Information Processing Letters 109 (2009) 585–588

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Conditional fault hamiltonian connectivity of the complete graph

Tung-Yang Ho [a,*], Yuan-Kang Shih [b], Jimmy J.M. Tan [b], Lih-Hsing Hsu [c]

[a] Department of Information Management, Ta Hwa Institute of Technology, Hsinchu, Taiwan 30740, ROC

[b] Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan 30010, ROC

[c] Department of Computer Science and Information Engineering, Providence University, Taichung, Taiwan 43301, ROC

ARTICLE INFO

Article history:
Received 22 August 2008
Received in revised form 15 January 2009
Available online 20 February 2009
Communicated by A.A. Bertossi

Keywords:
Complete graph
Hamiltonian
Hamiltonian connected
Fault tolerance

ABSTRACT

A path in G is a hamiltonian path if it contains all vertices of G. A graph G is hamiltonian connected if there exists a hamiltonian path between any two distinct vertices of G. The degree of a vertex u in G is the number of vertices of G adjacent to u. We denote by $\delta(G)$ the minimum degree of vertices of G. A graph G is conditional k edge-fault tolerant hamiltonian connected if $G - F$ is hamiltonian connected for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G - F) \geqslant 3$. The conditional edge-fault tolerant hamiltonian connectivity $\mathcal{HC}_e^3(G)$ is defined as the maximum integer k such that G is k edge-fault tolerant conditional hamiltonian connected if G is hamiltonian connected and is undefined otherwise. Let $n \geqslant 4$. We use K_n to denote the complete graph with n vertices. In this paper, we show that $\mathcal{HC}_e^3(K_n) = 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$.

1. Introduction

For the graph definitions and notations, we follow [1]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. The *complete graph* K_n is the graph with n vertices such that any two distinct vertices are adjacent. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is the number of vertices adjacent to u. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A *path* of length $m - 1$, $\langle v_0, v_1, \ldots, v_{m-1}\rangle$, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for $0 \leqslant i \leqslant m - 2$. We also write the path $\langle v_0, \ldots, v_k, P, v_l, \ldots, v_m\rangle$ for $P = \langle v_k, \ldots, v_l\rangle$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. A *hamiltonian path* is a path of length $V(G) - 1$.

A hamiltonian graph G is k *edge-fault tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge-fault tolerant hamiltonicity*, $\mathcal{H}_e(G)$, is defined as the maximum integer k such that G is k edge-fault tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Assume that G is a hamiltonian graph, and x is a vertex such that $\deg_G(x) = \delta(G)$. We arbitrary choose $\deg_G(x) - 1$ edges from those edges incident to x to form an edge faulty set F. Obviously, $\deg_{G-F}(x) = 1$; hence, $G - F$ is not hamiltonian. Therefore, $\mathcal{H}_e(G) \leqslant \delta(G) - 2$ if $\mathcal{H}_e(G)$ is defined. Assume that n is an integer with $n \geqslant 3$. It is proved by Ore [9] that any n-vertex graph with at least $C(n, 2) - (n - 3)$ edges is hamiltonian. Moreover, there exists a non-hamiltonian n-vertex graph with $C(n, 2) - (n - 2)$ edges. In other words, $\mathcal{H}_e(K_n) = n - 3$ for $n \geqslant 3$. In [5], it is proved that $\mathcal{H}_e(Q_n) = n - 2$ for $n \geqslant 2$ where Q_n is the n-dimensional hypercube. In [6], it is proved that $\mathcal{H}_e(S_n) = n - 3$ for $n \geqslant 3$ where S_n is the n-dimensional star graph.

Chan and Lee [2] began the study of the existence of hamiltonian cycle in a graph such that each vertex is incident to at least two fault-free edges. A graph G is *conditional k edge-fault tolerant hamiltonian* if $G - F$ is hamilto-

* Corresponding author.
E-mail address: hoho@thit.edu.tw (T.-Y. Ho).

0020-0190/$ – see front matter
doi:10.1016/j.ipl.2009.02.008

nian for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G-F) \geqslant 2$. The *conditional edge-fault tolerant hamiltonicity*, $\mathcal{H}_e^2(G)$, is defined as the maximum integer k such that G is conditional k edge-fault tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Chan and Lee [2] proved that $\mathcal{H}_e^2(Q_n) = 2n-5$ for $n \geqslant 3$. Recently, Fu [3] studies the conditional edge-fault tolerant hamiltonicity of the complete graph.

