國 立 交 通 大 學

電信工程研究所

博 士 論 文

基於搜尋演算法的不定長度錯誤更正前置碼之設計

Algorithmic Design of Variable-Length Error-Correcting Prefix Code

研 究 生：吳庭伊

指導教授：陳伯寧 教授

中 華 民 國 一〇二 年 七 月

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學生：吳庭伊　　　　　　　　　　　　　　　　　指導教授：陳伯寧

國立交通大學電信工程研究所博士班

摘　　　要

在這篇論文中，我們針對離散無記憶訊號源(discrete memoryless source)設計整合訊源-通道之不定長度錯誤更正前置碼(variable-length error correcting prefix code or VLECPC)。我們的研究成果包含：一、在給定自由距離(free distance)最小允許值的前提下，證明運用優先權搜尋演算法(priority first search algorithm)，於我們所新設計的搜尋樹結構中，可保證找到最低平均碼長的不定長度錯誤更正前置碼。二、為進一步降低解碼錯誤率，我們提出在所有可達最低平均碼長的不定長度錯誤更正前置碼中，可以使用錯誤率聯集上界(union bound)的主要項$B_{d_{free}}$為依據，選取使主要項最低的最低平均碼長之不定長度錯誤更正前置碼，獲取較佳的容錯能力。三、對於較大的自由距離最小允許值、或是較多的離散訊號源個數，前述所提的搜尋演算法因過於費時而不適用，因此在損失些微平均碼長的前提下，另提出簡化快速搜尋演算法。四、在解碼端，針對接收端另知不定長度碼的傳送個數的條件，設計了低複雜度的最大事後機率(maximum a posteriori)解碼演算法。模擬結果顯示，我們所提出的編碼演算法在平均碼長與效能上，皆優於現有文獻的方法。另外，與傳統的分散式訊源-通道編碼相比較，在相當的解碼複雜度下，我們所設計的整合式訊源-通道編解碼系統可達更低的傳輸錯誤率。

Algorithmic Design of Variable-Length Error-Correcting Prefix Code

Student: Ting-Yi Wu Advisor: Po-Ning Chen

Institute of Communications Engineering
National Chiao Tung University

ABSTRACT

A joint source-channel coding problem that combines the efficient compression of discrete memoryless sources with their reliable communication over memoryless channels via binary variable-length error-correcting prefix codes (VLECPCs) is considered. Under a fixed free distance constraint, a priority-first search algorithm is devised for finding an optimal VLECPC with minimal average codeword length. Two variations of the priority-first-search-based code construction algorithm are also provided. The first one improves the resilience of the developed codes against channel noise by additionally considering a performance parameter $B_{d_{free}}$ without sacrificing optimality in average codeword length. In the second variation, to accommodate a large free distance constraint as well as a large source alphabet such as the 26-symbol English data source, the VLECPC construction algorithm is modified with the objective of significantly reducing its search complexity while still yielding near-optimal codes. A low-complexity *sequence maximum a posteriori* (MAP) decoder for all VLECPCs (including our constructed optimal code) is then proposed under the premise that the receiver knows the number of codewords being transmitted. Simulations show that the realized optimal and suboptimal VLECPCs compare favorably with existing codes in the literature in terms of coding efficiency, search complexity and error rate performance.

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Chapter 1

Introduction

1.1 Overview

One of Shannon's key contributions in information theory is the separation principle for source-channel coding [27], which states that the source and channel coding operations can be separately designed and performed in tandem without affecting the system's optimality for reliably transmitting a data source over a noisy channel. However, this result hinges on the assumption that unlimited complexity and coding delay can be afforded by the system, which is unrealistic in today's resource constrained communication systems. It is indeed well-known via both analytical and empirical studies (e.g., see [1, 2, 14, 33] and the references therein) that joint source-channel coding (JSCC) can significantly outperform separate source-channel coding (SSCC), particularly when the system has stringent delay and complexity restrictions. JSCC, which may use codes of fixed or variable length, is typically realized in two ways: by coordinating the source and channel coding functions in tandem or by combining them within a single step (examples of various JSCC schemes can be found in [33]). In this dissertation, we focus on variable-length single-step JSCC with the objective of designing optimal or close-to-optimal variable-length error-correcting prefix codes (VLECPC) with low complexity for the efficient compression and communication of data sources in the presence of channel noise. Here optimality is interpreted as achieving minimal average codeword length among all VLECPC designs subject to a fixed free-distance constraint. The successful development of such VLECPCs, which play the dual role of good data compression and error-correcting codes, provides an interesting alternative to the classical SSCC scheme, particularly when the system's complexity can be significantly reduced without degrading its error performance.

First introduced in [17, 5, 6], VLECPCs were thoroughly investigated by Buttigieg in [7, 9] and were shown to exhibit properties akin to those of convolutional codes: they have a memory structure, which can naturally be represented via a trellis, and they are best suited for being decoded via a sequence maximum-likelihood (ML) or maximum a posteriori (MAP) Viterbi-like decoder (as opposed to decoding their codewords instantaneously). Furthermore, Buttigieg showed how the VLECPCs' distance spectrum and the union bound can be used to predict their error performance under hard-decision ML decoding for the binary symmetric channel (BSC) and identified the codes' free distance d_{free} as a key parameter which, when maximized, can improve the codes' performance. In related works, the error exponent of VLECPCs is analyzed [3] and conditions for the existence of VLECPCs are studied [31, 24].

In [7], Buttigieg originally proposed two techniques to construct VLECPCs with a given d_{free} value. They are respectively based on a greedy algorithm (GA) and a majority vote algorithm (MVA). Specifically, he employs either the GA or MVA procedure to select as many codewords as possible of the same length, where the selected codewords must satisfy certain minimum distance conditions in order to reach the required d_{free}. Later, Lamy and Paccaut [23] replaced Buttigieg's GA and MVA schemes with new algorithm designed to obtain a good trade-off between system complexity and coding efficiency. In [30], Wang *et al.* improved the coding efficiency of VLECPCs by iteratively replacing longer codewords with shorter ones. In [26], Savari and Kliewer focused on minimizing the average codeword length of VLECPCs. In their design, each codeword is required to have Hamming weight w, where w is a multiple of an integer greater or equal to 2, resulting in a class of VLECPCs with $d_{\text{free}} \geq 2$. In [11, 13, 18], Diallo *et al.* proposed several algorithms for obtaining VLECPCs with maximal d_{free} under the premise that all codeword lengths are known in advance. A similar approach was used in [12] for developing good error-correcting arithmetic codes.

With respect to VLECPC decoding, Buttigieg [7] used a trellis representation of

VLECPCs and modified the Viterbi algorithm (VA) to realize a sequence MAP decoder, which is optimal in terms of minimizing the VLECPCs' sequence error probability. Later in 2008, Huang *et al.* [19] proposed a trellis-based MAP priority-first search decoding algorithm for VLECPCs based on a suitable soft-decision MAP decoding criterion and empirically showed a significant complexity improvement over Buttigieg's MAP decoder. MAP decoding techniques using an extended trellis under the assumption that the receiver knows both the number of transmitted bits and the number of transmitted codewords were developed in [4, 21]. Other decoding methods for variable-length codes (VLC) that use other trellis VLC representations include the sequence MAP decoder of [3] and iterative (Turbo-like) decoders of [4, 22].

In this dissertation, we present a novel priority-first search algorithm that can construct VLECPCs with minimal average codeword length and free distance no less than a pre-given d^*_{free}. We next investigate how to select, among all obtained optimal[1] VLECPCs, the one with the best error correction capability. We observe that the codes' Levenshtein coefficient $B_{d_{\text{free}}}$ plays an important role in their error performance: choosing the optimal code with the smallest $B_{d_{\text{free}}}$ yields the best system error rate. Furthermore, we modify our construction algorithm to reduce its search complexity in order to accommodate large values of d_{free} and large source alphabets such as the 26-symbol English data source. We also propose a low-complexity two-phase sequence MAP decoder that can be applied to all VLECPCs (including our constructed optimal and suboptimal codes) under the assumption that the receiver knows both the number of transmitted bits and the number of transmitted codewords. We show by simulations that the resulting suboptimal VLECPCs outperform most existing VLECPCs in the literature in terms of compression efficiency, search complexity and error rate. We also compare our JSCC codes with traditional SSCCs.

The rest of this dissertation is organized as follows. In Chapter 2, we formulate our problem and present some background material about VLECPCs. In Chapter 3, we de-

[1] We emphasize that, throughout the dissertation, an "optimal VLECPC" is defined as a VLECPC with minimal average codeword length. In other words, an optimal VLECPC does not guarantee to yield the best error rate performance.

scribe our code construction which guarantees the development of optimal VLECPCs with a given free distance constraint. In Chapter 4, two VLECPC construction modifications are proposed respectively for the design of optimal codes with enhanced error correction capability and for the design of suboptimal VLECPCs for large d_{free} and large source alphabet sizes. In Chapter 5, a low-complexity two-phase sequence MAP decoder is introduced. Simulation results illustrating the performance of the constructed optimal and suboptimal VLECPCs are given in Chapter 6. Finally, conclusions are stated in Chapter 7.

1.2 Contributions

The main contributions of this thesis are briefed as follows.

- The first algorithm that guarantees the construction of an *optimal VLECPC* (in the sense of minimizing the average codeword length) subject to a free distance constraint is proposed.
- The error correction capability of the constructed optimal VLECPC is *enhanced* by choosing the optimal VLECPC with minimum $B_{d_{\text{free}}}$.
- Simplified suboptimal construction algorithm has a search complexity *superior to the state-of-the-art code construction algorithms* in the literature and can accommodate large source alphabets such as the 26-symbol English text source.
- An *efficient low-complexity sequence MAP decoder* for a receiver knowing the number of transmitted codewords is also proposed.

Chapter 2

Problem Formulation and Preliminaries

We consider the JSCC problem of efficient compression of a discrete memoryless (independent and identically distributed) source and its reliable communication over a noisy channel via a single binary VLECPC. We assume a binary phase-shift keying (BPSK) modulated additive white Gaussian noise (AWGN) channel (although other channel models can also be considered) and employ optimal sequence MAP decoding in the sense of minimizing the code's sequence error[1] probability. The VLECPC's free distance d_{free} has already been identified as a key error performance parameter, playing a similar role as for convolutional codes: the larger d_{free} is, the better is the code's error resilience particularly at high signal-to-noise ratios (SNRs) [7, 9]. Our objectives are four-fold:

- Designing an algorithm that guarantees the construction of an optimal (i.e., with minimal average codeword length) binary VLECPC for a given free distance bound d^*_{free}.
- Enhancing the error correction capability of the constructed optimal VLECPCs by optimizing an important performance parameter $B_{d_{\text{free}}}$.
- Ensuring that the construction algorithms have a search complexity superior to the state-of-the-art code construction algorithms in the literature so that they can accommodate large source alphabets such as the 26-symbol English data source.
- Designing an efficient low-complexity sequence MAP decoder under the premise

[1] A sequence error occurs when a decoded sequence of VLECPCs is not exactly the same as the transmitted one.

that the receiver knows the total number of transmitted VLECPC codewords (in addition to the total number of transmitted code bits).

The successful achievement of these objectives has interesting applications for the effective compression and error-resilient transmission of text documents over noisy channels.

In what follows, we present some preliminary background about VLECPCs. Consider a K-ary discrete memoryless source with alphabet $\mathcal{S} \triangleq \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ and respective symbol probabilities $p_1, p_2, \ldots, p_K$ (such that $\sum_{i=1}^{K} p_i = 1$). A (first-order) VLECPC encoder maps each symbol $\alpha_i \in \mathcal{S}$ to a binary variable-length codeword $\boldsymbol{c}_i$, where $i = 1, 2, \ldots, K$. The set of codewords is denoted by $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_K\}$ and the average codeword length for code $\mathcal{C}$ is given by

$$\overline{\mathcal{C}} \triangleq \sum_{i=1}^{K} p_i |\boldsymbol{c}_i|, \tag{2.1}$$

where $|\boldsymbol{c}_i|$ is the length of codeword $\boldsymbol{c}_i$.

