectively, and then

$$\sigma(\omega_1) = -c_1 e^{-2\eta_1 \omega_1} \sigma(w_1),$$

$$\sigma(\omega_2) = -c_2 e^{-2\eta_2 \omega_2} \sigma(w_2).$$

Consequently,

$$c_1 = -e^{2\eta_1 \omega_1},$$

$$c_2 = -e^{2\eta_2 \omega_2}.$$

4. (Properties of homogeneity)

$$\sigma(\lambda z; \lambda\omega_1, \lambda\omega_2) = \lambda\sigma(z; \omega_1, \omega_2)$$

4.3 The Theta-functions

4.3.1 Definition

Let τ be a (constant) complex number whose imaginary part is positive; and write $q = e^{\pi i \tau}$, so that $|q| < 1$.

Consider the function $\vartheta(z, q)$, defined by the series

$$\vartheta(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}.$$

It is evident that

$$\vartheta(z, q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz,$$

and that

$$\vartheta(z + \pi, q) = \vartheta(z, q);$$

further

$$\begin{aligned}\vartheta(z + \pi\tau, q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{2n} e^{2niz} \\ &= -q^{-1} e^{-2iz} \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz},\end{aligned}$$

and so

$$\vartheta(z+\pi\tau, q) = -q^{-1}e^{-2iz}\vartheta(z,q).$$

In consequence of these results, $\vartheta(z,q)$ is called a quasi doubly-periodic function of z, and accordingly 1 and $-q^{-1}e^{-2iz}$ are called the periodicity factors associated with the periods π and $\pi\tau$ respectively.

4.3.2 The four types of Theta-functions

It is customary to write $\vartheta_4(z,q)$ in place of $\vartheta(z,q)$; the other three types of Theta-functions are then defined as follows:

The function $\vartheta_3(z,q)$ is defined by the equation

$$\vartheta_3(z,q) = \vartheta_4(z+\frac{1}{2}\pi, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz.$$

Next, $\vartheta_1(z,q)$ is defined by the equation

$$\begin{aligned}\vartheta_1(z,q) &= -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{1}{2}\pi\tau, q)\\ &= -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)iz}\\ &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin{(2n+1)z}.\end{aligned}$$

Lastly, $\vartheta_2(z,q)$ is defined by the equation

$$\vartheta_2(z,q) = \vartheta_1(z+\frac{1}{2}\pi,q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos{(2n+1)z}.$$

Summary:

$$\vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin{(2n+1)z}$$
$$\vartheta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos{(2n+1)z}$$
$$\vartheta_3(z,q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz$$
$$\vartheta_4(z,q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz$$

For brevity, the parameter q will usually not be specified, so that $\vartheta_i(z)$ will be written for $\vartheta_i(z,q), i=1,2,3,4$.

4.3.3 Properties of $\vartheta_i(z)$

1. $\vartheta_1(z)$ is an odd function and the other Theta-functions are even functions.

2. The zeros of the Theta-functions:

$$
\begin{aligned}
\vartheta_1(z) = 0, &\quad \text{where} \quad z = 0 + m\pi + n\pi\tau \\
\vartheta_2(z) = 0, &\quad \text{where} \quad z = \frac{\pi}{2} + m\pi + n\pi\tau \\
\vartheta_3(z) = 0, &\quad \text{where} \quad z = \frac{\pi}{2} + \frac{\pi\tau}{2} + m\pi + n\pi\tau \\
\vartheta_4(z) = 0, &\quad \text{where} \quad z = \frac{\pi\tau}{2} + m\pi + n\pi\tau
\end{aligned}
$$

3. The identity $\vartheta_2^4(0) + \vartheta_4^4 = \vartheta_3^4(0)$.

4. Jacobi's expressions for the Theta-functions as infinite products:

$$
\begin{aligned}
\vartheta_1(z) &= 2q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2z + q^{4n}) \\
\vartheta_2(z) &= 2q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2z + q^{4n}) \\
\vartheta_3(z) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2z + q^{4n-2}) \\
\vartheta_4(z) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos 2z + q^{4n-2})
\end{aligned}
$$

5. The differential equation satisfied by the Theta-functions

$$
\frac{\partial^2 \vartheta_3(z|\tau)}{\partial z^2} = -\frac{4}{\pi i} \frac{\partial \vartheta_3(z|\tau)}{\partial \tau}
$$

where we may regard $\vartheta_3(z|\tau)$ as a function of two independent variables z and τ.

6. A relation between Theta-functions of zero argument

$$
\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0).
$$

7. Weierstrass-sigma function can express in terms of Theta-functions; in other words, there will exist expressions for any elliptic functions in terms of Theta-functions.

8. The differential equations satisfied by quotients of Theta-functions:

$$\frac{d}{dz}\left\{\frac{\vartheta_1(z)}{\vartheta_4(z)}\right\} = \vartheta_4^2\frac{\vartheta_2(z)}{\vartheta_4(z)}\frac{\vartheta_3(z)}{\vartheta_4(z)} \tag{4.2}$$

$$\frac{d}{dz}\left\{\frac{\vartheta_2(z)}{\vartheta_4(z)}\right\} = -\vartheta_3^2\frac{\vartheta_1(z)}{\vartheta_4(z)}\frac{\vartheta_3(z)}{\vartheta_4(z)} \tag{4.3}$$

$$\frac{d}{dz}\left\{\frac{\vartheta_3(z)}{\vartheta_4(z)}\right\} = -\vartheta_2^2\frac{\vartheta_1(z)}{\vartheta_4(z)}\frac{\vartheta_2(z)}{\vartheta_4(z)} \tag{4.4}$$