Fault tolerant hamiltonian connectivity is another important parameter for graphs [4]. A graph G is *hamiltonian connected* if there exists a hamiltonian path between any two distinct vertices of G. It is easy to see that a hamiltonian connected graph with at least three vertices is hamiltonian. It is proved by Moon [7] that the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3. A graph G is *k edge-fault tolerant hamiltonian connected* if $G-F$ remains hamiltonian connected for any $F \subset E(G)$ with $|F| \leqslant k$. The *edge-fault tolerant hamiltonian connectivity* of a graph G, $\mathcal{HC}_e(G)$, is defined as the maximum integer k such that G is k edge-fault tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise. Assume that G is a hamiltonian connected graph with at least four vertices and x is a vertex such that $\deg_G(x) = \delta(G)$. We arbitrary choose $\deg_G(x) - 2$ edges from those edges incident to x to form an edge faulty set F. Obviously, $\deg_{G-F}(x) = 2$; hence, $G-F$ is not hamiltonian connected. Therefore, $\mathcal{HC}_e(G) \leqslant \delta(G) - 3$ if $\mathcal{HC}_e(G)$ is defined. Again, Ore [8] proved that $\mathcal{HC}_e(K_n) = n-4$ for $n \geqslant 4$.

In this paper, we study the concept of conditional edge-fault tolerant hamiltonian connectivity. Since the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3, it is natural to assume that each vertex is incident to at least three fault-free edges. A graph G is *conditional k edge-fault tolerant hamiltonian connected* if $G-F$ is hamiltonian connected for every $F \subset E(G)$ with $|F| \leqslant k$ and $\delta(G-F) \geqslant 3$. The *conditional edge-fault tolerant hamiltonian connectivity*, $\mathcal{HC}_e^3(G)$, is defined to be the maximum integer k such that G is conditional k edge-fault tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise.

Assume that n is an integer with $n \geqslant 4$. In this paper, we prove that $\mathcal{HC}_e^3(K_n) = 2n-10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$. To reach this goal, we present some preliminary in the following section. In Section 3, we prove our main result.

2. Preliminary

Let F be a faulty edge set. We define $K_n(F)$ be a graph with $E(K_n(F)) = F$ and $V(K_n(F)) = V(K_n)$. The following statement is proved in [3]:

Suppose $F \subset E(K_n)$ and $\delta(K_n - F) \geqslant 2$, where $n \geqslant 4$. If $n \notin \{7, 9\}$ (respectively, $n \in \{7, 9\}$) then $K_n - F$ is hamiltonian, where $|F| \leqslant 2n-8$ (respectively, $|F| \leqslant 2n-9$).

In the conclusion of [3], it is claimed that the above statement is optimal. Using our terminology, we obtain the following statement.

$\mathcal{H}_e^2(K_n) = 2n-8$ for $n \notin \{7, 9\}$ and $n \geqslant 4$, $\mathcal{H}_e^2(K_7) = 5$, and $\mathcal{H}_e^2(K_9) = 9$.

Yet, it is easy to check that $\mathcal{H}_e^2(K_3)$ is 0 and $\mathcal{H}_e^2(K_4)$ is 2 (not 0.) Thus, we have the following theorem.

Theorem 1. *$\mathcal{H}_e^2(K_n) = 2n-8$ for $n \notin \{7, 9\}$ and $n \geqslant 5$, $\mathcal{H}_e^2(K_3) = 0$, $\mathcal{H}_e^2(K_4) = 2$, $\mathcal{H}_e^2(K_7) = 5$, and $\mathcal{H}_e^2(K_9) = 9$.*

Lemma 1. *Assume that n is an integer with $n \geqslant 6$ and F is any subset of $E(K_n)$ with $|F| = 2n-10$ if $n \notin \{8, 10\}$ and $|F| = 2n-11$ if $n \in \{8, 10\}$. There exists a vertex w in $K_n(F)$ such that $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$.*

