國立交通大學

Palindromic Quadratization and Structure-Preserving Algorithm for Palindromic Matrix Polynomials

保結構迴紋式二次化求解迴紋式方程

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摘 要

本文探討的主題是開發和利用高效率迴紋二次化的方法來解決結構迴紋多項式特徵值的問題。第 1 章將簡要介紹一些基本概念，數學符號和一般所謂的『迴紋多項式特徵值問題』。在第 2 章中，起源於德國的高速列車之計算與振動分析和表面聲波濾波器的研究，我們探討和分析高效率的方法去解決多項式特徵值問題。在第 3 章中，我們將提出一個保結構解決方法去對付奇次的迴紋矩陣多項式以及展示明確的遞迴係數矩陣。最後，這篇論文的結論和未來的工作將在第 4 章討論。

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Introduction and Preliminaries

Contents

The theme explored in this thesis is to develop and exploit efficient palindromic quadratization methods to solve the structure-palindromic polynomial eigenvalue problem. This chapter will briefly introduce some basic notions, mathematical notations and conventional methods of the so-called "palindromic polynomial eigenvalue problems". In Chapter 2, we investigate and analyze efficient methods for polynomial eigenvalue problems arising from computing in the vibration analysis for fast trains in Germany [20, 21] and then in the study of surface acoustic wave filters [65]. In Chapter 3, we will propose a structured-preserving method of palindromic matrix polynomial of odd degree and show the explicitly recursive coefficient matrices. Finally, conclusions and the future work of this thesis will be discussed in Chapter 4.

1.1 The Arnoldi Method for Standard Eigenvalue Problems

Given a large sparse matrix $A \in \mathbb{C}^{n\times n}$, the Arnoldi method [30] is prevalent and very widespread algorithm for solving the so-called standard eigenvalue problem (SEP):

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{1.1}$$

Then, to find a scalar λ (real or complex) and a nonzero n-vector $\mathbf{x}$ which satisfy the equation (1.1). In this case, we say that λ is an eigenvalue of A and $\mathbf{x}$ is called an eigenvector of A with respect to λ. Moreover, the eigenpair of A can be define as the pair $(\lambda, \mathbf{x})$.

Before we resolve the SEP, we define some notations below. The following notations are frequently used in this thesis. Some Other notations will be clearly defined whenever they are used.

- $\mathtt{i} = \sqrt{-1}$.

- $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.
- $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$, respectively, denote the real part and the complex part of the scalar $\lambda \in \mathbb{C}$.
- $\mathbb{C}^{n\times m}$ is the set of all $n \times m$ complex matrices, $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ and $\mathbb{C} = \mathbb{C}^1$.
- The direct sum of two matrices is denoted by "$\oplus$".
- $\mathbf{0}$ denotes zero vectors and matrices with appropriate size.
- I_n denotes the $n \times n$ identity matrix.
- We use $\cdot^\top$ and $\cdot^H$ to denote the transpose and conjugate transpose for vectors or matrices.
- $\otimes$ denotes the Kronecker product.
- We adopt the following MATLAB notations:

  $\mathbf{v}(i : j)$ denotes the subvector of the vector $\mathbf{v}$ that consists of the ith to the jth entries of $\mathbf{v}$;

  $A(i : j, k : \ell)$ denotes the submatrix of the matrix A that consists of the intersection of the rows i to j and the columns k to ℓ;

  $A(i : j, :)$ denotes the rows of A from i to j and $A(:, k : \ell)$ denotes the columns of A from k to ℓ.

Let us come back to consider the Arnoldi method now. Beginning from a unit vector $\mathbf{v}_1$, the Arnoldi method successively constructs a sequence of unitary vectors $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m$ which creates an unitary basis of the Krylov subspace $\mathcal{K}_m(A, \mathbf{v}_1) \equiv$

$\text{span}\{\mathbf{v}_1, A\mathbf{v}_1, \ldots, A^{m-1}\mathbf{v}_1\}$ with $m \ll n$ such that

$$\begin{cases} h_{j+1,j}\mathbf{v}_{j+1} = A\mathbf{v}_j - \sum_{i=1}^{j} h_{ij}\mathbf{v}_i, \quad j = 1, 2, \ldots, m, \\ \mathbf{v}_s^H \mathbf{v}_t = 0, \ \forall s \neq t \quad \text{and} \quad \mathbf{v}_s^H \mathbf{v}_s = 1, \ \forall s, \end{cases}$$

or we can rewrite to

$$\begin{cases} AV_m = V_m H_m + h_{m,m+1}\mathbf{v}_{m+1}\mathbf{e}_m^\top, \\ \begin{bmatrix} V_m^H \\ \mathbf{v}_{m+1}^H \end{bmatrix} [V_m \ \mathbf{v}_{m+1}] = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \end{cases} \tag{1.2}$$

where V_m is an $n \times m$ matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and H_m is an $m \times m$ upper Hessenberg matrix. Processing the factorization (1.2), called the Arnoldi decomposition, we can diminish A into the upper Hessenberg H_m by using the unitary transformation $V_m^H A V_m = H_m$. The eigenpairs of the reduced SEP $H_m\mathbf{z} = \mu\mathbf{z}$ can be solved by the classical eigenvalue techniques, for instance the QR algorithm [13, 14]. Besides, we see that if $(\theta, \mathbf{y})$ is an eigenpair of H_m then $(\theta, V_m\mathbf{y})$ is so-called a Ritz pair of A that is an approximate eigenpair of A with the residual norm

$$\|(A - \theta I_n)V_m\mathbf{y}\| = |h_{m+1,m}||\mathbf{e}_m^\top \mathbf{y}|.$$

Moreover, we refer to [11, 49, 59] for more details on the practical realization and theoretical analysis of the Arnoldi method.

1.2 The Generalized Arnoldi Method for Generalized Eigenvalue Problem

The generalized eigenvalue problem (GEP) for the matrix pencil $A - \lambda B$ of two square matrices A and B with size n is to determine scalars λ and n-vectors $\mathbf{x} \neq \mathbf{0}$

such that

$$A\mathbf{x} = \lambda B\mathbf{x}. \tag{1.3}$$

If B is nonsingular, the GEP (1.3) can be transformed into SEPs

$$(B^{-1}A)\mathbf{x} = \lambda\mathbf{x} \tag{1.4}$$

or

$$(AB^{-1})\mathbf{y} = \lambda\mathbf{y}, \quad \mathbf{y} = B\mathbf{x} \tag{1.5}$$

which following can be solved by the standard Arnoldi method. In addition, the QZ algorithm [39] present an analog of the QR algorithm for the GEP that is also a popular choice for dealing with the GEP (1.3) with small dense coefficient matrices.

However, when we are faced with large-scale GEPs, some pioneers came up with some solutions. Sorensen [50] proposed the truncated QZ method to approach the eigenpairs. For $m \ll n$, this method constructs a generalization of the standard Arnoldi decomposition (1.2),

$$\begin{cases} AZ_m = Y_m H_m + h_{m+1,m}\mathbf{y}_{m+1}\mathbf{e}_m^\top, \\ BZ_m = Y_m R_m, \\ Z_m^H Z_m = I_m, \; Y_m^H Y_m = I_m, \; Y_m^H \mathbf{y}_{m+1} = \mathbf{0}, \end{cases} \tag{1.6}$$

which is called the generalized Arnoldi reduction in [50]. It is efficient and artfully processing the small-sized GEP $H_m\mathbf{v} = \mu R_m\mathbf{v}$ of the $m \times m$ upper Hessenberg-triangular pair (H_m, R_m) to approximate eigenpairs of the original large-scale GEP (1.3).

1.3 Palindromic Eigenvalue Problem and Linearization

We consider the palindromic quadratic eigenvalue problem (PQEP) of the form

$$\mathcal{P}(\lambda) \equiv Q(\lambda)\mathbf{x} \equiv (\lambda^2 A_1^\top + \lambda A_0 + A_1)\mathbf{x} = \mathbf{0}, \tag{1.7}$$

where $\lambda \in \mathbb{C}$, $x \in \mathbb{C}\backslash\{0\}$ and $A_1, A_0 \in \mathbb{C}^{n\times n}$ with $A_0^\top = A_0$. The scalar λ and the nonzero vector x in (1.7) are the eigenvalue and the associated eigenvector of $\mathcal{P}(\lambda)$, respectively. A palindrome is a word or phrase which reads the same in both directions. The matrix polynomial $\mathcal{P}(\lambda)$ also has the property that reversing the order of the coefficients, then continued by taking the transpose, which leads back to the original matrix polynomial. Therefore, we can find that the eigenvalues of $\mathcal{P}(\lambda)$ satisfy the “symplectic" property by taking the transpose of (1.7). The word symplectic is, the eigenvalues λ and $1/\lambda$ both exist with respect to the unit circle (with 0 and ∞ considered to be reciprocal).

The “linearization" is a typical and frequently used technique to solve the (PQEP) in which the problem is reformulated into a linear one which doubles the order of the system. We select suitable matrices $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{2n\times 2n}$ and the vector $\boldsymbol{\varphi} \in \mathbb{C}^{2n}$ and transform (1.7) into the (GEP)

$$(\mathcal{A} - \lambda\mathcal{B})\boldsymbol{\varphi} = \mathbf{0} \tag{1.8}$$

satisfying the relation

$$\mathcal{E}(\lambda)(\mathcal{A} - \lambda\mathcal{B})\mathcal{F}(\lambda) = \begin{bmatrix} \mathcal{P}(\lambda) & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix},$$

where $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ are $2n \times 2n$ matrix polynomials in λ with constant nonzero determinants. In this case,

$$\det(\mathcal{A} - \lambda\mathcal{B}) = \det(\lambda^2 A_1^\top + \lambda A_0 + A_1)$$

shows that the eigenvalues of the original PQEP (1.7) are simultaneous with the eigenvalues of the enlarged GEP (1.8). Consequently, the linearization technique of PQEPs makes classical methods for GEPs as well as SEPs can be used.

There are many choices of $(\mathcal{A}, \mathcal{B})$'s, but probably the most famous ones in practice are the companion forms [16]: the first companion form

$$\mathcal{A} = \begin{bmatrix} -A_0 & -A_1 \\ I_n & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} A_1^\top & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$$

as well as the second companion form

$$\mathcal{A} = \begin{bmatrix} -A_0 & I_n \\ -A_1 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} A_1^\top & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}. \tag{1.9}$$

However, there are some drawbacks of the linearization technique to solve PQEPs. For example, it doubles the size of the problem dimension that increases the computational cost and the original structures of the coefficient matrices (A_0, A_1) such as palindromic and symplectic may be lost. In order to prevent these drawbacks, one may solve the PQEP directly to keep some original advantages. Methods of this type include the residual iteration method [22, 38, 44], the second-order Arnoldi method [1, 32, 62], the Jacobi-Davidson method [51, 52], the nonlinear Arnoldi method [60], a Krylov-type subspace method [31] and an iterated shift-and-invert Arnoldi method [64]. These methods use a similar projection process, but the selection of projection subspaces is the main difference between them.

In chapter 2, we propose a palindromic quadratization approach that transforms a palindromic matrix polynomial of even degree to a palindromic quadratic pencil. Based on the $(\mathcal{S}+\mathcal{S}^{-1})$-transform and Patel's algorithm, W.-W. Lin puts forward the structure-preserving algorithm [24] which can then be applied to solve the corresponding palindromic quadratic eigenvalue problem. Consequently, Numerical experiments show that the relative residuals for eigenpairs of palindromic polynomial eigenvalue problems computed by palindromic quadratized eigenvalue problems are better than those via palindromic linearized eigenvalue problems or `polyeig` in MATLAB. In chapter 3, We consider the structured factorization of a palindromic matrix polynomials of odd degree. To obtain such factorizations, there are some difficult nonlinear matrix equations that have to be solved. However, these equations are shown to be equivalent to the well known solvent equation, when the solution X is soluble. Without writing down the dreary NMEs, we provide a general version from the palindromic matrix polynomials of odd degree to a structure-preserving factorization by solving the solvent equation.

2

Palindromic Quadratization and Structure-Preserving Algorithm for Palindromic Matrix Polynomials of Even Degree

Contents

2.1 Introduction

In this section, we extend from PQEP(1.7) to $(\star, \varepsilon)$-palindromic matrix polynomials of even degree $2d$

$$\mathcal{P}(\lambda) \equiv \sum_{k=0}^{d-1} \lambda^{2d-k} A_{d-k}^{\star} + \lambda^{d} A_0 + \varepsilon \sum_{k=1}^{d} \lambda^{d-k} A_k, \tag{2.1}$$

where $d \geq 2$, $\varepsilon = \pm 1$ and $\star = \mathrm{H}$ (Hermitian) or T (transpose), $A_k \in \mathbb{C}^{n\times n}$ $(k = 0, 1, \ldots, d)$ and $A_0^{\star} = \varepsilon A_0$. The corresponding polynomial eigenvalue problem $\mathcal{P}(\lambda)x = 0$, with $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \backslash \{0\}$ being the eigenvalue and the associated eigenvector respectively, is called a $(\star, \varepsilon)$-palindromic polynomial eigenvalue problem ($(\star, \varepsilon)$-PPEP). The equation (2.1) is also called a $\star$-PPEP if $\varepsilon = 1$ or a $\star$-anti-PPEP if $\varepsilon = -1$.

