國 立 交 通 大 學

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碩 士 論 文

黎曼曲面與橢圓函數的理論
及其對正弦高登方程的應用

The Theories of
Riemann Surfaces and Elliptic Functions
with Application to the sine-Gordon Equation

研究生：陳建澤

指導教授：李榮耀 教授

中華民國 一〇二 年 六 月

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碩 士 論 文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
In Partial Fulfillment of the Requirements
for the Degree of Master
in
Applied Mathematics
June, 2013
Hsinchu, Taiwan, Republic of China

中華民國 一〇二 年 六 月

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摘要

我們有興趣的是，研究正弦高登方程的一些特殊解，正弦高登方程如下：

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0$$

其中 $-\infty < x < \infty$，而且 $t > 0$。

經由變數變換，我們可以將原本的方程式變成以下的形式：

$$u_{ss} + \sin[u(s)] = 0$$

這是一個對於時間 s 的單擺運動方程式，而且我們可以繼續推導變成 $u_s = \sqrt{2[E + \cos(u)]}$，其中 E 是一個常數。

可是 $\sqrt{2[E + \cos(u)]}$ 是一個複數上的雙值函數，所以我們介紹黎曼曲面 $\mathfrak{R}$ 的理論，使得這一個函數在這個曲面上變成了一個可以分析的單值函數。

接下來，我們介紹橢圓函數的古典理論，並且利用它去對 $u_{ss} + \sin[u(s)] = 0$ 求解，並分析相關的性質。

中華民國 一0二 年 六 月

The Theories of
Riemann Surfaces and Elliptic Functions
with Application to the sine-Gordon Equation

Student: Jian-Ze Chen   Advisor: Jong-Eao Lee

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Abstract

The Goal of this paper is to solve the sine-Gordon equation,

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0, \text{ where } -\infty < x < \infty \text{ and } t > 0.$$

By using the method of substitution, we get

$$u_{ss} + \sin[u(s)] = 0,$$

which is a simple pendulum motion at time s with the angular displacement u, and it implies $u_s = \sqrt{2[E + \cos(u)]}$, where E is constant.

But $\sqrt{2[E + \cos(u)]}$ is a two-valued function on $\mathbb{C}$, so we introduce the theory of the Riemann surface $\mathfrak{R}$ such that it comes to a single-valued analytic function on this surface.

Next, we introduce the classical theory of the elliptic functions, to solve $u_{ss} + \sin[u(s)] = 0$, and analyze the associated properties.

June, 2013

誌謝

在整篇論文的研究中，非常感謝指導教授李榮耀 老師。老師在給我們方向之後，給我們很大的空間去研究與發揮，對於我們的問題也會親切地解惑，並鼓勵我們學生互相協助與討論。回想研究的一年，過程與回憶是有收穫的，非常感謝。

在努力的過程中，同伴的協助對於我的研究，是非常重要的。感謝同一間研究室的范名宏，給我極大的鼓勵與協助，毫不保留地幫助我、聽我抱怨，非常感恩；感謝蔣宜津，這一路上我們時常在討論研究中所遇到的許多疑惑跟難題，在討論的過程中，我們都領悟了許多學術分析上的技巧，很謝謝妳這一年來的協助。

當然也要感謝我的父母陳協成、黃美芬，研究最後的這一年，我幾乎都待在新竹，沒有辦法時常回去陪伴您們，也謝謝您們的包容與鼓勵。

最後我要感謝無時無刻都陪伴我的黃政運，不管我在學校研究到多晚，你都願意來載我回去，就算我半夜還在用電腦寫論文，可能打擾到你的睡眠，你依然包容我、為我加油。這一年，我的成就，多虧有你，謝謝你。

最後謝謝我所有親朋好友的陪伴，感恩。

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Chapter 1

Introduction

The Goal of this paper is to solve a second-order partial differential equation, that of the sine-Gordon equation [1].

The sine-Gordon equation is

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0, \text{ where } -\infty < x < \infty \text{ and } t > 0,$$

and by letting $s = \alpha x - \beta t$ with $\beta^2 - \alpha^2 = 1$, the original equation comes to

$$u_{ss} + \sin[u(s)] = 0.$$

We multiply an intergral factor u_s on both sides, and integral it with respect to s, we get $\frac{1}{2}u_s^2 - \cos(u) = E$, where E is constant, then $u_s = \pm\sqrt{2[E + \cos(u)]}$.

Assume without loss of generality that $u_s = \sqrt{2[E + \cos(u)]}$, it is a two-valued function on complex plane $\mathbb{C}$, so we introduce the theory of the Riemann surface $\mathfrak{R}$ such that $\sqrt{2[E + \cos(u)]}$ is a single-valued analytic function on this surface [2].

Next, we discuss the situations of the modified value in the MATHEMATICA on the Riemann surface $\mathfrak{R}$ [3][4], so that we can use this skill to solve $u_s = \sqrt{2[E + \cos(u)]}$ by approximation of $\cos(u)$ by the Taylor series at 0.

We back to consider $u_s = \dfrac{du}{ds} = \sqrt{2[E + \cos(u)]}$, we get

$$s = \int_0^{u(s)} \frac{1}{\sqrt{2[E + \cos(U)]}} dU,$$

and in order to solve it, we introduce the classical theory of the elliptic functions [5]. The theory occurs on Riemann surfaces of genus 1.

Finally, we use this theory to find out the special solutions and the corresponding periods of above integral problem on Riemann surfaces of genus 1.

Chapter 2

The Riemann Surfaces of Genus N

To begin with, we have $z \in \mathbb{C}$, and by using the polar form,

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= |z|e^{i\theta}, \text{ for some } \theta, \text{ where } e \text{ is an Euler's number.} \end{aligned}$$

Now, let $f(z) = \sqrt{z}$, and notice that $z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}$, $\forall n \in \mathbb{Z}$, then

$$\begin{aligned} f(z) &= \sqrt{z} \\ &= |z|^{\frac{1}{2}} e^{i\frac{\theta+2n\pi}{2}} \\ &= \begin{cases} |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \text{if } n \text{ is even,} \\ -|z|^{\frac{1}{2}} e^{i\frac{\theta}{2}} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

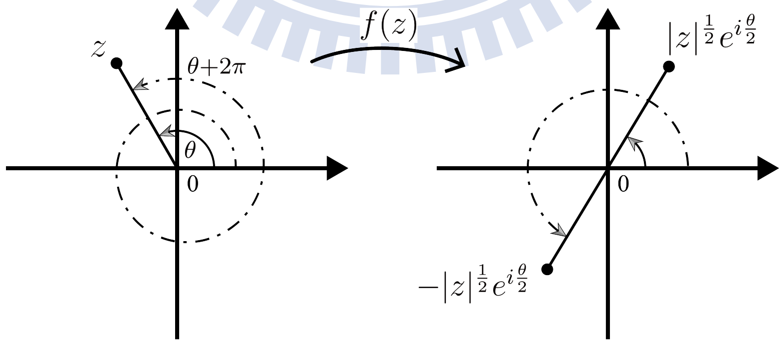

Fig. 2.1 $f(z) = \sqrt{z}$

Thus, $f(x)$ is a two-valued function, so we want to let $f(x)$ become a single-valued function. Now, we will modify its domain $\mathbb{C}$ to construct the corresponding new surface, *Riemann surface* $\mathfrak{R}$, such that $f(x)$ becomes a single-valued analytic function on $\mathfrak{R}$.

2.1 The Geometric Structure

Consider some nonzero complex number $z_0 = |z_0|e^{i\alpha}$, for some α, we have

$$\begin{aligned} f(z_0) &= \sqrt{z_0} \\ &= |z_0|^{\frac{1}{2}} e^{i\frac{\alpha}{2}}. \end{aligned}$$

Fixing $|z_0|$, and if α increases by 2π, $f(z_0)$ comes to the value $|z_0|^{\frac{1}{2}} e^{i\frac{\alpha+2\pi}{2}} = -|z_0|^{\frac{1}{2}} e^{i\frac{\alpha}{2}}$, which is precisely the negative of its original value. Continuing above way, then α increases by 2π, $f(z_0)$ comes to its original value.

In this sense, we cut the extended complex plane along the line, $re^{i\alpha}$, for all non-negative $r \in \mathbb{R}$, and restrict the angle, we get two single-valued branches of $f(z)$:

$$\begin{cases} f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}, \ \theta \in [\alpha - 2\pi, \alpha), \\ f(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}, \ \theta \in [\alpha, \alpha + 2\pi). \end{cases}$$

Now, we take these two cut planes of the complex plane, and call them <u>*sheet-I*</u> and <u>*sheet-II*</u>, and call the cut a <u>*branch cut*</u>.

Notice that, the cut on each sheet has two edges, we label the starting edge with a $+$, and label the ending edge with a $-$. Then attach the $+$ edge of the cut on one of these two sheets, to the $-$ edge of the cut on the other. Thus, whenever we cross the cut, we pass from one sheet to the other.

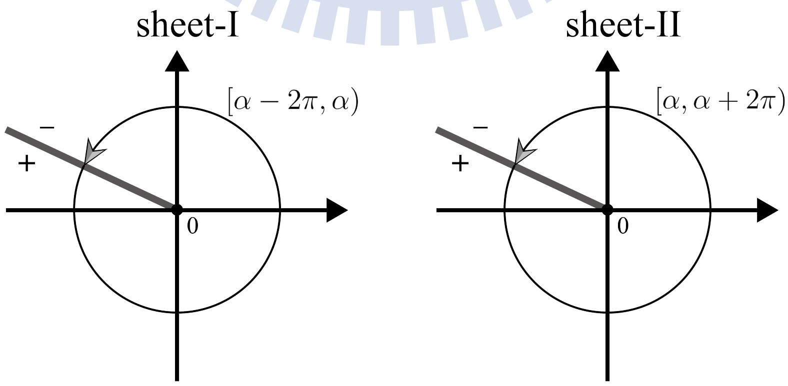

Fig. 2.2 <u>Sheet-I and sheet-II</u>

This two-sheeted surface we cannot be realized in three-dimensional Euclidean space, so we want to build a new surface.

To begin with, we use the stereographic projection, the two sheets to be the Riemann spheres (Fig. 2.3).

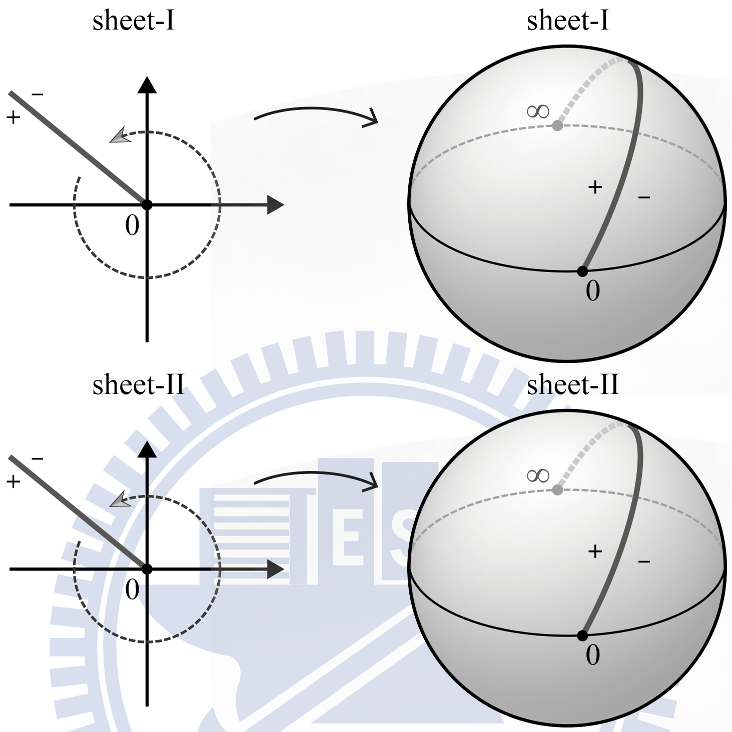

Fig. 2.3 Complex planes and the corresponding Riemann spheres

Furthermore, we pretend that the spheres are made of rubber, we separate the edges of cuts (Fig. 2.4).

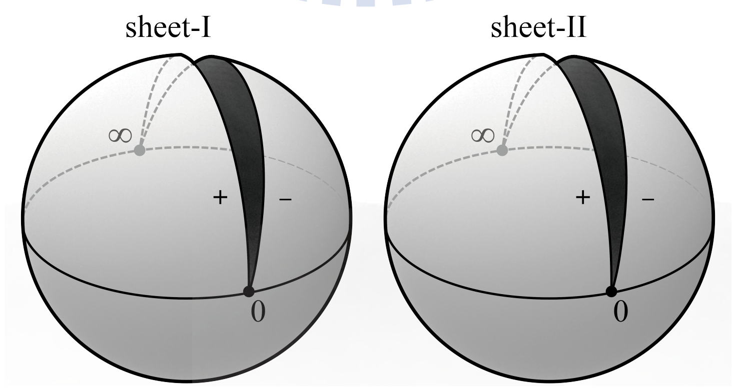

Fig. 2.4 Separating the edges of cuts on the spheres

Finally, we deform each sheet into a hemisphere, and rotated each sheet so that the opening of the hemipheres face each other (Fig. 2.5), and paste the edges marked + and − each other. Therefore, we glue the two hemishperes together to be a sphere and call it a *Riemann surface* $\mathfrak{R}$ of genus 0 (Fig. 2.6).

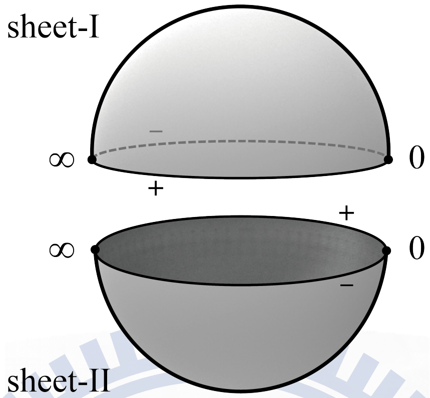

Fig. 2.5 Opening of the hemipheres face each other

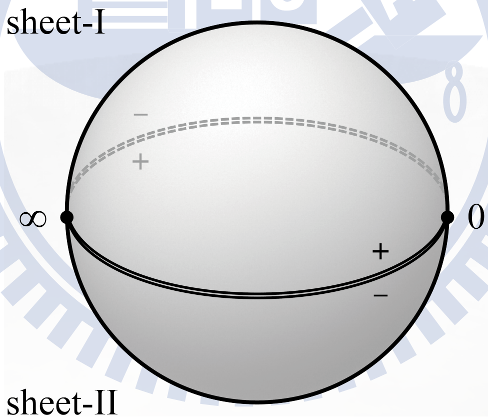

Fig. 2.6 Riemann surface $\mathfrak{R}$ of genus 0

Notice that, in this new surface $\mathfrak{R}$, the + edge of sheet-I is equivalent to the − edge of sheet-II, and the − edge of sheet-II is equivalent to the + edge of sheet-I.

Since $f(z) = \sqrt{z} = \sqrt{z-0}$, we call 0 a <u>*branch point*</u>.

Consider the different branch point z_1, in the manner of $f(z) = \sqrt{z}$, we also have the Riemann surface $\mathfrak{R}$ of genus 0 (Fig. 2.7).

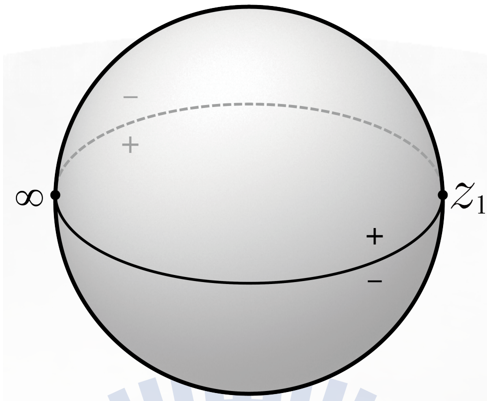

Fig. 2.7 <u>$\mathfrak{R}$ with branch point z_1</u>

Before we discuss the general case of the construction of $\mathfrak{R}$, consider the following situation.

We consider $f(z) = \sqrt{\prod(z - z_k)}$ for distinct $z_k \in \mathbb{C}$, all arguments of z_k are same, and $|z_1| < |z_2| < \ldots < |z_n|$, i.e., for all z_k on the same line orderly.

Now, we cut plane start from z_k to ∞ along this line. Recall that we pass to another sheet when we cross the cut, and so $f(z)$ is precisely the negative of its original value. Therefore, if the path cross even cuts, we pass to the same sheet, that is, there's no branch cut; on the contrary, if the path cross odd cuts, there's a branch cut (Fig. 2.8).

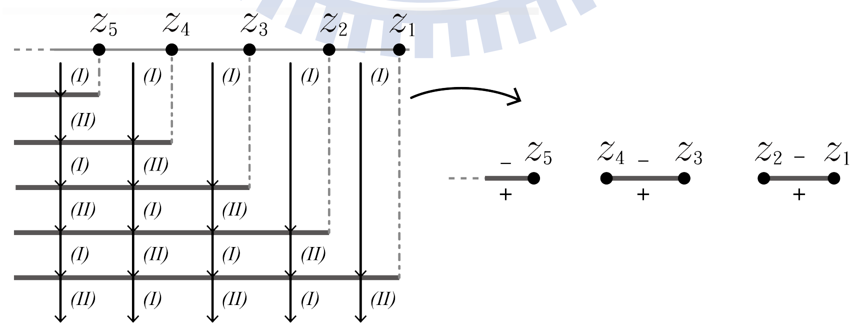

Fig. 2.8 <u>Cut plane and the corresponding branch cuts</u>

Now, we consider the general case,

$$
\begin{aligned}
f(z) &= \sqrt{\prod_{k=1}^{n}(z - z_k)} \\
&= \sqrt{(z - z_1)}\sqrt{(z - z_2)} \dots \sqrt{(z - z_n)},
\end{aligned}
$$

where z_k are distinct branch points, then we have m branch cuts:

$$
\begin{cases}
1.\ \overline{z_1 z_2}, \overline{z_3 z_4}, \dots, \overline{z_{2m-1} z_{2m}},\ \text{if } n = 2m, \\
2.\ \overline{z_1 z_2}, \overline{z_3 z_4}, \dots, z_{2m-1} \to \infty,\ \text{if } n = 2m-1.
\end{cases}
$$

In like manner, there are two sheets and the corresponding Riemann spheres, and we separate the edges of cuts (Fig. 2.9).