2.1 Sequence MAP Decoding Criterion

Let

$$\mathcal{X}_{L,N} \triangleq \{\boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{x}_3 \cdots \boldsymbol{x}_L : \forall \boldsymbol{x}_i \in \mathcal{C} \text{ and } \sum_{i=1}^{L} |\boldsymbol{x}_i| = N\} \tag{2.2}$$

be a set of bitstreams consisting of L (concatenated) codewords with overall length N. Define

$$\mathcal{X}_N \triangleq \bigcup_{i \geq 1} \mathcal{X}_{i,N} \tag{2.3}$$

as a set of bitstreams consisting of some (concatenated) codewords with overall length N. Assume that a sequence of VLECPC codewords of overall length N is transmitted over the binary-input AWGN channel and that $\boldsymbol{r} \triangleq (r_1, r_2, \ldots, r_N)$ is received at the channel output. The sequence MAP (soft-decision) decoder then outputs $\hat{\boldsymbol{v}} \triangleq (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_N)$ if $\hat{\boldsymbol{v}}$ satisfies [19]

$$\sum_{i=1}^{N} (y_i \oplus \hat{v}_i) \|\phi_i\|_1 - \ln \Pr(\hat{\boldsymbol{v}}) \leq \sum_{i=1}^{N} (y_i \oplus v_i) \|\phi_i\|_1 - \ln \Pr(\boldsymbol{v}) \tag{2.4}$$

for all

$$\boldsymbol{v} \in \begin{cases} \mathcal{X}_N & \text{if the receiver only knows } N, \\ \mathcal{X}_{L,N} & \text{if the receiver knows both } L \text{ and } N, \end{cases}$$

where $\oplus$ is modulo-2 addition, $\Pr(\cdot)$ denotes probability, $\|\cdot\|_1$ denotes absolute value, ϕ_i is a log-likelihood ratio given by

$$\phi_i \triangleq \ln\left[\frac{\Pr(r_i|0)}{\Pr(r_i|1)}\right] \tag{2.5}$$

and y_i is the hard decision of r_i given by

$$y_i \triangleq \begin{cases} 1 & \text{if } \phi_i < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

2.2 VLECPC Trellis Diagrams

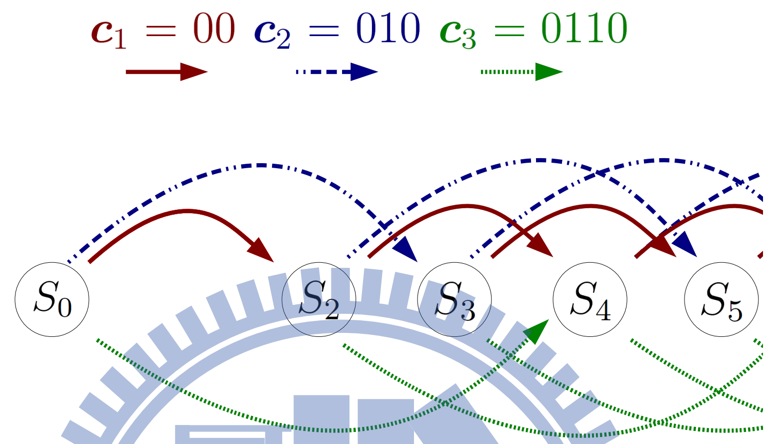
In [7, 9], Buttigieg employed a VLECPC decoding trellis $\mathcal{T}_N$ as exemplified in Figure 2.1(a) for $\mathcal{C} = \{00, 010, 0110\}$, in which state S_j denotes that the number of bits decoded thus far is j.

We can construct an extended trellis $\mathcal{T}_{L,N}$ as defined in [4, 21] under the assumption that the receiver knows both L and N. An example of such extended trellis for $\mathcal{C} = \{00, 010, 0110\}$ is shown in Figure 2.1(b), where $S_{i,j}$ denotes that the number of decoded symbols and the number of decoded bits thus far are i and j, respectively.

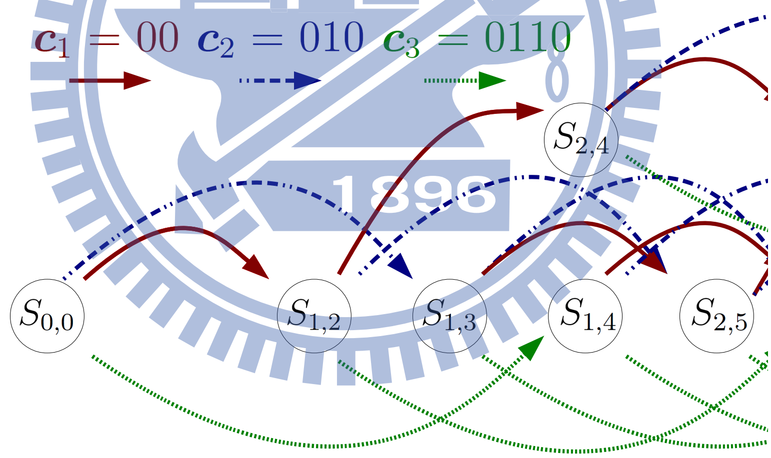
2.3 Free Distance

In [7], in order to analyze the error performance of a trellis-based VLECPC decoder, Buttigieg defined the free distance as the minimal Hamming distance between any two distinct paths converge at the same node in the trellis. Thus, the free distance d_{free} of $\mathcal{C}$ as defined in [7] depends on the structure of its decoding trellis diagram. For the computation of d_{free}, we will assume throughout the dissertation that the receiver knows both L and N. Therefore, d_{free} is defined based on $\mathcal{X}_{L,N}$ and is given by

$$d_{\text{free}}(\mathcal{C}) \triangleq \min\{d(\boldsymbol{a},\boldsymbol{b}) : \boldsymbol{a},\boldsymbol{b} \in \mathcal{X}_{L,N} \text{ for some } L, N \text{ and } \boldsymbol{a} \neq \boldsymbol{b}\}, \tag{2.7}$$



(a) Trellis $\mathcal{T}_N$



(b) Trellis $\mathcal{T}_{L,N}$

Figure 2.1: Trellis representations of a VLECPC. The red-color (solid), blue-color (dash-dot) and green-color (dotted) arrows correspond respectively to the transition of transmitting codewords $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and $\boldsymbol{c}_3$.

where $d(\boldsymbol{a}, \boldsymbol{b})$ denotes the Hamming distance between bitstreams $\boldsymbol{a}$ and $\boldsymbol{b}$. The following lower bound on $d_{\text{free}}(\mathcal{C})$ has been shown in [7, 9]

$$d_{\text{free}}(\mathcal{C}) \geq \min\{d_{\text{b}}(\mathcal{C}), d_{\text{c}}(\mathcal{C}) + d_{\text{d}}(\mathcal{C})\}, \tag{2.8}$$

where $d_{\text{b}}(\mathcal{C})$ is the "overall minimum block distance" defined as

$$d_{\text{b}}(\mathcal{C}) \triangleq \min\{d(\boldsymbol{c}_i, \boldsymbol{c}_j) : \boldsymbol{c}_i, \boldsymbol{c}_j \in \mathcal{C}, \boldsymbol{c}_i \neq \boldsymbol{c}_j \text{ and } |\boldsymbol{c}_i| = |\boldsymbol{c}_j|\}, \tag{2.9}$$

$d_{\text{c}}(\mathcal{C})$ is the "minimum converge distance" given by

$$d_{\text{c}}(\mathcal{C}) \triangleq \min\{d(\boldsymbol{c}_i, \boldsymbol{c}'_j) : \boldsymbol{c}_i, \boldsymbol{c}_j \in \mathcal{C}, |\boldsymbol{c}_i| < |\boldsymbol{c}_j|,\ \boldsymbol{c}'_j \text{ is the suffix of } \boldsymbol{c}_j \text{ and } |\boldsymbol{c}'_j| = |\boldsymbol{c}_i|\}, \tag{2.10}$$

and $d_{\text{d}}(\mathcal{C})$ is the "minimum diverge distance" defined as

$$d_{\text{d}}(\mathcal{C}) \triangleq \min\{d(\boldsymbol{c}_i, \boldsymbol{c}'_j) : \boldsymbol{c}_i, \boldsymbol{c}_j \in \mathcal{C}, |\boldsymbol{c}_i| < |\boldsymbol{c}_j|,\ \boldsymbol{c}'_j \text{ is the prefix of } \boldsymbol{c}_j \text{ and } |\boldsymbol{c}'_j| = |\boldsymbol{c}_i|\}. \tag{2.11}$$

Chapter 3

Optimal VLECPC Construction

We herein present a new search algorithm for constructing an optimal VLECPC with a given free-distance bound d^*_{free}. The search algorithm always outputs an optimal VLECPC with its $d_{\text{free}} \geq d^*_{\text{free}}$. This algorithm, which is a modification and extension of the algorithm introduced in [20] for finding optimal lossless data compression codes with reversible VLC structure, uses a new search tree and a priority-first search method.

To construct an optimal VLECPC with K codewords and $d_{\text{free}} \geq d^*_{\text{free}}$, we use a search tree in which each node $\mathtt{X}$ contains three components given by the triplet $\{\mathcal{C}_{\mathtt{X}}, \mathcal{A}_{\mathtt{X}}, f(\mathtt{X})\}$. Here, $\mathcal{C}_{\mathtt{X}} = \{\boldsymbol{c}^{\mathtt{X}}_1, \boldsymbol{c}^{\mathtt{X}}_2, \ldots, \boldsymbol{c}^{\mathtt{X}}_t\}$ denotes the set of t codewords that have been selected for the desired VLECPC, and $\mathcal{A}_{\mathtt{X}} = \{\boldsymbol{a}^{\mathtt{X}}_1, \boldsymbol{a}^{\mathtt{X}}_2, \ldots\}$ is the set of all bitstreams, which can be future candidate codewords and hence do not contain any bitstreams for which the codewords currently in $\mathcal{C}_{\mathtt{X}}$ are their prefixes. These bitstreams are listed in order of nondecreasing lengths: $|\boldsymbol{a}^{\mathtt{X}}_1| \leq |\boldsymbol{a}^{\mathtt{X}}_2| \leq \cdots$.[1] Finally, $f(\mathtt{X})$ denotes the metric employed for finding an optimal VLECPC and is given by

$$f(\mathtt{X}) \triangleq \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}^{\mathtt{X}}_i| + \sum_{i=t+1}^{K} p_i \cdot |\boldsymbol{a}^{\mathtt{X}}_{i-t}|. \tag{3.1}$$

The search tree is binary (i.e., each of its nodes except a leaf or terminal node has two children); the relation between a parent node and its children is illustrated in Figure 3.1. Specifically, for a parent node $\mathtt{P}$, its left child $\mathtt{L}$ is obtained by adding the next candidate codeword $\boldsymbol{a}^{\mathtt{P}}_1$ into $\mathcal{C}_{\mathtt{L}}$. Since $\boldsymbol{a}^{\mathtt{P}}_1$ is now a codeword in $\mathcal{C}_{\mathtt{L}}$, the set $\mathcal{A}_{\mathtt{L}}$ needs to be updated by removing all bitstreams in $\mathcal{A}_{\mathtt{P}}$ whose prefix is $\boldsymbol{a}^{\mathtt{P}}_1$. Hence, the triplet of the left child $\mathtt{L}$

[1] Recall that candidate codewords of equal length can be listed in any order without affecting the optimality of the output VLECPC of our construction algorithm. For programing convenience, we simply list candidate codewords of equal length alphabetically in $\mathcal{A}_{\mathtt{X}}$, e.g., see $\mathcal{A}_{\mathtt{root}}$ in (3.9).

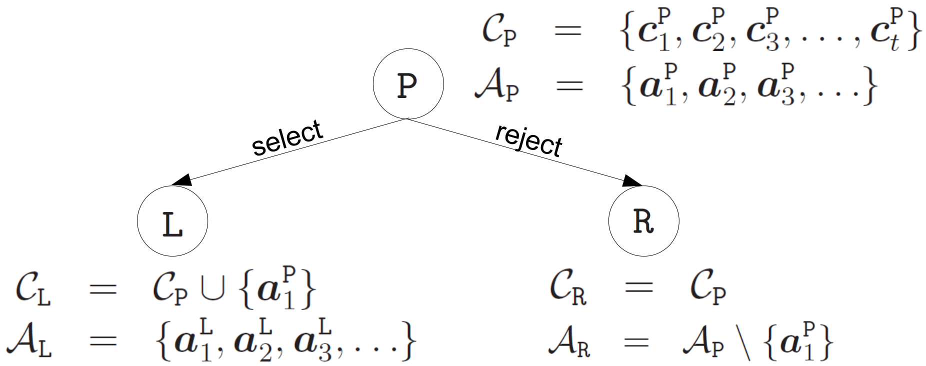


Figure 3.1: Relation between a parent node and its children in a search tree.

becomes

$$\mathcal{C}_{\mathtt{L}} = \mathcal{C}_{\mathtt{P}} \cup \{\boldsymbol{a}_1^{\mathtt{P}}\} \tag{3.2}$$

$$\begin{aligned}\mathcal{A}_{\mathtt{L}} &= \{\boldsymbol{a}_1^{\mathtt{L}}, \boldsymbol{a}_2^{\mathtt{L}}, \ldots\} \\ &= \{\boldsymbol{a} : \boldsymbol{a} \in \mathcal{A}_{\mathtt{P}} \text{ and } \boldsymbol{a}_1^{\mathtt{P}} \text{ is not a prefix of } \boldsymbol{a}\}\end{aligned} \tag{3.3}$$

$$f(\mathtt{L}) = \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\mathtt{P}}| + p_{t+1} \cdot |\boldsymbol{a}_1^{\mathtt{P}}| + \sum_{i=t+2}^{K} p_i \cdot |\boldsymbol{a}_{i-t-1}^{\mathtt{L}}|. \tag{3.4}$$

On the other hand, the right child $\mathtt{R}$ is obtained by rejecting the next candidate codeword $\boldsymbol{a}_1^{\mathtt{P}}$ from its parent node. So, the triplet of the right child $\mathtt{R}$ becomes

$$\mathcal{C}_{\mathtt{R}} = \mathcal{C}_{\mathtt{P}} \tag{3.5}$$

$$\mathcal{A}_{\mathtt{R}} = \{\boldsymbol{a}_2^{\mathtt{P}}, \boldsymbol{a}_3^{\mathtt{P}}, \ldots\} = \mathcal{A}_{\mathtt{P}} \setminus \{\boldsymbol{a}_1^{\mathtt{P}}\} \tag{3.6}$$

$$f(\mathtt{R}) = \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\mathtt{P}}| + \sum_{i=t+1}^{K} p_i \cdot |\boldsymbol{a}_{i-t+1}^{\mathtt{P}}|. \tag{3.7}$$

Finally, since the root node has not yet selected any codeword, all bitstreams are its candidates; thus its components are given by

$$\mathcal{C}_{\mathtt{root}} = \emptyset \tag{3.8}$$

$$\begin{aligned}\mathcal{A}_{\mathtt{root}} &= \{\boldsymbol{a}_1^{\mathtt{root}}, \boldsymbol{a}_2^{\mathtt{root}}, \ldots\} \\ &= \{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}\end{aligned} \tag{3.9}$$

$$f(\mathtt{root}) = \sum_{i=1}^{K} p_i \cdot |\boldsymbol{a}_i^{\mathtt{root}}|. \tag{3.10}$$

Since every possible VLECPC can be obtained by traversing the search tree from the root node to its corresponding leaf nodes, a priority-first search algorithm can be applied on the tree to find a VLECPC whose average codeword length is smallest among all VLECPCs with free distances no less than d_{free}^*. To reduce the search space, the average codeword length of any known VLECPC with free distance no less than d_{free}^* is denoted by U_b and used as an upper bound for the average codeword length to exclude obviously uncompetitive nodes during the search process. The search algorithm for finding an optimal VLECPC is described as follows.