The underlying matrix polynomial $\mathcal{P}(\lambda)$ in (2.1) has the property that reversing the order of coefficients, followed by taking the (conjugate) transpose, leads to the original matrix polynomial (anti-)invariant, which satisfies

$$\mathcal{P}(\lambda) = \varepsilon \lambda^{2d} \mathcal{P}(1/\lambda)^{\star} \tag{2.2}$$

and explains the world "(anti-)palindromic" [40]. Consequently, taking the (conjugate) transpose of (2.1), we easily see that the eigenvalues of $\mathcal{P}(\lambda)$ satisfy a "reciprocal" property, that is, they appear in the pairs of the form $(\lambda, 1/\lambda^{\star})$.

The $(\star, \varepsilon)$-PPEPs arise in solving higher order systems of ordinary or partial differential equations. In the beginning, a T-palindromic quadratic eigenvalue problem (T-PQEP) is raised in the vibration analysis for fast trains in Germany [20, 21] and then in the study of surface acoustic wave filters [65]. An H-palindromic quadratic eigenvalue problem (H-PQEP) arises in the computation of the Crawford number, for detecting definite Hermitian pairs or hyperbolic or elliptic quadratic eigenvalue problems [23]. Furthermore, a $\star$-PPEP of even degree is obtained by solving the linear quadratic discrete-time optimal control problem for higher order systems [2, 63].

A standard approach for computing the eigenpairs of $\mathcal{P}(\lambda)$ in (2.1) is to linearize it to a $2dn \times 2dn$ linear matrix pencil by the companion linearization and compute its generalized Schur form [55]. Nevertheless, the reciprocal property of eigenvalues of $\mathcal{P}(\lambda)$ is not preserved and this has resulted in large numerical errors [6, 24, 29]. Recently, in order to preserve the reciprocity of eigenvalues, some forerunner's works [40, 41] propose some good linearizations that linearize $\mathcal{P}(\lambda)$ to palindromic linear pencils of the form $\lambda Z^\star + Z$. This does lead to a great improvement over previous unstructured approaches, keeping the palindromic structure in the original polynomial and enabling structure-preserving numerical methods to be designed. Later, a QR-like algorithm [53] and a hybrid method [42] which combines Jacobi-type method with the Laub's trick, a postprocessing step of the generalized Schur form, are proposed for solving T-palindromic linear eigenvalue problems efficiently. The QR-like algorithm typically requires $O(n^4)$ flops and the hybrid method requires $O(n^3 log(n))$ flops. Recently, a URV-decomposition based structured method of cubic complexity is developed in [54] to solve T-palindromic linear eigenvalue problems, producing eigenvalues which are reciprocally paired to working precision. A new structure-preserving doubling algorithm with cubic complexity for solving $\star$-palindromic linear eigenvalue problems is developed in [7]. On the other hand, for solving a $(\star, \varepsilon)$-PQEP, a structure-preserving doubling algorithm is developed in [6, 8] via the computation of a solvent of a nonlinear matrix equation associated with the $(\star, \varepsilon)$-PQEP. Lately, a numerically stable structure-preserving algorithm (SPA), based on the $(\mathcal{S} + \mathcal{S}^{-1})$-transform [33] and Patel's algorithm [45], is proposed in [24] to solve the T-PQEP directly. The numerical results obtained by the SPA algorithm show much promise and the computational cost of SPA is about a half of that of the URV-based method.

The purpose of this section is to develop a palindromic quadratization which transforms a $(\star, \varepsilon)$-palindromic matrix polynomial of even degree with $(\star, \varepsilon) \neq (T, -1)$ into a $(\star, \varepsilon)$-palindromic quadratic pencil. If $\star = T$ and $\varepsilon = 1$, then we can apply the SPA algorithm in [24] to solve the associated quadratized T-PQEP directly. If $\star = H$ and $\varepsilon = \pm 1$, we first transform the associated quadratized (H, ε)-PQEP to an H-skew-Hamiltonian pencil by the $(\mathcal{S} + \mathcal{S}^{-1})$-transform and enlarge the H-skew-Hamiltonian pencil to a real skew-Hamiltonian pencil, to which the SPA algorithm is applicable. Note that for the case $(\star, \varepsilon) = (T, -1)$, the T-anti-PPEP can then be solved by applying the URV-based method [54, 58] to the linearized T-palindromic linear pencil.

This section is organized as follows. In Section 2.2, we propose a palindromic quadratization for a $(\star, \varepsilon)$-palindromic matrix polynomial of even degree. In Section 2.3, we develop a structure-preserving algorithm for solving the H-PQEPs. We consider the structured backward stability in Section 2.5. After that We develop balancing techniques for PPEPs and PQEPs in Section 2.5. Comparisons of numerical results computed by the palindromic quadratization, the palindromic linearization and the standard companion linearization are presented in Section 2.6. Conclusions are given in Section 2.7.

2.2 P-Quadratization of $(\star, \varepsilon)$-PPEP

In [24], a structure-preserving algorithm is well-developed for solving the T-PQEP. A similar structure-preserving algorithm for solving the H-PQEP will be introduced in Section 2.3. As the H-anti-PQEP can be easily transformed to the H-PQEP, all $(\star, \varepsilon)$-PQEPs with $(\star, \varepsilon) \neq (T, -1)$ can be solved by structure-preserving algorithms. Furthermore when $(\star, \varepsilon) \neq (T, -1)$, we shall propose a new palindromic quadratization (P-quadratization) which can be utilized to transform a $(\star, \varepsilon)$-PPEP into a $(\star, \varepsilon)$-PQEP so that the structure-preserving algorithm in [24] is applicable.

Next we present definitions of quadratization and P-quadratization of a general matrix polynomial and a palindromic matrix polynomial, respectively.

Definition 2.1. *(Quadratization/P-Quadratization)*

(i) Let $\mathcal{P}(\lambda)$ be an arbitrary $\nu \times \nu$ matrix polynomial of degree $p \geq 2$ with $p\nu = 2q$. A $q \times q$ quadratic matrix polynomial (quadratic pencil) $\mathcal{Q}(\lambda)$ is a quadratization of $\mathcal{P}(\lambda)$ if there are matrix rational functions $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ of size $q \times q$ with nonzero and constant determinants satisfying the two-sided factorization

$$\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \begin{bmatrix} \mathcal{P}(\lambda) & 0 \\ 0 & I_{q-\nu} \end{bmatrix}. \tag{2.3}$$

(ii) Let $\mathcal{P}(\lambda)$ be an arbitrary $\nu \times \nu$ $(\star, \varepsilon)$-palindromic matrix polynomial of degree $p \geq 2$ with $p\nu = 2q$ (i.e., $\mathcal{P}(\lambda) = \varepsilon\lambda^p\mathcal{P}(1/\lambda)^\star$ as in (2.2)). A quadratization $\mathcal{Q}(\lambda)$ of $\mathcal{P}(\lambda)$ having the $(\star, \varepsilon)$-palindromic structure is called a <u>P-quadratization</u> of $\mathcal{P}(\lambda)$.

Theorem 2.2. *Let $\mathcal{Q}(\lambda)$ be a $q \times q$ quadratization of a $\nu \times \nu$ matrix polynomial $\mathcal{P}(\lambda)$ of degree p with $p\nu = 2q$. Then*

(i) $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $\mathcal{Q}(\lambda)$ (i.e., det$(\mathcal{Q}(\lambda_0)) = 0$) if and only if λ_0 is a finite eigenvalue of $\mathcal{P}(\lambda)$ (i.e., det$(\mathcal{P}(\lambda_0)) = 0$).

(ii) ∞ is an eigenvalue of $\mathcal{Q}(\lambda)$ (i.e., det$\big([\lambda^2\mathcal{Q}(\frac{1}{\lambda})]\big|_{\lambda=0}\big) = 0$) if and only if ∞ is an eigenvalue of $\mathcal{P}(\lambda)$ (i.e., det$\big([\lambda^p\mathcal{P}(\frac{1}{\lambda})]\big|_{\lambda=0}\big) = 0$).

Proof. (i) The factorization (2.3) implies that $\det(\mathcal{Q}(\lambda)) = c \det(\mathcal{P}(\lambda))$ for some nonzero constant c, so that $\mathcal{Q}(\lambda)$ and $\mathcal{P}(\lambda)$ are singular or nonsingular for precisely the same values of λ_0.

(ii) Since

$$\begin{aligned}\det\left[\lambda^2 \mathcal{Q}\left(\frac{1}{\lambda}\right)\right] &= \lambda^{2q}\det\left[\mathcal{Q}\left(\frac{1}{\lambda}\right)\right] = c\lambda^{2q}\det\left[\mathcal{P}\left(\frac{1}{\lambda}\right)\right] \\ &= c\det\left[\lambda^p \mathcal{P}\left(\frac{1}{\lambda}\right)\right],\end{aligned}$$

both $\mathcal{Q}(\lambda)$ and $\mathcal{P}(\lambda)$ have or have no infinite eigenvalues. □

Since both $\det(\mathcal{E}(\lambda))$ and $\det(\mathcal{F}(\lambda))$ are nonzero and constant, it is easily seen that the two-sided factorization (2.3) implies the existence of a more wide class of one-sided factorization

$$\mathcal{Q}(\lambda)F(\lambda) \equiv \mathcal{Q}(\lambda)\mathcal{F}(\lambda)\begin{bmatrix} I_n \\ 0 \end{bmatrix} = \mathcal{E}(\lambda)^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}\mathcal{P}(\lambda) \equiv G(\lambda)\mathcal{P}(\lambda), \tag{2.4}$$

where $F(\lambda)$ and $G(\lambda)$ are matrix rational functions of size $q \times \nu$. From the factorization (2.4) a close connection between eigenpairs of $\mathcal{P}(\lambda)$ and eigenpairs of $\mathcal{Q}(\lambda)$ has been shown in [17].

Theorem 2.3. [17] *Assume that (2.4) holds at $\lambda_0 \in \mathbb{C}$ with $F(\lambda_0)$ and $G(\lambda_0)$ being of full column rank. Then $F(\lambda_0)z_1$ is an eigenvector of $\mathcal{Q}(\lambda)$ if and only if z_1 is an eigenvector of $\mathcal{P}(\lambda)$, both corresponding to eigenvalue λ_0.*

In Definition 2.1(i), we give a new definition of quadratization for a general matrix polynomial. In Theorems 2.2 and 2.3, we show the connection between eigenpairs of a general matrix polynomial and its quadratization. We next present a P-quadratization for a palindromic matrix polynomial of even degree explicitly.

Theorem 2.4. *Let $\mathcal{P}(\lambda)$ be an $n \times n$ $(\star, \varepsilon)$-palindromic matrix polynomial of degree $2d$ as in (2.1) with $(\star, \varepsilon) \neq (T, -1)$. Then $\mathcal{P}(\lambda)$ can be P-quadratized into a $(\star, \varepsilon)$-*

palindromic quadratic pencil of the form

$$\mathcal{Q}(\lambda) \equiv \lambda^2 \mathcal{A}_1^\star + \lambda \mathcal{A}_0 + \varepsilon \mathcal{A}_1 \tag{2.5}$$

with $\mathcal{A}_0^\star = \varepsilon \mathcal{A}_0$, *where*

(i) (For $2d = 4m$ *)* $\mathcal{A}_1$ *and* $\mathcal{A}_0$ *are given by*

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & d_1^\star I \\ \varepsilon\sqrt{\varepsilon} d_1 A_2 & 0 \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_2^\star A_2 & 0 \\ 0 & -\sqrt{\varepsilon} d_1 d_1^\star I \end{bmatrix} \tag{2.6}$$

if $m = 1$*; otherwise,*

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & \cdots & 0 & d_m^\star I \\ \varepsilon\sqrt{\varepsilon} d_1 A_{2m} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ A_{2m-1} & d_1^\star I & \ddots & & & & \vdots \\ 0 & & -\varepsilon d_2 I & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ A_3 & & & & d_{m-1}^\star I & 0 & \vdots \\ 0 & & & & & -\varepsilon d_m I & 0 \end{bmatrix}, \tag{2.7a}$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_{2m}^\star A_{2m} & 0 & A_{2m-2}^\star & 0 & \cdots & A_2^\star & 0 \\ 0 & -\sqrt{\varepsilon} d_1 d_1^\star I & & & & & \vdots \\ \varepsilon A_{2m-2} & & 0 & & & & \vdots \\ 0 & & & 0 & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \varepsilon A_2 & & & & & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{2.7b}$$

(ii) (For $2d = 4m+2$) $\mathcal{A}_1$ and $\mathcal{A}_0$ are given by

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & d_m^\star I \\ A_{2m+1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -\varepsilon d_1 I & \ddots & & & & & \vdots \\ A_{2m-1} & & d_1^\star I & \ddots & & & & \vdots \\ 0 & & & -\varepsilon d_2 I & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ A_3 & & & & & d_{m-1}^\star I & 0 & 0 \\ 0 & & & & & & -\varepsilon d_m I & 0 \end{bmatrix}, \tag{2.8a}$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & A_{2m}^\star & 0 & A_{2m-2}^\star & 0 & \cdots & A_2^\star & 0 \\ \varepsilon A_{2m} & 0 & & & & & & \vdots \\ 0 & & 0 & & & & & \vdots \\ \varepsilon A_{2m-2} & & & 0 & & & & \vdots \\ 0 & & & & 0 & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ \varepsilon A_2 & & & & & & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}; \tag{2.8b}$$

in which $d_1, \ldots, d_m$ are arbitrary nonzero constants.