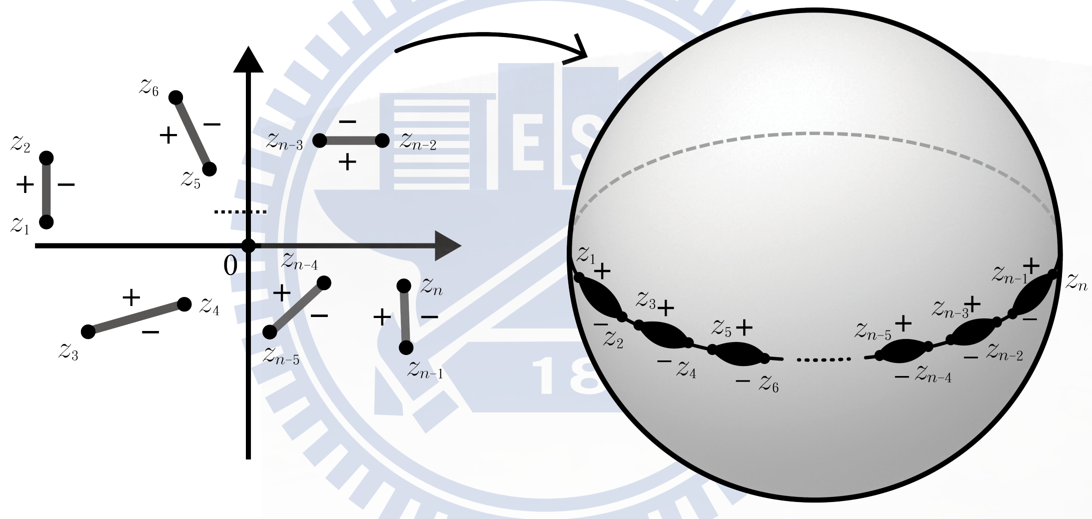

Fig. 2.9 Complex plane and the corresponding Riemann sphere

Finally, we deform and rorate each sheet so that the opening of them face each other (Fig. 2.10), and paste the edges marked + and − each other. Therefore, we glue them together to be a Riemann surface $\mathfrak{R}$ of genus $m-1$, that is, $m-1$ holes (Fig. 2.11).

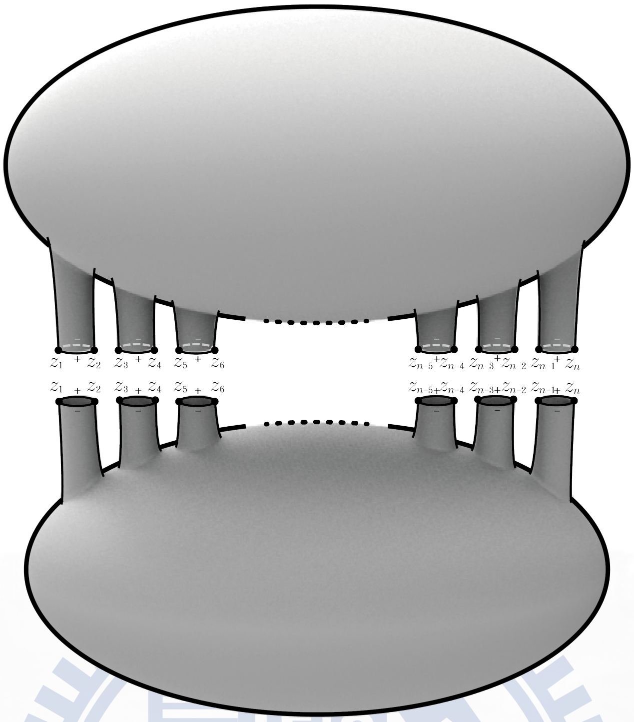

Fig. 2.10 Opening of the hemipheres face each other

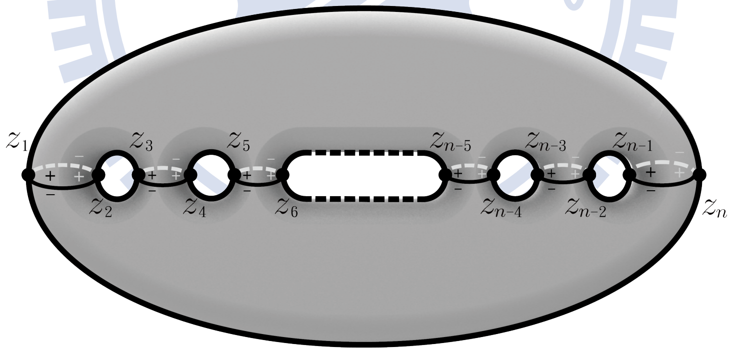

Fig. 2.11 Riemann surface $\mathfrak{R}$ of genus $m-1$

Notice that, in this case, if $f(z)$ has $2m$ or $2m-1$ branch points, we have m branch cuts, and construct the Riemann surface $\mathfrak{R}$ of genus $m-1$.

2.2 The Algebraic Structure

Now, in order to distinguish which sheet a path belongs when this path in the complex plane, we define that the path in the sheet-I is solid line, and the path in the sheet-II is dash line.

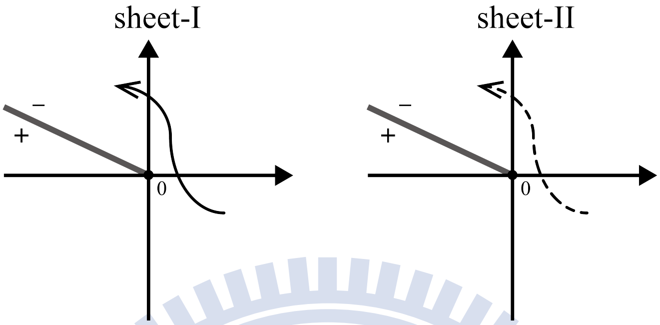

Fig. 2.12 Deffrence of the same path in sheet-I and sheet-II

The followings are some examples of those paths in $\mathbb{C}$ and the corresponding paths in $\mathfrak{R}$.

Example 2.1. *The path is start from point A on the $+$ edge in the sheet-I, to point B on the $-$ edge in the sheet-I, and cross the cut to point C on the $-$ edge in the sheet-II, shown in Fig. 2.13.*

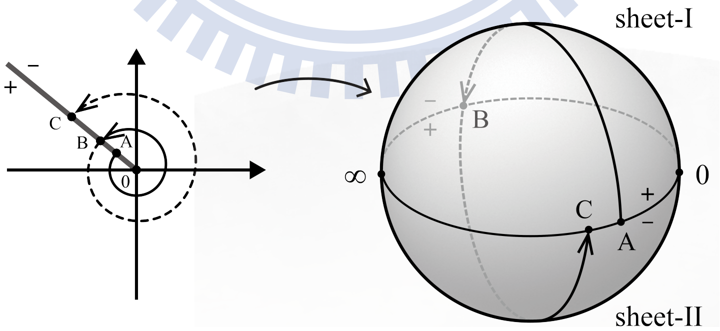

Fig. 2.13 Example 2.1, the path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$

Example 2.2. *The path is start from point A in the sheet-I, to point B on the − edge in the sheet-I, and cross the cut to point C in the sheet-II, shown in Fig. 2.14.*

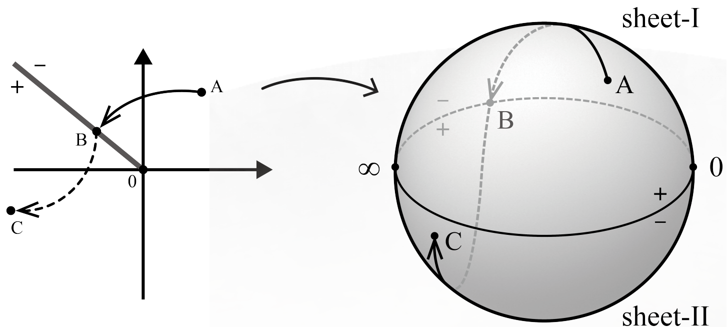

Fig. 2.14 Example 2.2, the path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$

Example 2.3. *The path is start from point A in the sheet-I, to point B on the + edge in the sheet-I, and cross the cut to point C in the sheet-II, shown in Fig. 2.15.*

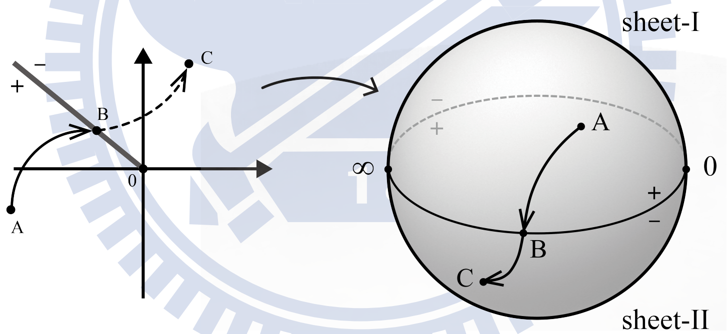

Fig. 2.15 Example 2.3, the path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$

The followings are the different situations of same path in $\mathbb{C}$ with the different branch points.

We consider the path is start from point A to point B, shown in Fig. 2.16.

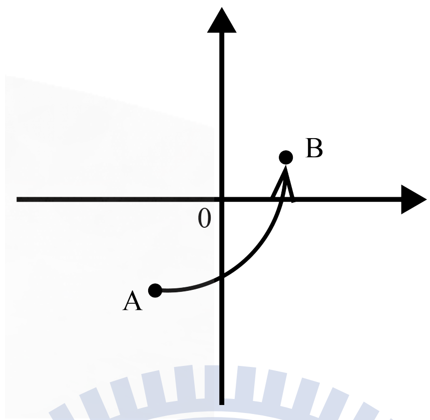

Fig. 2.16 Path for case of the different branch points

Then, we consider the branch points 0 and z_1, we get the different corresponding paths in $\mathfrak{R}$, shown in Fig. 2.17 and 2.18.

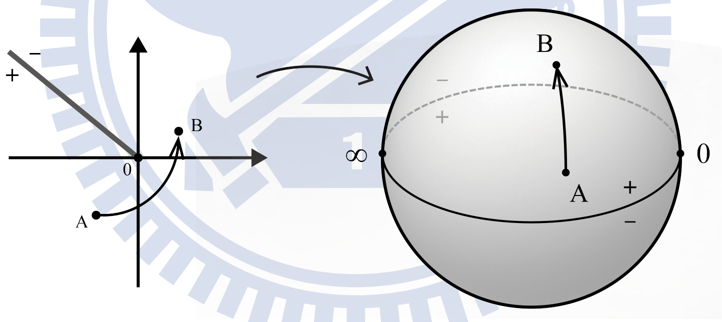

Fig. 2.17 Path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$, with the branch points 0

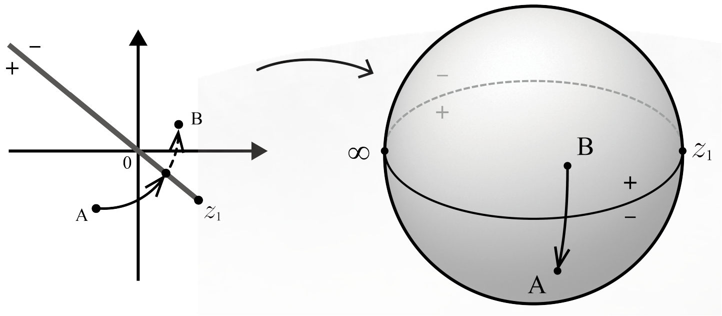

Fig. 2.18 Path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$, with the branch points z_1

And, the followings are the different situations of same path in $\mathbb{C}$ with the different branch cuts.

We consider the path is start from point A to point B, shown in Fig. 2.19.

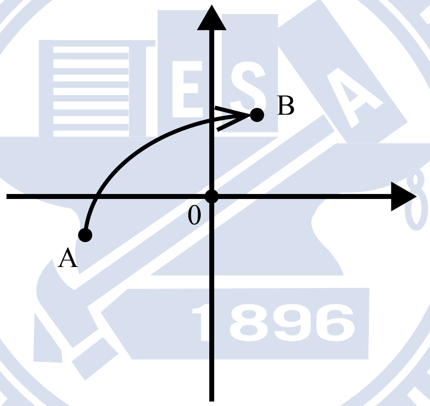

Fig. 2.19 Path for case of the different branch cuts

Then, we consider the different branch cuts, we get the different corresponding paths in $\mathfrak{R}$, shown in Fig. 2.20 and 2.21.

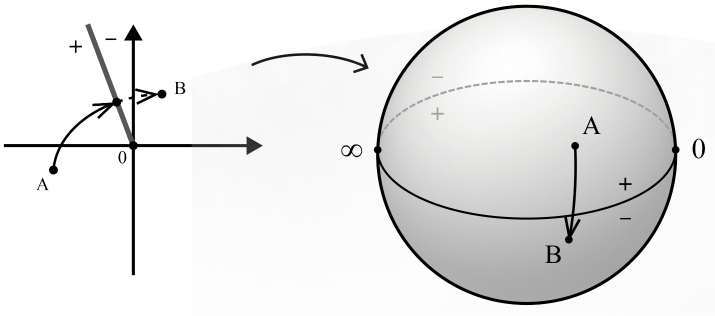

Fig. 2.20 Path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$, with the cut of first kind

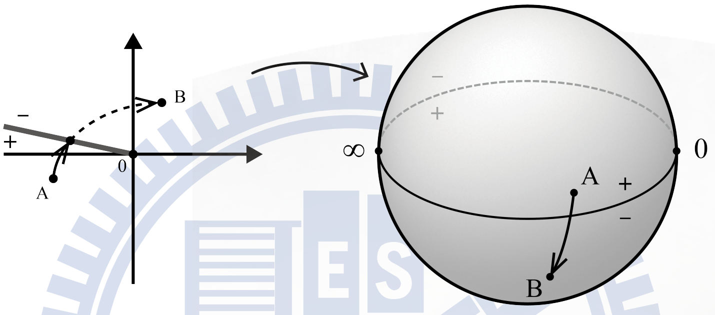

Fig. 2.21 Path in $\mathbb{C}$ and the corresponding path in $\mathfrak{R}$, with the cut of second kind

2.3 The Paths on the Riemann Surfaces of Genus N with Algebraic Structures

Notice that, by the Cauchy integral formula, for any closed path in $\mathfrak{R}$ of genus m is homotopic to an integral linear combination of the loop-cuts a_i and b_i, $i = 1, 2, \ldots, m$, so we will consider the integrals of $f(z)$ over a, b-cycles, in this paper, help us to obtain the integrals.

The followings are some examples of a, b-cycles in $\mathbb{C}$ and the corresponding cycles in $\mathfrak{R}$.

Example 2.4. *Consider* $f(z) = \sqrt{\prod_{k=1}^{5}(u - u_k)}$, *shown in Fig. 2.22.*

And, we consider the cycles a_1 *and* a_2 *in* $\mathbb{C}$, *and the corresponding cycles in* $\mathfrak{R}$ *of genus* 2.

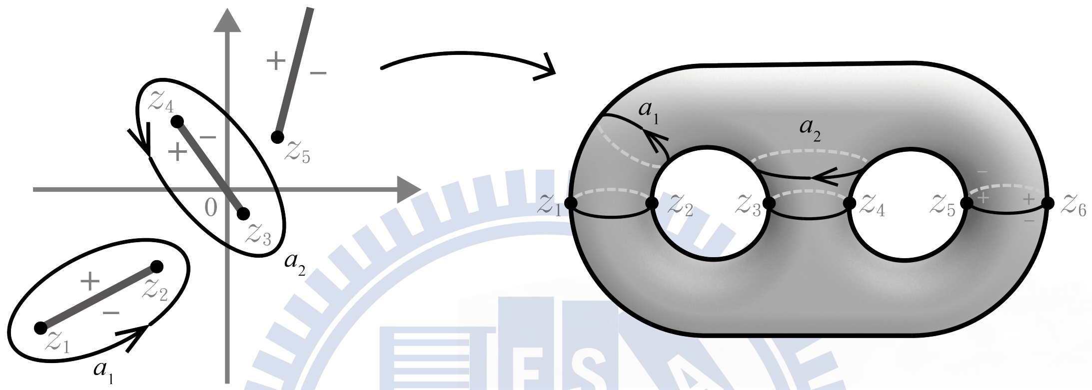

Fig. 2.22 Example 2.4, the cycles a_i in $\mathbb{C}$ and the corresponding cycles in $\mathfrak{R}$

Example 2.5. *Consider* $f(z) = \sqrt{\prod_{k=1}^{5}(u - u_k)}$, *shown in Fig. 2.23.*

And, we consider the cycle b_1 *and* b_2 *in* $\mathbb{C}$, *and the corresponding cycles in* $\mathfrak{R}$ *of genus* 2.

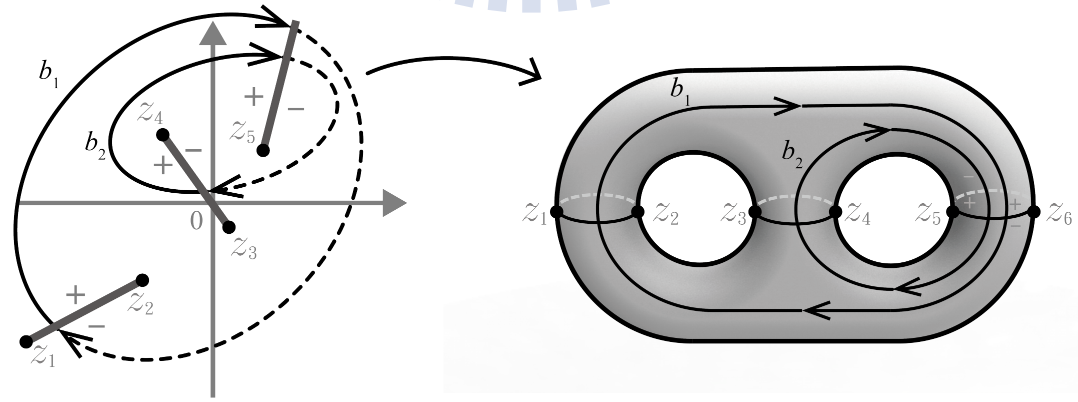

Fig. 2.23 Example 2.5, the cycles b_i in $\mathbb{C}$ and the corresponding cycles in $\mathfrak{R}$

2.4 The Path Integrals on Riemann Surfaces

Consider the line with the slope $m = \tan\alpha$, $0 < \alpha \leq \pi$, and for any cuts on this line, we define that

$$z = \begin{cases} re^{i\theta},\ \theta \in [\alpha - 2\pi, \alpha) & \text{iff } z \text{ is in the sheet-I,} \\ re^{i\theta},\ \theta \in [\alpha, \alpha + 2\pi) & \text{iff } z \text{ is in the sheet-II.} \end{cases}$$

To begin with, we consider the values in sheet-I and sheet-II.

