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New solutions to the constant-head test performed at a partially penetrating well

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A R T I C L E I N F O

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S U M M A R Y

The mathematical model describing the aquifer response to a constant-head test performed at a fully penetrating well can be easily solved by the conventional integral transform technique. In addition, the Dirichlet-type condition should be chosen as the boundary condition along the rim of wellbore for such a test well. However, the boundary condition for a test well with partial penetration must be considered as a mixed-type condition. Generally, the Dirichlet condition is prescribed along the well screen and the Neumann type no-flow condition is specified over the unscreened part of the test well. The model for such a mixed boundary problem in a confined aquifer system of infinite radial extent and finite vertical extent is solved by the dual series equations and perturbation method. This approach provides analytical results for the drawdown in the partially penetrating well and the well discharge along the screen. The semi-analytical solutions are particularly useful for the practical applications from the computational point of view.

Introduction

The constant-head test is generally performed in a confined aquifer to yield field scale characteristic parameters including hydraulic conductivity, the specific storativity and possibly the leakage factor. These parameters can be used to quantify groundwater water resources (Cassiani and Kabala, 1998). The analysis of data obtained from the test is used to determine the aquifer hydrogeological parameters. The test well may be fully or partially penetrated and the wellbore may have well skin. The fully penetrating well can be considered as a Dirichlet (also called first type) boundary condition in the constant-head test, and the resulting model can be solved by the conventional integral transform techniques (Hantush, 1964). If the effect of well skin is negligible, the Dirichlet boundary condition is suitable to describe the drawdown (or well water level) along the well screen and the Neuman (also called second type) boundary condition of zero flux is specified along the casing for a partially penetrating well. Thus, the boundary condition specified for the partially penetrating well is a mixed-type condition. The term "mixed-type" boundary is used to distinguish this boundary condition from the "uniform" conditions of Dirichlet and Neuman or a combination of Dirichlet and Neuman boundaries (Robin boundary). The well skin is usually of a finite thickness and thus should be considered as a different formation zone (see, e.g., Yang and Yeh, 2002; Yeh et al., 2003; Yeh and Yang, 2006a) instead of neglecting its thickness and using a factor to represent its effect.

Many physical problems can be described by partial differential equations with various types of initial and boundary conditions. At present time, the analytical solutions to the mixed-type boundary value problems in well hydraulics are very limited. The techniques used to solve the mixed-type boundary value problems analytically include the dual integral/series equation (Sneddon, 1966), Weiner-Hopf technique (Noble, 1958), and Green's function (Hung and Chang, 1984). However, most of solutions to the mixed-type boundary value problems are obtained numerically (Yedder and Bilgen, 1994) or by approximate methods such as asymptotic analysis (Bassani et al., 1987) or perturbation techniques (Wilkinson and Hammond, 1990).

For the mathematical model subject to the mixed-type boundary condition in a confined aquifer of semi-infinite thickness, Wilkinson and Hammond (1990) used the perturbation method to give an approximate solution for drawdown changes at the well. Cassiani and Kabala (1998) used the dual integral equation method to derive the Laplace-domain solutions for the constant rate pumping test and slug test performed at partially penetrating wells that account not only for wellbore storage, infinitesimal skin, and aquifer

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anisotropy, but also for the mixed-type boundary condition. Cassiani et al. (1999) further used the same mathematical method to derive the Laplace-domain solutions for constant-head pumping test and double packer test that treated as the mixed-type boundary value problems. Slim and Kirkham (1974) used the Gram–Schmidt orthonormalization method to find a steady state drawdown solution in a confined aquifer of finite horizontal extent. Furthermore, similar mixed-type value problems also arise in the field of heat conduction. Among others, Hung (1985) used the Weiner-Hopf technique to find the solution in a semi-infinite slab and Hung and Chang (1984) combined the Green's function with conformal mapping to develop the solution in an elliptic disk.

For the real world problem, the thickness of aquifer is generally finite. As mentioned above, Cassiani andKabala (1998) and Cassiani et al. (1999) developed the solutions to the mixed boundary problem by assuming infinite aquifer thickness. These solutions are appropriate for the case where the pressure change caused by the pumping test has not reached the bottom of the aquifer or the screen length is significantly smaller than the aquifer thickness. Chang and Chen (2002) considered an aquifer with a finite thickness and a skin factor accounting for the well skin effect. They treated the boundary along the well screen as a Cauchy (third type) boundary condition and handled the wellbore flux entering through the well screen as unknown. In addition, they changed the mixed boundary into homogeneous Neumann boundary and then discretized the screen length into M segments. Thus, their solution may be inaccurate for the case where the size of segments is coarse.

The purpose of this study is to develop a new solution to the constant-head test performed at a partially penetrating well in an aquifer with a finite thickness. The mathematical model with the mixed boundary condition at the well is directly solved via the methods of dual series equations and perturbation method. This solution contains single and double infinite series involving the summations of multiplication of integrals, trigonometric functions, and the modified Bessel functions of second kind, where the integrals are in terms of trigonometric functions multiplying the associated Legendre functions. The series in the solution developed in Laplace domain are difficult to accurately evaluate due to the oscillatory nature and slow convergence of the multiplied functions. Therefore, Shanks' transform method (Shanks, 1955) is used to accelerate the evaluation of the Laplace-domain solution and the numerical inversion scheme, Stehfest algorithm (Stehfest, 1970), is used to obtain the time domain solution.