We write $\xi \equiv \frac{\vartheta_1(z)}{\vartheta_4(z)}$ and use the results established above, there is

$$\left(\frac{d\xi}{dz}\right)^2 = (\vartheta_2^2 - \xi^2\vartheta_3^2)(\vartheta_3^2 - \xi^2\vartheta_2^2).$$

Write $\frac{\xi\vartheta_3}{\vartheta_2} = y, z\vartheta_3^2 = u, \frac{\vartheta_2}{\vartheta_3} = k^{\frac{1}{2}}$, and we get the equation determining y in terms of u is

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2). \tag{4.5}$$

This differential equation has the particular solution

$$y = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u\vartheta_3^{-2})}{\vartheta_4(u\vartheta_3^{-2})}. \tag{4.6}$$

4.4 Jacobian elliptic functions

4.4.1 Definition

From (4.5) and (4.6), we have the integral representation of y is

$$u = \int_0^y \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt,$$

so we defined $y = \mathrm{su}(u,k)$ or simply $y = \mathrm{sn}(u)$, when it is unnecessary to emphasize the modulus k.

Jacobian functions defined as follow:

$$\mathrm{sn}(u,k) = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u/\vartheta_3^2)}{\vartheta_4(u/\vartheta_3^2)}$$

$$\mathrm{cn}(u,k) = \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(u/\vartheta_3^2)}{\vartheta_4(u/\vartheta_3^2)}$$

$$\mathrm{dn}(u,k) = \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(u/\vartheta_3^2)}{\vartheta_4(u/\vartheta_3^2)}$$

From (4.2), (4.3), and (4.4) with $k^2 + k^{'2} = 1$, we get the solutions for the following integral equations:

$$\text{If} \quad u = \int_0^y \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt, \qquad \text{then} \quad y = \mathrm{sn}(u,k). \tag{4.7}$$

$$\text{If} \quad u = \int_y^1 \frac{1}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}dt, \qquad \text{then} \quad y = \mathrm{cn}(u,k). \tag{4.8}$$

$$\text{If} \quad u = \int_y^1 \frac{1}{\sqrt{(1-t^2)(t^2-k'^2)}}dt, \qquad \text{then} \quad y = \mathrm{dn}(u,k). \tag{4.9}$$

Moreover, the integrals (4.7), (4.8), and (4.9) are called the elliptic integrals of the first kind.

4.4.2 Glaisher's notation for quotients

A short and convenient notation has been invented by Glaisher to express reciprocals and quotients of the Jacobian elliptic functions.

$$\begin{array}{lll}
\mathrm{ns}(u) = 1/\mathrm{sn}(u) & \mathrm{nc}(u) = 1/\mathrm{cn}(u) & \mathrm{nd}(u) = 1/\mathrm{dn}(u) \\
\mathrm{sc}(u) = \mathrm{sn}(u)/\mathrm{cn}(u) & \mathrm{sd}(u) = \mathrm{sn}(u)/\mathrm{dn}(u) & \mathrm{cd}(u) = \mathrm{cn}(u)/\mathrm{dn}(u) \\
\mathrm{cs}(u) = \mathrm{cn}(u)/\mathrm{sn}(u) & \mathrm{ds}(u) = \mathrm{dn}(u)/\mathrm{sn}(u) & \mathrm{dc}(u) = \mathrm{dn}(u)/\mathrm{cn}(u)
\end{array}$$

We obtain the following results:

$$\begin{aligned}
u &= \int_0^{\mathrm{sn}(u)} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt & &= \int_{\mathrm{ns}(u)}^{\infty} \frac{1}{\sqrt{(t^2-1)(t^2-k^2)}}dt \\
&= \int_{\mathrm{cn}(u)}^{1} \frac{1}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}dt & &= \int_{1}^{\mathrm{nc}(u)} \frac{1}{\sqrt{(t^2-1)(k'^2t^2+k^2)}}dt \\
&= \int_{\mathrm{dn}(u)}^{1} \frac{1}{\sqrt{(1-t^2)(t^2-k'^2)}}dt & &= \int_{1}^{\mathrm{nd}(u)} \frac{1}{\sqrt{(t^2-1)(1-k'^2t^2)}}dt \\
&= \int_0^{\mathrm{sc}(u)} \frac{1}{\sqrt{(1+t^2)(1+k'^2t^2)}}dt & &= \int_{\mathrm{cs}(u)}^{\infty} \frac{1}{\sqrt{(t^2+1)(t^2+k'^2)}}dt \\
&= \int_0^{\mathrm{sd}(u)} \frac{1}{\sqrt{(1-k'^2t^2)(1+k^2t^2)}}dt & &= \int_{\mathrm{ds}(u)}^{\infty} \frac{1}{\sqrt{(t^2-k'^2)(t^2+k^2)}}dt \\
&= \int_{\mathrm{cd}(u)}^{1} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt & &= \int_{\mathrm{dc}(u)}^{1} \frac{1}{\sqrt{(t^2-1)(t^2-k^2)}}dt
\end{aligned}$$

4.4.3 Some relations among Jacobian elliptic functions

1.

$$\frac{d}{du}\mathrm{sn}(u) = \mathrm{cn}(u)\,\mathrm{dn}(u) \tag{4.10}$$

2.

$$\begin{aligned} \operatorname{sn}^2(u) + \operatorname{cn}^2(u) &= 1 &(4.11)\\ k^2 \operatorname{sn}^2(u) + \operatorname{dn}^2(u) &= 1 &(4.12)\\ \operatorname{cn}^2(0) + \operatorname{dn}^2(0) &= 1 &(4.13) \end{aligned}$$

3. By (4.10) and (4.11), there is

$$\frac{d}{du}\operatorname{cn}(u) = -\operatorname{sn}(u)\operatorname{dn}(u).$$

4. By (4.10) and (4.12), there is

$$\frac{d}{du}\operatorname{dn}(u) = -k^2 \operatorname{sn}(u)\operatorname{cn}(u).$$

4.4.4 Some properties of Jacobian functions

1. $\operatorname{sn}(u)$ is an odd function of $u, \operatorname{cn}(u)$ and $\operatorname{dn}(u)$ are an even functions of u.