If we have $f(z) = \sqrt{\sum_{k=1}^{n}(z - z_k)}$, and by using the polar form,

$$\sum_{k=1}^{n}(z - z_k) = re^{i\theta_1} = re^{i\theta_2},$$

where θ_1 and θ_2 is in sheet-I and sheet-II respectively, so that $\theta_2 = \theta_1 + 2\pi$.

Therefore,

$$\begin{aligned} f(z)|_{(II)} &= \sqrt{r}e^{i\frac{\theta_2}{2}} \\ &= \sqrt{r}e^{i\frac{\theta_1+2\pi}{2}} \\ &= \sqrt{r}e^{i\frac{\theta_1}{2}}e^{i\pi} \\ &= -\sqrt{r}e^{i\frac{\theta_1}{2}} = -f(z)|_{(I)}, \end{aligned}$$

where $f(z)|_{(I)}$ and $f(z)|_{(II)}$ denotes the value of $f(z)$ with z in sheet-I and sheet-II respectively.

Notice that the difference of argument between z in these two sheets is 2π, so the difference of argument between $f(z)|_{(I)}$ and $f(z)|_{(II)}$ is π, and this implies $f(z)|_{(II)} = -f(z)|_{(I)}$.

If we could find the equivalent straight paths of a, b-cycles, it would be easier to calculate the integrals over these cycles.

First, consider an a-cycle is shown in Fig. 2.24.

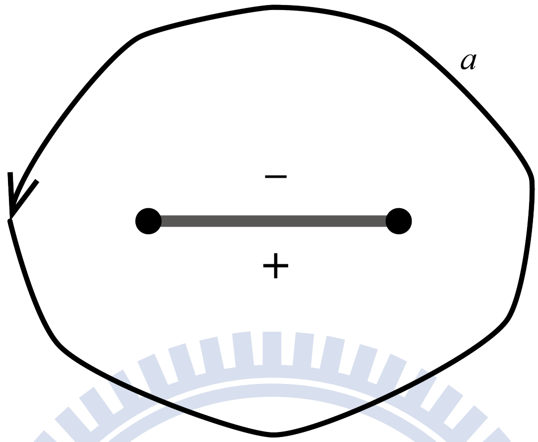

Fig. 2.24 Cycle a crossover a cut

We consider these paths are shown in Fig. 2.25.

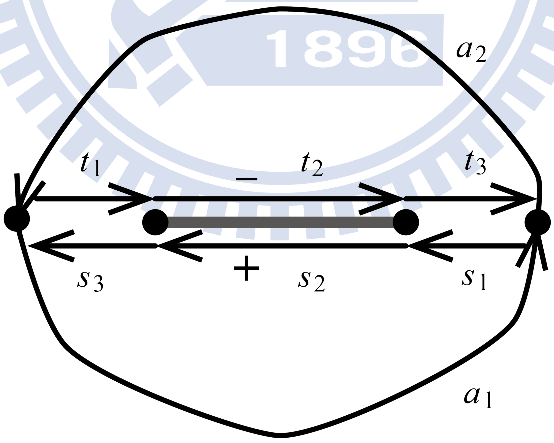

Fig. 2.25 Cycle $a = a_1 \cup a_2$ crossover a cut and some auxiliary paths

Notice that, by the Cauchy's integral theorem,

$$\int_{t_1} f dz + \int_{s_3} f dz = 0 \;\Rightarrow\; \int_{t_1} f dz = -\int_{s_3} f dz,$$

and

$$\int_{t_3} f dz + \int_{s_1} f dz = 0 \;\Rightarrow\; \int_{t_3} f dz = -\int_{s_1} f dz.$$

Again, by the Cauchy integral theorem,

$$\begin{cases} \int_{a_1} f dz + \int_{s_1} f dz + \int_{s_2} f dz + \int_{s_3} f dz = 0 \\ \int_{a_2} f dz + \int_{t_1} f dz + \int_{t_2} f dz + \int_{t_3} f dz = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \int_{a_1} f dz + \int_{s_1} f dz + \int_{s_2} f dz + \int_{s_3} f dz = 0 \\ \int_{a_2} f dz - \int_{s_3} f dz + \int_{t_2} f dz - \int_{s_1} f dz = 0 \end{cases}$$

$$\Rightarrow \int_{a_1} f dz + \int_{s_2} f dz + \int_{a_2} f dz + \int_{t_2} f dz = 0$$

$$\Rightarrow \int_{a} f dz = -\int_{s_2} f dz - \int_{t_2} f dz$$

$$\Rightarrow \int_{a} f dz = \int_{r_1} f dz + \int_{r_2} f dz, \text{ where } r_1 \text{ and } r_2 \text{ are shown in Fig. 2.26.}$$

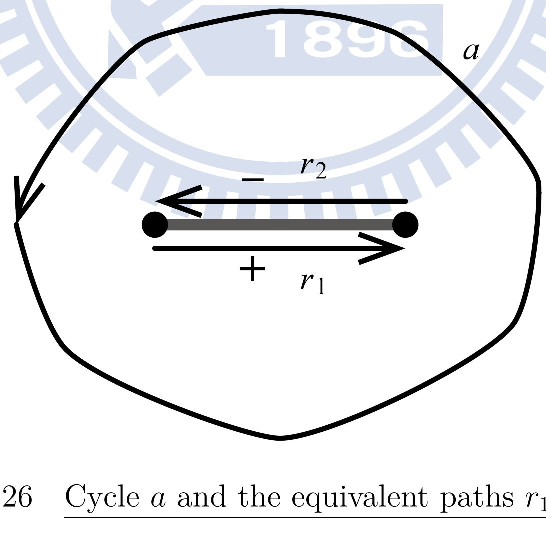

Fig. 2.26 Cycle a and the equivalent paths r_1 and r_2

So, we get that the paths r_1 and r_2 are the equivalent paths of cycle a, i.e., $a \approx r_1 \cup r_2$. □

Next, consider a b-cycle is shown in Fig. 2.27.

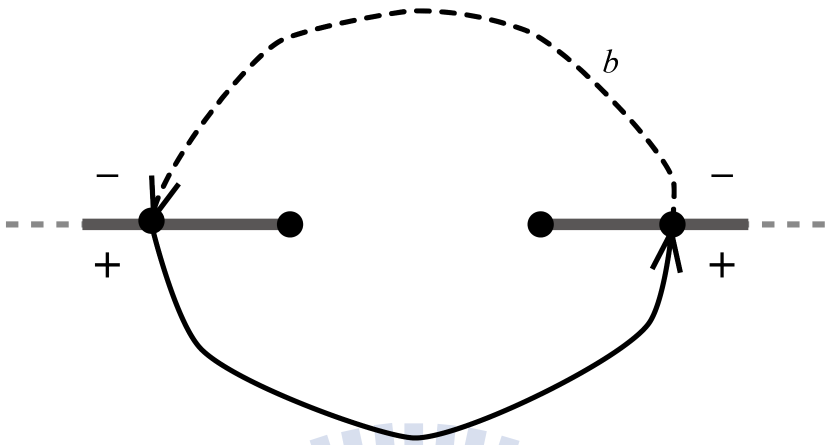

Fig. 2.27 Cycle b crossover two cuts

We consider these paths are shown in Fig. 2.28.

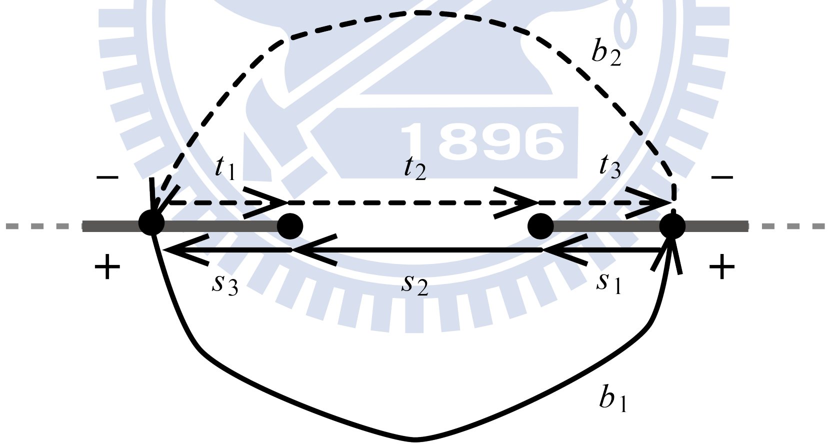

Fig. 2.28 Cycle $b = b_1 \cup b_2$ crossover two cuts and some auxiliary paths

Since

$$\begin{aligned} t_1 &\in -\text{ edge of sheet-II} \\ &= +\text{ edge of sheet-I } \Rightarrow \int_{t_1} f dz = -\int_{s_3} f dz, \end{aligned}$$

and

$$\begin{aligned} t_3 &\in -\text{ edge of sheet-II} \\ &= +\text{ edge of sheet-I } \Rightarrow \int_{t_3} f dz = -\int_{s_1} f dz. \end{aligned}$$

And, by the Cauchy integral theorem,

$$\begin{cases} \int_{b_1} f dz + \int_{s_1} f dz + \int_{s_2} f dz + \int_{s_3} f dz = 0 \\ \int_{b_2} f dz + \int_{t_1} f dz + \int_{t_2} f dz + \int_{t_3} f dz = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \int_{b_1} f dz + \int_{s_1} f dz + \int_{s_2} f dz + \int_{s_3} f dz = 0 \\ \int_{b_2} f dz - \int_{s_3} f dz + \int_{t_2} f dz - \int_{s_1} f dz = 0 \end{cases}$$

$$\Rightarrow \int_{b_1} f dz + \int_{s_2} f dz + \int_{b_2} f dz + \int_{t_2} f dz = 0$$

$$\Rightarrow \int_{b} f dz = -\int_{s_2} f dz - \int_{t_2} f dz$$

$$\Rightarrow \int_{b} f dz = \int_{r_1} f dz + \int_{r_2} f dz, \text{ where } r_1 \text{ and } r_2 \text{ are shown in Fig. 2.29.}$$

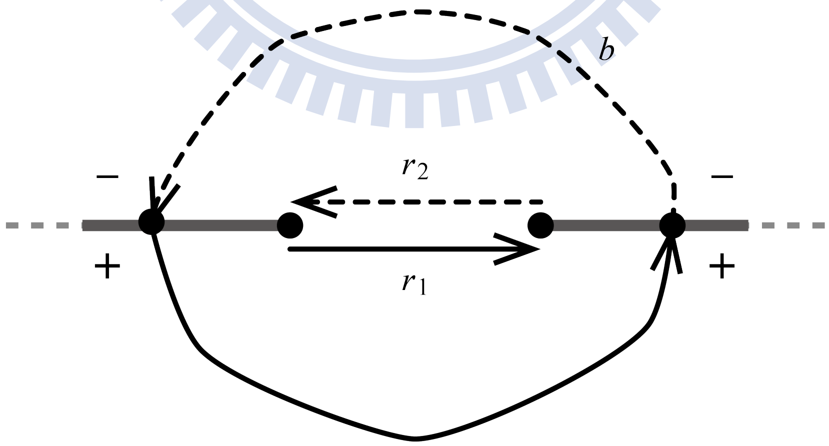

Fig. 2.29 Cycle b and the equivalent paths r_1 and r_2

So, we get that the paths r_1 and r_2 are the equivalent paths of cycle b, i.e., $b \approx r_1 \cup r_2$. □

Proposition 2.1. *If* $f(z) = \sqrt{\sum_{k=1}^{n}(z - z_k)}$, *where* z_k *are distinct, and the paths* r_1 *and* r_2 *are shown in Fig. 2.30.*

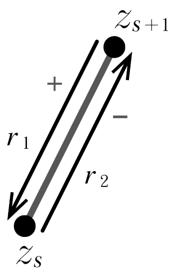

Fig. 2.30 Paths r_1 and r_2 for Pro. 2.1

Prove that

$$\int_{r_1} f(z)dz = \int_{r_2} f(z)dz.$$

Proof.

We know that

$$\begin{aligned} r_1 &\in + \text{ edge of sheet-I} \\ &= - \text{ edge of sheet-II,} \end{aligned}$$

so $\int_{r_1} f dz = \int_{r_1'} f dz$, where r_1' is a straight path from z_{s+1} to z_s on the $-$ edge of sheet-II.

And, since $f(z)|_{(II)} = -f(z)|_{(I)}$, $\int_{r_1'} f dz = -\int_{\tilde{r}_1} f dz$, where $\tilde{r}_1$ is a straight path from z_{s+1} to z_s on the $-$ edge of sheet-I.

Notice that, $\int_{\tilde{r}_1} f dz = -\int_{r_2} f(z)dz$, and this implies

$$\int_{r_1} f(z)dz = \int_{r_2} f(z)dz.$$

□

Proposition 2.2. *If* $f(z) = \sqrt{\sum_{k=1}^{n}(z - z_k)}$, *where* z_k *are distinct, and the paths* r_1 *and* r_2 *are shown in Fig. 2.31.*

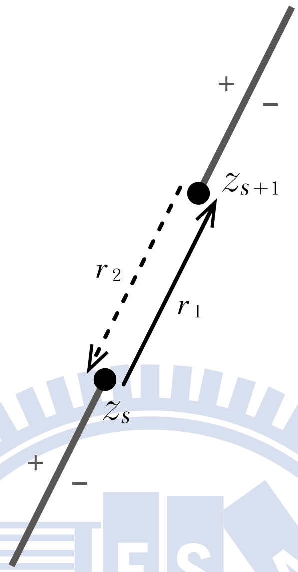

Fig. 2.31 Paths r_1 and r_2 for Pro. 2.2

Prove that

$$\int_{r_1} f(z)dz = \int_{r_2} f(z)dz.$$

Proof.

We know that $f(z)|_{(II)} = -f(z)|_{(I)}$, so

$$\int_{r_2} f(z)dz = -\int_{r_2'} f(z)dz, \text{ where } r_2' \text{ is a straight path from } z_{s+1} \text{ to } z_s \text{ in the sheet-I}$$
$$= \int_{r_1} f(z)dz.$$

□

Sometimes the integrals over some complicated paths are difficult to compute, so we want to use a computational software program, MATHEMATICA, to obtain the correct value of the integrals over cycles.

Now, we have known the difference between the value in sheet-I and sheet-II of theory, so we just discuss the difference between the value in the sheet-I of theory and in MATHEMATICA, then we can modify the computation in MATHEMATICA such that the numerical result we modified equals the numerical result of theory.

Notice that $\theta \in (-\pi, \pi]$ of $re^{i\theta}$ in MATHEMATICA, so if $\phi \notin (-\pi, \pi]$ of $re^{i\phi}$, MATHEMATICA will change $re^{i\phi}$ into $re^{i\theta^*}$, such that $re^{i\phi} = re^{i\theta^*}$ and $\theta^* \in (-\pi, \pi]$.

Lemma 2.1. *If z is in the sheet-I for a cut whose one of the end points is z_k.*
If this cut on the line with the slope $m = \tan\alpha$, $0 < \alpha \leq \pi$,

$$\sqrt{z - z_k}|_{(I)} = \begin{cases} \sqrt{z - z_k}|_{MATHEMATICA} & \text{if } arg(z - z_k) \in (-\pi, \alpha), \\ -\sqrt{z - z_k}|_{MATHEMATICA} & \text{if } arg(z - z_k) \in [\alpha - 2\pi, -\pi]. \end{cases}$$

Proof.
Let z be in the sheet-I, and using polar form, $z - z_k = re^{i\theta}$ for some $\theta \in [\alpha - 2\pi, \alpha)$.
Notice that MATHEMATICA will change $re^{i\theta}$ into $re^{i\theta^*}$ such that $re^{i\theta} = re^{i\theta^*}$, and $\theta^* \in (-\pi, \pi]$.

Case 1: $\arg(z - z_k) = \theta \in (-\pi, \alpha)$
Notice that $\theta \in (-\pi, \alpha) \subseteq (-\pi, \pi]$.
Then $\theta^* = \theta$, and this implies $\sqrt{z - z_k}|_{(I)} = \sqrt{z - z_k}|_{MATHEMATICA}$.

Case 2: $\arg(z - z_k) = \theta \in [\alpha - 2\pi, -\pi]$
Notice that $\theta \in [\alpha - 2\pi, -\pi] \nsubseteq (-\pi, \pi]$, so $\theta^* = \theta + 2\pi \in [\alpha, \pi] \subset (-\pi, \pi]$.
Then,

$$\begin{aligned} \sqrt{z - z_k}|_{(I)} &= \sqrt{r}e^{i\frac{\theta}{2}}, \textit{and} \\ \sqrt{z - z_k}|_{MATHEMATICA} &= \sqrt{r}e^{i\frac{\theta^*}{2}} \\ &= \sqrt{r}e^{i\frac{\theta+2\pi}{2}} \\ &= -\sqrt{r}e^{i\frac{\theta}{2}}, \end{aligned}$$

and this implies $\sqrt{z - z_k}|_{(I)} = -\sqrt{z - z_k}|_{MATHEMATICA}$. □

Definition 2.1.

Define that

$$\begin{cases} f(z) \overset{MATH.}{=} f(z) \text{ means } f(z)|_{(I)} = f(z)|_{MATHEMATICA}, \\ f(z) \overset{MATH.}{=} -f(z) \text{ means } f(z)|_{(I)} = -f(z)|_{MATHEMATICA}. \end{cases}$$

Next, we will discuss some situations of the domain such that the value in MATHEMATICA must be modified.

Case 1:

If z is in the sheet-I, and we consider a cut on the line $l_1(z) = 0$ with the slope $m = \tan\alpha$, $0 < \alpha < \pi$, from z_1 to ∞, where $z_1 = x_1 + iy_1 \in \mathbb{C}$ (Fig. 2.32).

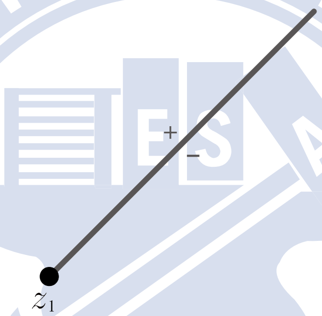

Fig. 2.32 Cut with end point z_1

1. $z \in +$ edge of this cut:

   $\arg(z - z_1) = \alpha - 2\pi$ implies $\sqrt{z - z_1} \overset{MATH.}{=} -\sqrt{z - z_1}$.