Mathematical model

Fig. 1 shows a partially penetrating well in a confined aquifer of finite extent with a thickness of b. The drawdown at the distance r from the well and the distance z from the bottom of the aquifer at time t is denoted as $s(r,z,t)$. The well screen extends from the top of the aquifer $(z = b)$ to $z = d$ with a length of l. The hydraulic parameters of the aquifer are horizontal hydraulic conductivity K_r, vertical hydraulic conductivity K_z, and specific storage S_s. The governing equation for the drawdown can be written as (Yang et al., 2006)

$$K_r\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)+K_z\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t} \tag{1}$$

A Dirichlet boundary condition for a fixed drawdown specified along the well screen is:

$$s(r_w,z,t)=s_w \quad d\leqslant z\leqslant b \tag{2a}$$

A Neumann boundary condition of zero flux is specified as:

$$\left.\frac{\partial s}{\partial r}\right|_{r=r_w}=0 \quad 0\leqslant z\leqslant d \tag{2b}$$

Moreover, the initial condition and other boundary conditions are:

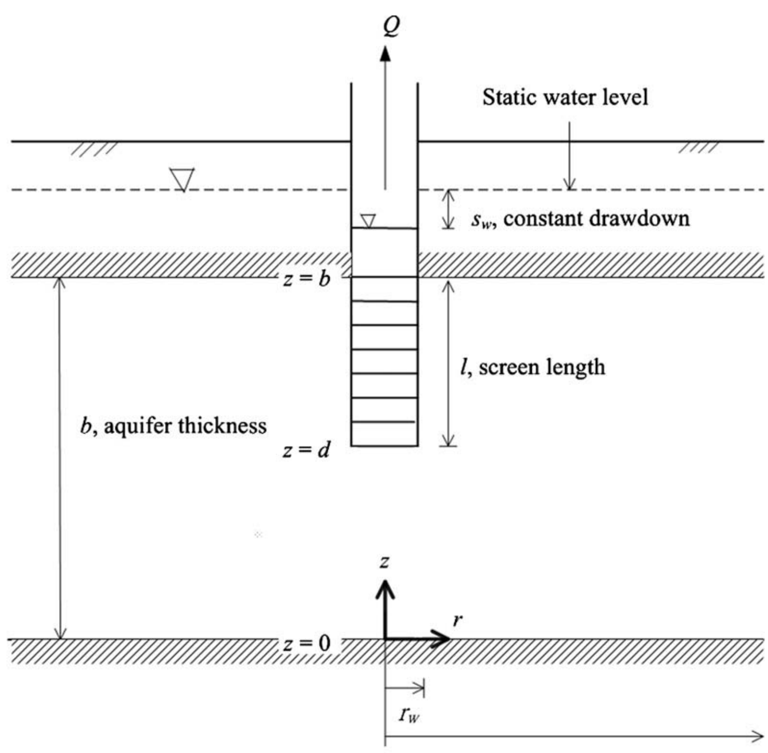

Fig. 1. The cross-section configuration of the aquifer system involving the mixed boundary value problem.

$$s(r,z,0) = 0 \tag{3}$$
$$s(\infty,z,t) = 0 \tag{4}$$

and

$$\frac{\partial s}{\partial z} = 0, \quad z = 0, \quad z = b \tag{5}$$

Eq. (1) may be expressed in dimensionless terms as:

$$\frac{\partial^2 s^*}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial s^*}{\partial \rho} + \alpha^2 \frac{\partial^2 s^*}{\partial \xi^2} = \frac{\partial s^*}{\partial \tau} \tag{6}$$

subject to the boundary and initial conditions written in dimensionless terms as

$$s^*(\rho,\xi,\tau=0) = 0 \tag{7}$$
$$s^*(\rho=\infty,\xi,\tau) = 0 \tag{8}$$

$$s^*(\rho=1,\xi,\tau) = 1, \quad \xi_d \leqslant \xi \leqslant \beta \tag{9a}$$
$$\left.\frac{\partial s^*}{\partial \rho}\right|_{\rho=1} = 0, \quad 0 \leqslant \xi \leqslant \xi_d \tag{9b}$$

$$\frac{\partial s^*}{\partial \xi} = 0, \quad \xi = 0, \quad \xi = \beta \tag{10}$$

where $s^* = s/s_w$ is the dimensionless drawdown, $\tau = tK_r/(S_s r_w^2)$ is the dimensionless time, $\alpha^2 = K_z/K_r$ is the anisotropy ratio of the aquifer, $\beta = b/r_w$ is the dimensionless aquifer thickness, $\rho = r/r_w$ and $\xi = z/r_w$ are dimensionless spatial coordinates, $\xi_d = d/r_w$ is the dimensionless depth at the bottom of the well screen. Note that Eqs. (6)–(10), construct a mixed-type boundary value problem.

The detailed development for the solution of Eq. (6) with Eqs. (7)–(10) using dual series equation and perturbation method is given in Appendix. The solution for the drawdown in Laplace domain can be written as:

$$\bar{s}^*(\rho,\xi,p) = \frac{1}{2}B(0,p)\frac{K_0(\sqrt{p}\rho)}{K_0(\sqrt{p})} + \sum_{n=1}^{\infty} B(n,p)\frac{K_0(\lambda_n\rho)}{K_0(\lambda_n)} \times \cos\left(n\frac{\xi}{\beta}\pi\right) \tag{11}$$

with

$$B(0,p) = {}^{(0)}B(0,p) + \sum_{k=1}^{\infty} I(k,p)B(k,p)C(0,k,p) \tag{12}$$

$$B(n,p) = {}^{(0)}B(n,p) + \sum_{k=1}^{\infty} I(k,p)B(k,p)C(n,k,p) \tag{13}$$

$${}^{(0)}B(0,p) = \frac{\frac{1}{p}\left[\frac{\sqrt{2}}{2\pi}\int_0^{\frac{\xi_d}{\beta}\pi} f_1(u)f_5(u)du + 2\left(1-\frac{\xi_d}{\beta}\right)\right]}{1+\frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)\int_0^{\frac{\xi_d}{\beta}\pi} f_2(u)du} \tag{14}$$