Proof. (i) (For $2d = 4m$) We define the $n \times n$ matrix rational functions for $j = 1, \ldots, m$:

$$F_{2j,1}(\lambda) = \left(\lambda^{2j-1} d_j^\star\right)^{-1} \left(\varepsilon\sqrt{\varepsilon}\lambda^{2m} I_n + \sum_{k=0}^{2j-2} \lambda^k A_{2m-k}\right), \tag{2.9a}$$

and for $j = 1, \dots, m-1$:

$$F_{2j+1,1}(\lambda) = \lambda^{2m-2j} I_n. \tag{2.9b}$$

Let

$$F_1(\lambda) = \left[\begin{array}{c|cccc} I_n & F_{2,1}(\lambda)^T & F_{3,1}(\lambda)^T & \cdots & F_{2m,1}(\lambda)^T \\ \hline 0 & & & I_{(2m-1)n} & \end{array}\right]^T. \tag{2.10}$$

Routine but tedious calculation in terms of $F_1(\lambda)$ in (2.10), $\mathcal{Q}(\lambda)$ in (2.5) and $\mathcal{A}_0, \mathcal{A}_1$ in (2.6) or (2.7) leads to

$$\mathcal{Q}(\lambda) F_1(\lambda) \begin{bmatrix} I_n \\ 0_{(d-1)n,n} \end{bmatrix} = \begin{bmatrix} Q_1(\lambda) \\ 0_{(d-1)n,n} \end{bmatrix}, \tag{2.11}$$

where

$$\begin{aligned} Q_1(\lambda) &= \lambda^2 A_1^\star + \lambda \left(A_0 - \sqrt{\varepsilon} I - \sqrt{\varepsilon} A_{2m}^\star A_{2m}\right) + \varepsilon A_1 \\ &+ \lambda \sqrt{\varepsilon} A_{2m}^\star \left(\lambda^{2m} \varepsilon \sqrt{\varepsilon} I + A_{2m}\right) + \sum_{k=1}^{m-1} \lambda^{2m-2k} \left(\lambda^2 A_{2m-2k+1}^\star + \lambda A_{2m-2k}^\star\right) \\ &+ \varepsilon \lambda^{1-2m} \left(\varepsilon \sqrt{\varepsilon} \lambda^{2m} I + \sum_{k=0}^{2m-2} \lambda^k A_{2m-k}\right) = \lambda^{1-2m} \mathcal{P}(\lambda). \end{aligned} \tag{2.12}$$

Next, we define for $j = 1, \dots, m$:

$$\begin{aligned} E_{1,2j+1}(\lambda) &= \lambda^{2j-2m} I_n, \quad \text{for } j = 1, \dots, m-1, \\ E_{1,2j}(\lambda) &= \left(\lambda^{2m-2j+1} d_j\right)^{-1} \left(\sum_{k=0}^{2j-2} \lambda^{2m-k} A_{2m-k}^\star + \sqrt{\varepsilon} I_n\right), \end{aligned}$$

and let

$$E_1(\lambda) = \left[\begin{array}{c|cccc} I_n & E_{1,2}(\lambda) & E_{1,3}(\lambda) & \cdots & E_{1,2m}(\lambda) \\ \hline 0 & \multicolumn{4}{c}{I_{(2m-1)n}} \end{array}\right]. \tag{2.13}$$

From (2.12) and the definition of $E_1(\lambda)$ in (2.13), we have

$$\begin{bmatrix} I_n & 0_{n,(d-1)n} \end{bmatrix} E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda) = \begin{bmatrix} \lambda^{1-2m}\mathcal{P}(\lambda) & 0_{n,(d-1)n} \end{bmatrix}.$$

Then from (2.10) and (2.13), it follows for $m = 1$ that

$$\begin{bmatrix} 0 & I_n \end{bmatrix} E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda) \begin{bmatrix} 0 \\ I_n \end{bmatrix} = -\lambda\sqrt{\varepsilon} d_1 d_1^\star I_n;$$

or for $m \geq 2$:

$$\begin{bmatrix} 0 & I_{(d-1)n} \end{bmatrix} E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda) \begin{bmatrix} 0 \\ I_{(d-1)n} \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda\sqrt{\varepsilon}d_1 d_1^\star I_n & \lambda^2 d_1 I_n & 0 & \cdots & \cdots & 0 \\ \varepsilon d_1^\star I_n & 0 & -\lambda^2 \varepsilon d_2^\star I_n & \ddots & & \vdots \\ 0 & -d_2 I_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 d_{m-1} I_n & 0 \\ \vdots & & \ddots & \varepsilon d_{m-1}^\star I_n & 0 & -\lambda^2 \varepsilon d_m^\star I_n \\ 0 & \cdots & \cdots & 0 & -d_m I_n & 0 \end{bmatrix}.$$

Using the factorizations

$$\begin{bmatrix} I_n & 0 \\ \frac{\sqrt{\varepsilon}}{\lambda d_j} I_n & I_n \end{bmatrix} \begin{bmatrix} -\lambda\sqrt{\varepsilon}d_j d_j^\star I_n & \lambda^2 d_j I_n \\ \varepsilon d_j^\star I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & \frac{\lambda}{\sqrt{\varepsilon} d_j^\star} I_n \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} -\lambda\sqrt{\varepsilon}d_j d_j^\star I_n & 0 \\ 0 & \lambda\sqrt{\varepsilon} I_n \end{bmatrix}$$

and

$$\begin{aligned}
&\begin{bmatrix} I_n & 0 \\ \frac{d_{j+1}}{\sqrt{\varepsilon}\lambda} I_n & I_n \end{bmatrix} \begin{bmatrix} \lambda\sqrt{\varepsilon} I_n & -\lambda^2 \varepsilon d_{j+1}^{\star} I_n \\ -d_{j+1} I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & \lambda\sqrt{\varepsilon} d_{j+1}^{\star} I_n \\ 0 & I_n \end{bmatrix} \\
= &\begin{bmatrix} \lambda\sqrt{\varepsilon} I_n & 0 \\ 0 & -\lambda\sqrt{\varepsilon} d_{j+1} d_{j+1}^{\star} I_n \end{bmatrix},
\end{aligned}$$

it holds that

$$\begin{aligned}
&E_{2m-1}(\lambda)\cdots E_2(\lambda)E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda)F_2(\lambda)\cdots F_{2m-1}(\lambda) \\
= &\operatorname{diag}\left\{\lambda^{1-2m}\mathcal{P}(\lambda), -\lambda\sqrt{\varepsilon} d_1 d_1^{\star} I_n, \lambda\sqrt{\varepsilon} I_n, \cdots, \lambda\sqrt{\varepsilon} I_n, -\lambda\sqrt{\varepsilon} d_m d_m^{\star} I_n\right\},
\end{aligned}$$

where

$$\begin{aligned}
E_{2j}(\lambda) &= I_{(2j-1)n} \oplus \begin{bmatrix} I_n & 0 \\ \frac{\sqrt{\varepsilon}}{\lambda d_j} I_n & I_n \end{bmatrix} \oplus I_{(2m-2j-1)n}, \\
E_{2j+1}(\lambda) &= I_{2jn} \oplus \begin{bmatrix} I_n & 0 \\ \frac{d_{j+1}}{\sqrt{\varepsilon}\lambda} I_n & I_n \end{bmatrix} \oplus I_{(2m-2j-2)n},
\end{aligned}$$

and

$$\begin{aligned}
F_{2j}(\lambda) &= I_{(2j-1)n} \oplus \begin{bmatrix} I_n & \frac{\lambda}{\sqrt{\varepsilon} d_j^{\star}} I_n \\ 0 & I_n \end{bmatrix} \oplus I_{(2m-2j-1)n}, \\
F_{2j+1}(\lambda) &= I_{2jn} \oplus \begin{bmatrix} I_n & \lambda\sqrt{\varepsilon} d_{j+1}^{\star} I_n \\ 0 & I_n \end{bmatrix} \oplus I_{(2m-2j-2)n},
\end{aligned}$$

for $j = 1, \ldots, m-1$. Finally, letting

$$\begin{aligned} E_{2m}(\lambda) &:= \operatorname{diag}\left\{\lambda^{2m-1} I_n, -(\lambda\sqrt{\varepsilon} d_1 d_1^\star)^{-1} I_n, (\lambda\sqrt{\varepsilon})^{-1} I_n, -(\lambda\sqrt{\varepsilon} d_2 d_2^\star)^{-1} I_n, \right. \\ &\qquad \left. \cdots, (\lambda\sqrt{\varepsilon})^{-1} I_n, -(\lambda\sqrt{\varepsilon} d_m d_m^\star)^{-1} I_n\right\}, \end{aligned}$$

$\mathcal{E}(\lambda) := E_{2m}(\lambda) \cdots E_1(\lambda)$ and $\mathcal{F}(\lambda) := F_1(\lambda) \cdots F_{2m-1}(\lambda)$, one can easily verify that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \operatorname{diag}(\mathcal{P}(\lambda), I_{(2m-1)n})$. Furthermore, it holds that $\det(\mathcal{E}(\lambda)) = (-\varepsilon)^m / (\sqrt{\varepsilon} \prod_{j=1}^m d_j d_j^\star)$ and $\det(\mathcal{F}(\lambda)) = 1$.

(ii) (For $2d = 4m+2$) Let

$$\Pi_{2j} = I_{(2j-1)n} \oplus \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \oplus I_{(2m-2j)n}, \quad \text{for } j = 1, \ldots, m.$$

Similar to part (i), we define $n \times n$ matrix rational functions $E_1(\lambda)$ and $F_1(\lambda)$ by

$$\begin{aligned} E_1(\lambda) &= \Pi_2 \left[\begin{array}{c|cccc} I_n & E_{1,2}(\lambda) & E_{1,3}(\lambda) & \cdots & E_{1,2m+1}(\lambda) \\ \hline 0 & & I_{2mn} & & \end{array}\right], \\ F_1(\lambda) &= \left[\begin{array}{c|cccc} I_n & F_{2,1}(\lambda)^T & F_{3,1}(\lambda)^T & \cdots & F_{2m+1,1}(\lambda)^T \\ \hline 0 & & I_{2mn} & & \end{array}\right]^T \end{aligned}$$

with

$$\begin{aligned} E_{1,2j}(\lambda) &= \lambda^{2j-2m-2} I_n, \quad E_{1,2j+1}(\lambda) = d_j^{-1} \sum_{k=0}^{2j-1} \lambda^{k+1} A_{2m+k-2j+2}^\star, \\ F_{2j,1}(\lambda) &= \lambda^{2m-2j+2} I_n, \quad F_{2j+1,1}(\lambda) = \left(d_j^\star \lambda^{2j}\right)^{-1} \sum_{k=0}^{2j-1} \lambda^k A_{2m-k+1} \qquad (2.14) \end{aligned}$$

for $j = 1, \ldots, m$. Via careful calculation we get

$$\mathcal{Q}(\lambda)F_1(\lambda)\begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda^{-2m}\mathcal{P}(\lambda) \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_n & 0 \end{bmatrix} E_1(\lambda)\mathcal{Q}(\lambda)F_1(\lambda) = \begin{bmatrix} \lambda^{-2m}\mathcal{P}(\lambda) & 0 \end{bmatrix}.$$

Letting

$$\begin{aligned} E_j(\lambda) &= \Pi_{2j}\left(I_{2(j-1)n} \oplus \begin{bmatrix} I_n & 0 \\ \lambda^{-2}I_n & I_n \end{bmatrix} \oplus I_{(2m-2j+1)n}\right), \\ F_j(\lambda) &= I_{(2j-3)n} \oplus \begin{bmatrix} I_n & 0 & \lambda^2 I_n \\ 0 & I_n & I_n \\ 0 & 0 & I_n \end{bmatrix} \oplus I_{(2m-2j+1)n}, \end{aligned}$$

for $j = 2, \ldots, m$, and

$$\begin{aligned} & E_{m+1}(\lambda) \\ = & \operatorname{diag}\left\{\lambda^{2m}I_n, \frac{-1}{d_1}I_n, \frac{-1}{\lambda^2\varepsilon d_1^\star}I_n, \frac{-1}{d_2}I_n, \frac{-1}{\lambda^2\varepsilon d_2^\star}I_n, \cdots, \frac{-1}{d_m}I_n, \frac{-1}{\lambda^2\varepsilon d_m^\star}I_n\right\}, \end{aligned}$$

one can also verify that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \operatorname{diag}(\mathcal{P}(\lambda), I_{2mn})$, where $\mathcal{E}(\lambda) = E_{m+1}(\lambda)\cdots E_1(\lambda)$ and $\mathcal{F}(\lambda) = F_1(\lambda)\cdots F_m(\lambda)$. Furthermore, it holds that $\det(\mathcal{E}(\lambda)) = \varepsilon^{-m}/(\prod_{j=1}^m(d_j d_j^\star))$ and $\det(\mathcal{F}(\lambda)) = 1$. □

Note that the P-quadratization of a $(\star, \varepsilon)$-palindromic matrix polynomial of odd degree with even matrix dimension can be defined as in Definition 2.1. However, to the best of our knowledge, a P-quadratization of this type has not been found.

We now show the relationship between eigenpairs of $\mathcal{Q}(\lambda)$ in (2.5) and $\mathcal{P}(\lambda)$ in (2.1).

Theorem 2.5. *Let $\mathcal{Q}(\lambda)$ in (2.5) be a P-quadratization of $\mathcal{P}(\lambda)$ in (2.1) with $(\star,\varepsilon) \neq (T,-1)$. Denote $z = \begin{bmatrix} z_1^T & \cdots & z_d^T \end{bmatrix}^T$ with $z_j \in \mathbb{C}^n$ $(j = 1,\ldots,d)$. Then*

(i) For $\lambda_0 \neq 0$, (λ_0, z_1) is an eigenpair of $\mathcal{P}(\lambda)$ if and only if (λ_0, z) is an eigenpair of $\mathcal{Q}(\lambda)$ with $z_j = F_{j,1}(\lambda_0)z_1$ $(j = 2,\ldots,d)$, where $\{F_{j,1}(\lambda)\}_{j=2}^d$ are given in (2.9) (for $d = 2m$) or (2.14) (for $d = 2m+1$).