Step 1: Push the root node into the *Encoding Stack*.[2] Set upper bound U_b as the average codeword length of an existing VLECPC with free distance no less than d^*_{free}.

Step 2: If the top node of the *Encoding Stack* has selected K codewords (i.e., $|\mathcal{C}_{\texttt{top}}| = K$) and $d_{\text{free}}(\mathcal{C}_{\texttt{top}}) \geq d^*_{\text{free}}$, then output $\mathcal{C}_{\texttt{top}}$ as the optimal VLECPC and stop the algorithm.

Step 3: Generate the two children of the top node as in Figure 3.1 and then delete the top node from the *Encoding Stack*. If the left child has selected K codewords with its free distance $\geq d^*_{\text{free}}$ and its associated metric f is smaller than U_b, then update $U_b = f$.

Step 4: Discard a child node which satisfies any of the following conditions:

1. It has selected more than K codewords for its $\mathcal{C}_{\texttt{child}}$;
2. There is no more candidate in $\mathcal{A}_{\texttt{child}}$ and the size of $\mathcal{C}_{\texttt{child}}$ is less than K (i.e., $\mathcal{A}_{\texttt{child}} = \emptyset$ and $|\mathcal{C}_{\texttt{child}}| < K$);
3. The metric $f(\texttt{child})$ is larger than U_b;
4. Its associated free distance $d_{\text{free}}(\mathcal{C}_{\texttt{child}})$ is less than d^*_{free}.[3]

Step 5: Insert the remaining children (those children which are not discarded in Step 4) into the *Encoding Stack*, and reorder the *Encoding Stack* in order of ascending metrics. Go to Step 2.

---

[2]The *Encoding Stack* can be implemented via the data structure named *HEAP* [10]. One important property of the *HEAP* structure is that it can access the node with the minimal metric (i.e., the top node in the *Encoding Stack*) within $O(\log(n))$ complexity, where n denotes the number of nodes in the *HEAP*.

[3]In order to check this condition efficiently, the lower bound on the free distance given in (2.8) is first computed; if it is less than d^*_{free}, then Dijkstra's algorithm [12] is adopted to determine the exact free distance. This is realized by transforming the finite-state VLECPC encoder into a pairwise distance graph and applying Dijkstra's algorithm to find the graph's shortest path, where the resulting shortest path yields the VLECPC's free distance. To our knowledge, Dijkstra's algorithm is the most efficient method to evaluate d_{free}.

It should be emphasized that the above construction algorithm focuses only on prefix-free VLECPCs as most previous works did [7, 9, 11, 12, 23, 26, 30]. Although non-prefix-free but uniquely decodable VLECPCs can also be constructed, they are not herein considered due to the added complexity in testing their unique decodability. The proof of the optimality of the above algorithm is provided in Appendix A.

Chapter 4

Modified VLECPC Constructions

In this chapter, two modifications on the optimal VLECPC construction algorithm introduced in Chapter 3 are proposed. The first modification further enhances the error-correcting capability of the found optimal VLECPC by examining the union bound coefficient $B_{d_{\text{free}}}$ of all equivalent[1] optimal VLECPCs satisfying the free distance constraint and then outputting the one with the smallest $B_{d_{\text{free}}}$, where $B_{d_{\text{free}}}$ is a Levenshtein parameter defined in Section 4.1 below. By targeting a suboptimal VLECPC instead of an optimal one, the second modification reduces considerably the search complexity of the optimal construction algorithm in order to make feasible the construction of VLECPCs for larger alphabet sizes (such as the 26-symbol English data source) along with a large d^*_{free} (such as $d^*_{\text{free}} = 10$).

4.1 Finding an optimal VLECPC with the smallest $B_{d_{\text{free}}}$

In [7, 9], Buttigieg found that under hard-decision ML decoding, the symbol error probability $P_e(\mathcal{C})$ of a VLECPC $\mathcal{C}$ transmitted over the BSC with crossover probability ϵ can be upper-bounded by

$$P_e(\mathcal{C}) \leq \sum_{h=d_{\text{free}}(\mathcal{C})}^{\infty} \tilde{B}_h P_h, \tag{4.1}$$

[1] Two VLECPCs are said to be *equivalent* if they have identical average codeword length.

where

$$\tilde{B}_h \triangleq \sum_{N=1}^{\infty} \sum_{\boldsymbol{a}\in\mathcal{X}_N} \Pr(\boldsymbol{a}) \cdot \left(\sum_{\boldsymbol{b}\,:\,\boldsymbol{b}\in\mathcal{X}_N \text{ and } d(\boldsymbol{a},\boldsymbol{b})=h} L(\boldsymbol{a},\boldsymbol{b}) \right) \tag{4.2}$$

and

$$P_h \triangleq \begin{cases} \sum_{e=(h+1)/2}^{h} \binom{h}{e} \epsilon^e (1-\epsilon)^{h-e} & \text{if } h \text{ is odd,} \\ \frac{1}{2}\binom{h}{h/2} \epsilon^{h/2}(1-\epsilon)^{h/2} + \sum_{e=\frac{h}{2}+1}^{h} \binom{h}{e} \epsilon^e (1-\epsilon)^{h-e} & \text{if } h \text{ is even.} \end{cases} \tag{4.3}$$

Note that in Buttigieg's derivation, the symbol errors are counted using the Levenshtein distance[2] $L(\cdot,\cdot)$ between transmitted sequence and decoded sequence, and the receiver decodes based on trellis $\mathcal{T}_N$ with N extending to infinity.

With a slight modification, a similar bound can be derived under the additional assumption that the receiver also knows the number of transmitted codewords L. In particular, (4.1) remains of the same form with $\tilde{B}_h$ replaced with B_h, where

$$B_h \triangleq \sum_{L=1}^{\infty} \sum_{N=1}^{\infty} \sum_{\boldsymbol{a}\in\mathcal{X}_{L,N}} \Pr(\boldsymbol{a}) \cdot \left(\sum_{\boldsymbol{b}\,:\,\boldsymbol{b}\in\mathcal{X}_{L,N} \text{ and } d(\boldsymbol{a},\boldsymbol{b})=h} L(\boldsymbol{a},\boldsymbol{b}) \right). \tag{4.4}$$

The coefficient B_h, as expressed in (4.4), can be regarded as the average Levenshtein distance between all converging path pairs that are at a Hamming distance h from each other in the extended trellis $\mathcal{T}_{L,N}$. Thus, it is evident that B_h plays a key role in the union bound (4.1), particularly the first term $B_{d_{\text{free}}} \triangleq B_{h_{\min}}$, where $h_{\min}$ is the smallest integer h no less than $d_{\text{free}}(\mathcal{C})$ such that B_h is positive. Accordingly, given a set of optimal VLECPCs, the one with the smallest $B_{d_{\text{free}}}$ is expected to have a better error performance. It should be mentioned that in this dissertation we use a soft-decision MAP decoder with respect to the AWGN channel. The simplified union bound for the BSC (not we used at (4.2)–(4.4)); however, can provide a much simplified view on the system performance and hence the parameters $d_{\text{free}}(\mathcal{C})$ and $B_{d_{\text{free}}}$ obtained from (4.1) are adopted in our code design.[3]

We now modify the algorithm in Chapter 3 to find the optimal VLECPC with the smallest $B_{d_{\text{free}}}$ among all optimal VLECPCs that has the minimum average codeword

[2] The Levenshtein distance, also called edit distance, between two sequences is the minimum number of character edits (including insertion, deletion and substitution) required to change one sequence into the other.

[3] We determine $B_{d_{\text{free}}}$ using the method proposed in [7, Section 3.5.1.1].

length. This can be achieved by continuing the algorithm, even if the top node of the *Encoding Stack* reaches the leaf node in Figure 3.1 (see Step 2 in Chapter 3), until the average codeword length of the new top node is greater than that of the optimal VLECPC. This continuation then guarantees that all optimal VLECPCs (of equal average codeword length) are examined and the one with the smallest $B_{d_{\text{free}}}$ can be selected. As a result, only the first two steps need to be modified:

Step 1′: Push the root node into the *Encoding Stack*. Set upper bound U_b as the average codeword length of an existing VLECPC with free distance no less than d^*_{free}, and initialize $B^*_{d_{\text{free}}} = \infty$.

Step 2′: If the metric f (namely, the average codeword length) of the top node is strictly greater than U_b, then output $\mathcal{C}^*$ and stop the algorithm; else if the top node of the *Encoding Stack* has selected K codewords (i.e., $|\mathcal{C}_{\texttt{top}}| = K$), and $d_{\text{free}}(\mathcal{C}_{\texttt{top}}) \geq d^*_{\text{free}}$, and $B_{d_{\text{free}}}(\mathcal{C}_{\texttt{top}}) < B^*_{d_{\text{free}}}$, then retain $\mathcal{C}^* = \mathcal{C}_{\texttt{top}}$ and $B^*_{d_{\text{free}}} = B_{d_{\text{free}}}(\mathcal{C}_{\texttt{top}})$. Delete the top node and reorder the *Encoding Stack* in order of ascending metrics.

4.2 Suboptimal code construction with parameters $(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$

The complexity and memory demand of the optimal code construction algorithm in Chapter 3 grows significantly when searching for VLECPCs corresponding to a large source alphabet size K and a large free distance requirement d^*_{free}. We herein alleviate the algorithm's complexity and memory demand by constructing a suboptimal VLECPC, which can accommodate higher free distance targets and larger source alphabet sizes. This is done based on four complexity reduction procedures.

First, we reduce the computational complexity incurred in examining the exact free distance of the top node by using its lower bound in (2.8) instead. Furthermore, Buttigieg recently observed [8] that good codes usually have converging and diverging distances (given in (2.10) and (2.11), respectively) that are equal (for even values of d_{free}) or differing by one (for odd values of d_{free}). Thus, we only focus on VLECPCs with the above property. In other words, the new suboptimal code construction only searches for the VLECPC $\mathcal{C}$ that satisfies the following conditions:

$$\begin{cases} \min\{d_{\text{b}}(\mathcal{C}), d_{\text{c}}(\mathcal{C}) + d_{\text{d}}(\mathcal{C})\} \geq d^*_{\text{free}}, \quad \text{and} \\ |d_{\text{c}}(\mathcal{C}) - d_{\text{d}}(\mathcal{C})| \leq 1. \end{cases} \tag{4.5}$$

With this modification, the actual free distance of the output VLECPC may be strictly larger than the required d^*_{free}; yet, this saves considerable computational effort in calculating the exact free distance for each node visited during the code search process.

Second, we adopt the early-elimination concept from [28], in which an efficient near-optimal sequential decoding algorithm for convolutional codes was proposed. In short, the authors in [28] propose to directly remove those nodes that are far behind the farthest node having been explored during the search process. Since the metric used in our code construction algorithm is also nondecreasing along every path in the trellis as in [28], these "far-behind" nodes are highly unlikely to result in a K-codeword offspring node whose average codeword length is small, and hence can be early-eliminated.

The third modification, also borrowed from [28], is to set a proper *Encoding Stack* size limitation in order to fix the memory demand and indirectly to reduce the search complexity.

In the last modification, we attempt to compensate for potential losses in coding efficiency (average codeword length) caused by the previous three modifications. Recall that the average codeword length of any existing VLECPC can be used as the upper bound U_b in our search algorithm. Hence, when our suboptimal approach results in a VLECPC whose average codeword length is smaller than the given U_b, we can update the value of U_b with this average codeword length and launch a new execution of our algorithm. This step can then be repeated in a number of iterations until no improvements in coding efficiency are realized or a prescribed maximal number of iterations is reached.

Four parameters $(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$ are accordingly added corresponding to the last three modifications.

1: *Early elimination window* Δ: Ignore the top node in the *Encoding Stack*, whose number of codewords $|\mathcal{C}_{\texttt{top}}|$ is less than $l_{\max} - \Delta$, where $l_{\max}$ is the largest $|\mathcal{C}|$ among all expanded nodes.

2: *Encoding Stack size* Γ: When the number of nodes in the *Encoding Stack* is larger than Γ, nodes are recursively deleted from the *Encoding Stack* according to one of the two criteria described below.

1. *Deletion criterion* $\mathcal{D} = \mathcal{D}_l$: Delete the node with the smallest code size $|\mathcal{C}|$.
2. *Deletion criterion* $\mathcal{D} = \mathcal{D}_m$: Delete the node with the largest metric f.

3: *The maximal number of iterations* $\mathcal{I}$.

The suboptimal algorithm, characterized by four parameters $(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$, can thus be obtained by modifying the optimal algorithm in Chapter 3 and adding a new Step 6 as follows:

Step 1″: Push the root node into the *Encoding Stack*. Set upper bound U_b as the average codeword length of an existing VLECPC with free distance no less than d^*_{free}. Alternatively for the followup iteration, set upper bound U_b as the average codeword length of the output VLECPC obtained from the previous iteration. Initialize the target VLECPC $\mathcal{C}^*$ as the empty set and $l_{\max} = 0$.