$$\begin{aligned}{}^{(0)}B(n,p) = {} & \frac{\sqrt{2}}{2\pi}\sqrt{p}H(0,p)B(0,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\left[\int_0^u f_2(v)dv\frac{d}{du}f_4(u)\right]du\right\} \\ & - \frac{\sqrt{2}}{2\pi}\sqrt{p}H(0,p)B(0,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi} f_2(u)du f_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & + \frac{2\sqrt{2}}{\pi^2}\frac{1}{p}\left\{\int_0^{\frac{\xi_d}{\beta}\pi} f_5(u)f_1(u)du f_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & - \frac{2\sqrt{2}}{\pi^2}\frac{1}{p}\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\int_0^u f_1(v)f_5(v)dv\frac{d}{du}f_4(u)du\right\} \\ & - \frac{2}{\pi}\frac{1}{p}\frac{1}{n}\sin\left(n\frac{\xi_d}{\beta}\pi\right)\end{aligned} \tag{15}$$

$$C(0,k,p) = \frac{\frac{1}{k}\frac{2\sqrt{2}}{\pi}\int_0^{\frac{\xi_d}{\beta}\pi} f_3(u,k)du}{1+\frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)\int_0^{\frac{\xi_d}{\beta}\pi} f_2(u)du} \tag{16}$$

$$\begin{aligned}C(n,k,p) = {} & \frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)C(0,k,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\left[\int_0^u f_2(v)dv\frac{d}{du}f_4(u)\right]du\right\} \\ & - \frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)C(0,k,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi} f_2(u)du f_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & + \frac{\sqrt{2}}{\pi}\frac{1}{k}\left\{\int_0^{\frac{\xi_d}{\beta}\pi} f_3(u,k)du f_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & - \frac{\sqrt{2}}{\pi}\frac{1}{k}\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\int_0^u f_3(v,k)dv\frac{d}{du}f_4(u)du\right\}\end{aligned} \tag{17}$$

$$f_1(a) = \frac{\sin(a/2)}{\sqrt{\cos(a)-\cos(\pi\xi_d/\beta)}} \tag{18}$$
$$f_2(a) = af_1(a) \tag{19}$$
$$f_3(a,b) = \sin(ab)f_1(a) \tag{20}$$
$$f_4(a) = [P_n(\cos a) + P_{n-1}(\cos a)] \tag{21}$$
$$f_5(a) = \ln\left(1-\cos\left(\frac{\xi_d}{\beta}+a\right)\right) - \ln\left(1-\cos\left(\frac{\xi_d}{\beta}-a\right)\right) \tag{22}$$
$$H(n,p) = K_1(\lambda_n)/K_0(\lambda_n) \tag{23}$$
$$I(n,p) = n - \lambda_n H(n,p) \tag{24}$$
$$\lambda_n = \sqrt{\left(\frac{n\pi\alpha}{\beta}\right)^2 + p} \tag{25}$$

where K_0 and K_1 are the modified Bessel functions of the second kind with order zero and one, respectively, and the $P_n(\cos a)$ is the associated Legendre function (Abramowitz and Stegun, 1970, p. 335).

The flux entering the well screen and the total well discharge obtained using Eq. (11) are, respectively, given as:

$$\begin{aligned}\bar{q}^*(1,\xi,p) &= -\left.\frac{\partial \bar{s}^*(\rho,\xi,p)}{\partial \rho}\right|_{\rho=1} \\ &= \frac{1}{2}B(0,p)\sqrt{p}\frac{K_1(\sqrt{p})}{K_0(\sqrt{p})} + \sum_{n=1}^{\infty} B(n,p)\lambda_n\frac{K_1(\lambda_n)}{K_0(\lambda_n)} \times \cos\left(n\frac{\xi}{\beta}\pi\right)\end{aligned} \tag{26}$$

and

$$\begin{aligned}Q(p) &= \int_{\xi_d}^{\beta} \bar{q}^*(1,\xi,p)d\xi/\lambda \\ &= \frac{1}{2}B(0,p)\sqrt{p}\frac{K_1(\sqrt{p})}{K_0(\sqrt{p})} - \sum_{n=1}^{\infty}\frac{\beta}{n\pi\lambda}B(n,p)\lambda_n\frac{K_1(\lambda_n)}{K_0(\lambda_n)} \times \sin\left(n\frac{\xi_d}{\beta}\pi\right)\end{aligned} \tag{27}$$

where $\lambda = l/r_w$ is the dimensionless length of screen.

Numerical evaluations

Eq. (11) contains single and double infinite series which consist of the summations of multiplication of integrals, trigonometric functions, and the modified Bessel functions of second kind. The integrals are in terms of trigonometric functions multiplying associated Legendre functions. This solution involves numerous complicated mathematical functions. Therefore, numerical approaches including the Gaussian quadrature (Gerald and Wheatley, 1989), Shanks' transform and Stehfest method are proposed to evaluate the solution. The Gaussian quadrature with six terms (Yang and Yeh, 2007) is first utilized to evaluate the integrals in Eq. (11). Since the oscillation and slow convergence of the multiplication terms, the summations are difficult to evaluate accurately and efficiently. Therefore, the Shanks' transform method (Shanks, 1955), a non-linear iterative algorithm based on the sequence of partial sums, is used to compute the summations in Eq. (11). This method has been successfully devoted to efficiently computing the solutions arisen in the groundwater area (see, e.g., Peng et al., 2002; Yeh and Yang, 2006b). In addition, the Stehfest algorithm (Stehfest, 1970) is further employed to inverse the Laplace-domain solution into time domain solution. The proposed numerical approaches can accurately evaluate the drawdown solution to the mixed-type boundary value problem for a flowing partially penetrating well and the results are demonstrated in the following section.