(ii) $(0, z_1)$ is an eigenpair of $\mathcal{P}(\lambda)$ if and only if $(0, z)$ is an eigenpair of $\mathcal{Q}(\lambda)$, where, for $j = 1,\ldots,m-1$:

$$\text{(for } d = 2m) \quad \begin{cases} z_{2m} = -(d_m^\star)^{-1}A_1 z_1, \ z_{2j+1} = 0, \\ z_{2j} = -(d_j^\star)^{-1}A_{2m-2j+1}z_1; \end{cases} \tag{2.15a}$$

$$\text{(or, for } d = 2m+1) \quad \begin{cases} z_{2m+1} = -(d_m^\star)^{-1}A_1 z_1, \ z_{2m} = 0, \\ z_{2j} = 0, \ z_{2j+1} = -(d_j^\star)^{-1}A_{2m-2j+1}z_1. \end{cases} \tag{2.15b}$$

(iii) (∞, z_2) is an eigenpair of $\mathcal{P}(\lambda)$ if and only if (∞, z) is an eigenpair of $\mathcal{Q}(\lambda)$, with $z_1 = z_3 = \cdots = z_d = 0$.

Proof. (i) From Theorem 2.4, there are $dn \times dn$ matrix rational functions $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$ with nonzero and constant determinants such that $\mathcal{E}(\lambda)\mathcal{Q}(\lambda)\mathcal{F}(\lambda) = \mathrm{diag}(\mathcal{P}(\lambda), I_{(d-1)n})$. Since $\lambda = 0$ is the only pole of $\mathcal{E}(\lambda)$ and $\mathcal{F}(\lambda)$, the matrices $F(\lambda_0)$ and $G(\lambda_0)$ defined in (2.4) are of full rank. The assertion in (i) follows immediately from Theorem 2.3 and the relation

$$F(\lambda_0)z_1 = \mathcal{F}(\lambda_0)\begin{bmatrix} z_1 \\ 0 \end{bmatrix} = F_1(\lambda_0)\begin{bmatrix} z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ F_{2,1}(\lambda_0)z_1 \\ \vdots \\ F_{d,1}(\lambda_0)z_1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{bmatrix}.$$

(ii) By the definition of $\mathcal{Q}(\lambda)$ in (2.5), we have $\mathcal{Q}(0)z = \varepsilon q \equiv \varepsilon \begin{bmatrix} q_1^T & \cdots & q_d^T \end{bmatrix}^T$, where for $d = 2m$:

$$\begin{cases} q_1 = A_1 z_1 + d_m^\star z_{2m},\ q_2 = \varepsilon\sqrt{\varepsilon} d_1 A_{2m} z_1,\ q_3 = A_{2m-1}z_1 + d_1^\star z_2, \\ q_4 = -\varepsilon d_2 z_3, \cdots, q_{2m-1} = A_3 z_1 + d_{m-1}^\star z_{2m-2},\ q_{2m} = -\varepsilon d_m z_{2m-1}; \end{cases} \tag{2.16a}$$

and for $d = 2m+1$:

$$\begin{cases} q_1 = A_1 z_1 + d_m^\star z_{2m+1},\ q_2 = A_{2m+1}z_1,\ q_3 = -\varepsilon d_1 z_2,\ q_4 = A_{2m-1}z_1 + d_1^\star z_3, \\ q_5 = -\varepsilon d_2 z_4, \cdots, q_{2m} = A_3 z_1 + d_{m-1}^\star z_{2m-1},\ q_{2m+1} = -\varepsilon d_m z_{2m}. \end{cases} \tag{2.16b}$$

From (2.16), we see that $(0, z_1)$ is an eigenpair of $\mathcal{P}(\lambda)$; i.e., $A_d z_1 = 0$ if and only if $(0, z)$ is an eigenpair of $\mathcal{Q}(\lambda)$; i.e., $\mathcal{Q}(0)z = \varepsilon q = 0$, where $\{z_j\}_{j=1}^d$ satisfy (2.15).

(iii) By the definition of $\mathcal{Q}(\lambda)$ in (2.5) again, we have $\left[\lambda^2 \mathcal{Q}\left(\frac{1}{\lambda}\right)\right]_{\lambda=0} z = q \equiv \begin{bmatrix} q_1^T & \cdots & q_d^T \end{bmatrix}^T$, where for $d = 2m$:

$$\begin{cases} q_1 = A_1^\star z_1 + \sqrt{\varepsilon} d_1^\star A_{2m}^\star z_2 + \sum_{k=1}^{m-1} A_{2m-2k+1}^\star z_{2k+1},\ q_2 = d_1 z_3, \\ q_3 = -\varepsilon d_2^\star z_4,\ \cdots,\ q_{2m-2} = d_{m-1} z_{2m-1},\ q_{2m-1} = -\varepsilon d_m^\star z_{2m},\ q_{2m} = d_m z_1; \end{cases} \tag{2.17a}$$

and for $d = 2m + 1$:

$$\begin{cases} q_1 = A_1^\star z_1 + \sum_{k=1}^{m} A_{2m-2k+3}^\star z_{2k}, \; q_2 = -\varepsilon d_1 z_3, \; q_3 = d_1^\star z_4, \\ \cdots, q_{2m} = -\varepsilon d_m z_{2m+1}, \; q_{2m+1} = d_m^\star z_1. \end{cases} \tag{2.17b}$$

From (2.17), it follows that (∞, z_2) is an eigenpair of $\mathcal{P}(\lambda)$; i.e., $A_d^\star z_2 = 0$ if and only if (∞, z) is an eigenpair of $\mathcal{Q}(\lambda)$, or $\left[\lambda^2 \mathcal{Q}\left(\frac{1}{\lambda}\right)\right]_{\lambda=0} z = q = 0$, with $z_1 = z_3 = \cdots = z_d = 0$. $\square$

Remark 2.6. *(i) Theorem 2.4 cannot be applied to the case $(\star, \varepsilon) = (T, -1)$. In fact, up to now, there is no structure-preserving algorithm to solve the T-anti-PQEP directly. So it is pointless to transform a T-anti-PPEP to a T-anti-PQEP. Thus, for a $(T, -1)$-palindromic matrix polynomial, we can apply the palindromic linearization [40] to transform it into a T-palindromic linear pencil $\lambda Z^T + Z$, and then solve it by the QR-like algorithm [53], the hybrid method [42], the URV-based method [54] or the doubling algorithm [7].*

(ii) On the other hand, if we rewrite $\lambda Z^T + Z$ to a T-palindromic quadratic pencil $\widehat{\mathcal{Q}}(\widehat{\lambda}) \equiv \widehat{\lambda}^2 Z^T + \widehat{\lambda} 0 + Z$ by letting $\widehat{\lambda}^2 = \lambda$, then the SPA algorithm [24] can also be used to solve its eigenpairs. It is shown in [24] that applying the SPA to solve $\widehat{\mathcal{Q}}(\widehat{\lambda}) y = 0$ is mathematically equivalent to applying the URV-based method to solve $\lambda Z^T + Z$.

Applying the P-quadratization in Theorem 2.4, an H-anti-PPEP can be quadratized into an H-anti-PQEP whose eigenpairs can then be computed from an H-PQEP by the following relationship.

Proposition 1. *Given an H-anti-PQEP: $(\lambda^2 A_1^H + \lambda A_0 - A_1)x = 0$, with $A_0^H = -A_0$. Then $(\imath\omega, x)$ is an eigenpair of the H-anti-PQEP if and only if (ω, x) is an eigenpair of the H-PQEP: $\left[\omega^2(-A_1)^H + \omega(\imath A_0) + (-A_1)\right] x = 0$.*

Proof. The result can be easily obtained by setting $\lambda = \imath\omega$ and using the fact $(\imath A_0)^H = \imath A_0$. □

We now consider the $\star$-even and $\star$-odd polynomial eigenvalue problems of even degree. Let $\mathcal{C}(\lambda) = \sum_{k=0}^{2d} \lambda^k C_k$, where $C_k \in \mathbb{C}^{n\times n}$ $(k = 0, 1, \dots, 2d)$ and $C_{2d} \neq 0$. The polynomial eigenvalue problem $\mathcal{C}(\lambda)x = 0$ is called a $\star$-even polynomial eigenvalue problem, if $C_{2k}^\star = C_{2k}$ $(k = 0, 1, \dots, d)$ and $C_{2k-1}^\star = -C_{2k-1}$ $(k = 1, \dots, d)$; and it is called a $\star$-odd polynomial eigenvalue problem, if $C_{2k}^\star = -C_{2k}$ $(k = 0, 1, \dots, d)$ and $C_{2k-1}^\star = C_{2k-1}$ $(k = 1, \dots, d)$. By the Cayley transformation, it was shown in [40] that a $\star$-even/odd polynomial eigenvalue problem can be transformed to a $(\star, \pm 1)$-PPEP, respectively.

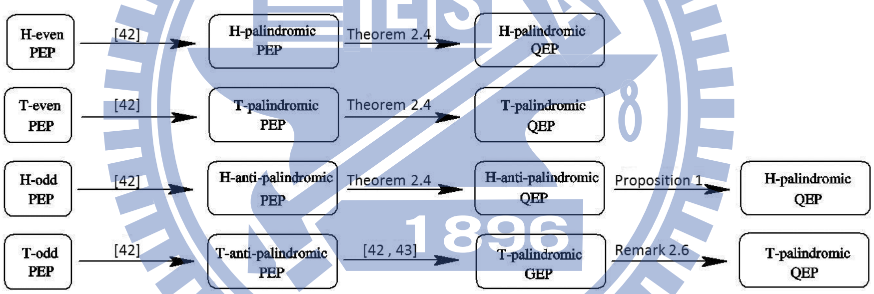

Figure 2.1: Relations between various structured polynomial eigenvalue problems (PEPs).

We illustrate the relationship among various structured polynomial eigenvalue problems in Figure 2.1. We see that T-even, T-odd, T-anti-palindromic and T-palindromic polynomial eigenvalue problems of even degree can be P-quadratized to T-PQEPs. Thus, the SPA algorithm in [24] can be applied to solve the associated T-PQEPs. On the other hand, we see that H-even, H-odd, H-anti-palindromic and H-palindromic polynomial eigenvalue problems of even degree can be P-quadratized to H-PQEPs. Therefore, we are motivated to develop a structure-preserving algorithm in the next section to solve the H-PQEP.

2.3 ⋆-palindromic Quadratic Eigenvalue Problems

Consider the ⋆-PQEP

$$\mathcal{Q}(\lambda)x \equiv (\lambda^2 A_1^\star + \lambda A_0 + A_1)x = 0, \tag{2.18}$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$ with $A_0^\star = A_0$. The eigenvalues of $\mathcal{Q}(\lambda)$ clearly appear in the "reciprocal" pairs of the form $(\lambda, 1/\hat{\lambda})$ where

$$\hat{\lambda} = \bar{\lambda} \text{ if } \star = H, \quad \hat{\lambda} = \lambda \text{ if } \star = \top.$$

Classical linearizations of (2.18) in a companion form, generally, do not preserve the symplectic structure. Fortunately, the special linearization of (2.18) (see [8] or [24])

$$(\mathcal{M} - \lambda\mathcal{L})z \equiv \left(\begin{bmatrix} A_1 & 0 \\ -A_0 & -I \end{bmatrix} - \lambda \begin{bmatrix} 0 & I \\ A_1^\star & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{2.19}$$

where $y = \frac{1}{\lambda}A_1 x$ and multiplying the second equation of (2.18) satisfies

$$\mathcal{M}\mathcal{J}\mathcal{M}^\star = \mathcal{L}\mathcal{J}\mathcal{L}^\star, \quad \mathcal{J} = \mathcal{J}_{2n} \equiv \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{2.20}$$

so that the matrix pair $(\mathcal{M}, \mathcal{L})$ has eigenvalues λ and $1/\hat{\lambda}$, preserving reciprocity. The pencil $\mathcal{M} - \lambda\mathcal{L}$ or the matrix pair $(\mathcal{M}, \mathcal{L})$ are called H-symplectic.

For any real symplectic matrix pair $(\mathcal{M}, \mathcal{L})$ satisfying (2.20), a structure-preserving $(\mathcal{S} + \mathcal{S}^{-1})$-transform for the computation of all eigenvalues was proposed by [33]. The $(\mathcal{S} + \mathcal{S}^{-1})$-transform $(\mathcal{M}_s, \mathcal{L}_s)$ of an ⋆-symplectic matrix pair $(\mathcal{M}, \mathcal{L})$ is defined by

$$\mathcal{M}_s \equiv \mathcal{M}\mathcal{J}\mathcal{L}^\star + \mathcal{L}\mathcal{J}\mathcal{M}^\star \equiv \mathcal{K}\mathcal{J}, \qquad \mathcal{L}_s \equiv \mathcal{L}\mathcal{J}\mathcal{L}^\star \equiv \mathcal{N}\mathcal{J}. \tag{2.21}$$

It is easily seen that $\mathcal{K}$ and $\mathcal{N}$ are both $\star$-skew-Hamiltonian, i.e., $\mathcal{K}\mathcal{J} = \mathcal{J}\mathcal{K}^\star$ and $\mathcal{N}\mathcal{J} = \mathcal{J}\mathcal{N}^\star$. Hence, if μ is an eigenvalue of $(\mathcal{K}, \mathcal{N})$, so is $\hat{\mu}$.

The relationship between eigenpairs of an $\star$-symplectic matrix pair and its $(\mathcal{S} + \mathcal{S}^{-1})$-transform has been given in [33].