2. The addition-theorems for Jacobian functions:

$$\begin{aligned} \operatorname{sn}(u+v) &= \frac{\operatorname{sn}(u)\operatorname{cn}(v)\operatorname{dn}(v) + \operatorname{sn}(v)\operatorname{cn}(u)\operatorname{dn}(u)}{1 - k^2 \operatorname{sn}^2(u)\operatorname{sn}^2(v)}\\ \operatorname{cn}(u+v) &= \frac{\operatorname{cn}(u)\operatorname{cn}(v) - \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{dn}(u)\operatorname{dn}(v)}{1 - k^2 \operatorname{sn}^2(u)\operatorname{sn}^2(v)}\\ \operatorname{dn}(u+v) &= \frac{\operatorname{dn}(u)\operatorname{dn}(v) - k^2 \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{cn}(u)\operatorname{cn}(v)}{1 - k^2 \operatorname{sn}^2(u)\operatorname{sn}^2(v)} \end{aligned}$$

3. The constants $K, K^{'}$:

a. Symbol K is a function of k such that $\operatorname{sn}(K, k) = 1$. In other words,

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$$

and $\mathrm{sn}(K) = 1$, $\mathrm{cn}(K) = 0$, $\mathrm{dn}(K) = k^{'}$.

b. Symbol $K^{'}$ is a function of $k^{'}$,

$$K^{'}(k^{'}) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k'^2t^2)}}dt.$$

c. Another form of K and $K^{'}$:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi$$

$$K^{'}(k^{'}) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k'^2\sin^2\phi}}d\phi$$

4. The periodic properties of the Jacobian elliptic functions:

a. associated with K :

$$\mathrm{sn}(u+2K) = -\mathrm{sn}(u) \qquad \mathrm{sn}(u+4K) = \mathrm{sn}(u)$$
$$\mathrm{cn}(u+2K) = -\mathrm{cn}(u) \qquad \mathrm{cn}(u+4K) = \mathrm{cn}(u)$$
$$\mathrm{dn}(u+2K) = \mathrm{dn}(u) \qquad \mathrm{dn}(u+4K) = \mathrm{dn}(u)$$

b. associated with $K + iK^{'}$:

$$\mathrm{sn}(u+2K+2iK^{'}) = -\mathrm{sn}(u) \qquad \mathrm{sn}(u+4K+4iK^{'}) = \mathrm{sn}(u)$$
$$\mathrm{cn}(u+2K+2iK^{'}) = \mathrm{cn}(u) \qquad \mathrm{cn}(u+4K+4iK^{'}) = \mathrm{cn}(u)$$
$$\mathrm{dn}(u+2K+2iK^{'}) = -\mathrm{dn}(u) \qquad \mathrm{dn}(u+4K+4iK^{'}) = \mathrm{dn}(u)$$

c. associated with $iK^{'}$:

$$\mathrm{sn}(u+2iK^{'}) = \mathrm{sn}(u) \qquad \mathrm{sn}(u+4iK^{'}) = \mathrm{sn}(u)$$
$$\mathrm{cn}(u+2iK^{'}) = -\mathrm{cn}(u) \qquad \mathrm{cn}(u+4iK^{'}) = \mathrm{cn}(u)$$
$$\mathrm{dn}(u+2iK^{'}) = -\mathrm{dn}(u) \qquad \mathrm{dn}(u+4iK^{'}) = \mathrm{dn}(u)$$

-	sn(u)	cn(u)	dn(u)
Periods	$4K, 2iK'$	$4K, 2K+2iK'$	$2K, 4iK'$
Zeros	0 mod $(2K, 2iK')$	K mod $(2K, 2iK')$	$K+iK'$ mod $(2K, 2iK')$
Poles	$iK', 2K+iK'$ mod $(4K, 2iK')$	$iK', 2K+iK'$ mod $(4K, 2K+2iK')$	$iK', 3K'$ mod $(2K, 4iK')$
Parity	odd	even	even
Derivative	cn(u) dn(u)	-sn(u) dn(u)	-k^2 sn(u) cn(u)

Table 4.1: Summary about sn(u), cn(u) and dn(u)

4.4.5 Elliptic integrals of the first kind

The function sn(u) satisfies the differential equation (4.5)

$$(\frac{dy}{du})^2 = (1-y^2)(1-k^2y^2),$$

we have the integral representation of sn(u) is

$$u = \int_0^y \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt, \quad \text{thus } y = \text{sn}(u,k).$$

A special case of the integral representation is

$$K = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt,$$

this is the complete elliptic integral of the first kind. Moreover, if we let $t = \sin\phi$, and we have at once