Results and discussion

When the well fully penetrates the entire thickness of the formation, i.e., ξ_d is zero, the drawdown and the well discharge can be obtained using Eqs. (11) and (17), respectively, as

$$\bar{s}^*(\rho,\xi,p) = \frac{1}{p}\frac{K_0(\sqrt{p}\rho)}{K_0(\sqrt{p})} \tag{28}$$

and

$$Q(p) = \frac{K_1(\sqrt{p})}{\sqrt{p}K_0(\sqrt{p})} \tag{29}$$

Eqs. (28) and (29) are identical to the solutions of drawdown and flow rate in Laplace domain given in Chen and Stone (1993); Yang and Yeh (2005). The solutions of the aquifer drawdown and well flux can be determined by inverting Eqs. (11) and (26) by the Stehfest (1970) method with eight weighting factors.

The validity of the proposed solutions can be assessed by examining the sensitivity of the boundary conditions in (9) in the calculation. Fig. 2 shows the drawdown for $\beta = 100$ and $\xi_d = 50$ with different ρ values at $\tau = 1,\ 100,\ 10^4$ and 10^6. As indicated in the figure, the drawdown is constant along the well screen and decreases with increasing radial distance at $\tau = 1$. In addition, the drawdown increases with dimensionless time along the unscreened part of the well. Fig. 3 shows the plots of the flux along the well screen for $\beta = 100$ and $\xi_d = 50$ at $\tau = 1,\ 100,\ 10^4$ and 10^6. The flux is non-uniformly distributed and larger at the screen edge, due to the vertical flow induced by the presence of well partial penetration.

In order to explore the effect of partial penetration on the well discharge, Fig. 4 illustrates four different penetration ratios $\omega = \lambda/\beta$ with $\lambda = 50$. The well discharges for those four cases are the same at the small time; however, it decreases with increasing penetration ratio at large time. If the penetration ratio is smaller than 0.01, the well discharge of this study agrees with that of constant-head pumping test in Cassiani et al. (1999) in an aquifer of semi-infinite thickness. In other words, if the aquifer thickness is

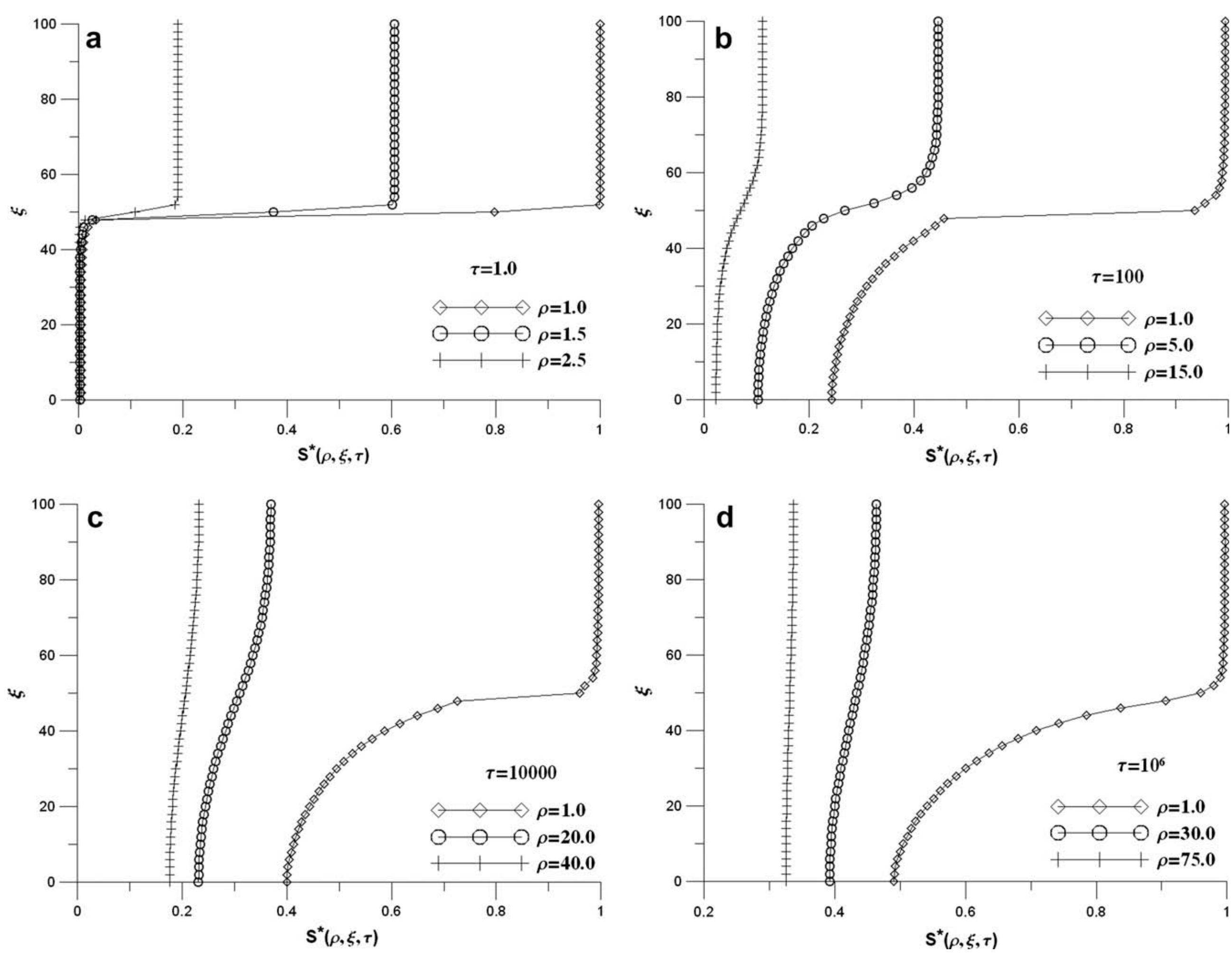

Fig. 2. The drawdown distribution at dimensionless time $\tau = 1, 100, 10^4$ and $\tau = 10^6$ for different ρ.