Step 2″: If the *Encoding Stack* is empty and $\mathcal{C}^* \neq \emptyset$, then output $\mathcal{C}^*$ as the optimal VLECPC and stop the algorithm; else if both the *Encoding Stack* and $\mathcal{C}^*$ are empty, then report a code search failure and stop the algorithm.[4]

If $|\mathcal{C}_{\texttt{top}}| < l_{\max} - \Delta$, then directly delete the top node from the *Encoding Stack* and redo Step 2″; else if $l_{\max} < |\mathcal{C}_{\texttt{top}}|$, update $l_{\max} = |\mathcal{C}_{\texttt{top}}|$.

If the top node of the *Encoding Stack* has selected K codewords (i.e., $|\mathcal{C}_{\texttt{top}}| = K$)

[4]Even if U_b is the average codeword length of an existing VLECPC, the search space could be forced to become empty due to extra node exclusions of the first three complexity reduction modifications, i.e., requiring the free distance lower bound to be no less than d^*_{free}, early eliminations, and node deletions for a fully filled *Encoding Stack*. Note that when a node is excluded, all of its offspring nodes can no longer be visited; hence, it is possible that all the valid nodes (i.e., all the valid VLECPCs) are removed after several recursions of Steps 2″–5″.

Since, in the two earlier optimal code construction algorithms, the nodes corresponding to optimal VLECPCs will never be excluded, the *Encoding Stack* can never be empty prior to finding the optimal VLECPC. Accordingly, it is not necessary to conduct an empty *Encoding Stack* check in these algorithms.

and $\mathcal{C}_{\texttt{top}}$ satisfies condition (4.5), then output $\mathcal{C}_{\texttt{top}}$ as the optimal VLECPC and stop the algorithm.

Step 3′′: Generate the two children of the top node as in Figure 3.1 and then delete the top node from the *Encoding Stack*. Then update U_b as the metric f of left child and put left child as $\mathcal{C}^*$ if left child satisfies all of the following conditions:

1. The left child has selected K codewords in his $\mathcal{C}_{\texttt{left}}$;
2. $\mathcal{C}_{\texttt{left}}$ satisfies condition (4.5);
3. Its associated metric f is smaller than U_b.

Step 4′′: Discard the child node which satisfies any of the following conditions:

1. It has selected more than K codewords for its $\mathcal{C}_{\texttt{child}}$;
2. There is no more candidate in $\mathcal{A}_{\texttt{child}}$ and the size of $\mathcal{C}_{\texttt{child}}$ is less than K (i.e., $\mathcal{A}_{\texttt{child}} = \emptyset$ and $|\mathcal{C}_{\texttt{child}}| < K$);
3. The metric $f(\texttt{child})$ is larger than U_b;
4. It disobeys condition (4.5).

Step 5′′: After inserting the remaining children into the *Encoding Stack*, recursively delete nodes from the *Encoding Stack* based on the chosen deletion criterion $\mathcal{D}$ until the *Encoding Stack* size is no greater than Γ. Reorder the *Encoding Stack* in order of ascending metrics. Go to Step 2′′.

Step 6 : Repeat Steps 1′′–5′′ until either the maximum number of iterations $\mathcal{I}$ is reached or the upper bound U_b remains the same as the previous iteration.

We end this chapter with a remark about the free distances of the VLECPCs found by the three code construction algorithms introduced in this dissertation.

Recall that the two optimal code construction algorithms, respectively introduced in Chapter 3 and Section 4.1, guarantee to output the VLECPC whose average codeword length is smallest among all VLECPCs with free distance never smaller than the target free distance. In all cases we have examined, however, the free distance of the resulting optimal VLECPCs is always equal to the target free distance; although we conjecture the validity of this observation, we could not confirm it with a formal proof.

As expected, the suboptimal code construction algorithm may produce a (suboptimal) VLECPC with free distance strictly larger than d^*_{free}. However, in the particular case of the 26-symbol English alphabet (as will be presented in Chapter 6), the suboptimal code construction algorithm also consistently deliver a (suboptimal) VLECPC with free distance equal to d^*_{free}, which indicates that the free distance lower bound in (2.8) is indeed tight for the found suboptimal VLECPC. It should be mentioned that the tightness of (2.8) depends on the distribution of the source. In [13] and [18], it is shown that the tightness of (2.8) may be weak when the source distribution is highly unbalanced. Details will be given in Chapter 6.

Chapter 5

Two-Phase Sequence MAP (TP-SMAP) Decoding

In [19], an efficient sequence MAP decoder with the assumption that the receiver knows only the number of transmitted bits N was proposed. This decoder therefore can only operate on the traditional trellis $\mathcal{T}_N$ shown in Figure 2.1(a). With the additional information about the number of transmitted symbols L, we herein propose a new two-phase sequence MAP (TP-SMAP) decoder, which can now operate on the extended trellis $\mathcal{T}_{L,N}$ (cf. Figure 2.1(b)), and whose average decoding complexity is only slightly greater than that for running the Viterbi algorithm (VA) on $\mathcal{T}_N$ (even if $\mathcal{T}_{L,N}$ has significantly more nodes and more transitions than $\mathcal{T}_N$). We next describe the TP-SMAP decoding scheme.

In trellis $\mathcal{T}_{L,N}$, as defined in Section 2.2 and illustrated in Figure 2.1(b), a path traversing from $S_{0,0}$ to $S_{i,j}$ can be labeled as $\boldsymbol{x}_{(0,0)}^{(i,j)} \triangleq \boldsymbol{x}_1\boldsymbol{x}_2\cdots\boldsymbol{x}_i \in \mathcal{X}_{i,j}$, where each $\boldsymbol{x}_i \in \mathcal{C}$. Then, by following the MAP decoding criterion described in Section 2.1, the path metric of $\boldsymbol{x}_{(0,0)}^{(i,j)}$ is defined as

$$\boldsymbol{g}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) = \sum_{\ell=1}^{j}(y_\ell \oplus b_\ell)\|\phi_\ell\|_1 - \ln \Pr\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right), \tag{5.1}$$

where $b_1b_2\cdots b_j$ denotes the binary representation of path $\boldsymbol{x}_{(0,0)}^{(i,j)}$. Based on this new notation, the objective of the MAP decoder that knows both L and N is to find a path whose metric is the smallest among all valid paths $\boldsymbol{x}_{(0,0)}^{(L,N)}$ from $S_{0,0}$ to $S_{L,N}$.

In short, the TP-SMAP scheme first performs backward VA on $\mathcal{T}_N$, whose size is significantly smaller than that of $\mathcal{T}_{L,N}$, and preserves the metric of each backward survivor path as $\boldsymbol{h}(S_j)$. The first phase of the TP-SMAP is described as follows.

Step 1: Associate a zero path metric to node S_N in $\mathcal{T}_N$, i.e., $\boldsymbol{h}(S_N) = 0$.

Step 2: Apply the backward VA with path metric given by (5.1) starting from S_N in $\mathcal{T}_N$, and record the metric and survivor path for each state as $\boldsymbol{h}(S_i)$ and $\boldsymbol{p}(S_i)$, respectively.

Step 3: If the number of codewords correspond to survivor path $\boldsymbol{p}(S_0)$ is equal to L, then output path $\boldsymbol{p}(S_0)$ as the MAP decision and stop the algorithm; otherwise, go to phase 2.

In the second phase, the TP-SMAP applies a priority-first search algorithm [15] on $\mathcal{T}_{L,N}$ with the decoding metric of path $\boldsymbol{x}_{(0,0)}^{(i,j)}$ being re-defined as

$$\boldsymbol{m}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) = \boldsymbol{g}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) + \boldsymbol{h}\left(S_j\right). \tag{5.2}$$

The second phase of the decoder is next described.

Step 1: Initialize the path metric of $\boldsymbol{x}_{(0,0)}^{(0,0)}$ as $\boldsymbol{m}(\boldsymbol{x}_{(0,0)}^{(0,0)}) = \boldsymbol{h}(S_0)$, and load it into the *Decoding Stack*.[1]

Step 2: If the top node of the *Decoding Stack* reaches the final state $S_{L,N}$ in $\mathcal{T}_{L,N}$, then output its associated path as the MAP decision and stop the algorithm.

Step 3: Mark the state of the top node as visited. Then extend the top node to all its successors and compute their metrics according to (5.2). Delete the top node from the *Decoding Stack*.

Step 4: Discard the successors if they had been marked as visited. Also, discard the successors for which the number of decoded symbols exceeds L or the number of decoded bits exceeds N.

Step 5: Insert the remaining successors (those successors which are not discarded in Step 4) into the *Decoding Stack* and reorder the *Decoding Stack* in order of ascending $\boldsymbol{m}$-metrics defined in (5.2). Go to Step 2.

[1] The role of the *Decoding Stack* is similar to that of the *Encoding Stack*, except that the *Decoding Stack* stores the nodes of $\mathcal{T}_{L,N}$ as its elements. It is also implemented via the data structure named *HEAP* [10] and accesses the node with minimal metric (i.e., its top node) within $O(\log(n))$ complexity, where n denotes its total number of nodes.

It can be noted that the second phase of the decoder follows similar procedures as the code construction algorithm introduced in Chapter 3, except that the priority-first algorithm is now applied on the trellis $\mathcal{T}_{L,N}$ instead of applying it on a search tree for code construction. Since some paths of the trellis $\mathcal{T}_{L,N}$ run across the same node, the priority-first algorithm must avoid expanding the same node on the trellis $\mathcal{T}_{L,N}$ more than once. We therefore need to mark the expanded node (top node) as visited in Step 3, and discard the successors which have already been marked as visited in Step 4. The proof of optimality for the above decoding algorithm is provided in Appendix B.

Chapter 6

Simulation Results

In this chapter, we assess via simulations the error performances of the found VLECPCs in terms of reconstructed source symbol error rate (SER).[1] In all simulations, the source is assumed memoryless and the channel is the BPSK-modulated AWGN channel. The decoding complexity of the proposed two-phase sequence MAP (TP-SMAP) decoder is also examined. Furthermore, comparisons with other systems in literature, including three known VLECPC schemes and a traditional SSCC system, are provided. For measuring the time to search for the optimal and suboptimal VLECPCs, the experiments were carried using the *C* programing language under a *64-bit* operation system *Linux (Ubuntu 10.04 LTS)* executed on a desktop computer with a *Intel-Core2 Duo E6600 2.4GHz CPU* and *4GB* memory. It should noted that the decoders of VLECPCs in the following simulations are assumed to be TP-SMAP, if they are not be specified.

As usual, the system signal-to-noise ratio (SNR) is given by SNR $\triangleq E/N_0$, where E is the signal energy per channel use and $N_0/2$ is the variance of the zero-mean additive channel noise sample. To account for the coding redundancy of systems with different code rates, SNR per source symbol is used in presenting the simulation results, which is given by

$$\mathrm{SNR_s} = \frac{E_\mathrm{s}}{N_0} = \frac{E}{N_0} \cdot \frac{1}{R}, \tag{6.1}$$

where E_s is the energy per source symbol, and R is the overall (average) system rate defined as the number of transmitted source symbols per channel use. For an SSCC

[1] As a convention, the SER here is the Levenshtein distance between the transmitted sequence and the decoded sequence divided by the number of transmitted source symbols (i.e., L).

system, the overall rate R satisfies $R = R_c/R_s$, where R_s is the source coding rate (in coded bits/source symbol) and R_c is the channel coding rate (in coded bits/channel use). Hence, for an SSCC system employing a kth-order Huffman VLC[2] followed by a tail-biting convolutional code, R_s is the average codeword length of the Huffman code divided by k, and R_c is the rate of the tail-biting convolutional code. Note that a VLECPC (or a single-step JSCC) can be regarded as having $R_c = 1$ with R_s being its averaged source coding rate, since no explicit channel coding is performed.

Table 6.1: Average codeword length per grouped symbol of a 8-ary alphabet generated from binary non-uniform memoryless sources with different p_0.

	Buttigieg's		Lamy's		Wang's		Opt. VLECPC	
p_0	0.7	0.8	0.7	0.8	0.7	0.8	0.7	0.8
$d^*_{\text{free}} = 3$	4.500	4.000	4.500	4.000	4.500	4.000	4.473	3.992
$d^*_{\text{free}} = 5$	6.443	5.912	6.443	5.912	6.443	5.912	6.340	5.592
$d^*_{\text{free}} = 7$	8.326	7.864	8.473	7.936	8.326	7.864	8.016	7.240

In Table 6.1, we compare the VLECPCs found by the proposed method in Chapter 3 with Buttigieg's codes [7], Lamy's codes [23] and the codes by Wang *et al.* [30]. Here, we group three information bits, generated from a binary non-uniform memoryless source with bit probability $p_0 \triangleq \Pr(0) \in \{0.7, 0.8\}$, as one source symbol; hence, the VLECPCs are 3rd order VLCs (i.e., $k = 3$), and the size of the source alphabet is $K = 2^3 = 8$. Since our proposed algorithm guarantees to find VLECPCs with minimal average codeword length under a fixed d^*_{free}, the resulting VLECPCs have a shorter average codeword length than any other code with identical free distance.

We then investigate the improvement in both error performance and decoding complexity of the proposed TP-SMAP decoder. In Figure 6.1, 30 information bits (i.e., 10 grouped symbols) are encoded by the optimal VLECPC with $d^*_{\text{free}} = 7$ and $p_0 = 0.8$ of Table 6.1, which is listed in Table 6.2. The dotted lines show the performance of the MAP decoder under the assumption that the receiver only knows the number of transmitted bits, N. The solid line portrays the MAP decoder's performance under the assumption that receiver knows both number of symbols, L, and transmitted bits, N. As shown in

[2]Recall that a kth order VLC maps a block of k source symbols onto a variable-length codeword. So its average source coding rate is given by the average codeword length divided by k.