2. $z \in -$ edge of this cut:

   $\arg(z - z_1) = \alpha$ implies $\sqrt{z - z_1} \overset{MATH.}{=} \sqrt{z - z_1}$.

3. z is not on this cut:

   By the Lemma 2.1, $\sqrt{z - z_1} \overset{MATH.}{=} -\sqrt{z - z_1}$ if $\arg(z - z_1) \in [\alpha - 2\pi, -\pi]$ (Fig. 2.33).

   Thus, if $z \in \{z | z \in \mathbb{C},\ l_1(z) < 0,\ y_1 \leq \mathrm{Im}(z)\}$,

   $$\sqrt{z - z_1} \overset{MATH.}{=} -\sqrt{z - z_1}.$$

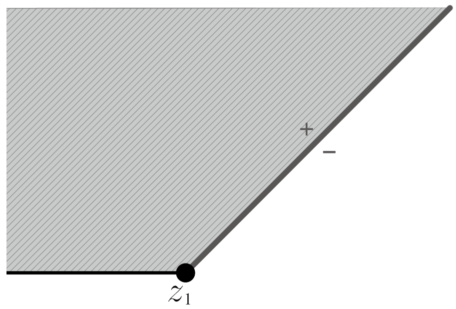

Fig. 2.33 Modified-Value Domain for the cut with end point z_1

Case 2:

We consider a cut on the line $l_1(z) = 0$ with the slope $m = \tan\alpha$, $0 < \alpha < \pi$.

Assume without loss of generality that end points of the cut are z_1 and z_2, $\forall z_k = x_k + iy_k \in \mathbb{C}$ (Fig. 2.34).

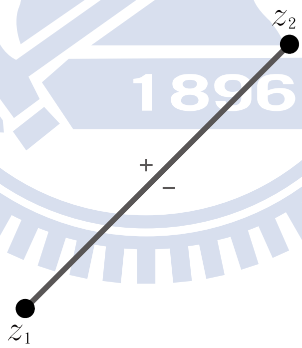

Fig. 2.34 Cut with end points z_1 and z_2

Now, we want to find the domain where z belongs, such that

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}.$$

1. $z \in +$ edge of this cut:

   (a) $\arg(z-z_1) = \alpha - 2\pi$ implies $\sqrt{z-z_1} \overset{MATH.}{=} -\sqrt{z-z_1}$.

   (b) $\arg(z-z_2) = \alpha - \pi$ implies $\sqrt{z-z_2} \overset{MATH.}{=} \sqrt{z-z_2}$.

   Therefore,

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}.$$

2. $z \in -$ edge of this cut:

   (a) $\arg(z-z_1) = \alpha$ implies $\sqrt{z-z_1} \overset{MATH.}{=} \sqrt{z-z_1}$.

   (b) $\arg(z-z_2) = \alpha - \pi$ implies $\sqrt{z-z_2} \overset{MATH.}{=} \sqrt{z-z_2}$.

   Therefore,

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2}.$$

3. z is not on this cut:

   By the Lemma 2.1, we know that

$$\sqrt{z-z_k} \overset{MATH.}{=} -\sqrt{z-z_k} \text{ if } \arg(z-z_k) \in [\alpha-2\pi, -\pi],$$

   and in this sense, consider

$$\begin{cases} S^+ = \{z|z \in \mathbb{C},\ \arg(z-z_1) \in (-\pi, \alpha),\ z \text{ is not in the cut}\}, \\ S^- = \{z|z \in \mathbb{C},\ \arg(z-z_1) \in [\alpha-2\pi, -\pi],\ z \text{ is not in the cut}\}, \\ T^+ = \{z|z \in \mathbb{C},\ \arg(z-z_2) \in (-\pi, \alpha),\ z \text{ is not in the cut}\}, \\ T^- = \{z|z \in \mathbb{C},\ \arg(z-z_2) \in [\alpha-2\pi, -\pi],\ z \text{ is not in the cut}\}. \end{cases}$$

$$\Rightarrow \begin{cases} \sqrt{z-z_1} \overset{MATH.}{=} \sqrt{z-z_1} \text{ if } z \in S^+, \\ \sqrt{z-z_1} \overset{MATH.}{=} -\sqrt{z-z_1} \text{ if } z \in S^-, \\ \sqrt{z-z_2} \overset{MATH.}{=} \sqrt{z-z_2} \text{ if } z \in T^+, \\ \sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_2} \text{ if } z \in T^-. \end{cases}$$

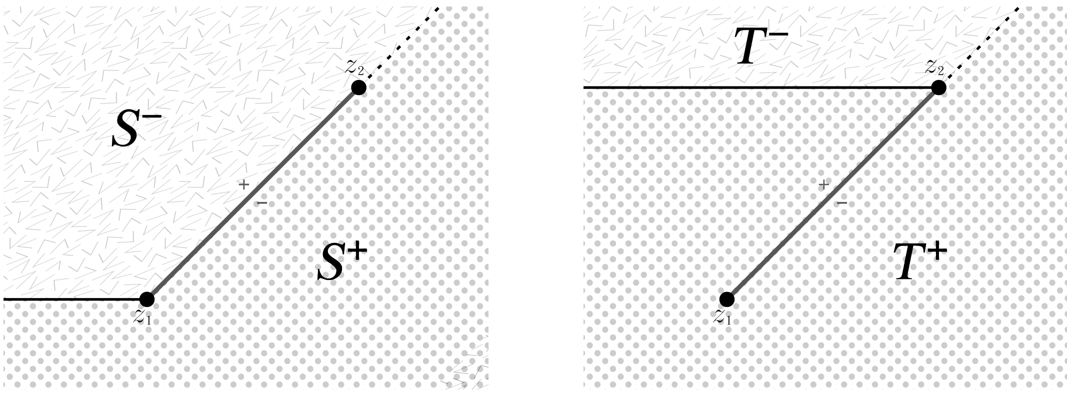

Fig. 2.35 S^+, S^-, T^+, and T^-

Then, $\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$ if and only if $z \in S^+ \cap T^-$ or $z \in S^- \cap T^+$.

Notice that $S^+ \cap T^- = \emptyset$, so $\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$ if and only if $z \in S^- \cap T^+$.

Therefore, if $z \in S^- \cap T^+ = \{z | z \in \mathbb{C},\ l_1(z) < 0,\ y_1 \leq \text{Im}(z) < y_2\}$, shown in Fig. 2.36,

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}.$$

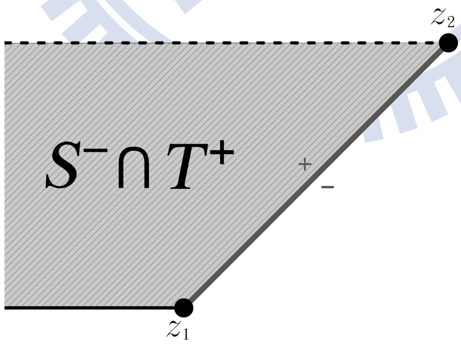

Fig. 2.36 $S^- \cap T^+$

Case 3:

We consider a horizontal cut from z_1 to ∞.

Since the cut is horizontal, its slope $m = \tan\alpha = 0$, and this implies $\alpha = \pi$ (Fig. 2.37).

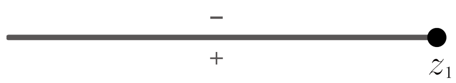

Fig. 2.37 Cut with end point z_1

1. $z \in +$ edge of this cut:

   $\arg(z - z_1) = -\pi$ implies $\sqrt{z - z_1} \overset{MATH.}{=} -\sqrt{z - z_1}$.

2. $z \in -$ edge of this cut:

   $\arg(z - z_1) = \pi$ implies $\sqrt{z - z_1} \overset{MATH.}{=} \sqrt{z - z_1}$.

3. z is not in this cut:

   Since z is not in this cut, $\arg(z - z_1) \in (-\pi, \pi)$.

   By the Lemma 2.1, we know that

$$\sqrt{z - z_1} \overset{MATH.}{=} \sqrt{z - z_1}.$$

<u>Case 4:</u>

We consider a horizontal cut with end points are z_1 and z_2.

Since the cut is horizontal, its slope $m = \tan\alpha = 0$, and this implies $\alpha = \pi$ (Fig. 2.38).

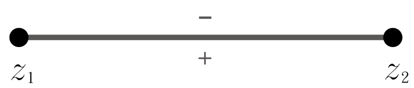

Fig. 2.38 <u>Cut with end points z_1 and z_2</u>

1. $z \in +$ edge of this cut:

   (a) $\arg(z - z_1) = 0$ implies $\sqrt{z - z_1} \overset{MATH.}{=} \sqrt{z - z_1}$.

   (b) $\arg(z - z_2) = -\pi$ implies $\sqrt{z - z_2} \overset{MATH.}{=} -\sqrt{z - z_2}$.

   Therefore,

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{MATH.}{=} -\sqrt{z - z_1}\sqrt{z - z_2}.$$

2. $z \in -$ edge of this cut:

   (a) $\arg(z - z_1) = 0$ implies $\sqrt{z - z_1} \overset{MATH.}{=} \sqrt{z - z_1}$.

   (b) $\arg(z - z_2) = \pi$ implies $\sqrt{z - z_2} \overset{MATH.}{=} \sqrt{z - z_2}$.

   Therefore,

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{MATH.}{=} \sqrt{z - z_1}\sqrt{z - z_2}.$$

3. z is not on this cut:

   Suppose that $\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$.

   By the Lemma 2.1, we know that

$$\sqrt{z-z_k} \overset{MATH.}{=} -\sqrt{z-z_k} \text{ if } \arg(z-z_k) = -\pi,$$

   and in this sense, $\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2}$ if and only if only one of $\arg(z-z_1)$ and $\arg(z-z_2)$ is $-\pi$.

   Then, z is on the cut, but it is a contradiction.

So, for a horizontal cut with end points are z_1 and z_2,

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} -\sqrt{z-z_1}\sqrt{z-z_2} \text{ if } z \in + \text{ edge of this cut.}$$

For these different cases, we know that where z belongs for different cuts such that the value in MATHEMATICA must be modified.

Now, the followings are some examples.

First, we consider $f(z) = \sqrt{\sum_{k=1}^{7}(z - z_k)}$, where

$$\begin{cases} z_1 = -1 - i, \\ z_2 = -1 + 3i, \\ z_3 = 0, \\ z_4 = 2, \\ z_5 = 3 - 2i, \\ z_6 = \dfrac{9}{2} + i, \text{ and} \\ z_7 = 6 - i. \end{cases}$$

And, we shown the branch cuts in Fg. 2.39.

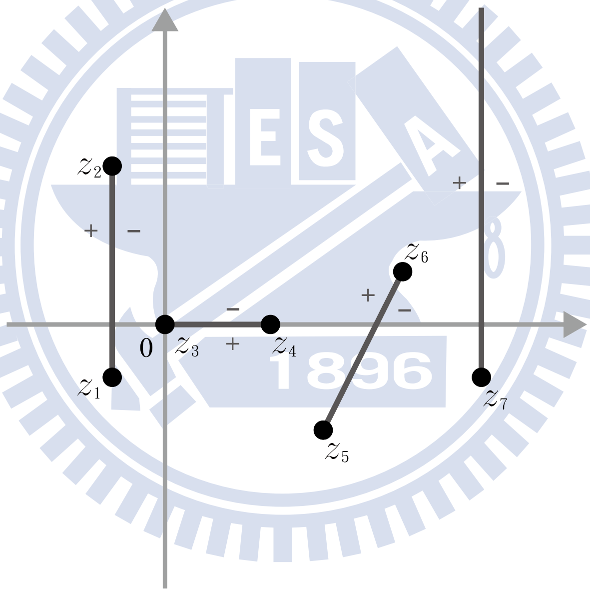

Fig. 2.39 Branch points and the branch cuts of $f(z)$

Example 2.6. *Evaluate* $\int_{c_1} f(z)dz$, c_1 *is shown in Fig. 2.40.*

Solution.

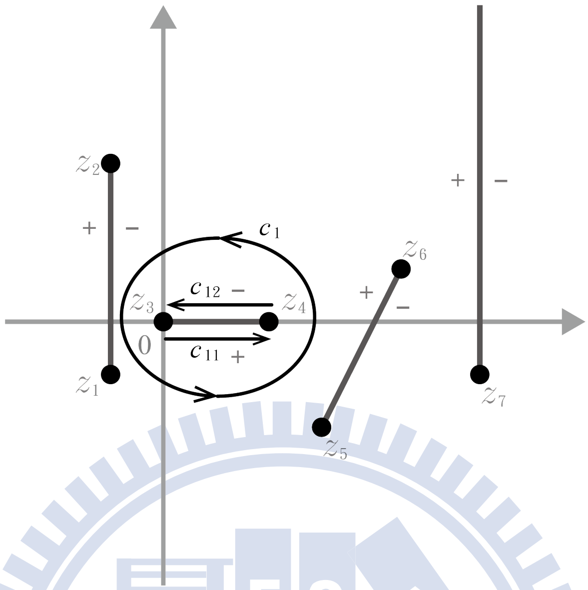

Fig. 2.40 Cycle c_1 and the equivalent paths c_{11} and c_{12}

We have $f(z) = \sqrt{\sum_{k=1}^{7}(z - z_k)} = \sum_{k=1}^{7} \sqrt{z - z_k}$, and notice that the cycle c_1 is simple connected, so we use the equivalent paths, say $c_1^* = c_{11} \cup c_{12}$, shown in Fig. 2.40, such that $c_1 \approx c_1^*$, where
c_{11} is the straight path on horizontal cut from z_3 to z_4 on the + edge of sheet-I, and
c_{12} is the straight path on horizontal cut from z_4 to z_3 on the − edge of sheet-I.

1. If $z \in c_{11}$:

   By the discussion in this subsection, we have

$$\sqrt{z - z_1}\sqrt{z - z_2} \overset{MATH.}{=} \sqrt{z - z_1}\sqrt{z - z_2},$$
$$\sqrt{z - z_3}\sqrt{z - z_4} \overset{MATH.}{=} -\sqrt{z - z_3}\sqrt{z - z_4},$$
$$\sqrt{z - z_5}\sqrt{z - z_6} \overset{MATH.}{=} -\sqrt{z - z_5}\sqrt{z - z_6}, \text{ and}$$
$$\sqrt{z - z_7} \overset{MATH.}{=} -\sqrt{z - z_7},$$

   so $f(z) \overset{MATH.}{=} -f(z)$.

   Thus,

$$\int_{c_{11}} f(z)dz \overset{MATH.}{=} -\int_0^2 f(z)dz.$$

2. If $z \in c_{12}$:

   By the Proposition 2.1,

$$\int_{c_{12}} f(z)dz = \int_{c_{11}} f(z)dz.$$

Therefore,

$$\begin{aligned}\int_{c_1} f(z)dz &= \int_{c_1^*} f(z)dz \\ &= \int_{c_{11}} f(z)dz + \int_{c_{12}} f(z)dz \\ &\overset{MATH.}{=} -2\int_0^2 f(z)dz.\end{aligned}$$

□

Example 2.7. *Evaluate* $\int_{c_2} f(z)dz$*,* c_2 *is shown in Fig. 2.41.*

Solution.

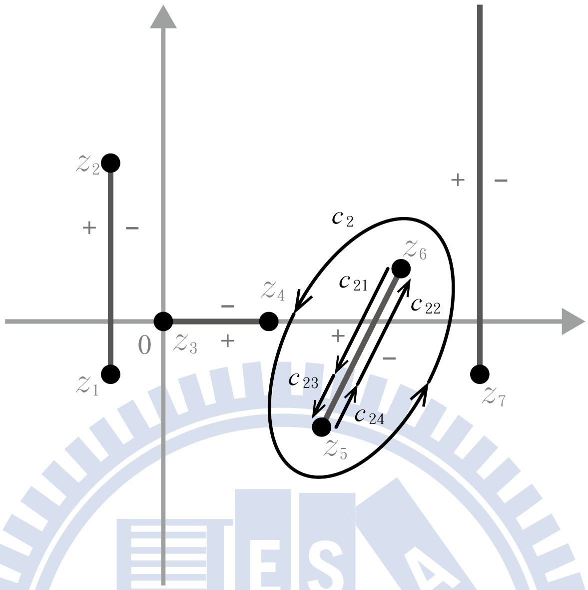

Fig. 2.41 Cycle c_2 and the equivalent paths c_{21}, c_{22}, c_{23}, and c_{24}

We consider the equivalent paths, say $c_2^* = c_{21} \cup c_{22} \cup c_{23} \cup c_{24}$, shown in Fig. 2.41, such that $c_2 \approx c_2^*$, where

c_{21} is the straight path on slant cut from z_6 to $\frac{7}{2} - i$ on the $+$ edge of sheet-I,

c_{22} is the straight path on slant cut from $\frac{7}{2} - i$ to z_6 on the $-$ edge of sheet-I,

c_{23} is the straight path on slant cut from $\frac{7}{2} - i$ to z_5 on the $+$ edge of sheet-I, and

c_{24} is the straight path on slant cut from z_5 to $\frac{7}{2} - i$ on the $-$ edge of sheet-I.

1. If $z \in c_{21}$:

   Let

$$
\begin{aligned}
z &= z_6 + [(\frac{7}{2} - i) - z_6]k \\
&= (\frac{9}{2} + i) + (-1 - 2i)k, \text{ where } k \text{ is from 0 to 1,}
\end{aligned}
$$

   then $dz = (-1 - 2i)dk$.

By the discussion in this subsection, we have

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{MATH.}{=} \sqrt{z-z_3}\sqrt{z-z_4},\\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{MATH.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}, \text{ and}\\
\sqrt{z-z_7} &\overset{MATH.}{=} -\sqrt{z-z_7},
\end{aligned}$$

so $f(z) \overset{MATH.}{=} f(z)$.

Thus,

$$\int_{c_{21}} f(z)dz \overset{MATH.}{=} \int_0^1 f((\frac{9}{2}+i)+(-1-2i)k)(-1-2i)dk.$$

2. If $z \in c_{22}$:

   By the Proposition 2.1,

$$\int_{c_{22}} f(z)dz = \int_{c_{21}} f(z)dz.$$

3. If $z \in c_{23}$:

   Let

$$\begin{aligned}
z &= (\frac{7}{2}-i)+[z_5-(\frac{7}{2}-i)]k\\
&= (\frac{7}{2}-i)+(-\frac{1}{2}-i)k, \text{ where } k \text{ is from 0 to 1},
\end{aligned}$$

then $dz = (-\frac{1}{2}-i)dk$.