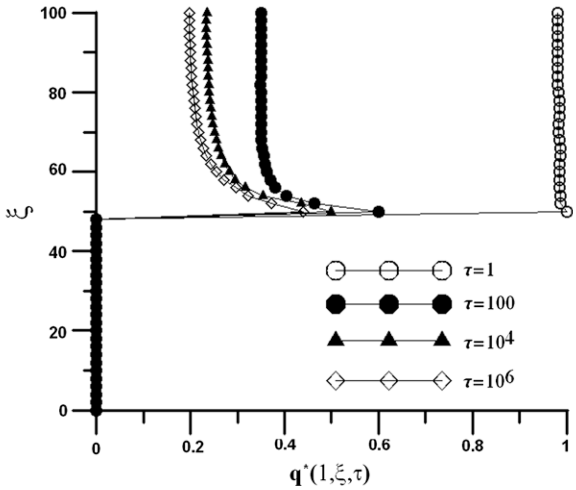

Fig. 3. The distribution of flux along the well screen at different dimensionless time.

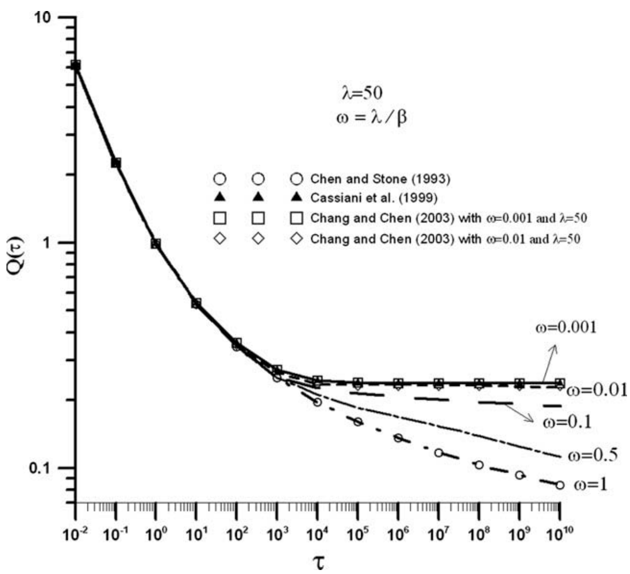

Fig. 4. The influence of the penetration ratio on the flux.

greater than 100 times of the screen length, the aquifer can be considered as semi-infinite. As the penetration ratio equals unity, the well discharge of this study is identical to that of Chen and Stone (1993) for a fully penetrating well. In addition, well discharges of this study agree with those of Chang and Chen (2003) for $\omega = 0.01$ and $\omega = 0.001$ when $\lambda = 50$. As indicated in Fig. 4, there are no obvious differences between the well discharges for different penetration ratios until $\tau = 10^4$. For the cases of $\omega = 0.1$ and $\omega = 0.01$, the flow caused by the partial penetrating well has not reached the bottom of the aquifer before $\tau = 10^4$. The aquifer thickness has an influence on groundwater flow after τ is greater than 10^4. The well discharge for $\omega = 0.01$ becomes steady as τ increases to 10^6 and the well discharge for $\omega = 0.5$ continues to decrease.

Conclusions

This paper developed a new semi-analytical solution for the aquifer system in response to the constant-head test at a partially penetrating well in a confined aquifer of infinite radial extent and finite vertical extent. The Laplace and finite cosine Fourier transforms is first used to reduce the original partial differential equation with mixed-type boundary and initial conditions for a partially penetrating well in an aquifer of finite thickness to the dual series equations. The dual series equations are then solved via the perturbation method.

The present solutions for a fully penetrating well in an aquifer of finite thickness are identical to the solutions of the drawdown and well discharge given in Chen and Stone (1993). It is found that the solution of Cassiani et al. (1999) for well response to a constant-head pumping test in a semi-infinite aquifer approximates the solution for the case where the aquifer thickness of a finite aquifer is 100 times greater than the length of well screen. In addition, the flux is non-uniformly distributed along the screen and with a local peak at the edge, due to the vertical flow induced by well partial penetration.

The new semi-analytical solutions provide accurate description of the response of the aquifer system to a constant-head pumping test performed at a partially penetrating well in a confined aquifer of infinite radial extent and finite vertical extent. In addition, those solutions are particularly attractive for practical applications since they can be used to evaluate the sensitivities of the input parameters in a mathematical model, to identify the hydraulic parameters if coupling with an optimization approach in the analysis of aquifer data, and to validate a numerical solution.

Acknowledgements

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Appendix

The Laplace and finite cosine Fourier transforms are first used to solve the mixed-type boundary value problem. The definition of Laplace transform is (Sneddon, 1972):

$$\bar{s}^*(\rho,\xi,p) = L_p[s^*(\rho,\xi,\tau);\tau \to p] = \int_0^\infty s^*(\rho,\xi,\tau)e^{-p\tau}d\tau \tag{A1}$$

where $\bar{s}^*(\rho,\xi,p)$ is the dimensionless drawdown in Laplace domain. Taking the Laplace transform of Eqs. (6) and (8)–(10) and using the initial condition in Eq. (7), the problem reads:

$$\frac{\partial^2\bar{s}^*}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{s}^*}{\partial\rho} + \alpha^2\frac{\partial^2\bar{s}^*}{\partial\xi^2} - p\bar{s}^* = 0 \tag{A2}$$

$$\bar{s}^*(\rho=\infty,\xi,p) = 0 \tag{A3}$$

$$\bar{s}^*(\rho=1,\xi,p) = \frac{1}{p}, \quad \xi_d \leqslant \xi \leqslant \beta \tag{A4a}$$

$$\left.\frac{\partial\bar{s}^*}{\partial\rho}\right|_{\rho=1} = 0, \quad 0 \leqslant \xi \leqslant \xi_d \tag{A4b}$$

$$\frac{\partial\bar{s}^*}{\partial\xi} = 0, \quad \xi = 0, \xi = \beta \tag{A5}$$