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}}dt.$$

4.4.6 The graphs of Jacobian functions

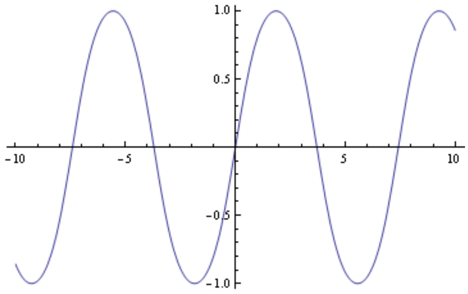

Figure 4.1: $\underline{\text{sn}(u, \frac{1}{2})}$

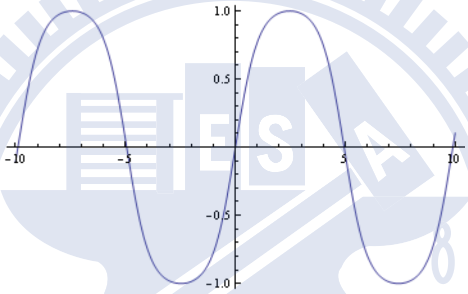

Figure 4.2: $\underline{\text{sn}(u, \frac{7}{8})}$

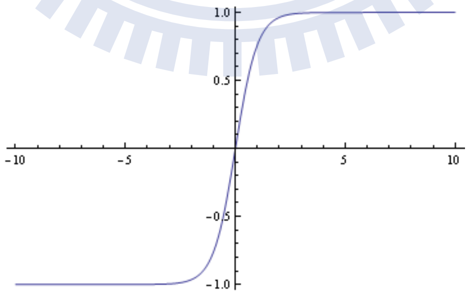

Figure 4.3: $\underline{\text{sn}(u, 1)}$

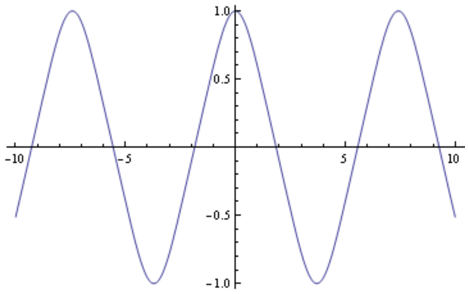

Figure 4.4: $\underline{\mathrm{cn}(u, \frac{1}{2})}$

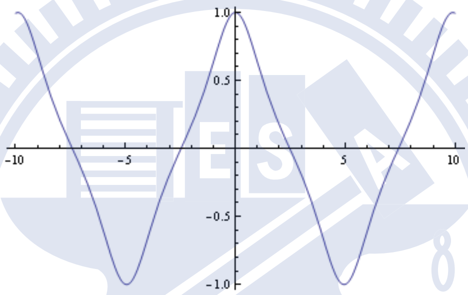

Figure 4.5: $\underline{\mathrm{cn}(u, \frac{7}{8})}$

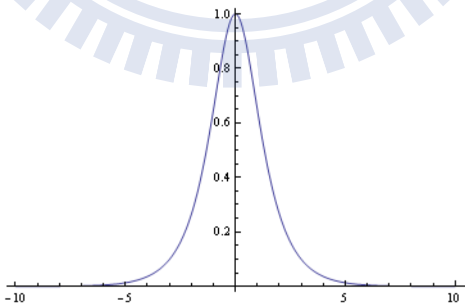

Figure 4.6: $\underline{\mathrm{cn}(u, 1)}$

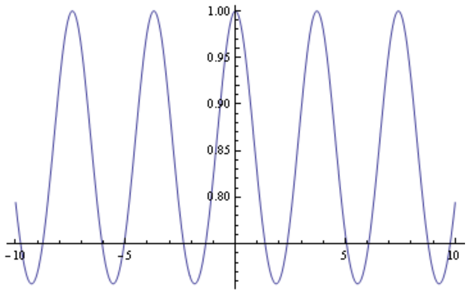

Figure 4.7: $\underline{\mathrm{dn}(u, \frac{1}{2})}$

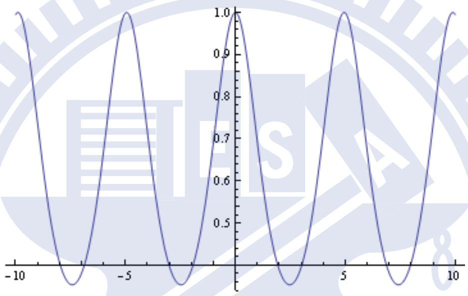

Figure 4.8: $\underline{\mathrm{dn}(u, \frac{7}{8})}$

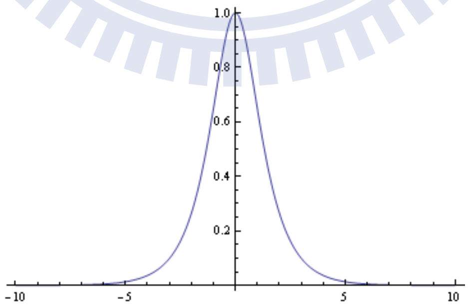

Figure 4.9: $\underline{\mathrm{dn}(u, 1)}$

Chapter 5

Exact Theory of the Simple Pendulum Motion

5.1 Introduction of the simple pendulum

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force combined with the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. A pendulum swings with a specific period which depends on its length mainly.

The simple pendulum is an idealized mathematical model of a pendulum. This is a weight (or bob) on the end of a massless cord suspended from a pivot, without friction. When given an initial push, it will swing back and forth at a constant amplitude.