Proof. Suppose that the lemma is false. Then $\deg_{K_n(F)}(w) \geqslant \lfloor \frac{n-1}{2} \rfloor$ for every vertices with $\deg_{K_n(F)}(w) \neq 0$. Obviously, there are at least $\lfloor \frac{n-1}{2} \rfloor + 1$ vertices with $\deg_{K_n(F)}(w) \neq 0$. Hence, $|F| \geqslant (\lfloor \frac{n-1}{2} \rfloor (\lfloor \frac{n-1}{2} \rfloor + 1))/2$. However, $(\lfloor \frac{n-1}{2} \rfloor \times (\lfloor \frac{n-1}{2} \rfloor + 1))/2 > 2n-10$ for $n \notin \{8, 10\}$ and $(\lfloor \frac{n-1}{2} \rfloor (\lfloor \frac{n-1}{2} \rfloor + 1))/2 > 2n-11$ for $n \in \{8, 10\}$. It is a contradiction. The lemma is proved. □

The following theorem can be found in [1].

Theorem 2. *(See [1].) Let $D = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence with $d_1 \geqslant 1$ and $d_i \geqslant 0$ for $2 \leqslant i \leqslant n$. We set $D' = (d'_1, d'_2, \ldots, d'_{n-1}) = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. Then there exists a graph G with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $\deg_G(x_i) = d_i$ for $1 \leqslant i \leqslant n$ if and only if there exists a graph G' with vertex set $\{y_1, y_2, \ldots, y_{n-1}\}$ such that $\deg_{G'}(y_j) = d'_j$ for $1 \leqslant j \leqslant n-1$.*

By the above theorem, we know that there is a graph G with degree sequence D if and only if there is a graph G' with degree sequence D'. If $d'_i < 0$ for some i, then D' is not the degree sequence of any graph, neither is D.

Lemma 2. *Let F be a subset of $E(K_9)$ with $|F| = 8$ and $\delta(K_9 - F) \geqslant 3$. Let u and v be any two distinct vertices in K_9 such that $\deg_{K_9(F)}(u) = 0$ and $\deg_{K_9(F)}(v) = 0$. Then there exists a vertex w with $\deg_{K_9(F)}(w) \in \{2, 3\}$.*

Proof. Let $\{x_1, x_2, \ldots, x_8 = u, x_9 = v\}$ be the vertex set of K_9 such that $\deg_{K_9(F)}(x_i) = d_i$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_9$. Obviously, $\sum_{i=1}^{9} d_i = 16$. Assume that the lemma is false. Then $\deg_{K_9(F)}(x_i) \in \{0, 1, 4, 5\}$ for $1 \leqslant i \leqslant 9$. By brute force, all such sequences are listed below: $(5, 5, 5, 1, 0, 0, 0, 0, 0)$, $(5, 5, 4, 1, 1, 0, 0, 0, 0)$, $(5, 4, 4, 1, 1, 1, 0, 0, 0)$, $(4, 4, 4, 4, 0, 0, 0, 0, 0)$, and $(4, 4, 4, 1, 1, 1, 1, 0, 0)$. By Theorem 2, we can check that such a graph does not exist. Hence, the lemma is proved. □

Lemma 3. *Let F be a subset of $E(K_{11})$ with $|F| = 12$ and $\delta(K_{11} - F) \geqslant 3$. Let u and v be any two distinct vertices in K_{11} such that $\deg_{K_{11}(F)}(u) = 0$ and $\deg_{K_{11}(F)}(v) = 0$. Then there exists a vertex w with $\deg_{K_{11}(F)}(w) \in \{2, 3, 4\}$.*

Proof. Let $\{x_1, x_2, \ldots, x_{10} = u, x_{11} = v\}$ be the vertex set of K_{11} such that $\deg_{K_{11}(F)}(x_i) = d_i$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant$