Theorem 2.7. [33] *Let $(\mathcal{M}, \mathcal{L})$ be $\star$-symplectic and $(\mathcal{M}_s, \mathcal{L}_s)$ be its $(\mathcal{S} + \mathcal{S}^{-1})$-transform as in* (2.21)*. Suppose z_s is an eigenvector of $(\mathcal{M}_s, \mathcal{L}_s)$ corresponding to $\mu = \nu + \frac{1}{\nu}$. If $(\mathcal{L}^\star - \frac{1}{\nu}\mathcal{M}^\star)z_s \neq 0$ or $(\mathcal{L}^\star - \nu\mathcal{M}^\star)z_s \neq 0$, then $(\nu, \mathcal{J}(\mathcal{L}^\star - \frac{1}{\nu}\mathcal{M}^\star)z_s)$ or $(1/\nu, \mathcal{J}(\mathcal{L}^\star - \nu\mathcal{M}^\star)z_s)$ is an eigenpair of $(\mathcal{M}, \mathcal{L})$, respectively.*

We have some different derivative theorems between $\star = \top$ [24] and $\star = H$ here. Then the relationship between eigenpairs of $(\mathcal{M}_s, \mathcal{L}_s)$ and $\mathcal{Q}(\lambda), \star = H$ is given as follows.

Theorem 2.8. *Let $(\mathcal{M}, \mathcal{L})$ be H-symplectic of the form in* (2.19) *and $(\mathcal{M}_s, \mathcal{L}_s)$ be its $(\mathcal{S}+\mathcal{S}^{-1})$-transform. Suppose that z_s is an eigenvector of $(\mathcal{M}_s, \mathcal{L}_s)$ corresponding to the eigenvalue $\mu \neq \pm 2$, and denote $z_s \equiv [z_{s1}^T,\ z_{s2}^T]^T$ with $z_{s1}, z_{s2} \in \mathbb{C}^n$. Let ν be a root of the quadratic equation $\lambda + \frac{1}{\lambda} = \mu$. Then*

(i) *at least one of vectors $z_{s1} + \frac{1}{\nu}z_{s2}$ and $z_{s1} + \nu z_{s2}$ is nonzero;*

(ii) *if $z_{s1} + \frac{1}{\nu}z_{s2} \neq 0$, then $z_{s1} + \frac{1}{\nu}z_{s2}$ is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to ν;*

(iii) *if $z_{s1} + \frac{1}{\nu}z_{s2} = 0$, then z_{s2} is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to $\frac{1}{\nu}$.*

Proof. (i) Suppose that $z_{s1} + \frac{1}{\nu}z_{s2} = 0$ and $z_{s1} + \nu z_{s2} = 0$. It implies that $(\nu - \frac{1}{\nu})z_{s2} = 0$. If $z_{s2} = 0$, then $z_{s1} = 0$ and $z_s = 0$ which contradicts the fact that z_s is an eigenvector. Hence, $\nu = \pm 1$ and then $\mu = \pm 2$ which contradicts the assumption that $\mu \neq \pm 2$. Therefore, $z_{s1} + \frac{1}{\nu} z_{s2} \neq 0$ or $z_{s1} + \nu z_{s2} \neq 0$.

(ii) Since $\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H$, by (2.21) it holds that

$$0 = (\mathcal{M}_s - \mu \mathcal{L}_s)\, z_s = (\mathcal{M} - \nu \mathcal{L})\mathcal{J}(\mathcal{L}^H - \frac{1}{\nu}\mathcal{M}^H) z_s. \tag{2.22}$$

From (2.22), we obtain

$$(\mathcal{M} - \nu \mathcal{L}) \begin{bmatrix} z_{s1} + \frac{1}{\nu} z_{s2} \\ x_\nu \end{bmatrix} = 0, \tag{2.23}$$

where

$$x_\nu \equiv \frac{1}{\nu} A_1^H z_{s1} - \frac{1}{\nu} A_0 z_{s2} - A_1 z_{s2}. \tag{2.24}$$

Substituting $(\mathcal{M}, \mathcal{L})$ of (2.19) into (2.23), we have

$$x_\nu = \frac{1}{\nu} A_1 (z_{s1} + \frac{1}{\nu} z_{s2}) \tag{2.25}$$

and

$$A_0 (z_{s1} + \frac{1}{\nu} z_{s2}) + x_\nu + \nu A_1^H (z_{s1} + \frac{1}{\nu} z_{s2}) = 0. \tag{2.26}$$

Substituting x_ν of (2.25) into (2.26) and multiplying (2.26) by ν, we get $\mathcal{Q}(\nu)(z_{s1} + \frac{1}{\nu} z_{s2}) = 0$.

(iii) Since $z_{s1} + \frac{1}{\nu} z_{s2} = 0$, it follows that $z_{s1} = -\frac{1}{\nu} z_{s2} \neq 0$ and $x_\nu = 0$ in (2.25). Substituting these results into (2.24), it holds that

$$0 = x_\nu = -\left(\frac{1}{\nu}\right)^2 A_1^H z_{s2} - \frac{1}{\nu} A_0 z_{s2} - A_1 z_{s2}.$$

Therefore, z_{s2} is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to the eigenvalue $\frac{1}{\nu}$. □

In [24], a structure-preserving algorithm (SPA) based on Patel's algorithm [45] has been developed for solving T-PQEPs. In order to solve an H-PQEP, we apply the $(\mathcal{S} + \mathcal{S}^{-1})$-transform to the H-symplectic pair $(\mathcal{M}, \mathcal{L})$ of the form (2.19) and get the generalized eigenvalue problem $\mathcal{K} z_s = \mu \mathcal{N} z_s$, where $\mathcal{K}$ and $\mathcal{N}$ are defined in (2.21). Substituting $(\mathcal{M}, \mathcal{L})$ in (2.19) into (2.21), the H-skew-Hamiltonian $\mathcal{K}$ and $\mathcal{N}$ can be represented as

$$\mathcal{K} = \begin{bmatrix} A_0 & A_1^H - A_1 \\ A_1 - A_1^H & A_0 \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} -A_1 & 0 \\ 0 & -A_1^H \end{bmatrix}. \tag{2.27}$$

However, Patel's algorithm can only be applied to $(\mathcal{K}, \mathcal{N})$ of (2.27) in the real case, but cannot be directly applied to $(\mathcal{K}, \mathcal{N})$ in the complex conjugate case. In the following, we convert $(\mathcal{K}, \mathcal{N})$ of (2.27) into an enlarged real T-skew-Hamiltonian pair so that Patel's algorithm can be applied. We extend $(\mathcal{K}, \mathcal{N})$ in (2.27) to a real $4n \times 4n$ matrix pair $(\mathcal{K}_2, \mathcal{N}_2)$ by

$$\mathcal{K}_2 = \begin{bmatrix} \mathcal{K}_R & -\mathcal{K}_I \\ \mathcal{K}_I & \mathcal{K}_R \end{bmatrix}, \quad \mathcal{N}_2 = \begin{bmatrix} \mathcal{N}_R & -\mathcal{N}_I \\ \mathcal{N}_I & \mathcal{N}_R \end{bmatrix} \in \mathbb{R}^{4n \times 4n}, \tag{2.28}$$

where $\mathcal{K} = \mathcal{K}_R + \imath \mathcal{K}_I$ and $\mathcal{N} = \mathcal{N}_R + \imath \mathcal{N}_I$. From (2.28), it is easily seen that if μ is an eigenvalue of $(\mathcal{K}, \mathcal{N})$, then μ and $\bar{\mu}$ are eigenvalues of $(\mathcal{K}_2, \mathcal{N}_2)$.

Theorem 2.9. *The multiplicities of eigenvalues of* $(\mathcal{K}_2, \mathcal{N}_2)$ *are all even.*

Proof. Define $\widetilde{\mathcal{K}}_2 \equiv \Pi \mathcal{K}_2 \Pi$ and $\widetilde{\mathcal{N}}_2 \equiv \Pi \mathcal{N}_2 \Pi$, where $\Pi = I_n \oplus \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \oplus I_n$. It is easy to check that $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{N}}_2$ are real skew-Hamiltonian; i.e., $(\widetilde{\mathcal{K}}_2 \mathcal{J}_{4n})^T = -\widetilde{\mathcal{K}}_2 \mathcal{J}_{4n}$ and $(\widetilde{\mathcal{N}}_2 \mathcal{J}_{4n})^T = -\widetilde{\mathcal{N}}_2 \mathcal{J}_{4n}$. Therefore, from the result of [33], it follows that the multiplicities of eigenvalues of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ are all even. □

From (2.21), we see that μ is an eigenvalue of $(\mathcal{K}, \mathcal{N})$ if and only if $\bar{\mu}$ is also an eigenvalue. We now give the relationship between eigenpairs of $(\mathcal{K}, \mathcal{N})$ and $(\mathcal{K}_2, \mathcal{N}_2)$.

Theorem 2.10. **(i)** *If* $(\alpha + \imath\beta, x + \imath y)$ *is an eigenpair of* $(\mathcal{K}, \mathcal{N})$*, then* $\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix}$ *are eigenvectors of* $(\mathcal{K}_2, \mathcal{N}_2)$ *corresponding to the eigenvalues* $\alpha \pm \imath\beta$*.*

(ii) *If* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \imath \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *is an eigenvector of* $(\mathcal{K}_2, \mathcal{N}_2)$ *corresponding to the eigenvalue* $\alpha + \imath\beta$*, then* $(x_1 - y_2) + \imath(x_2 + y_1)$ *is an eigenvector of* $(\mathcal{K}, \mathcal{N})$ *corresponding to* $\alpha + \imath\beta$*.*

Proof. (i) Since $\mathcal{K}(x + \imath y) = (\alpha + \imath\beta)\mathcal{N}(x + \imath y)$, comparing the real and the imaginary parts of both sides leads to

$$\mathcal{K}_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix} \right) = (\alpha \pm \imath\beta)\mathcal{N}_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \pm \imath \begin{bmatrix} y \\ -x \end{bmatrix} \right).$$

(ii) Since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \imath \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an eigenvector of $(\mathcal{K}_2, \mathcal{N}_2)$ corresponding to the eigenvalue $\alpha + \imath\beta$, it holds that

$$\begin{aligned} \mathcal{K}_R x - \mathcal{K}_I y &= \alpha(\mathcal{N}_R x - \mathcal{N}_I y) - \beta(\mathcal{N}_I x + \mathcal{N}_R y), \\ \mathcal{K}_I x + \mathcal{K}_R y &= \beta(\mathcal{N}_R x - \mathcal{N}_I y) + \alpha(\mathcal{N}_I x + \mathcal{N}_R y), \end{aligned}$$

by setting $x = x_1 - y_2$ and $y = x_2 + y_1$. Thus, $(\alpha + \imath\beta, x + \imath y)$ is an eigenpair of $(\mathcal{K}, \mathcal{N})$. □

From Theorem 2.10, the eigenpairs of $(\mathcal{K}, \mathcal{N})$ can be computed from the eigenpairs of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$. Since $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{N}}_2$ are both real skew-Hamiltonian, based on Patel's approach [24, 45], the pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ can be reduced to block upper triangular forms

$$\widetilde{\mathcal{K}}_2 := Q^T \widetilde{\mathcal{K}}_2 Z = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{11}^T \end{bmatrix}, \qquad \widetilde{\mathcal{N}}_2 := Q^T \widetilde{\mathcal{N}}_2 Z = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}, \tag{2.29}$$

where $Q, Z \in \mathbb{R}^{4n\times 4n}$ are orthogonal satisfying $Q = \mathcal{J}_{4n}^T Z \mathcal{J}_{4n}$, and $K_{11}, N_{11} \in \mathbb{R}^{2n\times 2n}$ are upper Hessenberg and upper triangular, respectively.

From Theorem 2.9 and (2.29), we see that the pair (K_{11}, N_{11}) has the same spectrum as $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$. We then apply the QZ algorithm to (K_{11}, N_{11}) to compute all its eigenpairs $\{(\mu_j, \tilde{z}_j)\}_{j=1}^{2n}$. Consequently, $\{(\mu_j, \Pi Z \begin{bmatrix} \tilde{z}_j \\ 0 \end{bmatrix})\}_{j=1}^{2n}$ are the $2n$ eigenpairs of $(\mathcal{K}_2, \mathcal{N}_2)$. Let $\mu_j = \alpha_j + \imath\beta_j$ and $\Pi Z[\tilde{z}_j^T, 0^T]^T \equiv \left[x_{1j}^T, x_{2j}^T\right]^T + \imath\left[y_{1j}^T, y_{2j}^T\right]^T$ with $\alpha_j, \beta_j \in \mathbb{R}$ and $x_{1j}, x_{2j}, y_{1j}, y_{2j} \in \mathbb{R}^{2n}$. From Theorem 2.10, $\{(\alpha_j + \imath\beta_j, \mathcal{J}^T(x_{1j} - y_{2j} + \imath(x_{2j} + y_{1j})))\}_{j=1}^{2n}$ are eigenpairs of $(\mathcal{M}_s, \mathcal{L}_s)$. Finally, we compute all eigenvalues and the associated eigenvectors of $\mathcal{Q}(\lambda)$ by Theorem 2.8. We present the structure-preserving algorithm for solving H-PQEP in Algorithm 2.1.

Algorithm 2.1 Structure-Preserving Algorithm (SPA) for H-PQEP

Input: An H-palindromic quadratic pencil $\mathcal{Q}(\lambda) \equiv \lambda^2 A_1^H + \lambda A_0 + A_1$ with $A_0, A_1 \in \mathbb{C}^{n\times n}$ and $A_0^H = A_0$;

Output: All eigenvalues and eigenvectors of $\mathcal{Q}(\lambda)$.