The figure below shows the simple pendulum:

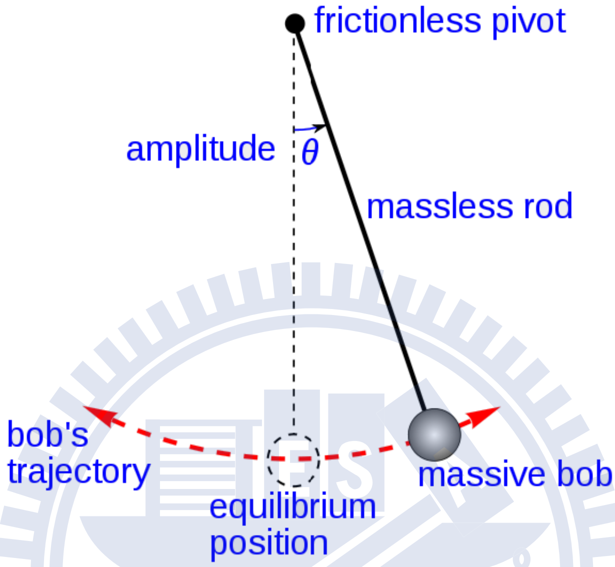

Figure 5.1: Simple pendulum

5.2 Analyze the derivation

Here we introduce two ways, via Newton's second law and conservation of energy, to obtain the differential equation

$$u'' + \sin u = 0. \tag{5.1}$$

1. By Newton's second law:

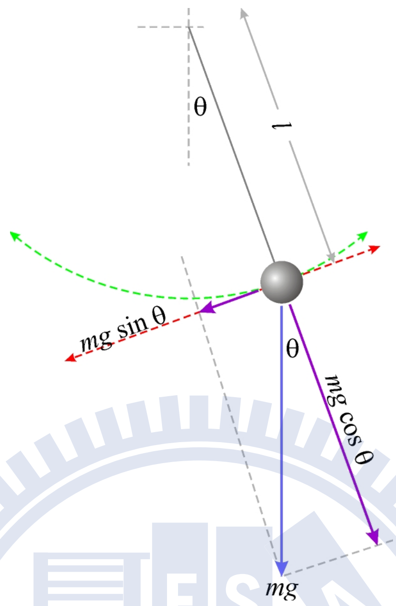

Figure 5.2: Analysis via Newton's second law

Consider Newton's second law,

$$F = ma$$

where F is the sum of forces on the object, m is mass, and a is the acceleration. Because the bob is forced to stay in a circular path, we apply Newton's equation to the tangential axis only,

$$F = -mg\sin\theta = ma$$
$$a = -g\sin\theta$$

where g is the acceleration due to gravity near the surface of the earth. The negative sign on the right hand side implies that θ and a always point in opposite directions. This makes sense because when a pendulum swings further to the left, we would expect it to accelerate back toward the right.

This linear acceleration a can be related to the change in angle θ by the arc length formulas; l is the length of the pendulum and s is the arc length:

$$\begin{aligned} s &= l\theta \\ v &= \frac{ds}{dt} = l\frac{d\theta}{dt} \\ a &= \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2} \end{aligned}$$

Thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{5.2}$$

2. By conservation of energy:

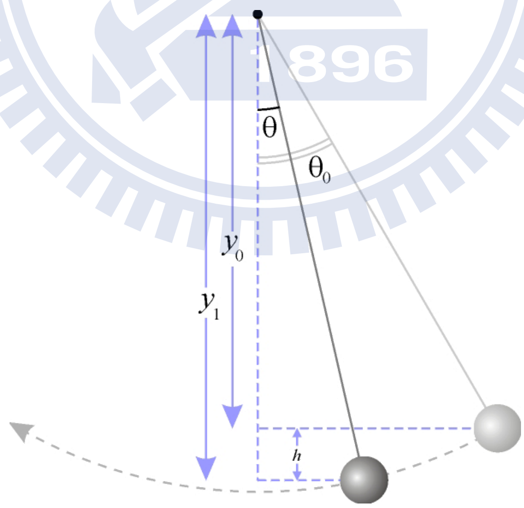

Figure 5.3: Analysis via conservation of energy

Any object falling a vertical distance h would acquire kinetic energy equal to that which it lost to the fall. In other words, gravitational potential energy is converted into kinetic energy. Change in potential energy is given by

$$\Delta U = mgh,$$

change in kinetic energy (body started from rest) is given by

$$\Delta K = \frac{1}{2}mv^2.$$

Since the conservation of energy, no energy is lost, those two must be equal

$$\frac{1}{2}mv^2 = mgh$$
$$v = \sqrt{2gh}.$$

Using the arc length formula above, this equation can be rewritten as

$$v = l\frac{d\theta}{dt} = \sqrt{2gh}$$
$$\frac{d\theta}{dt} = \frac{1}{l}\sqrt{2gh}$$

where h is the vertical distance the pendulum fell.

Look at Figure 5.3, which presents the trigonometry of a simple pendulum. If the pendulum starts its swing from some initial angle θ_0, then y_0, the vertical distance from the screw, is given by

$$y_0 = l\cos\theta_0,$$

similarly, for y_1, we have

$$y_1 = l\cos\theta,$$

then h is the difference of the two

$$h = l(\cos\theta - \cos\theta_0).$$

In terms of $\frac{d\theta}{dt}$ gives

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

We can differentiate, by applying the chain rule, with respect to time to get

$$\begin{aligned}
\frac{d^2\theta}{dt^2} = \frac{d}{dt}\frac{d\theta}{dt} &= \frac{d}{dt}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
&= \frac{1}{2}\frac{\frac{2g}{l}(-\sin\theta)}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}\frac{d\theta}{dt} \\
&= \frac{1}{2}\frac{\frac{2g}{l}(-\sin\theta)}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
&= -\frac{g}{l}\sin\theta.
\end{aligned}$$

Thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{5.3}$$

No matter which idea for derivation, there are the same results, (5.2) and (5.3). Letting $\frac{g}{l} = 1$ for convenience, there is (5.1). After the above pre-work, the following contents will recall the conclusion in Chapter 4 and get the exact theory of the simple pendulum motion.