Table 6.2: The optimal VLECPC with $d^*_{\text{free}} = 7$ and $p_0 = 0.8$ (the one with an average codeword length of 7.240) of Table 6.1.

Grouped Symbol	Probability	Optimal VLECPC with $d_{\text{free}} = 7$ and $p_0 = 0.8$
000	0.512	00100
001	0.128	01011010
010	0.128	100111001
100	0.128	1111111111
011	0.032	11010110011
101	0.032	000110010011
110	0.032	110011101011
111	0.008	1111110001011

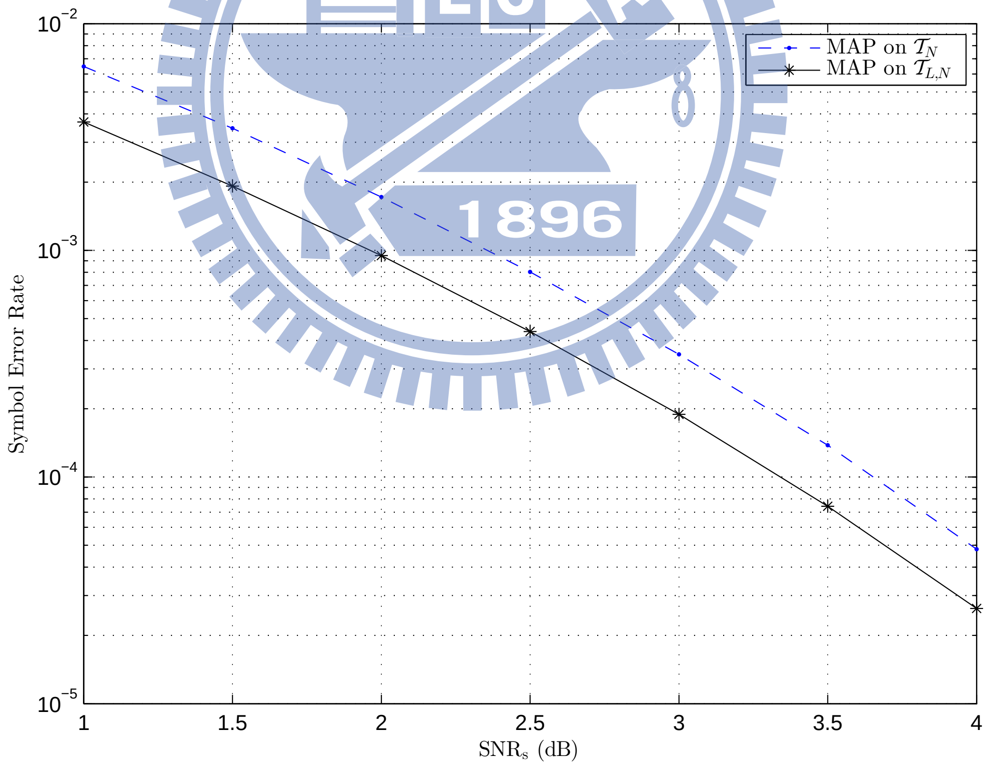


Figure 6.1: Error performances of using different decoders to decode the same VLECPC, which is encoded by the optimal VLECPC listed in Table 6.2. The number of 3-bit source symbols per transmission block is 10, which is equivalent to 30 source information bits.

Table 6.3: Average (AVG) and maximum (MAX) numbers of decoder branch metric computations for the codes of Figure 6.1.

E_b/N_0	1 dB		2 dB		3 dB		4 dB	
decoder	AVG	MAX	AVG	MAX	AVG	MAX	AVG	MAX
Viterbi on $\mathcal{T}_N$	459	768	459	768	459	768	459	768
Viterbi on $\mathcal{T}_{L,N}$	1651	2600	1651	2600	1651	2600	1651	2600
TP-SMAP $\mathcal{T}_{L,N}$	461	2970	460	1619	459	863	459	768

Figure 6.1, about 0.3 dB in coding gain is realized by knowing L (in addition to N). Table 6.3 summarizes the decoding complexities of different decoders in terms of the branch metric computations. From the table, we remark that the TP-SMAP decoder has a similar decoding complexity as the Viterbi algorithm on $\mathcal{T}_N$ while achieving about 0.3 dB coding gain in error performance. For identical error performance, the TP-SMAP decoding algorithm spends almost 4 times less in branch computations than the Viterbi algorithm on $\mathcal{T}_{L,N}$.

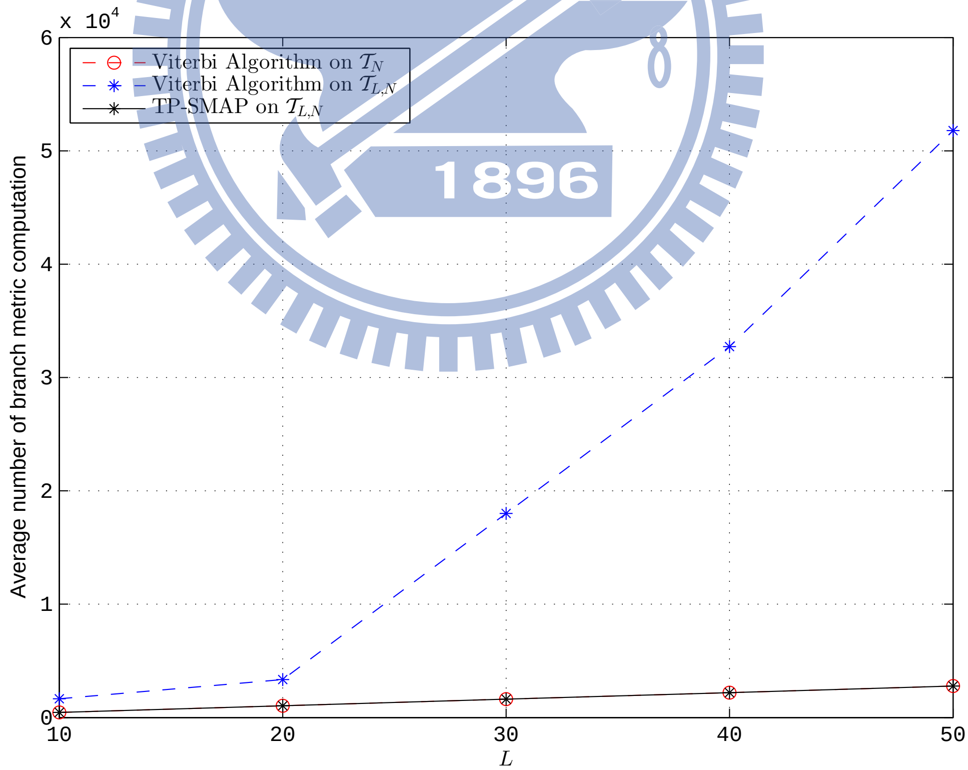


Figure 6.2: Average numbers of decoder branch metric computations of using different decoders to decode the same VLECPC for different L at $\text{SNR}_s = 3$ dB. The VLECPC is obtained from Table 6.2.

We further test the decoding complexities of different decoders for different L. In Figure 6.2, the optimal VLECPC of Table 6.2 is transmitted at $\mathrm{SNR_s} = 3.0$ dB. This figure indicates that the decoding complexities of TP-SMAP are similar to those of the Viterbi algorithm on $\mathcal{T}_N$. The result also shows that the decoding complexities of TP-SMAP decoder are proportional to the size of transmission block L. It should be emphasized that the decoding complexities of TP-SMAP are one order less than those of the Viterbi algorithm on $\mathcal{T}_{L,N}$, in which both have the same error performances.

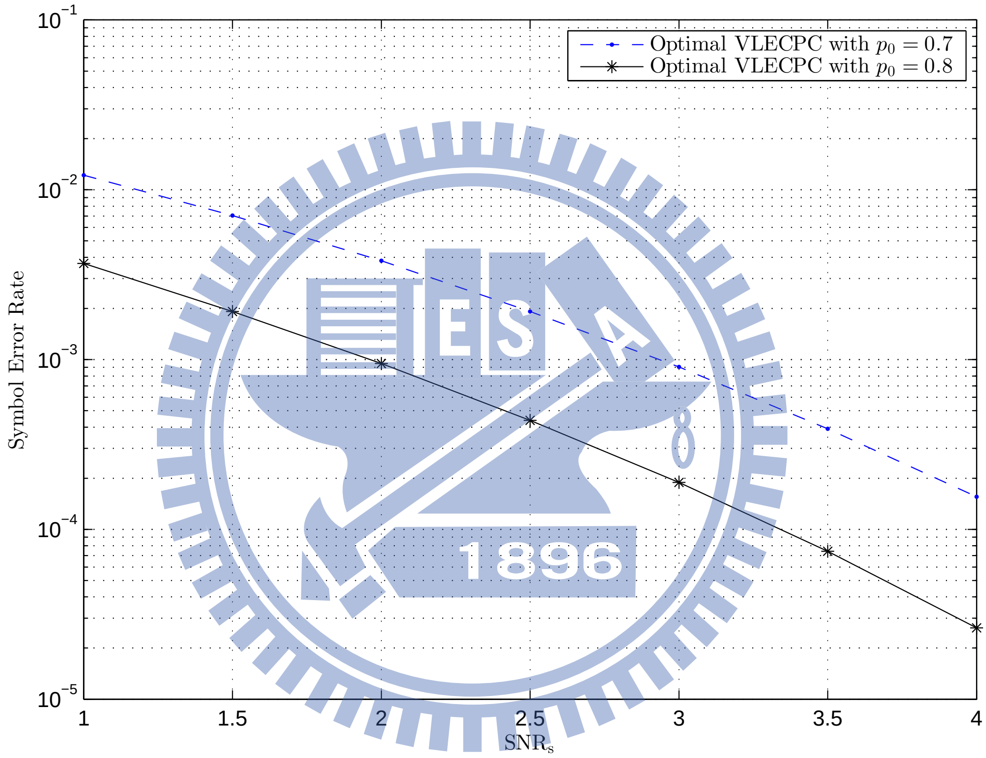


Figure 6.3: Error performances of optimal VLECPCs for different p_0. The VLECPCs are obtained from the optimal VLECPCs with $d^*_{\text{free}} = 7$ in Table 6.1. The number of 3-bit source symbols per transmission block is 10, which is equivalent to 30 source information bits.

We next investigate the error performances of optimal VLECPCs for different values of p_0 and L. Figure 6.3 shows that the optimal VLECPC for $p_0 = 0.8$ performs about 0.8 dB better than the optimal VLECPC for $p_0 = 0.7$ at SER of 10^{-3}. Figure 6.4 shows that the optimal VLECPC performs better when size of transmission block L is smaller. These two figures indicate that the optimal VLECPCs are better when the source distribution is more biased and the block length is shorter.

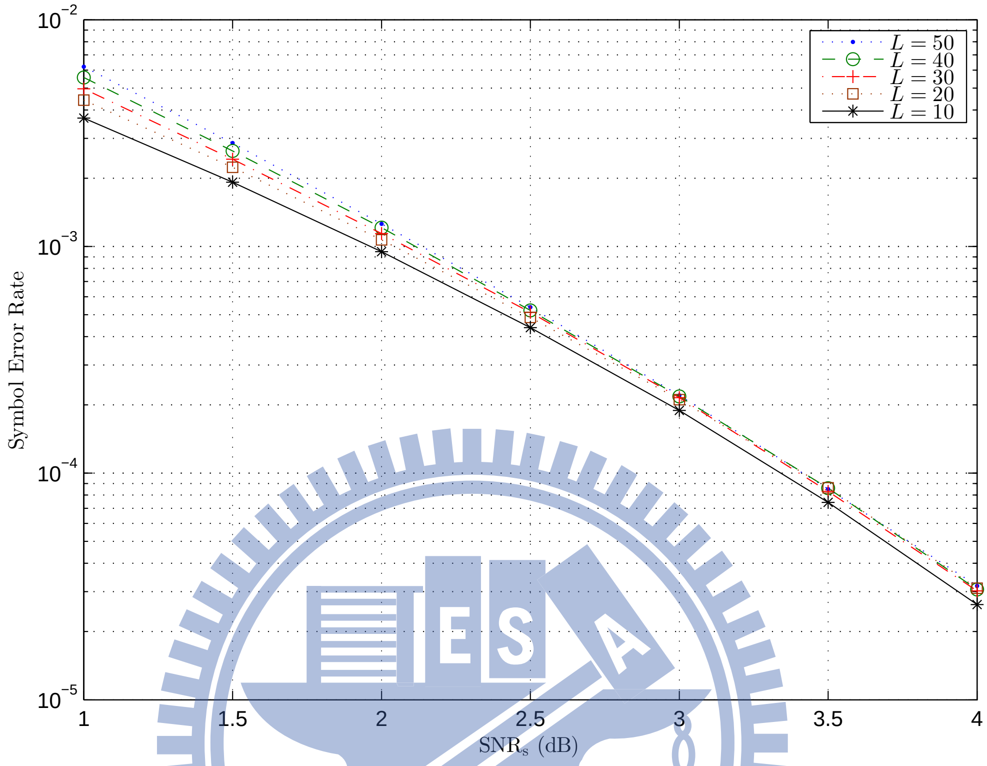


Figure 6.4: Error performances of the optimal VLECPC for different L. The optimal VLECPC is obtained from Table 6.2.

Table 6.4: Average (AVG) and maximum (MAX) numbers of decoder branch metric computations for the codes of Figure 6.5.