Form the matrix pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2) = (\Pi\mathcal{K}_2\Pi, \Pi\mathcal{N}_2\Pi)$ as in (2.28);

Reduce $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ to block upper triangular forms as in (2.29);

Compute eigenpairs $\{(\mu_j, \tilde{z}_j)\}_{j=1}^{2n}$ of (K_{11}, N_{11}) defined in (2.29) by the QZ algorithm;

Compute $\Pi Z \begin{bmatrix} \tilde{z}_j \\ 0 \end{bmatrix} \equiv \begin{bmatrix} x_{1j} \\ x_{2j} \end{bmatrix} + \imath \begin{bmatrix} y_{1j} \\ y_{2j} \end{bmatrix}, j = 1, 2, \ldots, 2n$;

Compute the eigenpair (μ_j, z_j), for $j = 1, 2, \ldots, 2n$, of $(\mathcal{M}_s, \mathcal{L}_s)$ by

$$z_j = \mathcal{J}^T(x_{1j} - y_{2j} + \imath(x_{2j} + y_{1j})) \equiv [z_{j1}^T, z_{j2}^T]^T;$$

Compute ν_j and $\frac{1}{\nu_j}$ by solving $\nu^2 - \mu_j\nu + 1 = 0$; Compute $x_{j1} \equiv z_{j1} + \frac{1}{\nu_j} z_{j2}$ and $x_{j2} \equiv z_{j1} + \nu_j z_{j2}$ for $j = 1, 2, \ldots, 2n$;

If $x_{j1} \neq 0$, then it is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to ν_j; If $x_{j2} \neq 0$, then it is an eigenvector of $\mathcal{Q}(\lambda)$ corresponding to $\frac{1}{\nu_j}$;

2.4 Structured Backward Perturbation Analysis

Typically, algorithms would approach the right solution in the limit, if there were no round-off or truncation errors. However, depending on the specific computational method, errors can be magnified and causing the error to grow exponentially.Let $\{\mu, z\}$ be a computed eigenpair of

$$\left(\sum_{\ell=0}^{d} A_\ell \lambda^\ell\right) x = 0.$$

Theoretically, we would like to have $\left(\sum_{\ell=0}^{d} A_\ell \mu^\ell\right) z = 0$. But practically we have $\left(\sum_{\ell=0}^{d} A_\ell \mu^\ell\right) z = -r$ with residual $r \neq 0$ but usually tiny. Backward error analysis asks if the computed eigenpair $\{\mu, z\}$ is an exact eigenpair of a nearby PEP such that

$$\left(\sum_{\ell=0}^{d} \left(A_\ell + \rho_\ell \triangle A_\ell\right) \mu^\ell\right) z = 0$$

where $\triangle A_\ell$ are called backward perturbation matrices and ρ_ℓ are scaling parameters. Tisseur [56] developed a backward error perturbation analysis for PEP generally, where $\triangle A_\ell$ is no structure. However, as A are Hermitian, we would like to enforcing that $\triangle A_\ell$ should be Hermitian, too. This is the reason why the structured backward perturbation analysis be developed. That is to say, we consider PPEP with

$$\left(\sum_{\ell=0}^{d} \left(A_\ell + \rho_\ell \triangle A_\ell\right) \mu^\ell\right) z = 0, \quad A_{d-\ell} = \varepsilon A_\ell^\star \quad \text{for } \ell = 0, 1, \ldots \lfloor d/2 \rfloor\,. \tag{2.30}$$

We have mentioned the fast train application [29, 43?] which yield a problem of this form with $d = 2, \star = \top$, and $\varepsilon = 1$ before. Let $\|\cdot\|$ be either the spectral norm $\|\cdot\|_2$ or the Frobenius norm $\|\cdot\|_F$.Now, we are interested in knowing the structrued backward error

$$\triangle = \min \sqrt{\sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle A_\ell\|^2},$$

$$\begin{aligned} \text{subject to } \left(\sum_{\ell=0}^{d} (A_\ell + \rho_\ell \triangle A_\ell)\, \mu^\ell\right) z &= 0, \\ \text{scaling parameters } \rho_{d-\ell} &= \rho_\ell \geq 0 \\ A_{d-\ell} + \rho_{d-\ell} \triangle A_{d-\ell} &= \varepsilon \left(A_\ell + \rho_\ell \triangle A_\ell\right)^\star \\ \text{for } \ell &= 0, 1, \ldots \lfloor d/2 \rfloor \end{aligned} \tag{2.31}$$

The constraints in (2.31) require $\rho_{d-\ell} = \rho_\ell$ and $\triangle A_{d-\ell} = \varepsilon \triangle A^\star_{d-\ell}$ just like the palindromic form, and we decompose (2.30) which is equivalent to

$$\left(\sum_{\ell=0}^{d} \rho_\ell \triangle A_\ell \mu^\ell\right) z = r \stackrel{\text{def}}{=} -\left(\sum_{\ell=0}^{d} A_\ell \mu^\ell\right) z, \tag{2.32}$$

where $\triangle A_\ell \in \mathbb{C}^{n\times n}$ $(\ell = 0, 1, \ldots d)$. We will seek if (2.32) has a solution $\{\triangle A_\ell, \ell = 0, 1, \ldots \lfloor d/2 \rfloor\}$ and if it does, we'll seek $\triangle A_\ell$ such that

$$\min \sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle A_\ell\|^2 \tag{2.33}$$

where $\|\cdot\|$ is either $\|\cdot\|_2$ or $\|\cdot\|_F$.

The key to solve (2.32) and (2.33) is a reduction technique that has been used in [4, 25, 35, 57]. The technique allows us to consider (2.32) in the case of 2×2 reduced matrix when we deal with backward errors in a 2-dimensional subspace spanned by $\{z, r\}$ or by $\{z, \bar{r}\}$.

Let $Q \in \mathbb{C}^{n\times n}$ be a unitary matrix ($QQ^H = I_n$), such that

$$Q^H \left(z \;\hat{r} \right) = \left(\begin{array}{cc} \alpha & \gamma \\ 0 & \beta \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array} \right), \tag{2.34}$$

where

$$r = r \quad \text{if } \star = H, \quad \text{and } \hat{r} = \bar{r} \quad \text{if } \star = \top.$$

Such Q always can be generated, and the equation (2.34) also implies

$$|\alpha| = \|z\|_2 \,, \gamma = \frac{z^H \hat{r}}{\bar{\alpha}}, |\beta| = \left\| \hat{r} - \frac{z^H \hat{r}}{\|z\|_2^2} z \right\|_2 = \frac{\sqrt{\|\hat{r}\|_2^2 \,\|z\|_2^2 - |z^H \hat{z}|^2}}{\|z\|_2} \tag{2.35}$$

We can rewrite (2.32) by multiplying Q , then

$$Q^\star \left(\sum_{\ell=0}^{d} \rho_\ell \triangle A_\ell \mu^\ell \right) QQ^H z = Q^\star r,$$

or equivalently

$$\left(\sum_{\ell=0}^{d} \rho_\ell \triangle B_\ell \mu^\ell \right) y = w, \rho_{d-\ell} = \rho_\ell \text{ and } \triangle B_{d-\ell} = \varepsilon \triangle B_\ell^\star, \text{for } \ell = 0, 1, ... \lfloor d/2 \rfloor \tag{2.36}$$

where

$$\triangle B_\ell = Q^\star \left(\triangle A_\ell \right) Q, y = Q^H z = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}, w = Q^\star r = \begin{pmatrix} \hat{r} \\ \hat{\beta} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\hat{\beta} = \beta \text{ and } \hat{\gamma} = \gamma \text{ if } \star = H, \text{ and } \hat{\beta} = \bar{\beta} \text{ and } \hat{\gamma} = \bar{\gamma} \text{ if } \star = \top.$$

Since Q is unitary, (2.31) and (2.36) have the same solvability property. It means that if the former is solvable, so is the latter, and moreover

$$\sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle B_\ell\|^2 = \sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle A_\ell\|^2 . \tag{2.37}$$

Thus we can focus on the optimal solution $\{\triangle B_\ell, \ell = 0, 1, ..., \lfloor d/2 \rfloor\}$ to (2.37) which means

$$\min \sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle B_\ell\|^2 . \tag{2.38}$$

That also generate one solution $\{\triangle A_\ell, \ell = 0, 1, ..., \lfloor d/2 \rfloor\}$ to (2.32) in the sence of (2.33), and vice versa. Furthemore, we transfrom (2.31) to the reduced structrued backward error

$$\triangle_p = \min \left\{ \sqrt{\sum_{\ell=0}^{\lfloor d/2 \rfloor} \|\triangle B_\ell\|_p^2 : (2.36) \text{ satisfied}} \right\} \text{ for } p =_{2,\mathrm{F}}$$

It follows from (2.35) that

$$|\delta_1| = \frac{|z^H \hat{r}|}{\|z\|_2^2}, |\delta_2| = \frac{\sqrt{\|\hat{r}\|_2^2 \|z\|_2^2 - |z^H \hat{r}|^2}}{\|z\|_2^2}, \sqrt{|\delta_1|^2 + |\delta_2|^2} = \frac{\|r\|_2}{\|z\|_2} \tag{2.39}$$

The complex calculations and technical operation in Theorem 2.12 [47] are omit and we present the significant results.We first define a few parameters in term of a given approximate eigenpair $\{\mu, z\}$ of PPEP (2.30): for even d

$$\begin{aligned}
\Phi_{\text{even}} &\stackrel{\text{def}}{=} \xi_{\text{even}} + \zeta_{\text{even}} = \sum_{\ell=0}^{d/2-1} \rho_\ell^2 \left(|\mu|^\ell + |\mu|^{d-\ell}\right)^2 + \rho_{d/2}^2 |\mu|^d, \\
\phi_{\text{even}} &\stackrel{\text{def}}{=} \xi_{\text{even}} - \zeta_{\text{even}} = \sum_{\ell=0}^{d/2-1} \rho_\ell^2 \left(|\mu|^\ell - |\mu|^{d-\ell}\right)^2, \\
\Psi_{\text{even}} &\stackrel{\text{def}}{=} \sum_{\ell=0}^{d/2-1} \rho_\ell^2 \left(|\mu|^{2\ell} + |\mu|^{2(d-\ell)}\right)^2 + \rho_{d/2}^2 |\mu|^d /2.
\end{aligned}$$

and for odd d

$$\begin{aligned}
\Phi_{\text{odd}} &\stackrel{\text{def}}{=} \xi_{\text{odd}} + \zeta_{\text{odd}} = \sum_{\ell=0}^{(d-1)/2} \rho_\ell^2 \left(|\mu|^\ell + |\mu|^{d-\ell}\right)^2, \\
\phi_{\text{odd}} &\stackrel{\text{def}}{=} \xi_{\text{odd}} - \zeta_{\text{odd}} = \sum_{\ell=0}^{(d-1)/2} \rho_\ell^2 \left(|\mu|^\ell - |\mu|^{d-\ell}\right)^2, \\
\Psi_{\text{odd}} &\stackrel{\text{def}}{=} \sum_{\ell=0}^{(d-1)/2} \rho_\ell^2 \left(|\mu|^{2\ell} + |\mu|^{2(d-\ell)}\right)^2.
\end{aligned}$$

Theorem 2.11. [47] *Let $\{\mu, z\}$ be a given approximate eigenpair of PPEP (2.30). Suppose $\star = H$ and $\varepsilon = \pm 1$ in (2.31), and δ_1 and δ_2 are as in (2.39) with $\hat{r} = r$ which is defined in (2.32). Let*

$$\phi = \begin{cases} \phi_{\text{even}} & \text{for even } d \\ \phi_{\text{odd}}, & \text{for odd } d \end{cases} \quad \Phi = \begin{cases} \Phi_{\text{even}} & \text{for even } d \\ \Phi_{\text{odd}}, & \text{for odd } d \end{cases} \quad \Psi = \begin{cases} \Psi_{\text{even}} & \text{for even } d \\ \Psi_{\text{odd}}, & \text{for odd } d \end{cases}$$

Theorem 2.12. *For the structure backward error $\triangle_F$ defined in (2.31), we have*

1. *If $|\mu| = 1$ and $z^H r/(\sqrt{\varepsilon}\mu^{d/2}) \notin \mathbb{R}$, then $\triangle_F = +\infty$*

2. If $|\mu| = 1$ *and* $z^H r/(\sqrt{\varepsilon}\mu^{d/2}) \in \mathbb{R}$*, then*

$$\triangle_F = \sqrt{\frac{|\delta_1|^2}{\Phi} + \frac{|\delta_2|^2}{\Psi}}$$

3. If $|\mu| \neq 1$*, then*

$$\triangle_F \leq \sqrt{\frac{|\delta_1|^2}{\phi} + \frac{|\delta_2|^2}{\Psi}}$$

Let us look back at our original problems (2.1) and apply the Theorem 2.12 in our PPEP. Taking $d = 4, \varepsilon = 1$ for example, we consider

$$\mathcal{P}(\lambda) = \lambda^4 A_2^H + \lambda^3 A_1^H + \lambda^2 A_0 + \lambda A_1 + A_2. \tag{2.40}$$

From the quadration we mentioned in Theorem 2.4 where $\mathcal{A}_1$ and $\mathcal{A}_0$ are given by

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & I \\ A_2 & 0 \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} A_0 - I - A_2^H A_2 & 0 \\ 0 & -I \end{bmatrix}$$

which satisties

$$Q(\lambda) = \lambda^2 \mathcal{A}_1^H + \lambda \mathcal{A}_0 + \mathcal{A}_1. \tag{2.41}$$

In section 2, we propose a method to solve the equation (2.41) with structure-preserving algorithm. In this case, it is quite reasonable to use structure-preserving backward error to estimate the stability. First, we select the equation (2.40) and transform it to (2.41). If (λ_0, z_1) is an eigenpair of $\mathcal{P}(\lambda)$, then (λ_0, z) is an eigenpair of $Q(\lambda)$ with $z = [z_1^\top, z_2^\top]^\top$ where

$$z_2 = \frac{1}{\lambda_0}\left(\lambda_0^2 \mathcal{A}_2^H z_1 + z_1\right).$$

By the structure-preserving backward error analysis, we use the equivalent notation on $Q(\lambda)$ and let $\{\mu, z\} = \left\{\lambda_0, [z_1^\top, z_2^\top]^\top\right\}$. The theorem 2.12 gives us an estimated backward error upper bound and the bound is $\triangle_F \leq \sqrt{\frac{|\delta_1|^2}{\phi} + \frac{|\delta_2|^2}{\Psi}}$. The terms ϕ and Ψ just relate to λ_0 which is preserved in our algorithm. In spite of this, the term δ_1 contains the eigenvector z which is changed in our process ($|\delta_1| = \frac{|z^H \hat{r}|}{\|z\|_2^2}$). More precisely, the norm of $\mathcal{A}_2$ cause considerable impact on our structure-preserving backward error. In general, the norm of $\mathcal{A}_\ell$, $\ell = 0, 1, ... \lfloor d/2 \rfloor$ become perturbative factors that we have to face it. Next section we provide a classical technique that modify our backward error and easily implement in our algorithm.