5.3 Apply Jacobian elliptic function to solve the simple pendulum motion

Consider the differential equation

$$u'' + \sin u = 0,$$

there is

$$\frac{1}{2}(u')^2 - \cos u = E \tag{5.4}$$

where E is the integration constant. Adding 1 to both sides yields

$$\frac{1}{2}(u')^2 + (1 - \cos u) = E + 1.$$

In the idea of energy, we can regard $\frac{1}{2}(u')^2$ as kinetic energy, $(1 - \cos u)$ as potential energy, and $E + 1$ as the total energy of this system. Certainly, u is a function of time t.

Since $(1 - \cos u)$ is regarded as the potential energy, $0 \le (1 - \cos u) \le 2$, and the kinetic energy $\frac{1}{2}(u')^2 \ge 0$, the total energy $E + 1$ must be greater than or equal to 0. Furthermore, when the potential energy reaches the maximum 2, it also means that the pendulum is right at the highest position in the circular path. So the total energy $E + 1 = 2$ will be the key factor of different types of pendulum motions.

$$\begin{aligned} 0 < E + 1 < 2 &\;\Rightarrow\; -1 < E < 1 \\ E + 1 = 2 &\;\Rightarrow\; E = 1 \\ E + 1 > 2 &\;\Rightarrow\; E > 1 \end{aligned}$$

Keeping on the above statement (5.4), since the equation

$$u' = \frac{du}{dt} = \sqrt{2(E + \cos u)} \tag{5.5}$$

is separable, we can obtain

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2(E + \cos u)}} du. \tag{5.6}$$

Our goal is to find out the solution of equation (5.6). That is, we must find the representation of $U(t)$ in terms of t. Discussing (5.6) in three different cases by given E is the next works.

1. $-1 < E < 1$:

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos u}} du \\ &= \int_0^{U(t)} \frac{1}{\sqrt{2E + (2 - 4\sin^2 \frac{u}{2})}} du \\ &= \frac{1}{\sqrt{2E+2}} \int_0^{U(t)} \frac{1}{\sqrt{1 - \frac{4}{2E+2}\sin^2 \frac{u}{2}}} du \end{aligned} \tag{5.7}$$

Let $k = \dfrac{\sqrt{2E+2}}{2}$, $z = \dfrac{1}{k}\sin\dfrac{u}{2}$, then

$$t = \int_0^{\frac{1}{k}\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-z^2}} \frac{1}{\sqrt{1-k^2z^2}} dz.$$

According to Jacobian function (4.7)

$$\text{sn}(t,k) = \frac{1}{k}\sin\frac{U(t)}{2},$$

i.e.

$$U(t) = 2\arcsin\left(k\,\text{sn}(t,k)\right) \tag{5.8}$$

where $k = \dfrac{\sqrt{2E+2}}{2}$.

2. $E = 1$:

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2+2\cos u}} du \\ &= \int_0^{U(t)} \frac{1}{\sqrt{4-4\sin^2\frac{u}{2}}} du \\ &= \frac{1}{2}\int_0^{U(t)} \frac{1}{\sqrt{1-\sin^2\frac{u}{2}}} du \end{aligned} \tag{5.9}$$

Let $z = \sin\dfrac{u}{2}$, then

$$t = \int_0^{\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}} dz.$$

So

$$\mathrm{sn}(t,1) = \sin\frac{U(t)}{2},$$

i.e.

$$U(t) = 2\arcsin\left(\mathrm{sn}(t,1)\right). \tag{5.10}$$

If we do not use Jacobian elliptic function, we can also get the solution by Calculus. There is given by

$$\begin{aligned} t &= \int_0^{\sin\frac{U(t)}{2}} \frac{1}{1-z^2} dz \\ &= \frac{1}{2}\ln\frac{1+\sin\frac{U(t)}{2}}{1-\sin\frac{U(t)}{2}}. \end{aligned}$$

So

$$\sin\frac{U(t)}{2} = \tanh t,$$

i.e.

$$U(t) = 2\arcsin(\tanh t). \tag{5.11}$$

3. $E > 1$:

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2E + 2\cos u}} du \\ &= \int_0^{U(t)} \frac{1}{\sqrt{2E + (2 - 4\sin^2 \frac{u}{2})}} du \\ &= \frac{1}{\sqrt{2E+2}} \int_0^{U(t)} \frac{1}{\sqrt{1 - \frac{4}{2E+2}\sin^2 \frac{u}{2}}} du \end{aligned} \tag{5.12}$$

Let $k = \dfrac{2}{\sqrt{2E+2}}$, $z = \sin\dfrac{u}{2}$, then

$$t = k \int_0^{\sin\frac{U(t)}{2}} \frac{1}{\sqrt{1-k^2z^2}} \frac{1}{\sqrt{1-z^2}} dz.$$

So

$$\operatorname{sn}(\frac{t}{k}, k) = \sin\frac{U(t)}{2},$$

i.e.

$$U(t) = 2\arcsin(\operatorname{sn}(\frac{t}{k}, k)) \tag{5.13}$$

where $k = \dfrac{2}{\sqrt{2E+2}}$.