By the discussion in this subsection, we have

$$\begin{aligned}
\sqrt{z-z_1}\sqrt{z-z_2} &\overset{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2},\\
\sqrt{z-z_3}\sqrt{z-z_4} &\overset{MATH.}{=} \sqrt{z-z_3}\sqrt{z-z_4},\\
\sqrt{z-z_5}\sqrt{z-z_6} &\overset{MATH.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}, \text{ and}\\
\sqrt{z-z_7} &\overset{MATH.}{=} \sqrt{z-z_7},
\end{aligned}$$

so $f(z) \overset{MATH.}{=} -f(z)$.

Thus,

$$\int_{c_{23}} f(z)dz \overset{MATH.}{=} -\int_0^1 f((\frac{7}{2}-i)+(-\frac{1}{2}-i)k)(-\frac{1}{2}-i)dk.$$

4. If $z \in c_{24}$:

   By the Proposition 2.1,

$$\int_{c_{24}} f(z)dz = \int_{c_{23}} f(z)dz.$$

   Therefore,

$$\begin{aligned}
\int_{c_2} f(z)dz &= \int_{c_2^*} f(z)dz \\
&= \int_{c_{21}} f(z)dz + \int_{c_{22}} f(z)dz + \int_{c_{23}} f(z)dz + \int_{c_{24}} f(z)dz \\
&\overset{MATH.}{=} 2\int_0^1 f((\frac{9}{2}+i)+(-1-2i)k)(-1-2i)dk \\
&\quad -2\int_0^1 f((\frac{7}{2}-i)+(-\frac{1}{2}-i)k)(-\frac{1}{2}-i)dk.
\end{aligned}$$

□

Example 2.8. *Evaluate* $\int_{d_1} f(z)dz$*,* d_1 *is shown in Fig. 2.42.*

Solution.

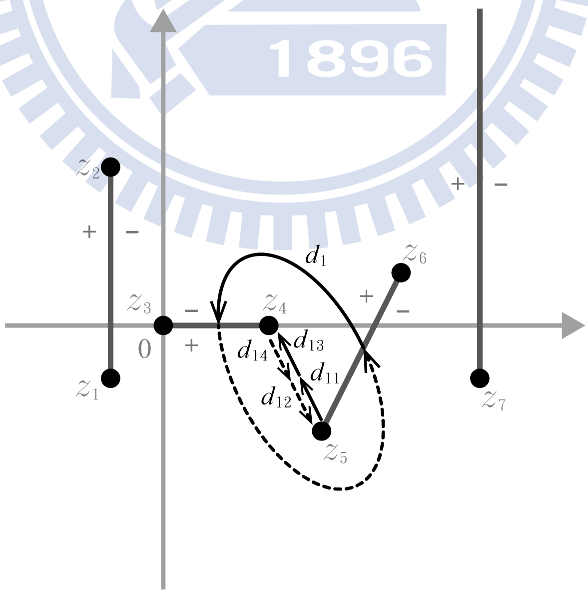

Fig. 2.42 Cycle d_1 and the equivalent paths d_{11}, d_{12}, d_{13}, and d_{14}

We consider the equivalent paths, say $d_1^* = d_{11} \cup d_{12} \cup d_{13} \cup d_{14}$, shown in Fig. 2.42, such that $d_1 \approx d_1^*$, where

d_{11} is the straight path from z_5 to $\frac{5}{2} - i$ in the sheet-I,
d_{12} is the straight path from $\frac{5}{2} - i$ to z_5 in the sheet-II,
d_{13} is the straight path from $\frac{5}{2} - i$ to z_4 in the sheet-I, and
d_{14} is the straight path from z_4 to $\frac{5}{2} - i$ in the sheet-II.

1. If $z \in d_{11}$:

   Let

$$\begin{aligned} z &= z_5 + [(\frac{5}{2} - i) - z_5]k \\ &= (3 - 2i) + (-\frac{1}{2} + i)k, \text{ where } k \text{ is from 0 to 1,} \end{aligned}$$

   then $dz = (-\frac{1}{2} + i)dk$.

   By the discussion in this subsection, we have

$$\begin{aligned} \sqrt{z - z_1}\sqrt{z - z_2} &\overset{MATH.}{=} \sqrt{z - z_1}\sqrt{z - z_2}, \\ \sqrt{z - z_3}\sqrt{z - z_4} &\overset{MATH.}{=} \sqrt{z - z_3}\sqrt{z - z_4}, \\ \sqrt{z - z_5}\sqrt{z - z_6} &\overset{MATH.}{=} -\sqrt{z - z_5}\sqrt{z - z_6}, \text{ and} \\ \sqrt{z - z_7} &\overset{MATH.}{=} \sqrt{z - z_7}, \end{aligned}$$

   so $f(z) \overset{MATH.}{=} -f(z)$.

   Thus,

$$\int_{d_{11}} f(z)dz \overset{MATH.}{=} -\int_0^1 f((3 - 2i) + (-\frac{1}{2} + i)k)(-\frac{1}{2} + i)dk.$$

2. If $z \in d_{12}$:

   By the Proposition 2.2,

$$\int_{d_{12}} f(z)dz = \int_{d_{11}} f(z)dz.$$

3. If $z \in d_{13}$:

   Let

$$z = (\frac{5}{2} - i) + [z_4 - (\frac{5}{2} - i)]k$$
$$= (\frac{5}{2} - i) + (-\frac{1}{2} + i)k, \text{ where } k \text{ is from 0 to 1,}$$

   then $dz = (-\frac{1}{2} + i)dk$.

   By the discussion in this subsection, we have

$$\sqrt{z-z_1}\sqrt{z-z_2} \stackrel{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \stackrel{MATH.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \stackrel{MATH.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}, \text{ and}$$
$$\sqrt{z-z_7} \stackrel{MATH.}{=} -\sqrt{z-z_7},$$

   so $f(z) \stackrel{MATH.}{=} f(z)$.

   Thus,

$$\int_{d_{13}} f(z)dz \stackrel{MATH.}{=} \int_0^1 f((\frac{5}{2} - i) + (-\frac{1}{2} + i)k)(-\frac{1}{2} + i)dk.$$

4. If $z \in d_{14}$:

   By the Proposition 2.2,

$$\int_{d_{14}} f(z)dz = \int_{d_{13}} f(z)dz.$$

Therefore,

$$\begin{aligned}\int_{d_1} f(z)dz &= \int_{d_1^*} f(z)dz \\ &= \int_{d_{11}} f(z)dz + \int_{d_{12}} f(z)dz + \int_{d_{13}} f(z)dz + \int_{d_{14}} f(z)dz \\ &\stackrel{MATH.}{=} -2\int_0^1 f((3-2i) + (-\frac{1}{2} + i)k)(-\frac{1}{2} + i)dk \\ &\quad + 2\int_0^1 f((\frac{5}{2} - i) + (-\frac{1}{2} + i)k)(-\frac{1}{2} + i)dk.\end{aligned}$$

□

Example 2.9. *Evaluate* $\int_{d_2} f(z)dz$, d_2 *is shown in Fig. 2.43.*

Solution.

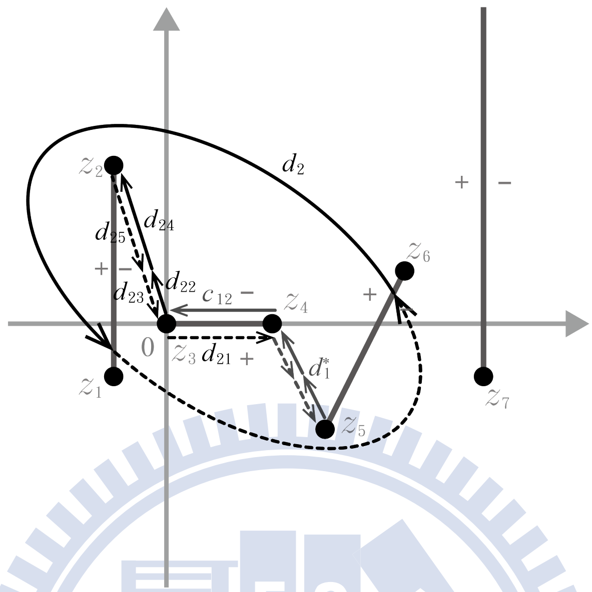

Fig. 2.43 Cycle d_2 and the equivalent paths d_1^*, c_{12}, d_{21}, d_{22}, d_{23}, d_{24}, and d_{25}

We consider the equivalent paths, say $d_2^* = d_1^* \cup c_{12} \cup d_{21} \cup d_{22} \cup d_{23} \cup d_{24} \cup d_{25}$, shown in Fig. 2.43, such that $d_2 \approx d_2^*$, where

d_{21} is the straight path on horizontal cut from z_3 to z_4 on the $+$ edge of sheet-II,

d_{22} is the straight path from z_3 to $-\dfrac{1}{3} + i$ in the sheet-I,

d_{23} is the straight path from $-\dfrac{1}{3} + i$ to z_3 in the sheet-II,

d_{24} is the straight path from $-\dfrac{1}{3} + i$ to z_2 in the sheet-I, and

d_{25} is the straight path from z_2 to $-\dfrac{1}{3} + i$ in the sheet-II.

To start with, we know that d_{21} is also the straight path on horizontal cut from z_3 to z_4 on the $-$ edge of sheet-I, that is, $d_{21} = -c_{12}$, and this implies

$$\int_{c_{12}} f(z)dz + \int_{d_{21}} f(z)dz = 0$$

1. If $z \in d_{22}$:

   Let

$$\begin{aligned} z &= z_3 + [(-\frac{1}{3} + i) - z_3]k \\ &= (-\frac{1}{3} + i)k, \text{ where } k \text{ is from 0 to 1,} \end{aligned}$$

   then $dz = (-\frac{1}{3} + i)dk$.

   By the discussion in this subsection, we have

$$\begin{aligned} \sqrt{z-z_1}\sqrt{z-z_2} &\overset{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2}, \\ \sqrt{z-z_3}\sqrt{z-z_4} &\overset{MATH.}{=} \sqrt{z-z_3}\sqrt{z-z_4}, \\ \sqrt{z-z_5}\sqrt{z-z_6} &\overset{MATH.}{=} -\sqrt{z-z_5}\sqrt{z-z_6}, \text{ and} \\ \sqrt{z-z_7} &\overset{MATH.}{=} -\sqrt{z-z_7}, \end{aligned}$$

   so $f(z) \overset{MATH.}{=} f(z)$.

   Thus,

$$\int_{d_{22}} f(z)dz \overset{MATH.}{=} \int_0^1 f((-\frac{1}{3} + i)k)(-\frac{1}{3} + i)dk.$$

2. If $z \in d_{23}$:

   By the Proposition 2.2,

$$\int_{d_{23}} f(z)dz = \int_{d_{22}} f(z)dz.$$

3. If $z \in d_{24}$:

   Let

$$\begin{aligned} z &= (-\frac{1}{3} + i) + [z_2 - (-\frac{1}{3} + i)]k \\ &= (-\frac{1}{3} + i) + (-\frac{2}{3} + 2i)k, \text{ where } k \text{ is from 0 to 1,} \end{aligned}$$

   then $dz = (-\frac{2}{3} + 2i)dk$.

By the discussion in this subsection, we have

$$\sqrt{z-z_1}\sqrt{z-z_2} \overset{MATH.}{=} \sqrt{z-z_1}\sqrt{z-z_2},$$
$$\sqrt{z-z_3}\sqrt{z-z_4} \overset{MATH.}{=} \sqrt{z-z_3}\sqrt{z-z_4},$$
$$\sqrt{z-z_5}\sqrt{z-z_6} \overset{MATH.}{=} \sqrt{z-z_5}\sqrt{z-z_6}, \text{ and}$$
$$\sqrt{z-z_7} \overset{MATH.}{=} -\sqrt{z-z_7},$$

so $f(z) \overset{MATH.}{=} -f(z)$.

Thus,

$$\int_{d_{24}} f(z)dz \overset{MATH.}{=} -\int_0^1 f((-\frac{1}{3}+i)+(-\frac{2}{3}+2i)k)(-\frac{2}{3}+2i)dk.$$

4. If $z \in d_{25}$:

   By the Proposition 2.2,

$$\int_{d_{25}} f(z)dz = \int_{d_{24}} f(z)dz.$$

Therefore,

$$\begin{aligned}
\int_{d_2} f(z)dz &= \int_{d_2^*} f(z)dz \\
&= \int_{d_1^*} f(z)dz + \int_{c_{12}} f(z)dz + \int_{d_{21}} f(z)dz + \int_{d_{22}} f(z)dz \\
&\quad + \int_{d_{23}} f(z)dz + \int_{d_{24}} f(z)dz + \int_{d_{25}} f(z)dz \\
&\overset{MATH.}{=} \int_{d_1^*} f(z)dz + 2\int_0^1 f((-\frac{1}{3}+i)k)(-\frac{1}{3}+i)dk. \\
&\quad - 2\int_0^1 f((-\frac{1}{3}+i)+(-\frac{2}{3}+2i)k)(-\frac{2}{3}+2i)dk.
\end{aligned}$$

□

Chapter 3

The Pendulum Motion on Riemann Surface

Recall that the sine-Gordon equation is

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0, \text{ where } -\infty < x < \infty \text{ and } t > 0.$$

If we let $s = \alpha x - \beta t$, by the Chain rule:

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial s}\dfrac{\partial s}{\partial x} = \alpha \dfrac{\partial}{\partial s}, \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial s}\dfrac{\partial s}{\partial t} = -\beta \dfrac{\partial}{\partial s}. \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x} u = \alpha^2 \dfrac{\partial^2 u}{\partial s^2}, \\ \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial t}\dfrac{\partial}{\partial t} u = \beta^2 \dfrac{\partial^2 u}{\partial s^2}. \end{cases}$$

Therefore, the Sine-Gorden equation comes to

$$\beta^2 u_{ss} - \alpha^2 u_{ss} + \sin[u(s)] = 0.$$

We let $\beta^2 - \alpha^2 = 1$, the equation is written as

$$u_{ss} + \sin[u(s)] = 0. \tag{3.1}$$

Notice that we can regard (3.1) as the simple pendulum motion at time s with the angular displacement u.

Multiplying an integral factor u_s to equation (3.1), and we integral it with respect to s:

$$
\begin{aligned}
& u_{ss} + \sin(u) = 0 \\
\Rightarrow\ & u_s u_{ss} + u_s \sin(u) = 0 \\
\Rightarrow\ & \int [u_s u_{ss} + u_s \sin(u)] ds = 0 \\
\Rightarrow\ & \frac{1}{2} u_s^2 - \cos(u) = E, \text{ where } E \text{ is constant}, \\
\Rightarrow\ & u_s = \pm\sqrt{2[E + \cos(u)]}.
\end{aligned}
$$

Since u_s is the angular velocity, the plus-minus sign means direction, and assume without loss of generality that

$$
u_s = \sqrt{2[E + \cos(u)]}. \tag{3.2}
$$

Next, using the skill in previous chapter, we will solve the problem of pendulum by approximation.

Notice that, by the Taylor series of $\cos u$ at 0,

$$
\cos u \approx 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!}.
$$

Then, the equation (3.2) comes to $u_s = \sqrt{2[E + 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!}]}$.

Now, we consider the situation of $E = 0$, and by using the MATHEMATICA, we have

$$
\begin{aligned}
u_s &= \sqrt{2[1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!}]} \\
&= \sqrt{\prod_{k=1}^{8} (u - u_k)},
\end{aligned}
$$

where

$$
\begin{cases}
u_1 = -4.8947 - 2.4869i, \\
u_2 = -4.8947 + 2.4869i, \\
u_3 = -4.2408, \\
u_4 = -1.5708, \\
u_5 = 1.5708, \\
u_6 = 4.2408, \\
u_7 = 4.8947 - 2.4869i, \text{ and} \\
u_8 = 4.8947 + 2.4869i.
\end{cases}
$$

Therefore, we get eight branch points and four branch cuts, shown in Fig. 3.1.

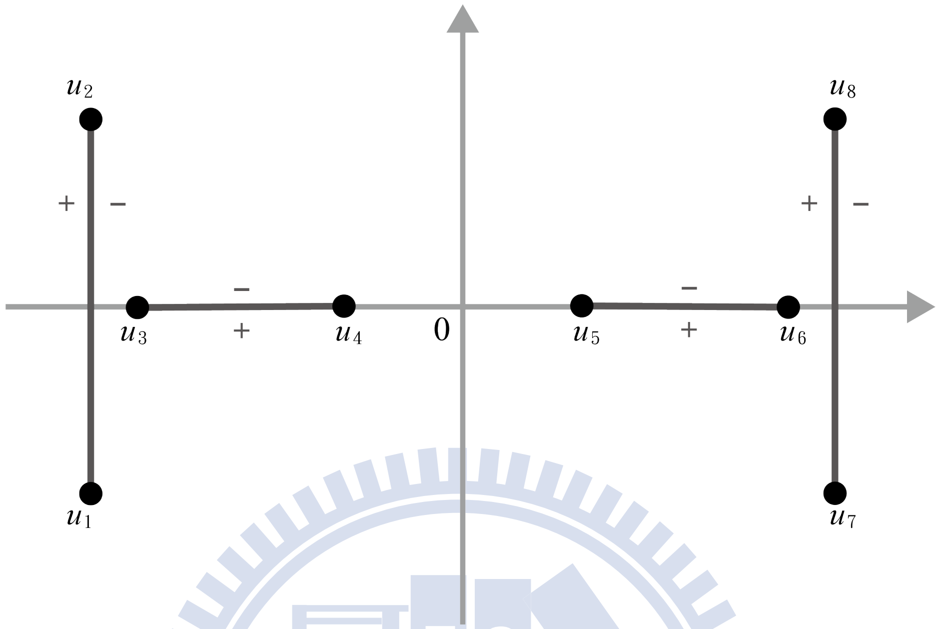

Fig. 3.1 Branch points and the branch cuts of u_s

For convience, we let $f(u) = \sqrt{\prod_{k=1}^{8}(u - u_k)}$.

Notice that, $u_s = \dfrac{du}{ds} = f(u)$, and

$$ds = \frac{1}{f(u)}du$$
$$\Rightarrow \quad s = \int \frac{1}{f(u)}du.$$

By the Cauchy integral formula, for any closed path in $\mathfrak{R}$ of genus 3 is homotopic to an integral combination of the loop-cuts a_i and b_i, $i = 1, 2, 3$, so we will consider the integrals of $\dfrac{1}{f(u)}$ over a, b-cycles.