In order to eliminate the ξ coordinate, the finite cosine Fourier transform is used as follows (Sneddon, 1972):

$$\hat{\bar{s}}^*(\rho,n,p) = F_c[\bar{s}^*(\rho,\xi,p);\xi \to n] = \int_0^\beta \bar{s}^*(\rho,\xi,p)\cos\left(\frac{n\pi\xi}{\beta}\right)d\xi \tag{A6}$$

where $\hat{\bar{s}}^*(\rho,n,p)$ is the dimensionless drawdown after finite cosine Fourier transform. Substituting Eq. (A6) into Eqs. (A2), (A3) and (A5) results in the Bessel differential equation as

$$\frac{\partial^2\hat{\bar{s}}^*}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\hat{\bar{s}}^*}{\partial\rho}-\lambda_n^2\hat{\bar{s}}^*=0 \tag{A7}$$

with the boundary condition

$$\hat{\bar{s}}^*(\rho=\infty,n,p)=0 \tag{A8}$$

where λ_n is defined in Eq. (25).

The general solution of Eq. (A7) with the boundary condition Eq. (A8) is (Carslaw and Jaeger, 1959, p. 193)

$$\hat{\bar{s}}^*(\rho,n,p)=A(n,p)K_0(\lambda_n\rho) \tag{A9}$$

where $A(n,p)$ can be found from using the mixed-type boundary condition Eq. (A4). The inverse of the finite cosine Fourier transform is (Sneddon, 1972, p. 425)

$$\bar{s}^*(\rho,\xi,p)=\frac{1}{\beta}\hat{\bar{s}}^*(\rho,0,p)+\frac{2}{\beta}\sum_{n=1}^{\infty}\hat{\bar{s}}^*(\rho,n,p)\cos\left(\frac{n\pi\xi}{\beta}\right) \tag{A10}$$

Thus, the solution in ξ domain obtained by inserting Eq. (A9) into Eq. (A10) is

$$\bar{s}^*(\rho,\xi,p)=\frac{1}{\beta}A(0,p)K_0(\sqrt{p}\rho)+\frac{2}{\beta}\sum_{n=1}^{\infty}A(n,p)K_0(\lambda_n\rho)\times\cos\left(\frac{n\pi\xi}{\beta}\right) \tag{A11}$$

with its derivative with respect to ρ given by

$$\frac{\partial\bar{s}^*}{\partial\rho}(\rho,\xi,p)=-\frac{1}{\beta}A(0,p)\sqrt{p}K_1(\sqrt{p}\rho)-\frac{2}{\beta}\times\sum_{n=1}^{\infty}A(n,p)\lambda_nK_1(\lambda_n\rho)\cos\left(\frac{n\pi\xi}{\beta}\right) \tag{A12}$$

Substituting Eq. (A11) into Eq. (A4a) and Eq. (A12) into Eq. (A4b) results in a system of the dual series equations (DSE)

$$\frac{1}{\beta}A(0,p)K_0(\sqrt{p})+\frac{2}{\beta}\sum_{n=1}^{\infty}A(n,p)K_0(\lambda_n)\cos\left(\frac{n\pi\xi}{\beta}\right)=\frac{1}{p},\quad \xi_d\leqslant\xi\leqslant\beta \tag{A13a}$$

$$\frac{1}{\beta}A(0,p)\sqrt{p}K_1(\sqrt{p})+\frac{2}{\beta}\sum_{n=1}^{\infty}A(n,p)\lambda_nK_1(\lambda_n)\cos\left(\frac{n\pi\xi}{\beta}\right)=0,\quad 0\leqslant\xi\leqslant\xi_d \tag{A13b}$$

We define that

$$B(n,p)=2A(n,p)K_0(\lambda_n)/\beta \tag{A14}$$

The DSE of (A13) can be arranged as (Sneddon, 1966, p. 161):

$$\frac{1}{2}B(0,p)+\sum_{n=1}^{\infty}B(n,p)\cos(nx)=\frac{1}{p},\quad \frac{\xi_d}{\beta}\pi<x\leqslant\pi \tag{A15a}$$

$$\frac{1}{2}B(0,p)\sqrt{p}H(0,p)+\sum_{n=1}^{\infty}nB(n,p)\cos(nx)=\sum_{k=1}^{\infty}I(k,p)B(k,p)\cos(kx),\quad 0\leqslant x\leqslant\frac{\xi_d}{\beta}\pi \tag{A15b}$$

Our goal is to determine the coefficients $B(0,p)$ and $B(n,p)$ appearing in Eq. (A15). The pair of DSE (A15) when $I(k,p)=0$ can be expressed as

$$\frac{1}{2}{}^{(0)}B(0,p)+\sum_{n=1}^{\infty}{}^{(0)}B(n,p)\cos(nx)=\frac{1}{p},\quad \frac{\xi_d}{\beta}\pi<x\leqslant\pi \tag{A16a}$$

$$\frac{1}{2}{}^{(0)}B(0,p)\sqrt{p}H(0,p)+\sum_{n=1}^{\infty}n\,{}^{(0)}B(n,p)\cos(nx)=0,\quad 0\leqslant x\leqslant\frac{\xi_d}{\beta}\pi \tag{A16b}$$

The coefficients of ${}^{(0)}B(0,p)$ and ${}^{(0)}B(n,p)$ can be determined by following the process given in Sneddon (1966). Assume that when $0\leqslant x\leqslant\xi_d\pi/\beta$

$$\frac{1}{2}{}^{(0)}B(0,p)+\sum_{n=1}^{\infty}{}^{(0)}B(n,p)\cos(nx)=\cos\left(\frac{x}{2}\right)\int_x^{\frac{\xi_d}{\beta}\pi}\frac{h_1(y)dy}{\sqrt{\cos x-\cos y}} \tag{A17}$$