d_{11}. Obviously, $\sum_{i=1}^{11} d_i = 24$. Assume that the lemma is false. Then $\deg_{K_{11}(F)}(x_i) \in \{0,1,5,6,7\}$ for $1 \leqslant i \leqslant 11$. By brute force, all such sequences are listed below: $(7,7,7,1,1,1,0,0,0,0,0)$, $(7,7,6,1,1,1,1,0,0,0,0)$, $(7,7,5,5,0,0,0,0,0,0,0)$, $(7,7,5,1,1,1,1,1,0,0,0)$, $(7,6,6,5,0,0,0,0,0,0,0)$, $(7,6,6,1,1,1,1,1,0,0,0)$, $(7,6,5,5,1,0,0,0,0,0,0)$, $(7,6,5,1,1,1,1,1,1,0,0)$, $(7,5,5,5,1,1,0,0,0,0,0)$, $(6,6,6,6,0,0,0,0,0,0,0)$, $(6,6,6,5,1,0,0,0,0,0,0)$, $(6,6,6,1,1,1,1,1,1,0,0)$, $(6,6,5,5,1,1,0,0,0,0,0)$, $(6,5,5,5,1,1,1,0,0,0,0)$, and $(5,5,5,5,1,1,1,1,0,0,0)$. By Theorem 2, we can check that such a graph does not exist. The lemma is proved. □

We can easily obtain the following lemma.

Lemma 4. *Let $k \geqslant 2$. Let G be a hamiltonian connected graph. Then deleting any set S of k vertices from G, the resulting graph $G - S$ contains at most $k-1$ connected components.*

By the above lemma, we have a simple observation.

Lemma 5. *Let $k \geqslant 2$. Let G be a graph. If there is a set S of k vertices such that $G - S$ contains k or more connected components, then G is not hamiltonian connected.*

3. Main result

Lemma 6. *Let $n \geqslant 4$ and $F \subset E(K_n)$ with $\delta(K_n - F) \geqslant 3$. Then $K_n - F$ is hamiltonian connected if $|F| \leqslant 2n - 10$ for $n \notin \{4,5,8,10\}$, $|F| = 0$ for $n = 4$, $|F| \leqslant 2$ for $n = 5$, and $|F| \leqslant 2n - 11$ for $n \in \{8,10\}$.*

Proof. We prove this lemma by induction on n. Yet, we should be very careful because the size of $|F|$ is depending on n. Without loss of generality, we assume that $|F| = 2n - 10$ for $n \notin \{4,5,8,10\}$, $|F| = 0$ for $n = 4$, $|F| = 2$ for $n = 5$, and $|F| = 2n - 11$ for $n \in \{8,10\}$. The induction bases are $n = 4$, $n = 5$, and $n = 6$. Suppose $n = 4$ and $|F| = 0$. It is easy to see that the complete graph K_4 is hamiltonian connected. Suppose $n = 5$ and $|F| = 2$. To keep $\delta(K_5 - F) \geqslant 3$, F forms two independent edges. By brute force, it is easy to check whether $K_5 - F$ is hamiltonian connected. Suppose that $n = 6$ and $|F| = 2$. Obviously, F is either two adjacent edges or two independent edges. Again, by brute force, we can check that $K_6 - F$ is hamiltonian connected.

Now, we assume that $n \geqslant 7$. Let u and v be any two vertices of K_n. The lemma follows if we can find a hamiltonian path of $K_n - F$ between u and v.

Case 1. $\deg_{K_n(F)}(u) \neq 0$ or $\deg_{K_n(F)}(v) \neq 0$. Without loss of generality, we assume that $\deg_{K_n(F)}(u) = k \neq 0$. Let $i_1, \ldots, i_k$ be the vertices such that $(u, i_j) \in F$ for $1 \leqslant j \leqslant k$. Let $F' = (F - \{(u,i_1),\ldots,(u,i_k)\}) \cup \{(v,i_1),\ldots,(v,i_k)\}$. Obviously, $|F'| \leqslant |F|$. Now, we consider $K_n - \{u\}$ as a complete graph of $(n-1)$ vertices with faulty edge set F'. Obviously, $|F'| \leqslant 2(n-1) - 8$ for $n \notin \{8,10\}$ and $|F'| \leqslant 2(n-1) - 9$ for $n \in \{8,10\}$. Moreover, $\delta(K_n - \{u\} - F') \geqslant 2$. Thus, we can apply Theorem 1 to obtain a hamiltonian cycle C in $K_n - \{u\} - F'$. Without loss of generality, we write C as $\langle v, x, \ldots, y, v\rangle$. Then, $\langle u, x, \ldots, y, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v.

Case 2. $\deg_{K_n(F)}(u) = 0$ and $\deg_{K_n(F)}(v) = 0$. By Lemmas 1, 2, and 3, there exists a vertex w such that $2 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \in \{9,11\}$ and $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \notin \{9,11\}$.