Note: There are no confusions with the patterns of k in (5.7), (5.9), (5.12). Since the definition of Jacobian elliptic functions, there is an identity $k^2 + k'^2 = 1$ for (4.7), (4.8), and (4.9). Hence, the patterns of k related to E must be determined with $k^2 \leq 1$.

5.4 Periods and phase portraits with different total energy

Since the solutions of $u'' + \sin u = 0$ had been found in terms of Jacobian elliptic function with different E, we want to further know the period of the solution if it is periodic.

Moreover, we try to plot the relation between U and U', i.e., the phase portrait. Before drawing the phase portrait, we see back to the equation (5.4) first. It shows that $\frac{1}{2}(u')^2 - \cos u$ is a constant. It can be regarded as a conservation law in the view point of mathematics since $-\cos u$ is not always larger than 0. (But this case can be transferred to the conservation law in the view point of physics by plus a constant 1 for equation (5.4).) This means that its total energy is a constant and the former part $\frac{1}{2}(u')^2$ can be regarded as kinetic energy and the latter part $-\cos u$ can be regarded as potential energy.

1. $-1 < E < 1$:
The solution with $-1 < E < 1$ is given by

$$U(t) = 2 \arcsin\left(k \,\mathrm{sn}(t, k)\right)$$

where $k = \dfrac{\sqrt{2E+2}}{2}$. Therefore, by subsection 4.4.4, the period is

$$\begin{aligned} T &= 4 \int_0^1 \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}} dz \\ &= 4K. \end{aligned}$$

Obviously, $K = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}} dz$.

By the assumption we defined before, there is

$$-1 < E_1 < E_2 < 1 \;\Rightarrow\; k_1 < k_2,$$

and, hence,

$$\frac{1}{\sqrt{(1-z^2)(1-k_1^2z^2)}} < \frac{1}{\sqrt{(1-z^2)(1-k_2^2z^2)}},$$

i.e. $K_1 < K_2$.

In short, if there are two different E_1 and E_2, where $-1 < E_1 < E_2 < 1$, the comparison with two periods is

$$T_1 < T_2.$$

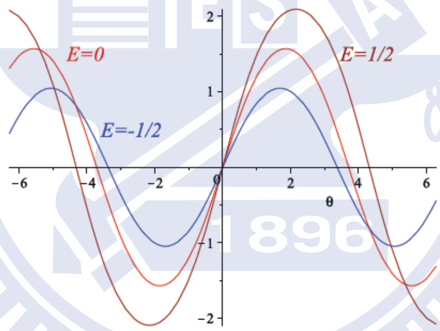

Figure 5.4: Solution curves with $E = -\frac{1}{2}$, $E = 0$, and $E = \frac{1}{2}$

In the sense of pendulum motion, the greater total energy means the higher initial position, and it is naturally that the time pendulum returns to the initial position is longer if the initial position is higher. Thus we have the result as above, $E_1 < E_2$ implies $T_1 < T_2$.

We set $E = 0$ to analyze the phase portrait. By the equation (5.4), we have $u' = \pm\sqrt{2\cos u}$. The following graphs are potential energy and phase portrait, respectively. Those graphs show the relation between u and $\cos u$ and the relation between u and u'.

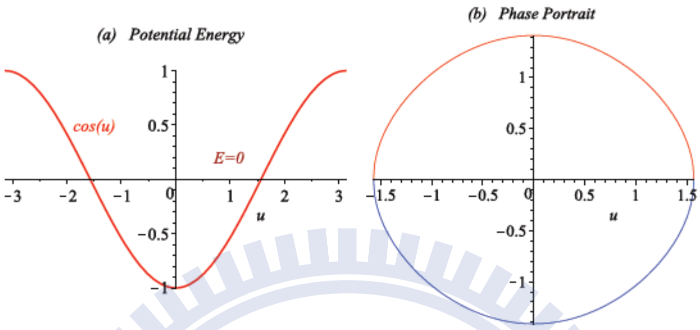

Figure 5.5: Potential energy and phase portrait with $E = 0$

From the graph of the phase portrait, the red curve means that the velocity at those position are positive and the blue curve means that the velocity at those position are negative. The positive velocity is defined by rotating counterclockwise and the negative velocity is defined by rotating clockwise.

2. $E = 1$:

The solution with $E = 1$ is given by

$$U(t) = 2\arcsin(\mathrm{sn}(t, 1)).$$

$$\begin{aligned} T &= 4\int_0^1 \frac{1}{\sqrt{(1-z^2)(1-z^2)}}dz \\ &= 4\int_0^1 \frac{1}{1-z^2}dz \\ &= \infty. \end{aligned}$$

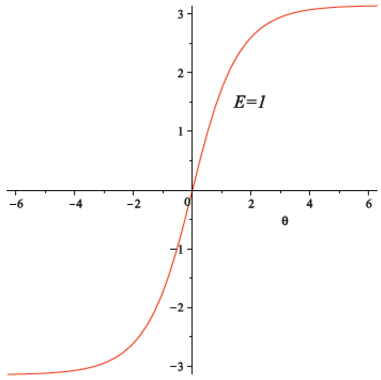

Figure 5.6: Solution curve with $E = 1$

In the sense of pendulum motion, since the total energy $E + 1 = 2$, the potential energy must be 2 and the kinetic energy must be 0 somewhere. By the language we used before, that is the greatest potential energy means the highest place in the pendulum motion, surely the top of the circular path. Therefore it implies that if we release the pendulum at the top of the circular path, it will return to the initial position after travelling the time infinity.