We shown a-cycles in Fg. 3.2.

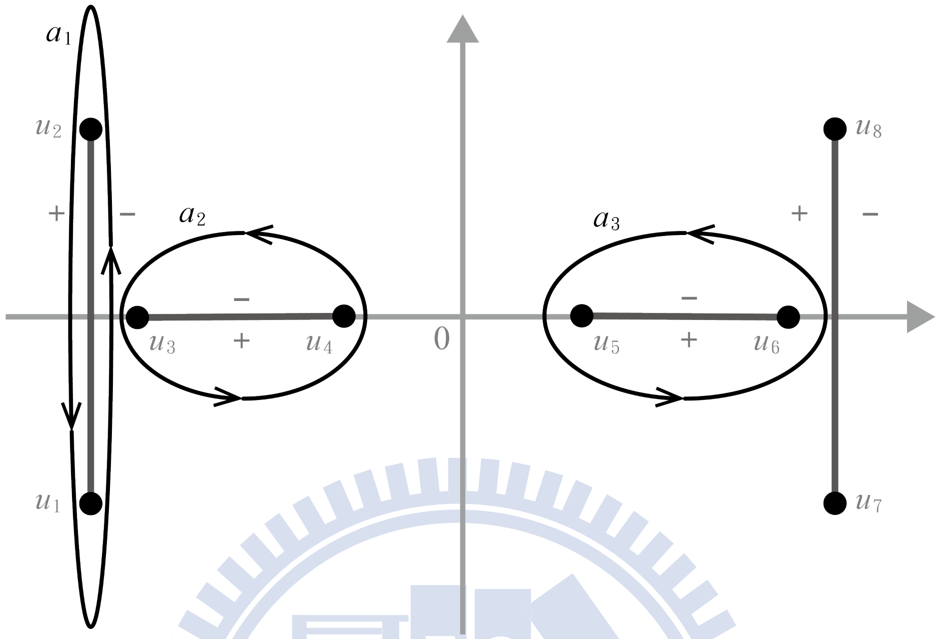

Fig. 3.2 a-cycles

a_1-cycle

We consider the equivalent paths, say $a_1^* = a_{11} \cup a_{12}$, shown in Fig. 3.3, such that $a_1 \approx a_1^*$, where
a_{11} is the straight path on vertical cut from u_2 to u_1 on the $+$ edge of sheet-I, and
a_{12} is the straight path on vertical cut from u_1 to u_2 on the $-$ edge of sheet-I.

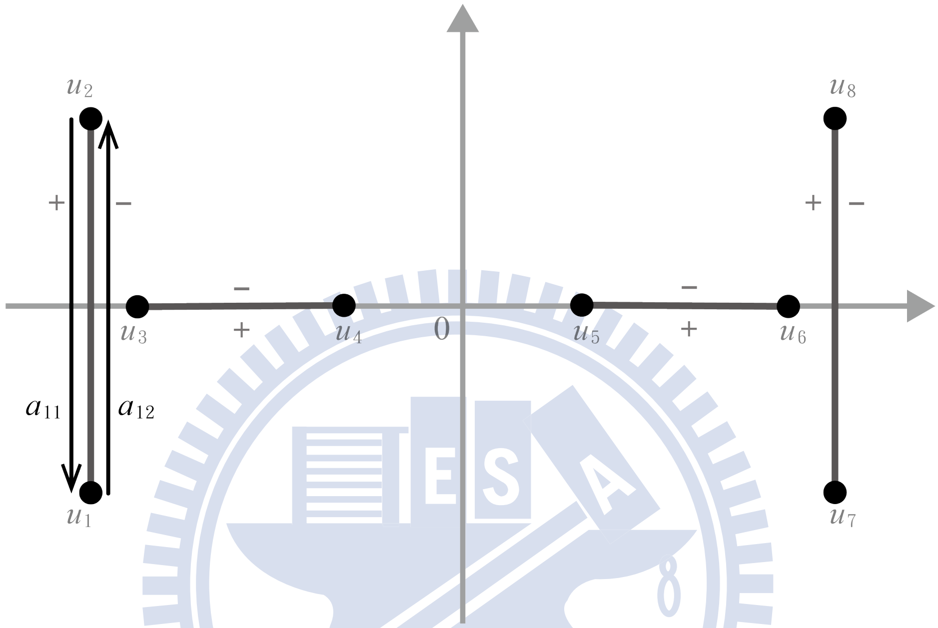

Fig. 3.3 The equivalent paths of a_1-cycle

We shown the detail of the integrals in Appendix A.1.
Therefore,

$$
\begin{aligned}
\int_{a_1} \frac{1}{f(u)} du &= \int_{a_1^*} \frac{1}{f(u)} du \\
&= \int_{a_{11}} \frac{1}{f(u)} du + \int_{a_{12}} \frac{1}{f(u)} du \\
&\overset{MATH.}{=} 2 \int_{u_2}^{u_1} \frac{1}{f(u)} du.
\end{aligned}
$$

□

a_2-cycle

We consider the equivalent paths, say $a_2^* = a_{21} \cup a_{22}$, shown in Fig. 3.4, such that $a_2 \approx a_2^*$, where
a_{21} is the straight path on horizontal cut from u_3 to u_4 on the + edge of sheet-I, and
a_{22} is the straight path on horizontal cut from u_4 to u_3 on the − edge of sheet-I.

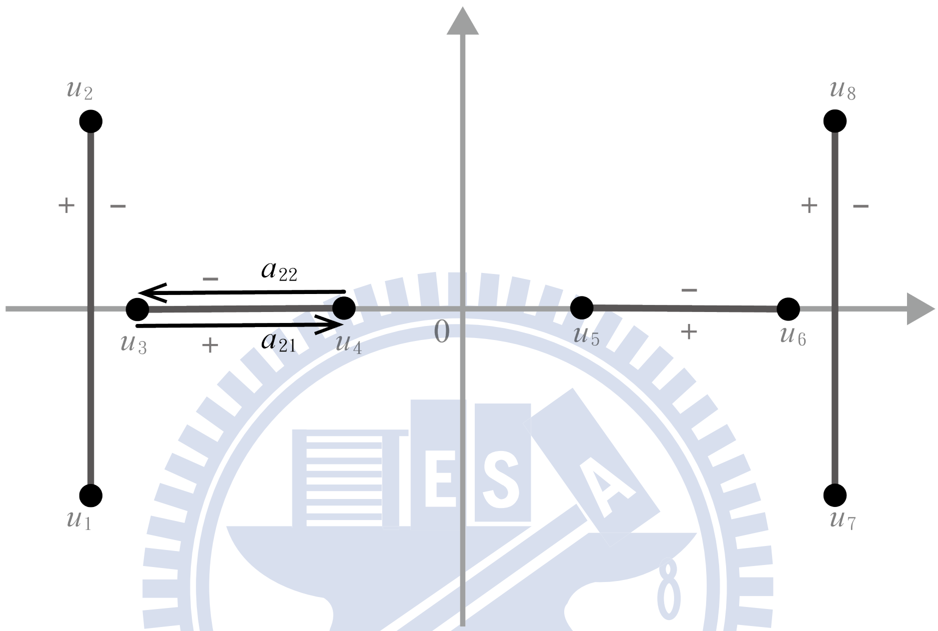

Fig. 3.4 The equivalent paths of a_2-cycle

We shown the detail of the integrals in Appendix A.2.
Therefore,

$$\begin{aligned} \int_{a_2} \frac{1}{f(u)} du &= \int_{a_2^*} \frac{1}{f(u)} du \\ &= \int_{a_{21}} \frac{1}{f(u)} du + \int_{a_{22}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} 2 \int_{u_3}^{u_4} \frac{1}{f(u)} du. \end{aligned}$$

□

a_3-cycle

We consider the equivalent paths, say $a_3^* = a_{31} \cup a_{32}$, shown in Fig. 3.5, such that $a_3 \approx a_3^*$, where
a_{31} is the straight path on horizontal cut from u_5 to u_6 on the + edge of sheet-I, and
a_{32} is the straight path on horizontal cut from u_6 to u_5 on the − edge of sheet-I.

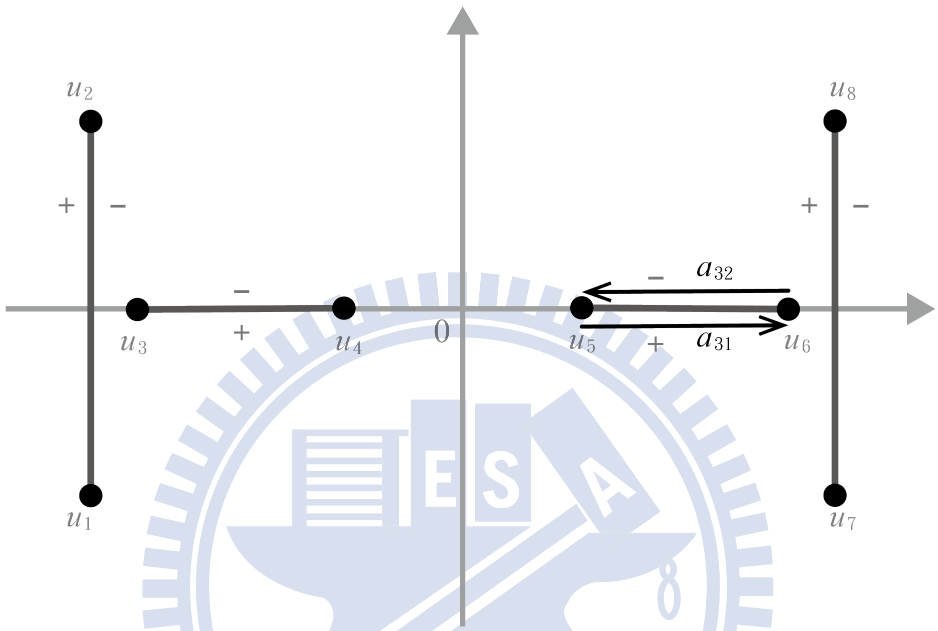

Fig. 3.5 The equivalent paths of a_3-cycle

We shown the detail of the integrals in Appendix A.3.
Therefore,

$$\begin{aligned}\int_{a_3} \frac{1}{f(u)} du &= \int_{a_3^*} \frac{1}{f(u)} du \\ &= \int_{a_{31}} \frac{1}{f(u)} du + \int_{a_{32}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} 2 \int_{u_5}^{u_6} \frac{1}{f(u)} du.\end{aligned}$$

□

We shown b-cycles in Fg. 3.6.

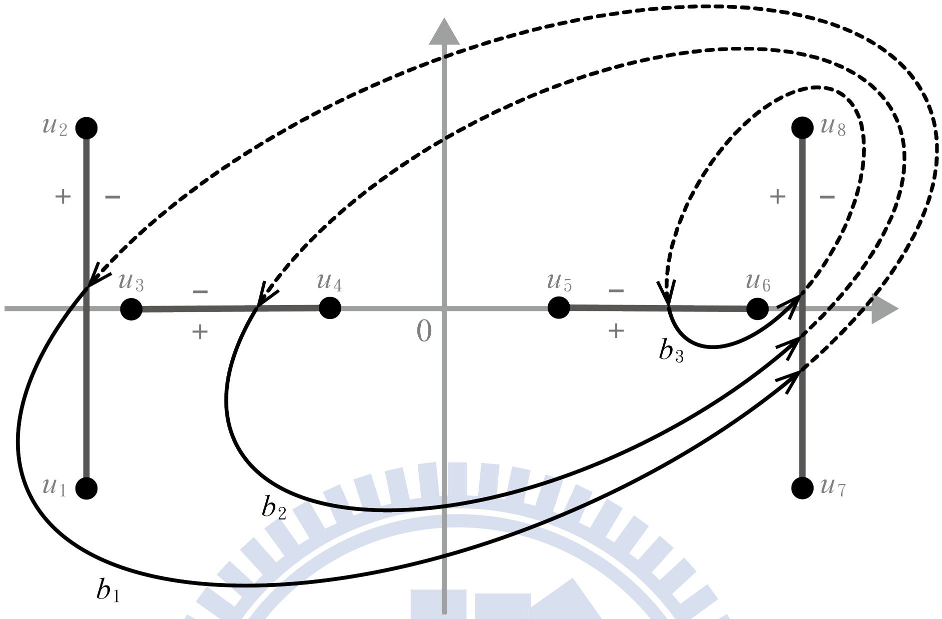

Fig. 3.6 b-cycles

b_3-cycle

We consider the equivalent paths, say $b_3^* = b_{31} \cup b_{32}$, shown in Fig. 3.7, such that $b_3 \approx b_3^*$, where
b_{31} is the straight path from u_6 to u_8 in the sheet-I, and
b_{32} is the straight path from u_8 to u_6 in the sheet-II.

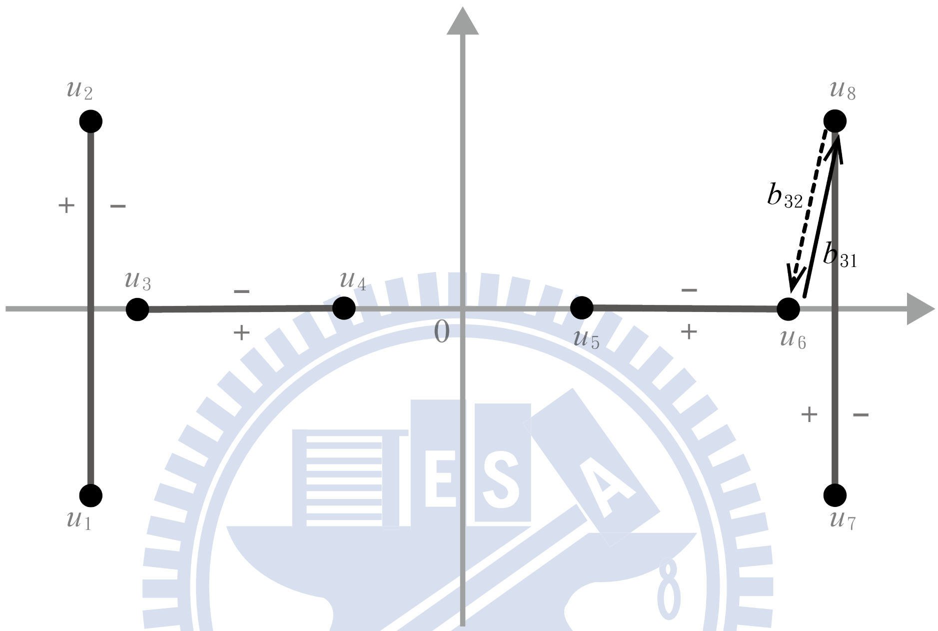

Fig. 3.7 The equivalent paths of b_3-cycle

We shown the detail of the integrals in Appendix A.4.
Therefore,

$$\begin{aligned}\int_{b_3} \frac{1}{f(u)} du &= \int_{b_3^*} \frac{1}{f(u)} du \\ &= \int_{b_{31}} \frac{1}{f(u)} du + \int_{b_{32}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} -2 \int_{u_6}^{u_8} \frac{1}{f(u)} du.\end{aligned}$$

□

b_2-cycle

We consider the equivalent paths, say $b_2^* = b_3^* \cup a_{31} \cup b_{21} \cup b_{22} \cup b_{23}$, shown in Fig. 3.8, such that $b_2 \approx b_2^*$, where
b_{21} is the straight path on horizontal cut from u_6 to u_5 on the $-$ edge of sheet-II,
b_{22} is the straight path from u_4 to u_5 in the sheet-I, and
b_{23} is the straight path from u_5 to u_4 in the sheet-II.

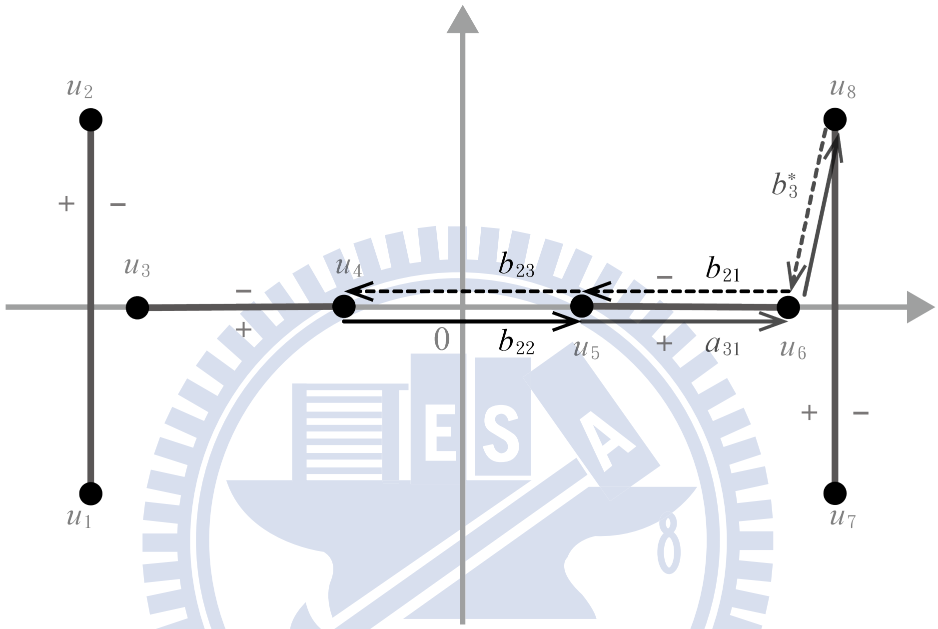

Fig. 3.8 The equivalent paths of b_2-cycle

We shown the detail of the integrals in Appendix A.5.
Therefore,

$$\begin{aligned}
\int_{b_2} \frac{1}{f(u)} du &= \int_{b_2^*} \frac{1}{f(u)} du \\
&= \int_{b_3^*} \frac{1}{f(u)} du + \int_{a_{31}} \frac{1}{f(u)} du + \int_{b_{21}} \frac{1}{f(u)} du + \int_{b_{22}} \frac{1}{f(u)} du + \int_{b_{23}} \frac{1}{f(u)} du \\
&\overset{MATH.}{=} \int_{b_3^*} \frac{1}{f(u)} du - 2 \int_{u_4}^{u_5} \frac{1}{f(u)} du.
\end{aligned}$$

□

b_1-cycle

We consider the equivalent paths, say $b_1^* = b_2^* \cup a_{21} \cup b_{11} \cup b_{12} \cup b_{13}$, shown in Fig. 3.9, such that $b_1 \approx b_1^*$, where
b_{11} is the straight path on horizontal cut from u_4 to u_3 on the $-$ edge of sheet-II,
b_{12} is the straight path from u_1 to u_3 in the sheet-I, and
b_{13} is the straight path from u_3 to u_1 in the sheet-II.