SNR$_s$	1 dB		2 dB		3 dB		4 dB	
	AVG	MAX	AVG	MAX	AVG	MAX	AVG	MAX
Lamy's VLECPC	511	3631	510	1858	510	970	510	731
Buttigieg's and Wang's VLECPCs	500	3439	499	1303	499	720	499	670
Optimal VLECPC	461	2970	460	1119	459	719	459	668
Optimal VLECPC with smallest $B_{d_{\rm free}}$	462	3040	460	1144	459	712	459	668

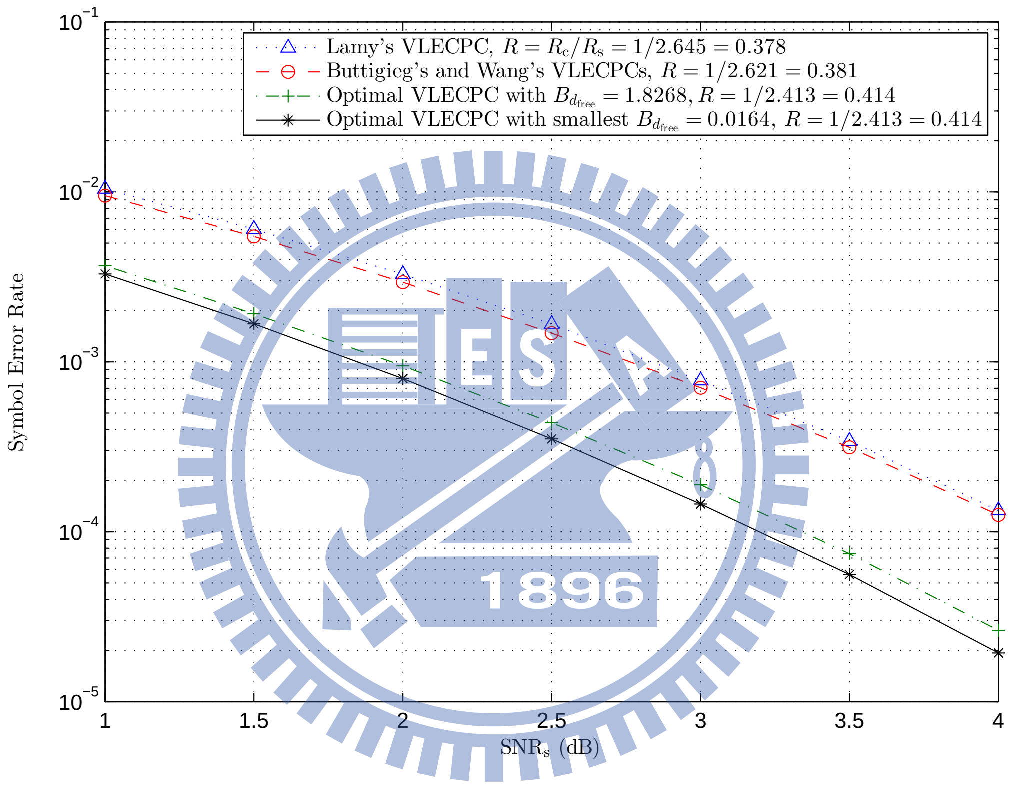


Figure 6.5: Error performances of different (3rd order) VLECPCs for a binary non-uniform source with $p_0 = 0.8$. The number of 3-bit source symbols per transmission block is 10, which is equivalent to 30 source information bits. The free distance d_{free} for all VLECPCs is $d_{\text{free}} = 7$.

We next examine in Figure 6.5 the improvement in error performance between the optimal code construction in Chapter 3 and the modified optimal one (that guarantees to output the optimal VLECPC with the smallest $B_{d_{\text{free}}}$) in Section 4.1. Here, we group three information bits, generated from a binary non-uniform memoryless source with bit probability $p_0 \triangleq \Pr(0) = 0.8$, as one source symbol; hence, the VLECPCs are 3rd order VLCs (i.e., $k = 3$), and the size of the source alphabet is $K = 2^3 = 8$. Also shown in the same figure are the error performances of three VLECPCs respectively obtained by Buttigieg's [7], Lamy's [23] and Wang's [30] code construction algorithms, which have the same free distance $d_{\text{free}} = 7$ as the optimal VLECPCs we constructed, where Buttigieg's and Lamy's algorithms coincidentally yield an identical code in this case. In each simulation, 10 source symbols (equivalently, 30 source information bits) are encoded and transmitted as a block. All codes are decoded using the TP-SMAP decoder of Chapter 5. Figure 6.5 shows that our optimal VLECPC constructed by the algorithm proposed in Chapter 3 has around 0.8 dB coding gain over the three existing VLECPCs; it also indicates that minimizing $B_{d_{\text{free}}}$ can further pick up another 0.1 dB in performance gain.

Table 6.4 summarizes the decoding complexity of the TP-SMAP for the VLECPCs of Figure 6.5. We notice that a VLECPC with higher average codeword length requires a higher decoding complexity. This is somehow anticipated since the decoding trellis is larger for a VLECPC with higher average codeword length. Along this observation, the optimal VLECPC and the optimal VLECPC with the smallest $B_{d_{\text{free}}}$ have expectedly similar decoding complexity because they have identical average codeword length. In addition, with a smaller (actually, the minimum) average codeword length, our optimal VLECPC decodes faster via the TP-SMAP than the other three VLECPCs.

We next test the performance of the suboptimal code construction algorithm of Section 4.2 for the 26-symbol English data source. Since there are two different distributions for the English alphabet that are generally used in the literature for constructing VLECPCs (e.g., compare [25, 30, 20, 26] with [7, 9, 13, 18]), we provide simulation results for both distributions; we will refer to them as Distributions 1 and 2, respectively. The

VLECPCs we obtain via our suboptimal code construction algorithm are presented in Tables 6.5 and 6.6 for Distributions 1 and 2, respectively.

In Table 6.7(a), we list, for different values of d_{free}, the average codeword lengths (ALs) of the resulting VLECPCs under Distribution 1 as well as the execution time needed for their construction via our suboptimal algorithm and the three algorithms referred above. For the sake of completeness, the parameters used in each algorithm are reported in Table 6.7(b).[3] These parameters are chosen through a number of trials in targeting a VLECPC with smaller average codeword length. The results indicate that by manipulating the parameters, the VLECPCs obtained by our suboptimal code construction algorithm can outperform all other three VLECPCs in average codeword length. Table 6.7(a) also shows that our suboptimal code construction algorithm is worse than Lamy's or Wang's algorithms in terms of execution time for $d_{\text{free}} \leq 9$; however, we can prevent the construction complexity of our algorithm from growing too quickly for $d_{\text{free}} \geq 10$ by properly adjusting its parameters under the premise that our algorithm can still yield a better code than the other three algorithms. Similar conclusions can be drawn about the performance of the above algorithms under Distribution 2; the results are presented in Table 6.8.

Analogously to other schemes, many combinations of parameters need to be tested in our suboptimal algorithm to arrive at a good VLECPC construction. The main parameters that control the algorithm's complexity are the early-elimination window Δ and the *Encoding Stack* size Γ. Usually, complexity increases when either Δ or Γ increase, albeit with the benefit of improving the VLECPC average codeword length. In general, it is not straightforward to decide on the right choice of values for these parameters before testing them. Despite this inconvenience, the proposed suboptimal approach is efficient enough to test many combinations of parameters in reasonable time. For example, to get the suboptimal VLECPC with $d_{\text{free}} = 3$ in Table 6.5, we simulated all combinations of the following parameters: $\Delta = \{1, 3, 5, 7, 9, 11, 13\}$, $\Gamma = \{20, 40, 60, 80, 100, 200, 300, 400, 500, 1000\}$

[3] Buttigieg's algorithm (specifically, MVA in [7]) and Wang's algorithm [30] are characterized by two parameters, L_1 and $L_{\max}$. An additional parameter L_s is needed for Lamy's algorithm (specifically, noHole+L_s in [23]).

Table 6.5: The VLECPCs for the English alphabet with Distribution 1 obtained by the suboptimal code construction algorithm for different values of free distance.

Alphabet	Probability	$d_{\text{free}} = 3$	$d_{\text{free}} = 5$	$d_{\text{free}} = 7$	$d_{\text{free}} = 9$	$d_{\text{free}} = 10$	$d_{\text{free}} = 11$
E	0.14878610	0111	00001	00000000	00101101	000100000	0000000000
T	0.09354149	00101	011110	11111111	111111100	0000011110	00001011111
A	0.08833733	11011	0101011	000011111	1111000111	00101100111	000111101001
O	0.07245769	000110	1010000	111100001	11001000100	11011011000	0011010101111
R	0.06872164	010011	00110100	0011010100	110001111011	010111101100	00111100111001
N	0.06498532	101111	10010011	1100110011	0101010010100	101010010011	11101011100110
H	0.05831331	111010	11101111	01011010010	1001001100011	0110111000010	010101110010101
I	0.05644515	0001011	011001011	10101010101	00010000010001	1111100111101	111011101111010
S	0.05537763	1000100	101111100	11000101001	10100010101010	10110011110101	0111110110110011
D	0.04376834	1011001	110000100	001111001100	001100101001000	11001100001011	1100011111011100
L	0.04123298	1110010	1011110111	010101100010	100000110110011	011010110110011	01101101110011010
U	0.02762209	00000011	1101000010	101010010001	0001101010110111	100111011101111	11010010111110110
P	0.02575393	00000100	11000100111	110000111100	0100011011001010	111101101010100	101001001101110100
F	0.02455297	10001111	11110101000	0101001110110	1000000001110000	0110001101111001	1101011110110111100
M	0.02361889	10010101	110001010011	0110011000011	01000010011110111	1011110110000101	1110100001100110110
C	0.02081665	10100001	110111001100	0110100111001	01011011101011001	01001001110110110	01110010111111001010
W	0.01868161	10100110	1100010101000	1001011011001	10000110110001010	10100010110010001	01111000101111010011
G	0.01521216	11000000	1101110000010	1001110000110	000101101101110010	11010101111101001	011001111011111000011
Y	0.01521216	010000011	11011101010111	1010001101100	010000110101010101	010010011101010011	110101000101110010110
B	0.01267680	010000100	11011101101000	00110011110010	100110111010001000	110000101110001100	1010011010111111000101
V	0.01160928	100100000	110111000010111	01011001100111	0001101011011010001	111111100111111010	1110110000000111101001 0
K	0.00867360	110001101	110111010111001	01100101111100	0100001011101101010	1000110101001001101	01000110110011111000011
X	0.00146784	1000001001	110111011001100	01101100001011	0100001100000010000	1100101011110000111	10111001000011111010010
J	0.00080064	1100001111	1101110001111001	10011001011001	00010110111110110001	1110010001011110110	11101100000100011011011 0
Q	0.00080064	1100011100	1101110101100100	10100110010101	00011010110010011110	11100010011011000111	010101010100011111000011
Z	0.00053376	10000010100	1101110110111111	001101101111001	01000011010011001000	11100101110110101011	101011001000100111011001 0

Table 6.6: The VLECPCs for the English alphabet with Distribution 2 obtained by the suboptimal code construction algorithm for different values of free distance.

Alphabet	Probability	$d_{\text{free}} = 3$	$d_{\text{free}} = 5$	$d_{\text{free}} = 7$	$d_{\text{free}} = 9$	$d_{\text{free}} = 10$	$d_{\text{free}} = 11$
E	0.1270	0111	000000	0011111	00000101	000100000	0000000001
T	0.0906	00011	111111	01000110	001110011	0000011110	00001111101
A	0.0817	11101	0001110	000010000	0101101000	00101100111	011100011010
O	0.0751	001010	1111000	111101101	01110111001	11011011000	0111011101010
I	0.0697	010011	00101001	0001001001	00111100110	010111101100	10110110111000
N	0.0674	101111	11010110	1110111000	110010011000	101010010011	11011101000111
S	0.0633	110110	010110100	00000101100	010011011111	0110111000010	101100101001110
H	0.0609	0010010	101100110	10001110011	01110110110110	1111100111101	111011010110000
R	0.0599	0100000	110010011	11110000001	10001011011100	10110011110101	1011100111010011
D	0.0425	1000110	111001101	010110100010	100011100011010	11001100001011	1110011000101100
L	0.0403	1011001	0100010101	100111010001	110100010101110	011010110110011	11011010110010100
C	0.0278	1101011	0101001011	101001111010	0000101011001010	100111011101111	110110100010011110
U	0.0276	10001011	1000110010	111000100101	101011100111110	111101101010100	111011101101100010
M	0.0241	10010100	1010011001	0001001110101	110101111010000	0110001101111001	1011101101111000000
W	0.0236	10100001	1011100101	1000011101010	01010000110010010	1011110110000101	1101101111000101111
F	0.0223	10100110	01001010101	1100100100001	10011111111001011	01001001110110110	1110010010011010010
G	0.0202	11000010	01011001011	00100010100011	11111011010111110	10100010110010001	10111011000010101110
Y	0.0197	11000101	10001100011	10110011110100	010111111101011110	11010101111101001	11101010111101110100
P	0.0193	000000100	10110100101	11001111110011	100111001010001010	010010011101010011	111001001001011110110
B	0.0149	100000001	011001000111	11011000101010	111000010011001110	011101110110101011	111010110000101101010
V	0.0098	100001111	100001010011	001001000100011	0010111010011001110	111000001010001101	111110100111111011101
K	0.0077	100100010	111010100011	011011001111100	1001000011010101010	1000110101001001101	10111011000000110000110
J	0.0015	0000001111	0011111100011	100111011100000	1110011011110010010	1011011101101111010	11101011011111111101101
X	0.0014	0000011010	00100111100011	0010001011100000	01101000011111001011	1110001001011110101	10111011011011111101100
Q	0.0010	00000111010	11000010100011	1101000011110100	10100000011010111110	11100111011011000111	11101000100101100010111
Z	0.0007	000001110010	011101111100011	1110001100100011	11011101111011001110	11111100100110111001	11111010001100010111010

Table 6.7: List of the VLECPCs obtained by three existing code construction schemes and the VLECPCs obtained by our suboptimal code construction algorithm for the 26-symbol English alphabet with Distribution 1 given in Table 6.5: (a) Average codeword lengths (ALs) of the found codes and execution time for each code construction algorithm; (b) Parameters used in each algorithm. The suboptimal algorithm is initialized with U_b set to equal the smallest of the average codeword lengths of the VLECPCs by Buttigieg, Lamy and Wang.