The coefficient ${}^{(0)}B(0,p)$ and ${}^{(0)}B(n,p)$ in Eq. (A17) are, respectively, given by the equations (Sneddon, 1966, p. 161, Eqs. (5.4.56) and (5.4.57))

$$^{(0)}B(0,p)=\frac{2}{\pi}\left[\frac{\pi}{\sqrt{2}}\int_0^{\frac{\xi_d}{\beta}\pi}h_1(y)dy+\int_{\frac{\xi_d}{\beta}\pi}^{\pi}\frac{1}{p}dy\right] \tag{A18}$$

$$^{(0)}B(n,p)=\frac{2}{\pi}\left\{\frac{\pi}{2\sqrt{2}}\int_0^{\frac{\xi_d}{\beta}\pi}h_1(y)[P_n(\cos y)+P_{n-1}(\cos y)]dy+\int_{\frac{\xi_d}{\beta}\pi}^{\pi}\frac{1}{p}\cos(ny)dy\right\} \tag{A19}$$

where $P_n(\cos u)$ is the associated Legendre function.

Substituting Eq. (A19) for the coefficients in the integrated equivalent of Eq. (A16b) obtains

$$\frac{1}{2}{}^{(0)}B(0,p)\sqrt{p}H(0,p)x+\sum_{n=1}^{\infty}{}^{(0)}B(n,p)\sin(nx)=0 \tag{A20}$$

We can find that $h_1(y)$ satisfies the following equation: (Sneddon, 1966, p. 161, Eq. (5.4.58))

$$\int_0^{\frac{\xi_d}{\beta}\pi}h_1(y)\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}[P_n(\cos y)+P_{n-1}(\cos y)]\sin nxdy=-\frac{1}{2}\sqrt{p}H(0,p)B(0,p)x-\frac{2}{\pi}\sum_{n=1}^{\infty}\int_{\frac{\xi_d}{\beta}\pi}^{\pi}\frac{1}{p}\cos(ny)dy\sin nx \tag{A21}$$

The summation term on the left-hand side of Eq. (A21) can be expressed as (Sneddon, 1966, p. 59, Eq. (2.6.31))

$$\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}[P_n(\cos y)+P_{n-1}(\cos y)]\sin nx=\frac{\cos\left(\frac{x}{2}\right)H(x-y)}{\sqrt{\cos y-\cos x}} \tag{A22}$$

where $H(X)$ is the Heaviside unit step function which has different value for different range of X as

$$H(X)=\begin{cases}0 & X<0\\ 1/2 & X=0\\ 1 & X>0\end{cases} \tag{A23}$$

Substituting Eq. (A22) into Eq. (A21) one gets

$$\int_0^{\frac{\xi_d}{\beta}\pi}\frac{h_1(y)H(x-y)}{\sqrt{\cos y-\cos x}}dy=\sec\frac{x}{2}\left\{-\frac{1}{2}\sqrt{p}H(0,p)^{(0)}B(0,p)x-\frac{2}{\pi}\sum_{n=1}^{\infty}\int_{\xi_d\pi/\beta}^{\pi}\frac{1}{p}\cos(ny)dy\sin nx\right\} \tag{A24}$$

Using the property of Heaviside unit step function in Eq. (A23) an equivalent integral equation of Eq. (A24) can be obtained

$$\int_0^x \frac{h_1(y)}{\sqrt{\cos y - \cos x}}dy = \sec\frac{x}{2}\left\{-\frac{1}{2}\sqrt{p}H(0,p)^{(0)}B(0,p)x - \frac{2}{\pi}\sum_{n=1}^{\infty}\int_{\xi_d\pi/\beta}^{\pi}\frac{1}{p}\cos(ny)dy\sin nx\right\} \quad 0 \leqslant x < \frac{\xi_d}{\beta}\pi \tag{A25}$$

Then, the function $h_1(y)$ can be obtained based on Sneddon (1966, p. 41, Eq. (2.3.5)) as

$$h_1(y) = -\frac{1}{\pi}\sqrt{p}H(0,p)^{(0)}B(0,p)\frac{d}{dy}\int_0^y \frac{x\sin(x/2)}{\sqrt{\cos x - \cos y}}dx + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{pn}\sin\left(n\frac{\xi_d\pi}{\beta}\right)\frac{d}{dy}\int_0^y \frac{\sin(x/2)\sin(nx)}{\sqrt{\cos x - \cos y}}dx \tag{A26}$$

By integrating Eq. (A26) and substituting it into Eqs. (A18) and (A19), the coefficients $B(0,p)$ and $B(n,p)$ can be expressed as Eqs. (14) and (15), respectively.

Based on the perturbation method (Boridy, 1990), one needs to find the coefficients of the following DSE when $I(k,p)\neq 0$

$$\frac{1}{2}B'(0,p) + \sum_{n=1}^{\infty}B'(n,p)\cos(nx) = 0, \quad \frac{\xi_d}{\beta}\pi < x \leqslant \pi \tag{A27a}$$

$$\frac{1}{2}B'(0,p)\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}nB'(n,p)\cos(nx) = \sum_{k=1}^{\infty}I(k,p)B(k,p)\cos(kx), \quad 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \tag{A27b}$$

The DSE above can be further separated into infinite pairs of DSE as (Boridy, 1990, Eqs. (11) and (12))

$$\begin{cases} \frac{1}{2}B'(0,p) + \sum_{n=1}^{\infty}B'(n,p)\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}B'(0,p)\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}nB'(n,p)\cos(nx) = I(1,p)B(1,p)\cos(x) \\ 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases}$$
$$\begin{cases} \frac{1}{2}B'(0,p) + \sum_{n=1}^{\infty}B'(n,p)\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}B'(0,p)\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}nB'(n,p)\cos(nx) = I(2,p)B(2,p)\cos(2x) \\ 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases}$$
$$\vdots$$
$$\begin{cases} \frac{1}{2}B'(0,p) + \sum_{n=1}^{\infty}B'(n,p)\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}B'(0,p)\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}nB'(n,p)\cos(nx) = I(k,p)B(k,p)\cos(kx) \\ 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases} \tag{A28}$$