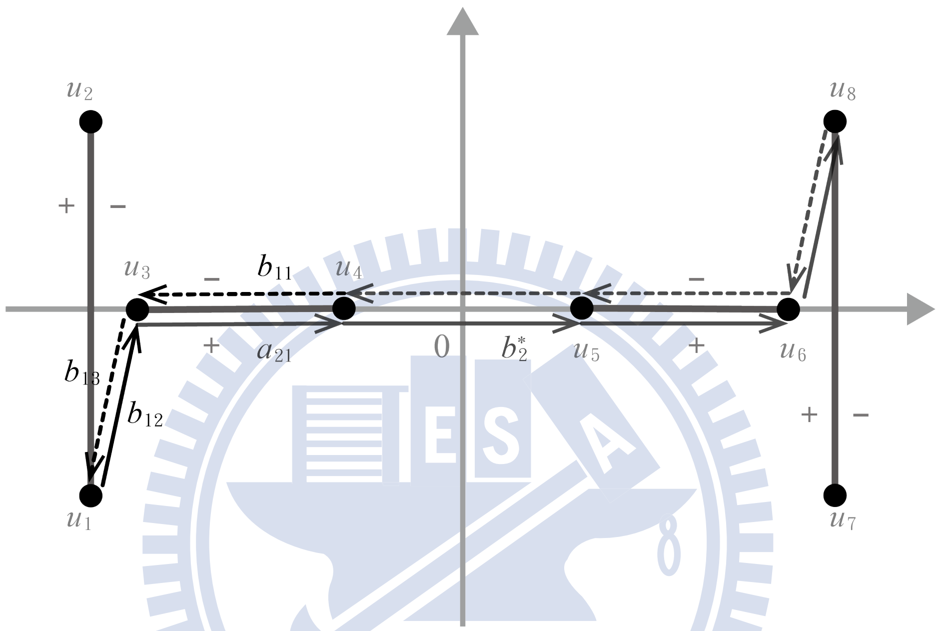

Fig. 3.9 The equivalent paths of b_1-cycle

We shown the detail of the integrals in Appendix A.6.
Therefore,

$$\begin{aligned}\int_{b_1} \frac{1}{f(u)} du &= \int_{b_1^*} \frac{1}{f(u)} du \\ &= \int_{b_2^*} \frac{1}{f(u)} du + \int_{a_{21}} \frac{1}{f(u)} du + \int_{b_{11}} \frac{1}{f(u)} du + \int_{b_{12}} \frac{1}{f(u)} du + \int_{b_{13}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} \int_{b_2^*} \frac{1}{f(u)} du - 2 \int_{u_1}^{u_3} \frac{1}{f(u)} du.\end{aligned}$$

□

Chapter 4

The Elliptic Functions

4.1 Definition and Properties of Elliptic Functions

Definition 4.1. ***(Doubly-Period Function and Elliptic Function)***

We call a function $f : \mathbb{C} \to \mathbb{C}$, a doubly-period function with periods $2\omega_1$ and $2\omega_2$, where ω_1 and ω_2 in $\mathbb{C}$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, if

$$f(z + 2\omega_1) = f(z), \text{ and } f(z + 2\omega_2) = f(z), \text{ for any } z \in \mathbb{C}.$$

And, a doubly-period function which is analytic except at poles, and which has no singularities other than poles in finite part of the plane, is called an elliptic function.

We consider a parallelogram with vertices 0, $2\omega_1$, $2\omega_2$, and $2\omega_1 + 2\omega_2$. If there is no point ω inside or on the boundary except vertices of this parallelogram such that

$$f(z + \omega) = f(z), \text{ for any } z \in \mathbb{C},$$

this parallelogram is called a *fundamental period-parallelogram* for an elliptic function with period $2\omega_1$ and $2\omega_2$.

In addition, the whole plane may be covered with a network of parallelograms equal to the fundamental period-parallelogram and similarly situated, notice that, all vertices in the form of $2m\omega_1 + 2n\omega_2$, $\forall m, n \in \mathbb{Z}$, then these parallelograms are called *period-parallelograms*. If there is no point ω inside or on the boundary except vertices of all parallelogram such that $f(z + \omega) = f(z), \forall z \in \mathbb{C}$, these parallelograms are called *cells*.

Proposition 4.1.

1. *The number of poles of an elliptic function in any cell is finite.*
2. *The number of zeros of an elliptic function in any cell is finite.*
3. *The sum of the residues of an elliptic function, $f(z)$, at its poles in any cell is zero.*
4. ***(Liouville's theorem)*** *An elliptic function, $f(z)$, with no poles in a cell is merely a constant.*

4.2 The Theta-Functions

Let τ be a constant complex number with $\mathrm{Im}(\tau)$ is positive, and let $q = e^{\pi i \tau}$, so that $|q| < 1$.

To begin with, we let

$$\begin{aligned}\vartheta_0(z,q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \\ &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2nz),\end{aligned}$$

and we use this function to define the <u>*Theta-functions*</u>.

Definition 4.2. ***(The Four Types of the Theta-Functions)***
Define that

$$\begin{cases} \vartheta_1(z,q) &= \vartheta_0(z + \frac{1}{2}\pi, q), \\ \vartheta_2(z,q) &= -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_0(z + \frac{1}{2}\pi\tau, q), \ \textit{and} \\ \vartheta_3(z,q) &= \vartheta_2(z + \frac{1}{2}\pi, q). \end{cases}$$

Thus, we have

$$\begin{cases} \vartheta_0(z,q) &= 1 + 2\sum\limits_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz), \\ \vartheta_1(z,q) &= 1 + 2\sum\limits_{n=1}^{\infty} q^{n^2}\cos(2nz), \\ \vartheta_2(z,q) &= 2\sum\limits_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z, \ \textit{and} \\ \vartheta_3(z,q) &= 2\sum\limits_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)z. \end{cases}$$

Since the parameter q will usually not be specified, $\vartheta_i(z)$ will be written for $\vartheta_i(z,q)$, $i = 0,1,2,3$. But if τ is specified, it will be written as $\vartheta(z|\tau)$. We also denote $\vartheta_i(0) = \vartheta_i$, and $\vartheta_i'(0) = \vartheta_i'$, $i = 0,1,2,3$.

It is obvious that

$$\begin{aligned}\vartheta_0(z+\pi) &= \vartheta_0(z), \text{ and}\\ \vartheta_0(z+\pi\tau) &= -q^{-1}e^{-2iz}\vartheta_0(z),\end{aligned}$$

and so $\vartheta_0(z)$ is a quasi doubly-periodic function of z with multipliers 1 and $-q^{-1}e^{-2iz}$, associated with the periods π and $\pi\tau$ respectively.

In like manner, we get that

$$\begin{aligned}
\vartheta_0(z+\pi) &= \vartheta_0(z), & \vartheta_0(z+\pi\tau) &= -q^{-1}e^{-2iz}\vartheta_0(z),\\
\vartheta_1(z+\pi) &= \vartheta_1(z), & \vartheta_1(z+\pi\tau) &= q^{-1}e^{-2iz}\vartheta_1(z),\\
\vartheta_2(z+\pi) &= -\vartheta_2(z), & \vartheta_2(z+\pi\tau) &= -q^{-1}e^{-2iz}\vartheta_2(z),\\
\vartheta_3(z+\pi) &= -\vartheta_3(z), \text{ and} & \vartheta_3(z+\pi\tau) &= q^{-1}e^{-2iz}\vartheta_3(z).
\end{aligned}$$

It is clear that if z_0 be any zero of any one of these Theta-functions, so is

$$z_0 + m\pi + n\pi\tau,$$

for all integral values of m and n, and by the Def. 4.2, we know that

$$\begin{cases}
\vartheta_0(z) = 0, \text{ if } z = \frac{\pi\tau}{2} & +m\pi + n\pi\tau,\\
\vartheta_1(z) = 0, \text{ if } z = \frac{\pi}{2} + \frac{\pi\tau}{2} & +m\pi + n\pi\tau,\\
\vartheta_2(z) = 0, \text{ if } z = 0 & +m\pi + n\pi\tau, \text{ and}\\
\vartheta_3(z) = 0, \text{ if } z = \frac{\pi}{2} & +m\pi + n\pi\tau.
\end{cases}$$

Proposition 4.2.

1. $\vartheta_3^2(z)\vartheta_0^2 = \vartheta_0^2(z)\vartheta_3^2 - \vartheta_2^2(z)\vartheta_1^2$

2. $\vartheta_2^2(z)\vartheta_0^2 = \vartheta_1^2(z)\vartheta_3^2 - \vartheta_3^2(z)\vartheta_1^2$

Now, we consider $\dfrac{\vartheta_2(z)}{\vartheta_0(z)}$ and $\dfrac{\vartheta_3(z)\vartheta_1(z)}{\vartheta_0^2(z)}$, it is obvious that they both has multipliers -1 and 1 associated with the periods π and $\pi\tau$ respectively, so do the derivative of the former,

$$\frac{d}{dz}[\frac{\vartheta_2(z)}{\vartheta_0(z)}] = \frac{\vartheta_2'(z)\vartheta_0(z) + \vartheta_2(z)\vartheta_0'(z)}{\vartheta_0^2(z)}.$$

Notice that $\dfrac{\frac{d}{dz}[\frac{\vartheta_2(z)}{\vartheta_0(z)}]}{\frac{\vartheta_3(z)\vartheta_1(z)}{\vartheta_0^2(z)}}$ is a doubly-periodic function with periods π and $\frac{1}{2}\pi\tau$, then by the Liouville's theorem and as $z \to 0$, it is a constant, that is ϑ_0^2.

Therefore, $\dfrac{d}{dz}[\dfrac{\vartheta_2(z)}{\vartheta_0(z)}] = \vartheta_0^2[\dfrac{\vartheta_3(z)\vartheta_1(z)}{\vartheta_0^2(z)}]$, by letting $\xi = \dfrac{\vartheta_2(z)}{\vartheta_0(z)}$, and using the proposition 4.2, it comes to

$$(\frac{d\xi}{dz})^2 = (\vartheta_3^2 - \xi^2\vartheta_1^2)(\vartheta_1^2 - \xi^2\vartheta_3^2). \tag{4.1}$$

Finally, we write $y = \dfrac{\xi\vartheta_1}{\vartheta_3}$, $u = z\vartheta_1^2$, and $k = (\dfrac{\vartheta_3}{\vartheta_1})^2$, the equation (4.1) with the particular solution $\dfrac{\vartheta_2(z)}{\vartheta_0(z)}$ may be written

$$(\frac{dy}{du})^2 = (1-y^2)(1-k^2y^2)$$

with the particular solution $y = \dfrac{\vartheta_1}{\vartheta_3}\dfrac{\vartheta_2(u\vartheta_1^{-2})}{\vartheta_0(u\vartheta_1^{-2})}$.

4.3 The Jacobian Elliptic Functions

Notice that, in above section, the differential equation $(\frac{dy}{du})^2 = (1-y^2)(1-k^2y^2)$ has the particular solution

$$y = \frac{\vartheta_1}{\vartheta_3}\frac{\vartheta_2(u\vartheta_1^{-2})}{\vartheta_0(u\vartheta_1^{-2})}.$$

The differential equation may be written

$$u = \int_0^y \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt,$$

and we let $y = \mathrm{sn}(u,k)$ or simply $y = \mathrm{sn}(u)$. Notice that, $\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}$ occurs on Riemann surface of genus 1.

The function $y = \mathrm{sn}(u)$ is known as a *Jacobian elliptic function* of u.

Chapter 5

The sine-Gordon Equation

5.1 The Special Solutions of the sine-Gordon Equation

Recall that the sine-Gordon equation is

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0, \text{ where } -\infty < x < \infty \text{ and } t > 0.$$

In chapter 3, we use the method of substitution such that it comes to

$$u_{ss} + \sin[u(s)] = 0,$$

and it implies

$$\frac{1}{2}u_s^2 - \cos(u) = E, \text{ where } E \text{ is constant,} \tag{5.1}$$

$$\Rightarrow u_s = \pm\sqrt{2[E + \cos(u)]}.$$

We focus on $u_s = \sqrt{2[E + \cos(u)]}$, and since $\cos(u) = 1 - 2\sin^2(\frac{u}{2})$, we have

$$u_s = \sqrt{2[E + 1 - 2\sin^2(\frac{u}{2})]}$$

$$= \sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}. \tag{5.2}$$

Then we can obtain

$$s = \int_0^{u(s)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{U}{2})}} dU. \tag{5.3}$$

By the equation (5.1), we have

$$\frac{1}{2}u_s^2 + [1 - \cos(u)] = E + 1,$$

and regard it as a energy system with the kinetic energy $\frac{1}{2}u_s^2$, the potential energy $1 - \cos(u)$, and the total energy $E+1$. Obvious that, all of them are greater than 0, but, since $0 \leq [1 - \cos(u)] \leq 2$, so we will discuss in four cases by given E in different conditions,

$$\begin{cases} E+1=0, \\ 0<E+1<2, \\ E+1=2, \text{ and} \\ E+1>2. \end{cases}$$

<u>**Case 1:** $E + 1 = 0$</u>

When $E + 1 = 0$, means the total energy is 0, so the pendulum is stationary, and it implies $u(s) = 0$.

<u>**Case 2:** $0 < E + 1 < 2$</u>

By the equation (5.3),

$$\begin{aligned} s &= \int_0^{u(s)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{U}{2})}} dU \\ &= \int_0^{u(s)} \frac{1}{\sqrt{2(E+1)}} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{U}{2})}} dU \\ &= \frac{1}{\sqrt{2(E+1)}} \int_0^{u(s)} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{U}{2})}} dU. \end{aligned}$$

Now, we let $t = \sqrt{\frac{2}{E+1}}\sin(\frac{U}{2})$, so that $\frac{1}{\sqrt{1-\frac{2}{E+1}\sin^2(\frac{U}{2})}} = \frac{1}{\sqrt{1-t^2}}$, and

$$\begin{aligned}
dt &= \frac{1}{2}\sqrt{\frac{2}{E+1}}\cos(\frac{U}{2})dU \\
&= \frac{1}{2}\sqrt{\frac{2}{E+1}}\sqrt{1-\sin^2(\frac{U}{2})}dU \\
&= \frac{1}{2}\sqrt{\frac{2}{E+1}-\frac{2}{E+1}\sin^2(\frac{U}{2})}dU \\
&= \frac{1}{2}\sqrt{\frac{2}{E+1}-t^2}dU \\
\Rightarrow dU &= \frac{2}{\sqrt{\frac{2}{E+1}-t^2}}dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
s &= \frac{1}{\sqrt{2(E+1)}}\int_0^{u(s)}\frac{1}{\sqrt{1-t^2}}dU \\
&= \frac{1}{\sqrt{2(E+1)}}\int_0^{\sqrt{\frac{2}{E+1}}\sin(\frac{u(s)}{2})}\frac{1}{\sqrt{1-t^2}}\frac{2}{\sqrt{\frac{2}{E+1}-t^2}}dt \\
&= \int_0^{\sqrt{\frac{2}{E+1}}\sin(\frac{u(s)}{2})}\frac{1}{\sqrt{1-t^2}}\frac{2}{\sqrt{4-2(E+1)t^2}}dt \\
&= \int_0^{\sqrt{\frac{2}{E+1}}\sin(\frac{u(s)}{2})}\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-\frac{E+1}{2}t^2}}dt.
\end{aligned}$$

And, let $k = \sqrt{\frac{E+1}{2}}$, then

$$s = \int_0^{\frac{1}{k}\sin(\frac{u(s)}{2})}\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt. \tag{5.4}$$

Finally, by the Jacobian elliptic function, $\text{sn}(s,k) = \frac{1}{k}\sin(\frac{u(s)}{2})$ with $0 < k < 1$, and this implies that

$$u(s) = 2\arcsin[k\text{sn}(s,k)].$$

Case 3: $E + 1 = 2$

By the equation (5.3),

$$s = \frac{1}{2}\int_0^{u(s)} \frac{1}{\sqrt{1-\sin^2(\frac{U}{2})}}dU.$$

Let $t = \sin(\dfrac{U}{2})$, so that $\dfrac{1}{\sqrt{1-\sin^2(\frac{U}{2})}} = \dfrac{1}{\sqrt{1-t^2}}$, and

$$\begin{aligned} dt &= \frac{1}{2}\cos(\frac{U}{2})dU \\ &= \frac{1}{2}\sqrt{1-\sin^2(\frac{U}{2})}dU \\ &= \frac{1}{2}\sqrt{1-t^2}dU \\ \Rightarrow dU &= \frac{2}{\sqrt{1-t^2}}dt. \end{aligned}$$

Therefore,

$$\begin{aligned} s &= \frac{1}{2}\int_0^{u(s)} \frac{1}{\sqrt{1-t^2}}dU \\ &= \frac{1}{2}\int_0^{\sin(\frac{u(s)}{2})} \frac{1}{\sqrt{1-t^2}}\frac{2}{\sqrt{1-t^2}}dt \\ &= \int_0^{\sin(\frac{u(s)}{2})} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-t^2}}dt. \end{aligned} \tag{5.5}$$

Finally, by the Jacobian elliptic function for $k = 1$, $\text{sn}(s, 1) = \sin(\dfrac{u(s)}{2})$, and this implies that

$$u(s) = 2\arcsin[\text{sn}(s, 1)].$$

Case 4: $E+1>2$

By the equation (5.3),

$$s = \frac{1}{\sqrt{2(E+1)}} \int_0^{u(s)} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{U}{2})}} dU.$$

Let $k = \sqrt{\dfrac{2}{E+1}}$ and $t = \sin(\dfrac{U}{2})$, so that $\dfrac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{U}{2})}} = \dfrac{1}{\sqrt{1-k^2t^2}}$, and

$$\begin{aligned} dt &= \frac{1}{2}\cos(\frac{U}{2})dU \\ &= \frac{1}{2}\sqrt{1-\sin^2(\frac{U}{2})}dU \\ &= \frac{1}{2}\sqrt{1-t^2}dU \\ \Rightarrow dU &= \frac{2}{\sqrt{1-t^2}}dt. \end{aligned}$$

Therefore,

$$\begin{aligned} s &= \frac{1}{\sqrt{2(E+1)}} \int_0^{u(s)} \frac{1}{\sqrt{1-k^2t^2}} dU \\ &= \frac{k}{2} \int_0^{\sin(\frac{u(s)}{2})} \frac{1}{\sqrt{1-k^2t^2}} \frac{2}{\sqrt{1-t^2}} dt \\ &= k \int_0^{\sin(\frac{u(s)}{2})} \frac{1}{\sqrt{1-t^2}} \frac{1}{\sqrt{1-k^2t^2}} dt \\ \Rightarrow \quad \frac{s}{k} &= \int_0^{\sin(\frac{u(s)}{2})} \frac{1}{\sqrt{1-t^2}} \frac{1}{\sqrt{1-k^2t^2}} dt. \end{aligned}$$

Finally, by the Jacobian elliptic function, $\mathrm{sn}(\frac{s}{k}, k) = \sin(\frac{u(s)}{2})$ with $0<k<1$, and this implies that

$$u(s) = 2\arcsin[\mathrm{sn}(\frac{s}{k}, k)].$$

5.2 The Periods of Those Solutions

To begin with, suppose that the pendulum released with no initial velocity, and has the period T.