(a)

Algorithm	Buttigieg's		Lamy's		Wang's		Suboptimal	
	AL	Time	AL	Time	AL	Time	AL	Time
$d_{\text{free}} = 3$	6.272617	2m2s	6.309980	4s	6.266612	<1s	6.189350	18s
$d_{\text{free}} = 5$	8.378035	6m42s	8.400986	44s	8.378035	12s	8.333866	2m27s
$d_{\text{free}} = 7$	10.559646	4h31m	10.599945	5m43s	10.488923	27s	10.302508	8m41s
$d_{\text{free}} = 9$	12.737255	6h27m	12.806644	9m52s	12.737255	2m30s	12.532291	5m29s
$d_{\text{free}} = 10$	12.757672	11h45m	12.867893	17m54s	12.757672	47m46s	12.593140	9m35s
$d_{\text{free}} = 11$	14.876166	19h14m	15.354549	21m43s	15.024952	2h15m	14.580329	14m53s

(b)

Algorithm	Buttigieg's	Lamy's	Wang's	Suboptimal
Parameters	$(L_1, L_{\max})$	$(L_1, L_{\max}, L_s)$	$(L_1, L_{\max})$	$(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$
$d_{\text{free}} = 3$	$(4, 13)$	$(4, 13, 10)$	$(4, 13)$	$(5, 300, \mathcal{D}_m, 2)$
$d_{\text{free}} = 5$	$(6, 15)$	$(6, 15, 12)$	$(6, 15)$	$(3, 500, \mathcal{D}_l, 1)$
$d_{\text{free}} = 7$	$(7, 16)$	$(7, 16, 13)$	$(7, 16)$	$(5, 2000, \mathcal{D}_m, 1)$
$d_{\text{free}} = 9$	$(9, 18)$	$(9, 18, 15)$	$(9, 18)$	$(1, 60, \mathcal{D}_m, 1)$
$d_{\text{free}} = 10$	$(10, 19)$	$(10, 19, 15)$	$(10, 19)$	$(1, 40, \mathcal{D}_l, 2)$
$d_{\text{free}} = 11$	$(12, 21)$	$(12, 21, 17)$	$(12, 21)$	$(1, 4, \mathcal{D}_l, 1)$

Table 6.8: List of the VLECPCs obtained by three existing code construction schemes and the VLECPCs obtained by our suboptimal code construction algorithm for the 26-symbol English alphabet with Distribution 2 given in Table 6.6: (a) Average codeword lengths (ALs) of the found codes and execution time for each code construction algorithm; (b) Parameters used in each algorithm. The suboptimal algorithm is initialized with U_b set to equal the smallest of the average codeword lengths of the VLECPCs by Buttigieg, Lamy and Wang.

(a)

Algorithm	Buttigieg's		Lamy's		Wang's		Suboptimal	
	AL	Time	AL	Time	AL	Time	AL	Time
$d_{\text{free}} = 3$	6.4038	20s	6.4047	14s	6.3574	<1s	6.2560	7s
$d_{\text{free}} = 5$	8.4740	5m16s	8.5049	47s	8.4740	9s	8.3223	1m13s
$d_{\text{free}} = 7$	10.5388	1h55m	10.5110	12m01s	10.5388	47s	10.3615	12m13s
$d_{\text{free}} = 9$	12.8898	3h14m	12.9644	13m04s	12.8898	4m22s	12.6647	6m03s
$d_{\text{free}} = 10$	12.8959	9h10m	13.0095	58m29s	12.8959	19m41s	12.7507	8m49s
$d_{\text{free}} = 11$	15.0345	17h37m	15.0846	38m53s	15.0345	1h20m	14.6521	16m12s

(b)

Algorithm	Buttigieg's	Lamy's	Wang's	Suboptimal
Parameters	$(L_1, L_{\max})$	$(L_1, L_{\max}, L_s)$	$(L_1, L_{\max})$	$(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$
$d_{\text{free}} = 3$	$(4, 13)$	$(4, 13, 13)$	$(4, 13)$	$(6, 200, \mathcal{D}_m, 1)$
$d_{\text{free}} = 5$	$(6, 15)$	$(6, 15, 13)$	$(6, 15)$	$(2, 250, \mathcal{D}_m, 1)$
$d_{\text{free}} = 7$	$(7, 18)$	$(7, 18, 15)$	$(7, 18)$	$(1, 3000, \mathcal{D}_m, 1)$
$d_{\text{free}} = 9$	$(9, 18)$	$(9, 18, 16)$	$(9, 18)$	$(1, 20, \mathcal{D}_l, 1)$
$d_{\text{free}} = 10$	$(10, 20)$	$(10, 20, 17)$	$(10, 20)$	$(3, 40, \mathcal{D}_l, 1)$
$d_{\text{free}} = 11$	$(11, 21)$	$(11, 21, 18)$	$(11, 21)$	$(1, 12, \mathcal{D}_l, 1)$

and $\mathcal{D} = \{\mathcal{D}_m, \mathcal{D}_l\}$. It took us only about 29 minutes to simulate all these 140 combinations within a single computer experiment.

Table 6.9: The complexities and performances of some different suboptimal code construction for $d_{\text{free}} = 4$ for the 26-symbol English alphabet (Distribution 2 given in Table 6.6).

$(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$	U_b	AL	# of node computations	Time
$(3, 200, \mathcal{D}_l, 1)$	∞	6.4794	2916	2s
$(3, 200, \mathcal{D}_l, 1)$	7.3375	6.4794	2916	2s
$(3, 200, \mathcal{D}_l, 1)$	6.4800	6.4794	3214	2s
$(3, 200, \mathcal{D}_m, 2)$	∞	6.7760	45236	10s
$(3, 200, \mathcal{D}_m, 2)$	7.3375	6.7973	44516	9s

We next provide efficiency comparisons with the recent works of Diallo *et al.* [13] and Hijazi *et al.* [18]. [4] Notably different from our work and also the main referenced works in this dissertation (i.e., Buttigieg's [7], Lamy's [23] and Wang's [30]), Diallo *et al.* and Hijazi *et al.* do not construct codes for a given distribution but for a pre-specified set of codeword lengths. The distributions assumed in their papers are therefore primarily for the computation of the resulting average codeword length. To compare with the VLECPC of [13], we simulated our suboptimal code construction for $d_{\text{free}} = 4$ under the same used distribution for the 26-symbol English alphabet (Distribution 2 given in Table 6.6). The VLECPC designed in [13, Table IV] has an average codeword length of 7.3375 and an execution time of 310 hours. Our suboptimal code construction algorithm, when initialized by an upper bound given by the average length of the code in [13] (i.e., with $U_b = 7.3375$) and parameters $(\Delta = 3, \Gamma = 200, \mathcal{D}_l, \mathcal{I} = 1)$ yields an improved VLECPC with an average codeword length of 6.4794 within only 2 seconds of execution. Furthermore, a similar result can be obtained without making use of the U_b parameter (i.e., by setting $U_b = \infty$). The (3,200,$\mathcal{D}_l$,1) suboptimal algorithm still yields a VLECPC with an average codeword length of 6.4794 within only 2 seconds of execution. This further shows that the proposed suboptimal algorithm is highly efficient. The complexities and performances of other found suboptimal VLECPCs are summarized in Table 6.9.

[4] It should be mentioned that we did not actually implement the systems of [13] and [18]; instead, the efficiency results of these systems are directly retrieved from each paper. Due to differences in the experimental platforms, the comparisons between our system and those of [13] and [18], especially in terms of execution time, may not be on a fully equal footing. They are however herein provided for the sake of reference.

In [18, Table 3], Hijazi *et al.* provide a VLECPC for $d_{\text{free}} = 7$ within an execution time of 13 minutes and 31 seconds for a given set of codeword lengths. For Distribution 2 in Table 6.6, the resulting average codeword length is 10.4213. In [18, Table 4], they provide another VLECPC for $d_{\text{free}} = 7$, resulting in a better average codeword length of 10.1138 under Distribution 2, but no execution time is given.

Table 6.10: The complexities and performances of some other suboptimal code construction for $d_{\text{free}} = 7$ for the 26-symbol English alphabet (Distribution 2 given in Table 6.6).

$(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$	U_b	AL	# of node computations	Time
$(1, 3000, \mathcal{D}_m, 1)$	∞	10.3615	459403	12m37s
$(1, 3000, \mathcal{D}_m, 1)$	10.5110	10.3615	452237	12m13s

In contrast, our best to-date suboptimal code construction, as shown in Table 6.8 with parameters $(\Delta = 1, \Gamma = 3000, \mathcal{D}_m, \mathcal{I} = 1)$ and $U_b = 10.5110$, outputs a VLECPC for $d_{\text{free}} = 7$ with an average codeword length of 10.3615, which is in between 10.4213 [18, Table 3] and 10.1138 [18, Table 4], under an execution time of 12 minutes and 13 seconds. On the other hand, our current suboptimal code construction algorithm, when initialized with $U_b = 10.4213$ (and also $U_b = 10.1138$), either reports a code search failure or cannot converge to a solution in reasonable time, depending on the choice of parameters $(\Delta, \Gamma, \mathcal{D}, \mathcal{I})$. It should be pointed out however, that unlike our suboptimal algorithm, the scheme of [18] requires a priori knowledge of all codeword lengths before it is run. Hence arriving at the right choice of codeword lengths for any given d_{free} and alphabet size requires additional trials (whose execution duration are not reported in [18]). Nonetheless, it is certainly of interest, to further improve the efficiency of our algorithm and assess whether or not the average codeword length of 10.1138 is optimal or not for $d_{\text{free}} = 7$. The complexities and performances of other found suboptimal VLECPCs are summarized in Table 6.10.

Figure 6.6 illustrates the SER performances of the VLECPCs presented in Table 6.7 with $d_{\text{free}} = 11$. Again, 10 source symbols are encoded and transmitted as a block in each simulation, and all codes are decoded using the TP-SMAP decoder in Chapter 5. We observe from the figure that the VLECPC obtained by our suboptimal code construction

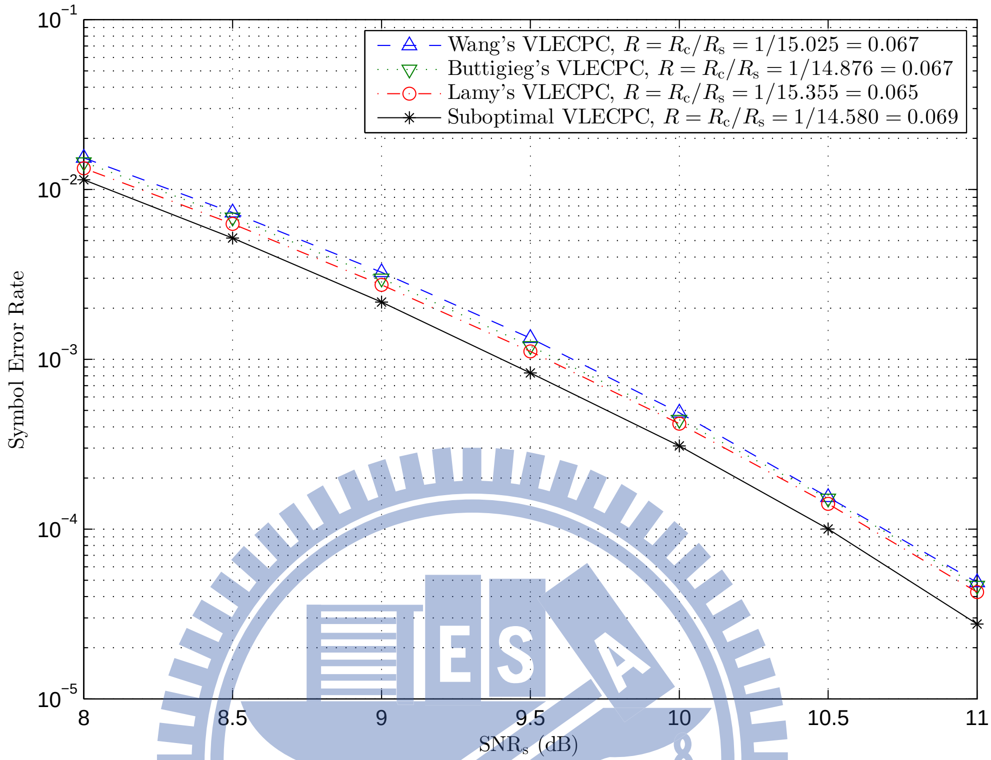


Figure 6.6: Error performances of the VLECPCs of Table 6.7 with $d_{\text{free}} = 11$ for the 26-symbol English alphabet (with Distribution 1). The number of source symbols per transmission block is $L = 10$.

Table 6.11: Average (AVG) and maximum (MAX) numbers of decoder branch metric computations for the codes of Figure 6.6.