Now we focus on the phase portrait with $E = 1$. By the equation (5.4), we have $u' = \pm\sqrt{2(1 + \cos u)}$, and phase portrait as following.

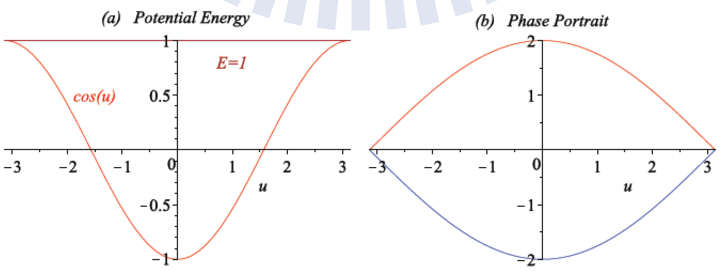

Figure 5.7: Potential energy and phase portrait with $E = 1$

3. $E > 1$:

By (5.5), there is $u' > 0$ if $E > 1$. This means that for any time t, the velocity of pendulum is always greater than 0. That is, the pendulum will never stop. So the motion is no periodicity.

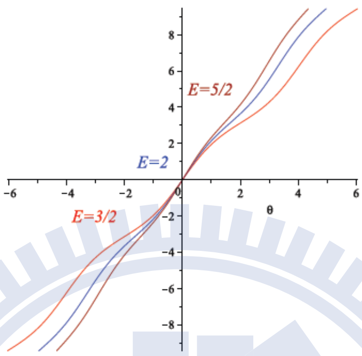

Figure 5.8: Solution curves with $E = \frac{3}{2}$, $E = 2$, and $E = \frac{5}{2}$

Last, we see the phase portrait with $E = \frac{3}{2}$. By the equation (5.4), we have $u' = \pm\sqrt{2(\frac{3}{2} + \cos u)}$, and phase portrait as following.

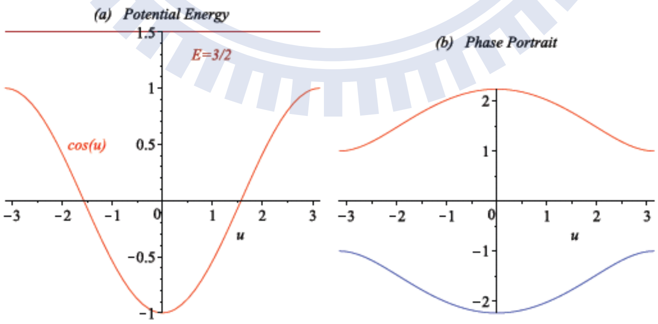

Figure 5.9: Potential energy and phase portrait with $E = \frac{3}{2}$

From the graph of the phase portrait, we know that the pendulum of this case will never stop since the phase portrait has no intersection with the u-axis. And by the graph of potential energy, we observe that the kinetic energy is never equal to 0. This implies that the case has no periodic solution and the result is corresponded to the property which we had discussed.

By our discussion, there are three kinds of the phase portraits. Before finishing the section, we combine the three phase portraits and the vector field together.

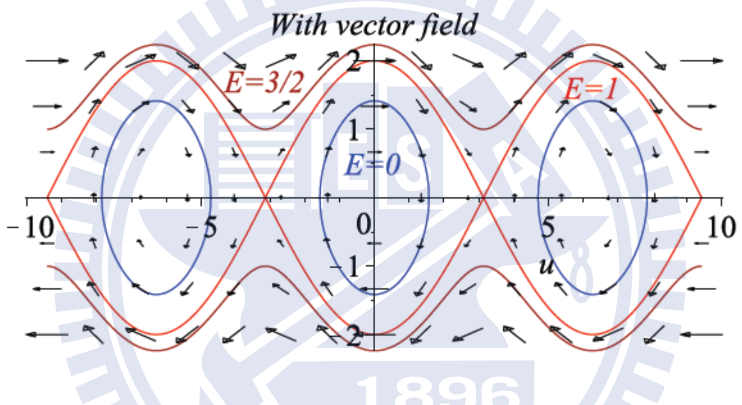

Figure 5.10: Global phase portrait

There are three different kinds of phase portraits with different energy E. The outer curve corresponds to larger energy E. They are separated by the phase portrait with $E = 1$ and the phase curve is called the separatrix with periods ∞. The phase curves outer the separatrix are called the wave train and they has no period. The phase curves inside the separatrix are periodic and their period T satisfies $2\pi < T < \infty$.

5.5 Summary

The mathematical model of the simple pendulum motion is a nonlinear second order differential equation. There are k corresponding to the given E, and thus the solutions of the simple pendulum motion is expressed by Jacobian elliptic function $\mathrm{sn}(t,k)$ within different cases of E.

Together the consequences in all cases we considered, there is the following table.

-	$-1 < E < 1$	$E = 1$	$E > 1$
Modulus k	$\frac{\sqrt{2E+2}}{2}$	1	$\frac{2}{\sqrt{2E+2}}$
Solution $U(t)$	$2\arcsin\left(k\,\mathrm{sn}(t,k)\right)$	$2\arcsin\left(\mathrm{sn}(t,1)\right)$	$2\arcsin\left(\mathrm{sn}(\frac{t}{k},k)\right)$
Period T	$4K$	∞	No periodicity

Table 5.1: Summary about the simple pendulum motion within different E

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