Rearranging Eq. (A28) obtains

$$\begin{cases} \frac{1}{2}\frac{B'(0,p)}{I(1,p)B(1,p)} + \sum_{n=1}^{\infty}\frac{B'(n,p)}{I(1,p)B(1,p)}\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}\frac{B'(0,p)}{I(1,p)B(1,p)}\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}n\frac{B'(n,p)}{I(1,p)B(1,p)}\cos(nx) = \cos(x) & 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases}$$
$$\begin{cases} \frac{1}{2}\frac{B'(0,p)}{I(2,p)B(2,p)} + \sum_{n=1}^{\infty}\frac{B'(n,p)}{I(2,p)B(2,p)}\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}\frac{B'(0,p)}{I(2,p)B(2,p)}\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}n\frac{B'(n,p)}{I(2,p)B(2,p)}\cos(nx) = \cos(2x) & 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases}$$
$$\vdots$$
$$\begin{cases} \frac{1}{2}\frac{B'(0,p)}{I(k,p)B(k,p)} + \sum_{n=1}^{\infty}\frac{B'(n,p)}{I(k,p)B(k,p)}\cos(nx) = 0 & \frac{\xi_d}{\beta}\pi < x \leqslant \pi \\ \frac{1}{2}\frac{B'(0,p)}{I(k,p)B(k,p)}\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}n\frac{B'(n,p)}{I(k,p)B(k,p)}\cos(nx) = \cos(kx) & 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \end{cases} \tag{A29}$$

Eq. (A29) can be expressed as a general pair of DSE by defining new coefficients $C(0,k,p)$ and $C(n,k,p)$ as

$$\frac{1}{2}C(0,k,p) + \sum_{n=1}^{\infty}C(n,k,p)\cos(nx) = 0, \quad \frac{\xi_d}{\beta}\pi < x \leqslant \pi \tag{A30a}$$

$$\frac{1}{2}C(0,k,p)\sqrt{p}H(0,p) + \sum_{n=1}^{\infty}nC(n,k,p)\cos(nx) = \cos(kx), \quad 0 \leqslant x \leqslant \frac{\xi_d}{\beta}\pi \tag{A30b}$$

where

$$C(0,k,p) = \frac{B'(0,p)}{I(k,p)B(k,p)} \tag{A31}$$

and

$$C(n,k,p) = \frac{B'(n,p)}{I(k,p)B(k,p)} \tag{A32}$$

Thus the coefficients $B(0,p)$ and $B(n,p)$ when $I_n \neq 0$ in Eq. (A15) can be, respectively, written as (Boridy, 1990, Eq. (13))

$$B(0,p) = {}^{(0)}B(0,p) + \sum_{k=1}^{\infty}I(k,p)B(k,p)C(0,k,p) \tag{A33}$$

and

$$B(n,p) = {}^{(0)}B(n,p) + \sum_{k=1}^{\infty}I(k,p)B(k,p)C(n,k,p) \tag{A34}$$

Consequently, the coefficients of $C(0,k,p)$ and $C(n,k,p)$ are, respectively, given by (Sneddon, 1966, p. 161, Eqs. (5.4.56) and (5.4.57))

$$C(0,k,p) = \frac{\frac{1}{k}\frac{2\sqrt{2}}{\pi}\int_0^{\frac{\xi_d}{\beta}\pi}f_3(u,k)du}{1 + \frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)\int_0^{\frac{\xi_d}{\beta}\pi}f_2(u)du} \tag{A35}$$

and

$$\begin{aligned} C(n,k,p) = {} & \frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)C(0,k,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\left[\int_0^u f_2(v)dv\frac{d}{du}f_4(u)\right]du\right\} \\ & - \frac{\sqrt{2}}{\pi}\sqrt{p}H(0,p)C(0,k,p)\left\{\int_0^{\frac{\xi_d}{\beta}\pi}f_2(u)duf_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & + \frac{\sqrt{2}}{\pi}\frac{1}{k}\left\{\int_0^{\frac{\xi_d}{\beta}\pi}f_3(u,k)duf_4\left(\frac{\xi_d}{\beta}\pi\right)\right\} \\ & - \frac{\sqrt{2}}{\pi}\frac{1}{k}\left\{\int_0^{\frac{\xi_d}{\beta}\pi}\int_0^u f_3(v,k)dv\frac{d}{du}f_4(u)du\right\} \end{aligned} \tag{A36}$$

where the function f_3 is defined in (20).

Series in Eqs. (A33) and (A34) can be considered as a perturbation series in $I(n,p)$. In the zeroth-order approximation, the second terms on the right-hand side of Eqs. (A33) and (A34) are ignored so that coefficients $B(0,p)$ and $B(n,p)$ are simply given by ${}^{(0)}B(0,p)$ and ${}^{(0)}B(n,p)$. In the first-order approximation $B(0,p)$ and $B(n,p)$ of the second member of Eqs. (A33) and (A34) are replaced by ${}^{(0)}B(0,p)$ and ${}^{(0)}B(n,p)$, respectively. Thus, in this approximation, coefficients $B(0,p)$ and $B(n,p)$ are denoted by ${}^{(1)}B(0,p)$ and ${}^{(1)}B(n,p)$, respectively, as (Boridy, 1990, Eq. (16))

$${}^{(1)}B(0,p) = {}^{(0)}B(0,p) + \sum_{k=1}^{\infty}I(k,p)^{(0)}B(k,p)C(0,k,p) \tag{A37}$$

and

$${}^{(1)}B(n,p) = {}^{(0)}B(n,p) + \sum_{k=1}^{\infty}I(k,p)^{(0)}B(k,p)C(n,k,p) \tag{A38}$$

Based on Eqs. (A11) and (A14), the solution for dimensionless drawdown in Laplace domain can be written as Eq. (11).

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