SNR$_s$	8 dB		9 dB		10 dB		11 dB	
VLECPC system	AVG	MAX	AVG	MAX	AVG	MAX	AVG	MAX
Wang's VLECPC	3124	11131	3123	4524	3123	4093	3123	4000
Buttigieg's VLECPC	3112	16433	3111	5950	3111	4001	3111	4001
Lamy's VLECPC	3211	14675	3210	5959	3209	4391	3209	4391
Suboptimal VLECPC	3108	10096	3104	4349	3104	3995	3104	3995

algorithm outperforms the other three VLECPCs by at least 0.15 dB. The decoding complexities of these systems are summarized in Table 6.11. As anticipated, the VLECPC obtained by our suboptimal code construction algorithm has the smallest average codeword length and hence its decoding complexity is smaller than those of the other three VLECPCs, particularly in the maximum number of branch metric computations.

Finally, we compare the SER performance of one suboptimal VLECPC shown in Table 6.7 with that of a traditional SSCC system for the situation where the source is the memoryless 26-symbol English data. The SSCC system consists of a Huffman source coder and a tail-biting convolutional channel (TBCC) coder. We use $(3, 1, 3)$, $(3, 1, 4)$, $(3, 1, 5)$ and $(3, 1, 6)$ TBCCs respectively with generator polynomial $[54, 64, 74]$, $[52, 66, 76]$, $[47, 53, 75]$ and $[564, 624, 754]$ (in octal) [29] such that the resulting SSCC systems have approximately the same code rate $R \approx 0.08$ as the VLECPC to be compared with. Also, the d_{free} of the chosen VLECPC is 10, while the largest minimum Hamming distances $d_{\min}$ for $(3, 1, 3)$, $(3, 1, 4)$, $(3, 1, 5)$ and $(3, 1, 6)$ TBCCs are 10, 12, 13 and 15, respectively. Both the VLECPC and the TBCCs are decoded by sequence decoders, where the one for the VLECPC is the TP-SMAP proposed in Chapter 5, and the one for the TBCCs is the priority-first search decoding algorithm (PFSA) introduced in [16]. The results are illustrated in Figure 6.7.

Table 6.12: Average (AVG) and maximum (MAX) numbers of decoder branch metric computations for the codes of Figure 6.7. The parameter λ used in PFSA is indicated inside the parentheses.

SNR_s		8 dB		9 dB		10 dB		11 dB	
Scheme									
Code	Decoder	AVG	MAX	AVG	MAX	AVG	MAX	AVG	MAX
$(3, 1, 3)$ TBCC $[54, 64, 74]$	PFSA(3)	753	2049	739	1518	731	1483	730	1253
$(3, 1, 4)$ TBCC $[52, 66, 76]$	PFSA(4)	1466	4192	1444	3298	1435	2916	1432	2528
$(3, 1, 5)$ TBCC $[47, 53, 75]$	PFSA(5)	2907	8909	2875	6437	2865	4851	2862	4661
$(3, 1, 6)$ TBCC $[564, 624, 754]$	PFSA(6)	5773	21062	5734	12814	5724	9063	5721	8687
Suboptimal VLECPC	TP-SMAP	2698	8322	2695	7362	2694	3840	2694	3840

We remark from Figure 6.7 that for almost all simulated SNRs, the suboptimal VLECPC outperforms the SSCC using a TBCC of memory order no larger than 5. In comparison with the SSCC equipped with the $(3, 1, 6)$ TBCC, the suboptimal VLECPC still performs better when SNR_s is less than 9 dB. Table 6.12 summarizes the decoding

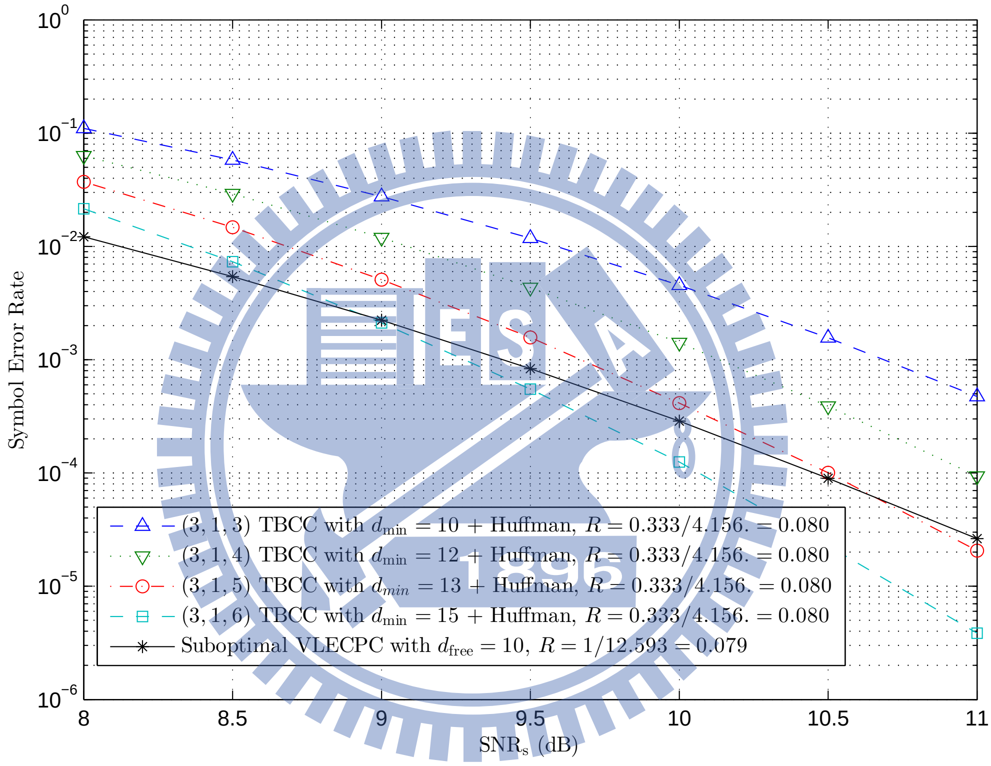


Figure 6.7: Error performances of the SSCC (specifically, first order Huffman + TBCC) and the VLECPC of Table 6.7 with $d_{\text{free}} = 10$ for the 26-symbol English alphabet (with Distribution 1). The number of source symbols per transmission block is $L = 10$.

complexities of the suboptimal VLECPC and the TBCCs in terms of the branch metric computations. It indicates that the VLECPC system is more efficient than the SSCC system using a TBCC of memory orders 5 and 6. Note that in this table, the decoding complexity of the Huffman coder is not even included. We can then conclude that the VLECPC system can achieve a better performance than an SSCC system of comparable decoding complexity. We end the discussion by pointing out again that the VLECPC system only requires one encoder and one decoder, while the SSCC system needs separate source coder and channel coder at both transmitter and receiver sides. This can be considered another advantage of the VLECPC system over the SSCC system.

Chapter 7

Conclusion

In this dissertation, a novel search algorithm is proposed for constructing optimal prefix-free VLECPCs for the effective joint source-channel coding of memoryless sources over memoryless channels. The optimal construction algorithm is modified to construct optimal VLECPCs with improved resilience against channel noise through a critical union bound parameter $B_{d_{\text{free}}}$. A suboptimal but much more efficient construction algorithm is next presented to construct VLECPCs with large free distances and for large source alphabets such as the 26-symbol English data source. A low-complexity two-phase sequence MAP (TP-SMAP) decoder for the VLECPCs is also proposed. Simulations show that the developed optimal and suboptimal VLECPCs can have evident gains over most existing VLECPCs of identical free distance in terms of average codeword length, error rate performance and decoding complexity. Also shown in this dissertation is that our VLECPC system outperforms traditional separate source/channel coding systems of similar overall rate at low to medium SNRs with the benefit of considerably smaller decoding complexity. Future research directions may include further improving the efficiency of our sub-optimal algorithm, extending our design to Markov sources as well as investigating powerful VLECPC iterative decoding methods (e.g., cf. [4, 22]) with manageable complexity.

Appendix A

Optimality of Constuction Algorithm in Chapter 3

To show that the proposed algorithm can always find a VLECPC with minimal average codeword length and free distance d^*_{free}, the following lemma is needed.

Lemma 1. *The metric f of each node is not greater than its children:*

$$f(\texttt{P}) \leq f(\texttt{L}) \text{ and } f(\texttt{P}) \leq f(\texttt{R}), \tag{A.1}$$

where node P *is the parent of* L, *and* R *as shown in Figure 3.1.*

Proof. The candidate codewords of each node are listed in order of nondecreasing lengths (i.e., $\mathcal{A} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots\}$ with $|\boldsymbol{a}_1| \leq |\boldsymbol{a}_2| \leq |\boldsymbol{a}_3| \cdots$). For the left child L, $\mathcal{A}_{\texttt{L}}$ is a subset of $\mathcal{A}_{\texttt{P}} \setminus \{\boldsymbol{a}_1^{\texttt{P}}\}$. Hence $|\boldsymbol{a}_{i+1}^{\texttt{P}}| \leq |\boldsymbol{a}_i^{\texttt{L}}|$ for all integers $i \geq 1$. Therefore,

$$f(\texttt{P}) = \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\texttt{P}}| + \sum_{i=t+1}^{K} p_i \cdot |\boldsymbol{a}_{i-t}^{\texttt{P}}| \tag{A.2}$$

$$= \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\texttt{P}}| + p_{t+1} \cdot |\boldsymbol{a}_1^{\texttt{P}}| + \sum_{i=t+2}^{K} p_i \cdot |\boldsymbol{a}_{i-t}^{\texttt{P}}| \tag{A.3}$$

$$\leq \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\texttt{P}}| + p_{t+1} \cdot |\boldsymbol{a}_1^{\texttt{P}}| + \sum_{i=t+2}^{K} p_i \cdot |\boldsymbol{a}_{i-t-1}^{\texttt{L}}| \tag{A.4}$$

$$= f(\texttt{L}). \tag{A.5}$$

Since $|\boldsymbol{a}_i^{\texttt{P}}| \leq |\boldsymbol{a}_{i+1}^{\texttt{P}}|$ for $i \geq 1$, for the right child R, we have

$$f(\texttt{P}) = \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\texttt{P}}| + \sum_{i=t+1}^{K} p_i \cdot |\boldsymbol{a}_{i-t}^{\texttt{P}}| \tag{A.6}$$

$$\leq \sum_{i=1}^{t} p_i \cdot |\boldsymbol{c}_i^{\texttt{P}}| + \sum_{i=t+1}^{K} p_i \cdot |\boldsymbol{a}_{i-t+1}^{\texttt{P}}| \tag{A.7}$$

$$= f(\texttt{R}). \tag{A.8}$$

The proposed algorithm repeatedly pops out the node with smallest f from the Stack. Suppose that the algorithm encounters the first top node which has selected K codewords and its free distance equals d^*_{free}; then by the above Lemma, no matter how the algorithm continues, extending any node in the Stack will generate a node with metric f no smaller then the top node. Hence, the algorithm yields an optimal VLECPC. □

Appendix B

Optimality of TP-SMAP Decoder in Chapter 5

The second phase of the TP-SMAP decoder is basically identical to the optimal code construction algorithm, except that the sequential search algorithm is applied on the trellis $\mathcal{T}_{L,N}$ instead of the search tree. Therefore, proving optimality here is similar to the proof provided in Appendix A, except that we need to prove that the path metric is nondecreasing along any path on $\mathcal{T}_{L,N}$.

Lemma 2. *In the second phase, the decoding metric is nondecreasing along any path on trellis $\mathcal{T}_{L,N}$, i.e.,*

$$\boldsymbol{m}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) \leq \boldsymbol{m}\left(\boldsymbol{x}_{(0,0)}^{(i+1,j+l)}\right), \tag{B.1}$$

if there exists a codeword $\boldsymbol{c} \in \mathcal{C}$ and $|\boldsymbol{c}| = l$.

Proof. Based on the backward VA of the first phase, $\boldsymbol{h}(S_j)$ is the minimal metric among all paths from level j to the final node; i.e.,

$$\boldsymbol{h}(S_j) = \min_{i:\sum_{k=i+1}^{L}|\boldsymbol{x}_k|=N-j \text{ with each } \boldsymbol{x}_k\in\mathcal{C}} \boldsymbol{g}\left(\boldsymbol{x}_{(i,j)}^{(L,N)}\right). \tag{B.2}$$

When there is a codeword $\boldsymbol{c} \in \mathcal{C}$ and $|\boldsymbol{c}| = l$, then

$$\boldsymbol{h}(S_j) \leq \boldsymbol{g}\left(\boldsymbol{x}_{(i,j)}^{(i+1,j+l)}\right) + \boldsymbol{h}\left(S_{j+l}\right).$$

Therefore,

$$\boldsymbol{m}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) = \boldsymbol{g}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) + \boldsymbol{h}(S_j) \tag{B.3}$$

$$\leq \boldsymbol{g}\left(\boldsymbol{x}_{(0,0)}^{(i,j)}\right) + \boldsymbol{g}\left(\boldsymbol{x}_{(i,j)}^{(i+1,j+l)}\right) + \boldsymbol{h}\left(S_{j+l}\right) \tag{B.4}$$

$$= \boldsymbol{g}\left(\boldsymbol{x}_{(0,0)}^{(i+1,j+l)}\right) + \boldsymbol{h}(S_{j+l}) \tag{B.5}$$

$$= \boldsymbol{m}\left(\boldsymbol{x}_{(0,0)}^{(i+1,j+l)}\right). \tag{B.6}$$

The second phase of the TP-SMAP repeatedly pops out the node with smallest $\boldsymbol{m}(\cdot)$ from the Decoding Stack. Suppose that the algorithm encounters the first top node which reaches the final state $S_{L,N}$; then by the above Lemma, no matter how the algorithm continues, extending any node in the Decoding Stack will generate a node with decoding metric $\boldsymbol{m}(\cdot)$ no smaller then the top node. Hence, according to the MAP decision in (2.4), the TP-SMAP certainly outputs the MAP decision. □

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