Obviously, $\delta(K_n - F - \{w\}) \geqslant 2$. Suppose that $\delta(K_n - F - \{w\}) = 2$. Let x be any vertex in $K_n - \{w\}$ such that $\deg_{K_n - \{w\} - F}(x) = 2$. Obviously, $(x,w) \notin F$, $\deg_{K_n - F}(x) = 3$, and $\deg_{K_n(F)}(x) = n - 4$. We claim that x is the only vertex in $K_n - \{w\}$ with $\deg_{K_n - \{w\} - F}(x) = 2$. If otherwise, let z be another vertex in $K_n - \{w\}$ with $\deg_{K_n - \{w\} - F}(z) = 2$. Then $|F| \geqslant \deg_{K_n(F)}(x) + \deg_{K_n(F)}(z) - 1 = 2n - 9$. This is impossible because $|F| \leqslant 2n - 10$. Thus, x is the only vertex in $K_n - \{w\}$ such that $\deg_{K_n - \{w\} - F}(x) = 2$. Thus, $\delta(K_n - F - \{u, x\}) \geqslant 3$.

Let $F' = F - \{(x,i) \mid i \in V(K_n)\}$. We consider $K_n - \{u,x\}$ as a complete graph of $(n-2)$ vertices with faulty edge set F'. Obviously, $|F'| = 1 \leqslant 2$ for $n = 7$, $|F'| = n - 7 \leqslant 2(n-2) - 10$ for $n \notin \{10,12\}$, and $|F'| = n - 7 \leqslant 2(n-2) - 11$ for $n \in \{10,12\}$. By induction, we have a hamiltonian path P of $K_n - \{u,x\} - F'$ joining w to v. So $\langle u, x, w, P, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v.

Now, we consider $\delta(K_n - \{w\} - F) \geqslant 3$. Since $2 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \in \{9,11\}$ and $1 \leqslant \deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ for $n \notin \{9,11\}$, there exists $(x,y) \in F$ such that $\{(w,x),(w,y)\} \cap F = \emptyset$. We set F' as $F - \{(w,z) \mid (w,z) \in F\} - \{(x,y)\}$ and consider $K_n - \{w\}$ with faulty set F'. We have $|F'| = 2n - 10 - \deg_{K_n(F)}(w) - 1 \leqslant 2n - 13$ for $n \in \{9,11\}$ and $|F'| = 2n - 10 - \deg_{K_n(F)}(w) - 1 \leqslant 2n - 12$ for $n \notin \{9,11\}$. By induction, there exists a hamiltonian path $P = \langle u = x_1, x_2, \ldots, x_{n-1} = v\rangle$ of $K_n - \{w\} - F'$ joining u to v. Suppose that $(x,y) \in P$. There exists an integer i such that $\{x_i, x_{i+1}\} = \{x, y\}$ for some i. Suppose that $(x,y) \notin P$. Since $\deg_{K_n(F)}(w) \leqslant \lfloor \frac{n-1}{2} \rfloor - 1$ and $\deg_{K_n(F)}(w) + \deg_{K_n - F}(w) = n - 1$, $\deg_{K_n - F}(w) \geqslant \lfloor \frac{n}{2} \rfloor + 1$. Hence, there exists an integer i such that $(x_i, x_{i+1}) \in P$ and $\{(w,x_i),(w,x_{i+1})\} \cap F = \emptyset$. Therefore, $\langle u = x_1, x_2, \ldots, x_i, w, x_{i+1}, x_{i+2}, \ldots, v\rangle$ forms a hamiltonian path of $K_n - F$ joining u to v. □

Theorem 3. *Let $n \geqslant 4$. Then $\mathcal{HC}_e^3(K_n) = 2n - 10$ for $n \notin \{4,5,8,10\}$, $\mathcal{HC}_e^3(K_4) = 0$, $\mathcal{HC}_e^3(K_5) = 2$, $\mathcal{HC}_e^3(K_8) = 5$, and $\mathcal{HC}_e^3(K_{10}) = 9$.*

Proof. Let F be any subset of $E(K_n)$ with $\delta(K_n - F) \geqslant 3$. Since $\delta(K_n - F) \geqslant 3$, $|F| = 0$ for $n = 4$ and $|F| \leqslant 2$ for $n = 5$. Thus, $\mathcal{HC}_e^3(K_4) = 0$ and $\mathcal{HC}_e^3(K_5) = 2$.