2.5 Balancing of $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$

Scaling [3, 9, 15, 34] is a commonly used technique for standard eigenvalue problems for the improvement of the sensitivity of eigenvalues. In this section, we first propose a diagonal scaling for $\mathcal{P}(\lambda)$ in (2.1). Then, we determine the free parameters $d_1, \ldots, d_m$ in (2.7) and (2.8) to improve the backward errors of eigenpairs for $\mathcal{P}(\lambda)$ as in [26, 27, 47].

In order to balance the entries of coefficient matrices in $\mathcal{P}(\lambda)$, we define a complex diagonal matrix

$$D \equiv \operatorname{diag}(2^{\alpha_1}, 2^{\alpha_2 + \imath\beta_2}, \cdots, 2^{\alpha_n + \imath\beta_n})$$

with $\alpha_j, \beta_j \in \mathbb{R}$ so that the magnitudes of entries of coefficient matrices in the new $(\star, \varepsilon)$-palindromic matrix polynomial

$$D \left(\sum_{k=0}^{d-1} \lambda^{2d-k} A_{d-k}^\star + \lambda^d A_0 + \varepsilon \sum_{k=1}^{d} \lambda^{d-k} A_k \right) D^\star$$

are close to one as much as possible. That is, we determine $\alpha_1, \ldots, \alpha_n$ and $\beta_2, \ldots, \beta_n$

so that

$$2^{\alpha_j+\imath\beta_j} A_k(j,\ell) 2^{\alpha_\ell-\imath\beta_\ell} \approx 1, \tag{2.42}$$

for $j,\ell = 1,2,\ldots,n$ and $k = 0,1,\ldots,d$, where $A_k(j,\ell)$ is the (j,ℓ)-th entry of A_k. By taking logarithm of (2.42), the parameters, $\alpha_1,\ldots,\alpha_n$ and $\beta_2,\ldots,\beta_n$ can be determined by solving the least square problems

$$\alpha_j + \alpha_\ell = -\mathfrak{Re}(\log_2(A_k(j,\ell))), \quad \beta_j - \beta_\ell = -\mathfrak{Im}(\log_2(A_k(j,\ell))),$$

where $\mathfrak{Re}(c)$ and $\mathfrak{Im}(c)$ represent the real and imaginary parts of c, respectively. Then, the parameters $\alpha_1,\ldots,\alpha_n$ and $\beta_2,\ldots,\beta_n$ are determined by the associated normal equations

$$B^T B\,[\alpha_1,\cdots,\alpha_n]^T = B^T b, \quad C^T C\,[\beta_2,\cdots,\beta_n]^T = C^T c.$$

We now determine $d_1,\ldots,d_m$ in (2.7) or (2.8), to balance the magnitudes of entries of $\mathcal{A}_0$ and $\mathcal{A}_1$ in $\mathcal{Q}(\lambda)$. For convenience, we define

$$d_i = \begin{cases} d_i^{(1)}, & \text{if } 2d = 4m, \\ d_i^{(2)}, & \text{if } 2d = 4m+2; \end{cases} \qquad \text{for } i = 1,\ldots,m.$$

From the row balancing of $\mathcal{A}_1$ in (2.7a) or (2.8a), we first set

$$\eta_i^{(s)} = \max\left\{1,\ \max\{\|A_{2m-k+s-1}\|_1 : k = 0,1,\ldots,2i-3+s\}\right\}.$$

Then we take $\delta_i^{(s)}$ to be the geometric average of $\eta_i^{(s)}$ and the average of the absolute magnitudes of entries of $A_{2m-2i+1}$ in $\mathcal{A}_1$; i.e.,

$$\delta_i^{(s)} = \sqrt{\eta_i^{(s)} \left(\sum_{j=1}^{n} \sum_{\ell=1}^{n} |A_{2m-2i+1}(j,\ell)|/n^2 \right)}$$

for $i = 1, \ldots, m$ and $s = 1, 2$. Although the value of $d_i^{(s)}$ can be set to $\delta_i^{(s)}$ to balance the entries of $\mathcal{A}_1$, we also need to consider the balance of the entries of both $\mathcal{A}_0$ and $\mathcal{A}_1$ in (2.7) or (2.8). As a result, we take the values of $d_1^{(s)}, \ldots, d_m^{(s)}$ to be the geometric average of the maximal values of $\delta_1^{(s)}, \ldots, \delta_m^{(s)}$ and the maximal average for the absolute magnitudes of entries of A_k ($k = 0, \ldots, d$); i.e., for $i = 1, \ldots, m$ and $s = 1, 2$, we set

$$d_i^{(s)} := \sqrt{\rho^{(s)} \max_{0 \le k \le d} \left\{ \sum_{j=1}^{n} \sum_{\ell=1}^{n} |A_k(j,\ell)|/n^2 \right\}}$$

with

$$\rho^{(s)} = \max\{\delta_i^{(s)}; i = 1, \ldots, m\}.$$

2.6 Numerical Results

In [24], an SPA is proposed for solving T-PQEPs. Numerical experiments show that SPA performs well on the T-PQEP arising from a finite element model of high-speed trains and rails. In this section, we shall focus on the numerical comparison of the performance and accuracy for solving H-PPEP of even degree by using structure-preserving algorithms and companion linearization.

For solving an $n \times n$ H-PPEP of even degree $2d$, we apply the P-quadratization in Section 2 to transform it into a $dn \times dn$ H-PQEP. We then apply the SPA (Algorithm 2.1) in Section 3 to solve the H-PQEP. The combination of the P-quadratization and SPA is called the PQ_SPA algorithm. On the other hand, we can also use the "good" linearization [40, 41] to transform the H-PPEP into a palindromic linear pencil $\lambda Z^H + Z$, and then utilize SPA to solve the H-PQEP: $(\hat{\lambda}^2 Z^H + \hat{\lambda} 0 + Z)x = 0$ with $\lambda = \hat{\lambda}^2$. The combination of the "good" linearization and SPA is called the PL_SPA algorithm. As mentioned in Remark 2.6 (ii), we see that applying the SPA to $\hat{\lambda}^2 Z^T + \hat{\lambda} 0 + Z$ is mathematically equivalent to applying the URV-based method [54] to $\lambda Z^T + Z$.

2.6.1 Computational Cost

For making PQ_SPA more efficient, we reorder the submatrices of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ in step 1 of Algorithm 2.1 by the permutations

$$\Pi_1 = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{(d-1)n} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{(d-2)n} \\ 0 & 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & I_{(d-2)n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 & I_{(d-2)n} & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We substitute $\mathcal{A}_1$ in Theorem 2.4 into $\widetilde{\mathcal{N}}_2$ and get

$$\begin{aligned} \widetilde{\mathcal{N}}_2 &:= \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \widetilde{\mathcal{N}}_2 \begin{bmatrix} \Pi_2^T & 0 \\ 0 & \Pi_1^T \end{bmatrix} \\ &= \begin{bmatrix} D_1 & 0 & -V_1 & V_2 \\ 0 & D_1 & V_2 & V_1 \\ 0 & 0 & V_3 & -V_4 \\ 0 & 0 & V_4 & V_3 \end{bmatrix} \oplus \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \\ -V_1^T & V_2^T & V_3^T & V_4^T \\ V_2^T & V_1^T & -V_4^T & V_3^T \end{bmatrix}, \end{aligned}$$

where D_1 is a $(2d-2)n \times (2d-2)n$ diagonal matrix and $V_3, V_4 \in \mathbb{R}^{n\times n}$. Set

$$\widetilde{\mathcal{K}}_2 := \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \widetilde{\mathcal{K}}_2 \begin{bmatrix} \Pi_2^T & 0 \\ 0 & \Pi_1^T \end{bmatrix}.$$

Now, we compare the computational costs for PQ_SPA and PL_SPA:

- (QR factorization and updating with real arithmetic operations) Compute Q_1 and Z_1 such that $Q_1^T \widetilde{\mathcal{N}}_2 Z_1 = \operatorname{diag}\left(N_{11}^{(1)}, (N_{11}^{(1)})^T\right)$, where $N_{11}^{(1)}$ is upper triangular, and update $Q_1^T \widetilde{\mathcal{K}}_2 Z_1$. It requires $(80/3n^3 + 32dn^3)$ and $341\frac{1}{3}d^3n^3$ flops for PQ_SPA and PL_SPA, respectively.

- (Given's rotations and updating with real arithmetic operations) Reducing the new pair $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ produced by above step to block upper triangular forms of (2.29), it requires $232d^3n^3 - (296d^2 - 24d)n^2$ and $1856d^3n^3 - 800d^2n^2$ flops for PQ_SPA and PL_SPA, respectively.

- (Computing eigenvalues of (K_{11}, N_{11})) Computing eigenvalues of the real upper Hessenberg and triangular pair (K_{11}, N_{11}) by QZ algorithm, it requires $176d^3n^3$ and $1408d^3n^3$ flops for PQ_SPA and PL_SPA, respectively, to obtain the upper quasi-triangular and triangular pair.

- The eigenvectors of $(\widetilde{\mathcal{K}}_2, \widetilde{\mathcal{N}}_2)$ can be computed by an additional $(408d^3 + 32d)n^3 - (332d^2 - 16d)n^2$ and $1920d^3n^3 - 1088d^2n^2$ flops for PQ_SPA and PL_SPA, respectively.

We summarize the computational flops of PQ_SPA and PL_SPA in Table 2.1.

	Eigenvalues	Eigenvectors	Total
PQ_SPA	$(408d^3 + 32d + 80/3)n^3$	$(408d^3 + 32d)n^3$	$(816d^3 + 64d + 80/3)n^3$
PL_SPA	$3605\frac{1}{3}d^3n^3$	$1920d^3n^3$	$5525\frac{1}{3}d^3n^3$

Table 2.1: Computational flops of PQ_SPA and PL_SPA

2.6.2 Numerical Experiments

For an approximate eigenpair (λ, x) of the palindromic matrix polynomial $\mathcal{P}(\lambda)$, we define the associated relative residual by

$$\text{RRes} \equiv \text{RRes}(\lambda, x) := \frac{\|\mathcal{P}(\lambda)x\|_2}{\left[\sum_{k=1}^{d} \left(|\lambda|^{d+k} + |\lambda|^{d-k}\right) \|A_k\|_2 + |\lambda|^d \|A_0\|_2\right] \|x\|_2}.$$

We will show numerical results of RRes and the reciprocal property of eigenpair (λ, x) for the H-PPEPs, computed by PQ_SPA, PL_SPA and `polyeig` in MATLAB (applied directly to (2.1)).

As mentioned before, theoretically, the eigenvalues of H-PPEP appear in reciprocal pairs $(\lambda, 1/\bar{\lambda})$. So, if we sort the eigenvalues in ascending order by modulus, the product of the i-th eigenvalue and the conjugate of the $(2dn+1-i)$-th eigenvalue should be one. Therefore, we define the reciprocities of the computed eigenvalues by

$$r_i \equiv |\lambda_i \bar{\lambda}_{2dn+1-i} - 1| \quad (i = 1, \ldots, dn).$$

All numerical experiments are carried out using MATLAB 2008b with machine precision $eps \approx 2.22 \times 10^{-16}$.

Let $\mathcal{C}_{n,b}$ denote the set of $n \times n$ complex matrices which real and imaginary parts are randomly generated by the normal distribution with zero mean and standard deviation b.

Example 2.1. *Consider the H-PPEP with* $d = 5$ *and* $A_k \in \mathcal{C}_{n,100}$ $(k = 0, \dots, 5)$.