Notice that, when this pendulum first passes through to bottom equilibrium position, it spent the time $\frac{T}{4}$, and so the initial angle is $u(\frac{T}{4})$.

Case 1: $E + 1 = 0$

Since the pendulum is stationary, $T = 0$.

Case 2: $0 < E + 1 < 2$

Notice that, in the discussion of above section, $k = \sqrt{\frac{E+1}{2}}$.

We consider the initial position, the energy system is

$$\begin{aligned} & 1 - \cos[u(\frac{T}{4})] = E + 1 \\ \Rightarrow \quad & 2\sin^2[\frac{u(\frac{T}{4})}{2}] = 2k^2 \\ \Rightarrow \quad & u(\frac{T}{4}) = \pm 2\arcsin k. \end{aligned}$$

Assume without loss of generality that $u(\frac{T}{4}) = 2\arcsin k$.

By the equation (5.4),

$$\begin{aligned} T &= 4(\frac{T}{4}) \\ &= 4\int_0^{\frac{1}{k}\sin(\frac{u(\frac{T}{4})}{2})} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt \\ &= 4\int_0^{\frac{1}{k}\sin(\arcsin k)} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt \\ &= 4\int_0^{1} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt. \end{aligned}$$

So, the period is $4\int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt$.

<u>**Case 3:** $E+1=2$</u>

We consider the initial position, the energy system is

$$\begin{aligned} 1-\cos[u(\frac{T}{4})] &= E+1 \\ &= 2 \\ \Rightarrow \cos[u(\frac{T}{4})] &= -1 \\ \Rightarrow u(\frac{T}{4}) &= \pm\pi. \end{aligned}$$

Assume without loss of generality that $u(\frac{T}{4})=\pi$.

By the equation (5.5),

$$\begin{aligned} T &= 4(\frac{T}{4}) \\ &= 4\int_0^{\sin(\frac{u(\frac{T}{4})}{2})} \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-t^2}}dt \\ &= 4\int_0^{\sin(\frac{\pi}{2})} \frac{1}{1-t^2}dt \\ &= 4\int_0^1 \frac{1}{1-t^2}dt \\ &= \infty. \end{aligned}$$

So, the period is ∞.

<u>**Case 4:** $E+1>2$</u>

Now, we want to find out the exact period, but the equation (5.2) always positive for $E+1>2$, and this means that the pendulum doesn't change the direction, and always have velocity for every position. Thus, the pendulum will never stop, and never swing back, so this implies it have no periodicity.

Chapter 6

Conclusion

Notice that the sine-Gordon equation,

$$u_{tt} - u_{xx} + \sin[u(x,t)] = 0, \text{ where } -\infty < x < \infty \text{ and } t > 0,$$

can be transformed to $u_{ss} + \sin[u(s)] = 0$, a simple pendulum motion at time s.

In above chapters, we have introduced the theory of the Riemann surface, and solved the problem of pendulum motion by approximation, which precisely occurs on Riemann surface.

Next, still on Riemann surface, we study the classical theory of elliptic functions, and solve the special case of the sine-Gordon equation.

The following list is of the special solutions and the periods.

	$0 < E+1 < 2$	$E+1=2$	$E+1>2$
Modulus k	$\sqrt{\frac{E+1}{2}}$	1	$\sqrt{\frac{2}{E+1}}$
Solution $U(x,t)$	$2\arcsin[k\,\mathrm{sn}(\alpha x - \beta t, k)]$	$2\arcsin[\mathrm{sn}(\alpha x - \beta t, 1)]$	$2\arcsin[\mathrm{sn}(\frac{\alpha x - \beta t}{k}, k)]$
Period	$4\int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-k^2t^2}}dt$	∞	No periodicity

Appendix A

The Integrals over a, b-cycles

By Chapter 3, we have the a, b-cycles in Fg. 3.2 and 3.6.
The follwing are the detail of the integrals over a, b-cycles.

A.1 a_1-cycle

We consider the equivalent paths, say $a_1^* = a_{11} \cup a_{12}$, shown in Fig. 3.3, such that $a_1 \approx a_1^*$, where
a_{11} is the straight path on vertical cut from u_2 to u_1 on the $+$ edge of sheet-I, and
a_{12} is the straight path on vertical cut from u_1 to u_2 on the $-$ edge of sheet-I.

1. If $z \in a_{11}$:

   By the discussion in previous chapter, we have

$$\begin{aligned}\sqrt{u-u_1}\sqrt{u-u_2} &\overset{MATH.}{=} -\sqrt{u-u_1}\sqrt{u-u_2},\\ \sqrt{u-u_3}\sqrt{u-u_4} &\overset{MATH.}{=} \sqrt{u-u_3}\sqrt{u-u_4},\\ \sqrt{u-u_5}\sqrt{u-u_6} &\overset{MATH.}{=} \sqrt{u-u_5}\sqrt{u-u_6}, \text{ and}\\ \sqrt{u-u_7}\sqrt{u-u_8} &\overset{MATH.}{=} -\sqrt{u-u_7}\sqrt{u-u_8},\end{aligned}$$

   so $f(u) \overset{MATH.}{=} f(u)$.

   Thus,

$$\int_{a_{11}} \frac{1}{f(u)}du \overset{MATH.}{=} \int_{u_2}^{u_1} \frac{1}{f(u)}du.$$

2. If $z \in a_{12}$:

   By the Proposition 2.1,

$$\int_{a_{12}} \frac{1}{f(u)} du = \int_{a_{11}} \frac{1}{f(u)} du.$$

   Therefore,

$$\begin{aligned} \int_{a_1} \frac{1}{f(u)} du &= \int_{a_1^*} \frac{1}{f(u)} du \\ &= \int_{a_{11}} \frac{1}{f(u)} du + \int_{a_{12}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} 2 \int_{u_2}^{u_1} \frac{1}{f(u)} du. \end{aligned}$$

□

A.2 a_2-cycle

We consider the equivalent paths, say $a_2^* = a_{21} \cup a_{22}$, shown in Fig. 3.4, such that $a_2 \approx a_2^*$, where
a_{21} is the straight path on horizontal cut from u_3 to u_4 on the + edge of sheet-I, and
a_{22} is the straight path on horizontal cut from u_4 to u_3 on the − edge of sheet-I.

1. If $z \in a_{21}$:

   By the discussion in previous chapter, we have

$$\begin{aligned} \sqrt{u-u_1}\sqrt{u-u_2} &\overset{MATH.}{=} \sqrt{u-u_1}\sqrt{u-u_2}, \\ \sqrt{u-u_3}\sqrt{u-u_4} &\overset{MATH.}{=} -\sqrt{u-u_3}\sqrt{u-u_4}, \\ \sqrt{u-u_5}\sqrt{u-u_6} &\overset{MATH.}{=} \sqrt{u-u_5}\sqrt{u-u_6}, \text{ and} \\ \sqrt{u-u_7}\sqrt{u-u_8} &\overset{MATH.}{=} -\sqrt{u-u_7}\sqrt{u-u_8}, \end{aligned}$$

   so $f(u) \overset{MATH.}{=} f(u)$.

   Thus,

$$\int_{a_{21}} \frac{1}{f(u)} du \overset{MATH.}{=} \int_{u_3}^{u_4} \frac{1}{f(u)} du.$$

2. If $z \in a_{22}$:

   By the Proposition 2.1,

$$\int_{a_{22}} \frac{1}{f(u)} du = \int_{a_{21}} \frac{1}{f(u)} du.$$

Therefore,

$$\begin{aligned} \int_{a_2} \frac{1}{f(u)} du &= \int_{a_2^*} \frac{1}{f(u)} du \\ &= \int_{a_{21}} \frac{1}{f(u)} du + \int_{a_{22}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} 2\int_{u_3}^{u_4} \frac{1}{f(u)} du. \end{aligned}$$

□

A.3 a_3-cycle

We consider the equivalent paths, say $a_3^* = a_{31} \cup a_{32}$, shown in Fig. 3.5, such that $a_3 \approx a_3^*$, where
a_{31} is the straight path on horizontal cut from u_5 to u_6 on the + edge of sheet-I, and
a_{32} is the straight path on horizontal cut from u_6 to u_5 on the − edge of sheet-I.

1. If $z \in a_{31}$:

   By the discussion in previous chapter, we have

$$\begin{aligned} \sqrt{u-u_1}\sqrt{u-u_2} &\overset{MATH.}{=} \sqrt{u-u_1}\sqrt{u-u_2}, \\ \sqrt{u-u_3}\sqrt{u-u_4} &\overset{MATH.}{=} \sqrt{u-u_3}\sqrt{u-u_4}, \\ \sqrt{u-u_5}\sqrt{u-u_6} &\overset{MATH.}{=} -\sqrt{u-u_5}\sqrt{u-u_6}, \text{ and} \\ \sqrt{u-u_7}\sqrt{u-u_8} &\overset{MATH.}{=} -\sqrt{u-u_7}\sqrt{u-u_8}, \end{aligned}$$

   so $f(u) \overset{MATH.}{=} f(u)$.

   Thus,

$$\int_{a_{31}} \frac{1}{f(u)} du \overset{MATH.}{=} \int_{u_5}^{u_6} \frac{1}{f(u)} du.$$

2. If $z \in a_{32}$:

   By the Proposition 2.1,

$$\int_{a_{32}} \frac{1}{f(u)} du = \int_{a_{31}} \frac{1}{f(u)} du.$$

Therefore,

$$\begin{aligned} \int_{a_3} \frac{1}{f(u)} du &= \int_{a_3^*} \frac{1}{f(u)} du \\ &= \int_{a_{31}} \frac{1}{f(u)} du + \int_{a_{32}} \frac{1}{f(u)} du \\ &\stackrel{MATH.}{=} 2\int_{u_5}^{u_6} \frac{1}{f(u)} du. \end{aligned}$$

□

A.4 b_3-cycle

We consider the equivalent paths, say $b_3^* = b_{31} \cup b_{32}$, shown in Fig. 3.7, such that $b_3 \approx b_3^*$, where
b_{31} is the straight path from u_6 to u_8 in the sheet-I, and
b_{32} is the straight path from u_8 to u_6 in the sheet-II.

1. If $z \in b_{31}$:

   By the discussion in previous chapter, we have

$$\begin{aligned} \sqrt{u-z_1}\sqrt{u-z_2} &\stackrel{MATH.}{=} \sqrt{u-z_1}\sqrt{u-z_2}, \\ \sqrt{u-z_3}\sqrt{u-z_4} &\stackrel{MATH.}{=} \sqrt{u-z_3}\sqrt{u-z_4}, \\ \sqrt{u-z_5}\sqrt{u-z_6} &\stackrel{MATH.}{=} \sqrt{u-z_5}\sqrt{u-z_6}, \text{ and} \\ \sqrt{u-z_7} &\stackrel{MATH.}{=} -\sqrt{u-z_7}, \end{aligned}$$

   so $f(z) \stackrel{MATH.}{=} -f(z)$.

   Thus,

$$\int_{b_{31}} \frac{1}{f(u)} du \stackrel{MATH.}{=} -\int_{u_6}^{u_8} \frac{1}{f(u)} du.$$

2. If $z \in b_{32}$:

   By the Proposition 2.2,

$$\int_{b_{32}} \frac{1}{f(u)} du = \int_{b_{31}} \frac{1}{f(u)} du.$$

   Therefore,

$$\begin{aligned} \int_{b_3} \frac{1}{f(u)} du &= \int_{b_3^*} \frac{1}{f(u)} du \\ &= \int_{b_{31}} \frac{1}{f(u)} du + \int_{b_{32}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} -2 \int_{u_6}^{u_8} \frac{1}{f(u)} du. \end{aligned}$$

□

A.5 b_2-cycle

We consider the equivalent paths, say $b_2^* = b_3^* \cup a_{31} \cup b_{21} \cup b_{22} \cup b_{23}$, shown in Fig. 3.8, such that $b_2 \approx b_2^*$, where
b_{21} is the straight path on horizontal cut from u_6 to u_5 on the $-$ edge of sheet-II,
b_{22} is the straight path from u_4 to u_5 in the sheet-I, and
b_{23} is the straight path from u_5 to u_4 in the sheet-II.

To start with, we know that b_{21} is also the straight path on horizontal cut from u_6 to u_5 on the $+$ edge of sheet-I, that is, $b_{21} = -a_{31}$, and this implies

$$\int_{a_{31}} \frac{1}{f(u)} du + \int_{b_{21}} \frac{1}{f(u)} du = 0$$

1. If $z \in b_{22}$:

   By the discussion in previous chapter, we have

$$\begin{aligned} \sqrt{u-u_1}\sqrt{u-u_2} &\overset{MATH.}{=} \sqrt{u-u_1}\sqrt{u-u_2}, \\ \sqrt{u-u_3}\sqrt{u-u_4} &\overset{MATH.}{=} \sqrt{u-u_3}\sqrt{u-u_4}, \\ \sqrt{u-u_5}\sqrt{u-u_6} &\overset{MATH.}{=} \sqrt{u-u_5}\sqrt{u-u_6}, \text{ and} \\ \sqrt{u-u_7} &\overset{MATH.}{=} -\sqrt{u-u_7}, \end{aligned}$$

   so $f(z) \overset{MATH.}{=} -f(z)$.

Thus,

$$\int_{b_{22}} \frac{1}{f(u)} du \overset{MATH.}{=} -\int_{u_4}^{u_5} \frac{1}{f(u)} du.$$

2. If $z \in b_{23}$:

   By the Proposition 2.2,

$$\int_{b_{23}} \frac{1}{f(u)} du = \int_{b_{22}} \frac{1}{f(u)} du.$$

Therefore,

$$\begin{aligned}
\int_{b_2} \frac{1}{f(u)} du &= \int_{b_2^*} \frac{1}{f(u)} du \\
&= \int_{b_3^*} \frac{1}{f(u)} du + \int_{a_{31}} \frac{1}{f(u)} du + \int_{b_{21}} \frac{1}{f(u)} du + \int_{b_{22}} \frac{1}{f(u)} du + \int_{b_{23}} \frac{1}{f(u)} du \\
&\overset{MATH.}{=} \int_{b_3^*} \frac{1}{f(u)} du - 2\int_{u_4}^{u_5} \frac{1}{f(u)} du.
\end{aligned}$$

□

A.6 b_1-cycle

We consider the equivalent paths, say $b_1^* = b_2^* \cup a_{21} \cup b_{11} \cup b_{12} \cup b_{13}$, shown in Fig. 3.9, such that $b_1 \approx b_1^*$, where
b_{11} is the straight path on horizontal cut from u_4 to u_3 on the $-$ edge of sheet-II,
b_{12} is the straight path from u_1 to u_3 in the sheet-I, and
b_{13} is the straight path from u_3 to u_1 in the sheet-II.

To start with, we know that b_{11} is also the straight path on horizontal cut from u_4 to u_3 on the $+$ edge of sheet-I, that is, $b_{11} = -a_{21}$, and this implies

$$\int_{a_{21}} \frac{1}{f(u)} du + \int_{b_{11}} \frac{1}{f(u)} du = 0$$

1. If $z \in b_{12}$:

   By the discussion in previous chapter, we have

$$\begin{aligned}
\sqrt{u-u_1}\sqrt{u-u_2} &\overset{MATH.}{=} \sqrt{u-u_1}\sqrt{u-u_2}, \\
\sqrt{u-u_5}\sqrt{u-u_6} &\overset{MATH.}{=} \sqrt{u-u_5}\sqrt{u-u_6}, \\
\sqrt{u-u_8}\sqrt{u-u_6} &\overset{MATH.}{=} \sqrt{u-u_8}\sqrt{u-u_6}, \text{ and} \\
\sqrt{u-u_7} &\overset{MATH.}{=} -\sqrt{u-u_7},
\end{aligned}$$

so $f(z) \overset{MATH.}{=} -f(z)$.

Thus,

$$\int_{b_{12}} \frac{1}{f(u)} du \overset{MATH.}{=} -\int_{u_1}^{u_3} \frac{1}{f(u)} du.$$

2. If $z \in b_{13}$:

By the Proposition 2.2,

$$\int_{b_{13}} \frac{1}{f(u)} du = \int_{b_{12}} \frac{1}{f(u)} du.$$

Therefore,

$$\begin{aligned} \int_{b_1} \frac{1}{f(u)} du &= \int_{b_1^*} \frac{1}{f(u)} du \\ &= \int_{b_2^*} \frac{1}{f(u)} du + \int_{a_{21}} \frac{1}{f(u)} du + \int_{b_{11}} \frac{1}{f(u)} du + \int_{b_{12}} \frac{1}{f(u)} du + \int_{b_{13}} \frac{1}{f(u)} du \\ &\overset{MATH.}{=} \int_{b_2^*} \frac{1}{f(u)} du - 2\int_{u_1}^{u_3} \frac{1}{f(u)} du. \end{aligned}$$

□

References

[1] R. Rajaraman (1982). *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*. Amsterdam: North-Holland Personal Library.

[2] George Springer (1981). *Introduction to Riemann Surfaces*. New York: Chelsea.

[3] 洪維恩 (2001)。數學運算大師 Mathematica 4.0。臺北市: 碁峰。

[4] Yun-Ting Wu (2010). *Theory of Riemann Surfaces and Its Applications to Differential Equations*. Unpublished master's thesis, National Chiao Tung University, Hsinchu, Taiwan, Republic of China.

[5] E. T. Whittaker and G. N. Watson (1962). *A Course of Modern Analysis*. Cambridge: Cambridge University Press.

[6] Jong-Eao Lee. *Topics on Periodic One and Two-Phase Solutions of Certain Nonlinear Evolution Equations*. National Chiao Tung University, Hsinchu, Taiwan, Republic of China.

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