Suppose $n = 8$. Let $V(K_8) = \{x_1, x_2, \ldots, x_8\}$. We set $R = \{x_1, \ldots, x_4\}$, $S = \{x_5, \ldots, x_8\}$, and $F = \{(u,v) \mid u, v \in R\}$. We can check that $\delta(K_8 - F) \geqslant 3$, $|F| = 6$ and $(K_8 - F) - S$ has four connected components. By Lemma 5, $K_8 - F$ is not hamiltonian connected. See Fig. 1(a) for illustration. Thus, $\mathcal{HC}_e^3(K_8) < 6$. By Lemma 6, $\mathcal{HC}_e^3(K_8) = 5$.

Suppose $n = 10$. Let $V(K_{10}) = \{x_1, x_2, \ldots, x_{10}\}$. We set $R = \{x_1, \ldots, x_5\}$, $S = \{x_6, \ldots, x_{10}\}$, and $F = \{(u,v) \mid u, v \in$

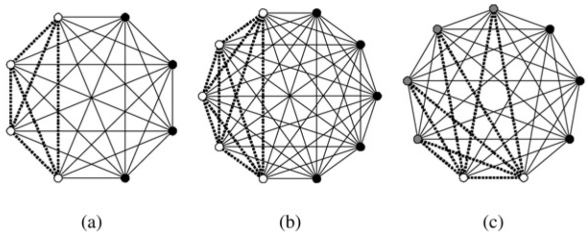

Fig. 1. All white vertices are in R, all black vertices are in S, and all gray vertices are in T. All dashed lines are in F.

$R\}$. Then, $\delta(K_{10} - F) \geqslant 3$, $|F| = 10$, and $(K_{10} - F) - S$ has five connected components. By Lemma 5, $K_{10} - F$ is not hamiltonian connected. See Fig. 1(b) for illustration. Thus, $\mathcal{HC}_e^3(K_{10}) < 10$. By Lemma 6, $\mathcal{HC}_e^3(K_{10}) = 9$.

Suppose that $n \in \{6, 7, 9\} \cup \{i \mid i \geqslant 11\}$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. We set $R = \{x_1, x_2\}$, $S = \{x_3, x_4, x_5\}$, $T = \{x_6, \ldots, x_n\}$, and $F = \{(u, v) \mid u \in R, v \in R \cup T\}$. Obviously, $\delta(K_n - F) \geqslant 3$, $|F| = 2(n-5)+1 = 2n-9$, and $(K_n - F) - S$ has three connected components. See Fig. 1(c) for illustration for case $n = 9$. By Lemma 5, $K_n - F$ is not hamiltonian connected. Thus, $\mathcal{HC}_e^3(K_n) < 2n - 9$. By Lemma 6, $\mathcal{HC}_e^3(K_n) = 2n - 10$.

The theorem is proved. □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, New York, 1980.

[2] M.Y. Chan, S.J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM Journal on Discrete Mathematics 4 (1991) 511–527.

[3] J.S. Fu, Conditional fault hamiltonicity of the complete graph, Information Processing Letters 107 (2008) 110–113.

[4] W.T. Huang, J.J.M. Tan, C.N. Hung, L.H. Hsu, Fault-tolerant hamiltonicity of twisted cubes, Journal of Parallel and Distributed Computing 62 (2002) 591–604.

[5] S. Latifi, S.Q. Zheng, N. Bagherzadeh, Optimal ring embedding in hypercubes with faulty links, in: Proceedings of the IEEE Symposium on Fault-Tolerant Computing, 1992, pp. 178–184.

[6] T.K. Li, J.J.M. Tan, L.H. Hsu, Hyper Hamiltonian laceability on the edge fault star graph, Information Sciences 165 (2004) 59–71.

[7] J.W. Moon, On a problem of Ore, Math. Gaz. 49 (1965) 40–41.

[8] O. Ore, Hamilton connected graphs, Journal of Mathematic Pures Application 42 (1963) 21–27.

[9] O. Ore, Note on Hamilton circuits, The American Mathematical Monthly 67 (1960) 55.