Example 2.2. *Consider the H-PPEP with* $d = 4$ *and* $A_1, A_3, A_4 \in \mathcal{C}_{n,100}$, *and* A_0 *and* A_2 *being defined as*

$$A_{k-1} = B_{1k} \cdot diag\left\{\varphi_1^{(k)}, \cdots, \varphi_n^{(k)}\right\} \cdot B_{2k} \in \mathbb{C}^{n\times n} \quad (k = 1, 3)$$

where $B_{1k}, B_{2k} \in \mathcal{C}_{n,1}$, *and*

$$\begin{cases} \varphi_i^{(k)} = 4^{i+k-\ell}, \ \varphi_{\ell+i}^{(k)} = 4^{i-k} \quad (i = 1, \dots, \ell), \\ \varphi_n^{(k)} = 4^{n/2-k} \quad \text{if } n \ \text{ is odd;} \end{cases} \tag{2.43}$$

with $\ell = n/2$ *(if* n *is even) or* $\ell = (n-1)/2$ *(otherwise).*

Example 2.3. *Consider the H-PPEP with* $d = 4$ *and* $A_0, A_1, A_3, A_4 \in \mathcal{C}_{n,100}$, *and* A_2 *being defined as*

$$A_2 = B_{13} \cdot diag\left\{\varphi_1^{(3)}, \cdots, \varphi_n^{(3)}\right\} \cdot B_{23} \in \mathbb{C}^{n\times n},$$

where $B_{13}, B_{23} \in \mathcal{C}_{n,1}$, *and* $\varphi_i^{(3)}$ *is defined in* (2.43) *with* $k = 3$.

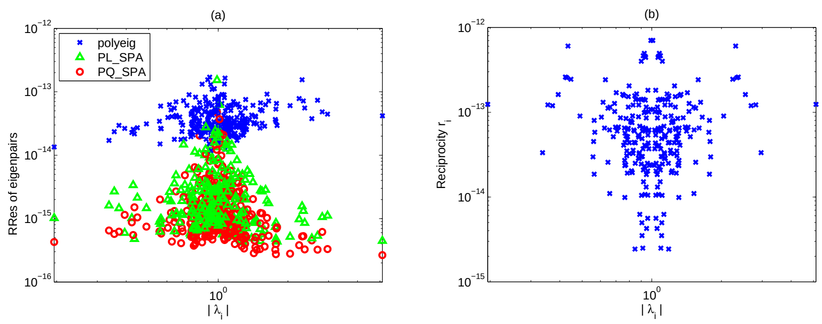

Figure 2.2: Relative residuals of eigenpairs and the associated reciprocity for Example 2.1.

We present the relative residuals (RRes) and the reciprocities of eigenpairs computed by the `polyeig`, PL_SPA and PQ_SPA for Examples 2.1–2.3, using the

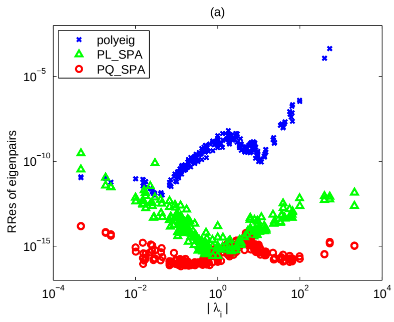

Figure 2.3: Relative residuals of eigenpairs and the associated reciprocity for Example 2.2 with larger $\|A_0\|_2$ and $\|A_2\|_2$.

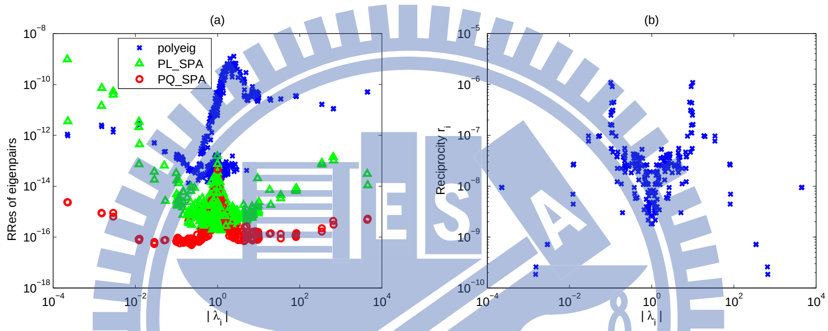

Figure 2.4: Relative residuals of eigenpairs and the associated reciprocity for Example 2.3 with larger $\|A_2\|_2$.

balancing technique in Section 2.5 with $n = 30$. Numerical results are shown in Figures 2.2–2.4. We indicate the results computed by `polyeig`, PL_SPA and PQ_SPA by "×", "△" and "o", respectively. For the PL_SPA and PQ_SPA, all reciprocities of eigenvalues are preserved to machine accuracy, which are ignored in Figures 2.2 (b)–2.4 (b). From Figures 2.2–2.4, we see that most of relative residuals of eigenpairs computed by the PQ_SPA are better than that computed by the PL_SPA, only a few exceptions. Overall, we conclude that applying P-quadratization and SPA (Algorithm 2.1) to solve PPEPs not only preserves the reciprocal property but also provides higher accuracy than that by PL_SPA and `polyeig` in MATLAB.

2.7 Conclusions

In this section, we mainly propose a palindromic quadratization to transform the $(\star, \varepsilon)$-palindromic matrix polynomial of even degree with $(\star, \varepsilon) \neq (T, -1)$ to a $(\star, \varepsilon)$-palindromic quadratic pencil, instead of the orthodox palindromic linearization approach. The structure-preserving algorithm for solving palindromic quadratic eigenvalue problem based on $(\mathcal{S}+\mathcal{S}^{-1})$-transform and Patel's algorithm can then be applied. Numerical experiments show that relative residuals of approximate eigenpairs for the palindromic polynomial eigenvalue problem computed by the PQ_SPA are better than those by the PL_SPA and `polyeig` in MATLAB. Moreover, the computational cost for PQ_SPA is much cheaper than that for PL_SPA.

3

Structured Factorization of Palindromic Matrix Polynomials of Odd Degree

Contents

3.1 Introduction

We are interesting in the palindromic matrix polynomial of degree d:

$$P_d(\lambda) \equiv \sum_{i=0}^{d} A_i \lambda^i, \quad A_i^\star = A_{d-i} \in \mathbb{C}^{n\times n} \quad (i = 0, \cdots, d) \tag{3.1}$$

with $\star = T, H$, and the associated palindromic eigenvalue problem (PEVP)

$$P_d(\lambda)x = 0, \quad x \neq 0. \tag{3.2}$$

Throughout the section, we assume that A_d is nonsingular and there is no zero or infinite eigenvalues for $P_d(\lambda)$, possibly after deflation.

With $\mu = \lambda^{1/2}$, any matrix polynomial of odd degree d can be re-written as a matrix polynomial of even degree $2d$ in μ, for which the palindromic quadratization approach in [27] can be applied. Interestingly, we only manage to factorize $P_d(\lambda)$ when the degree d is odd in this paper . Consequently, the factorization in this paper (for odd d) and the quadratization in [27] (for even d) complement each other, when apply to the solution of (3.2).

The main contribution of this paper is as follows. It has long been known, e.g. from [36, Theorem 3.7], that a matrix polynomial $P_d(\lambda)$ of degree d has a linear factor $(\lambda I - X)$ from the right where the solvent X is a solution of $P_d(X) = 0$; i.e., $P_d(\lambda) = P_{d-1}(\lambda)(\lambda I - X)$ for a matrix polynomial $P_{d-1}(\lambda)$ of degree $d - 1$. In order to preserving the "symplectic" property, we shall show for a palindromic matrix polynomial $P_d(\lambda)$ (which satisfies $[P_d(\lambda)]^\star = \lambda^d P_d(\lambda^{-1})$) of odd degree d can be factorized as

$$P_d(\lambda) = (\lambda I - X^{-\star})P_{d-2}(\lambda)X^{-1}(\lambda I - X) \tag{3.3}$$

where $P_{d-2}(\lambda)$ is a palindromic matrix polynomial of degree $d-2$ and X is a soluble solution of the solvent equation $P_d(X) = 0$. There may well be useful meaning on the structure-preserving solution of palindromic eigenvalue problems associated with $P_d(\lambda)$.

We shall explore the theoretical aspects of structured factorization of palindromic matrix polynomials. Numerical aspects, such as the solution of the associated PEVPs and nonlinear matrix equations, error analysis, operation counts, numerical experiments and comparison with other approaches, will be attempted elsewhere.

3.2 Palindromic Factorization for $d = 3$

For illustration and motivation, we first consider the cases $d = 3, 5$, before considering the general case.

In [?], the structure-preserving doubling algorithm (SDA) was based on the factorization for a palindromic matrix quadratic:

$$A_1^T\lambda^2 + A_0\lambda + A_1 = (\lambda A_1^T - X)X^{-1}(\lambda X - A_1), \quad A_0 = A_0^T \tag{3.4}$$

with X satisfying the nonlinear matrix equation [37]:

$$A_1^T X^{-1} A_1 + X + A_0 = 0. \tag{3.5}$$

Similar approaches involving nonlinear matrix equations or solvent equations, having been used by many others, can be found in, e.g., [5, 12, 18, 19, 28]. We first generalize this factorization matrix equation approach for $P_3(\lambda)$. We can verify that

$$P_3(\lambda) \equiv A_2^\star\lambda^3 + A_1^\star\lambda^2 + A_1\lambda + A_2 = (\lambda I - X^{-\star})(\lambda A_2^\star + X^\star A_2 X^{-1})(\lambda I - X) \tag{3.6}$$

with X satisfying the nonlinear matrix equation (NME)

$$X^{-\star}A_2^{\star}X - A_2X^{-1} - X^{\star}A_2 - A_1 = 0. \tag{3.7}$$

For the trivial case where $n = 1$, $A_2 = 0$ forces $A_1 = 0$. However, as we have assumed that A_2 is nonsingular (and thus cannot vanish), such a case will not occur.

After solving (3.7), the PEVP (3.2) can be solved, we may apply the QR algorithm to X and some other structure-preserving algorithm to the middle pencil in (3.6) (or the equivalent $\lambda A_2^{\star}X + (A_2^{\star}X)^{\star}$) in a structure-preserving way. Some eigenvectors may be directly obtained while others may have to be computed using inverse iteration.

The NME (3.7) is nonstandard and does not appear to be easy to solve. However, under mild conditions, (3.7) is equivalent to the more familiar solvent equation (SE) for $P_3(\lambda)$ in (3.6):

$$P_d(X) = \sum_{i=0}^{d} A_iX^i = 0. \tag{3.8}$$

Rewrite the PEVP (3.2) in the form

$$A_dV\Lambda^d + A_{d-1}V\Lambda^{d-1} + \cdots + A_1V\Lambda + A_0V = 0$$

with $\Lambda \in \mathbb{C}^{dn\times dn}$ in Jordan form containing all the eigenvalues and $V \in \mathbb{C}^{n\times dn}$ containing the corresponding eigenvectors. Consider a partition (possibly after a re-ordering of the eigenvalues and the corresponding eigenvectors)

$$\Lambda = \Lambda_1 \oplus \Lambda_2, \quad V = [V_1, V_2]; \quad \Lambda_1, V_1 \in \mathbb{C}^{n\times n}. \tag{3.9}$$

Recall that we have assumed that all Jordan blocks in λ are less than n in size, thus enabling the partition in (3.9). It is easy to check that $X = V_1\Lambda_1V_1^{-1}$ satisfy the SE

(3.8). In other words, the existence of the solution X of the SE is then guaranteed. We shall show that any solution of the NME (3.7) satisfies the SE (3.8), and any solved solution of the SE satisfies the NME.

Theorem 3.1. *A soluble X satisfies the nonlinear matrix equation* (3.7)*:*

$$X^{-\star}A_2^\star X - A_2X^{-1} - X^\star A_2 - A_1 = 0$$

if and only if it satisfies the solvent equation (3.8) *for $d = 3$: $P_3(X) = A_2^\star X^3 + A_1^\star X^2 + A_1X + A_2 = 0$.*

Proof. For necessity, the sum of (3.7) and its transpose/hermitian $(3.7)^\star$ times X implies

$$S(R(X)) \equiv R(X) + R(X)^\star X = 0 \tag{3.10}$$

where $R(x)$ denotes the operator on the left-hand-side of (3.7):

$$R(X) \equiv X^{-\star}A_2^\star X - A_2X^{-1} - X^\star A_2 - A_1. \tag{3.11}$$

We expand the equation, and (3.10) is equivalent to

$$(X^{-\star}A_2^\star X - A_2X^{-1} - X^\star A_2 - A_1) + (X^\star A_2X^{-1} - X^{-\star}A_2^\star - A_2^\star X - A_1^\star)X = 0$$

and, after simplification, is equivalent to

$$A_2^\star X^2 + A_1^\star X + A_1 + A_2X^{-1} = 0. \tag{3.12}$$

Equation (3.12) in turn is equivalent to the SE (3.1). (Other linear combinations of (3.7) and its transpose/hermitian, such as (3.7) $\times X$ + $(3.7)^\star$, $X^\star\times$ (3.7) + $(3.7)^\star$ or (3.7) + $X^{-\star}\times$ $(3.7)^\star$, lead to the same conclusion.)

For sufficiency, start from (3.12), which is equivalent to the SE (3.1) as well as (3.10), a $\star$-Sylvester [10]. The $\star$-Sylvester operator $S(\cdot)$ is invertible if and only if the eigenvalues λ_i $(i = 1, \cdots, n)$ of X satisfy [10, Theorem 2.1] (see Appendix I):

$$\lambda_i \neq -1 \quad (\forall i), \quad \lambda_i \neq \lambda_j^{-\star} \quad (\forall i \neq j).$$

These conditions are satisfied when X is solved with all λ_i inside the unit circle. Consequently, (3.10) implies that

$$R(X) = S^{-1}(0) = 0$$

or the NME (3.7). □

3.2.1 Solution of NME, SE or PEVP

If $\lambda_i = -1$, the problem can be transformed to have $\lambda_i = 1$ and the solution of the NME (3.7) can be obtained through the SE (3.1). Nonetheless, if we have both $\lambda = \pm 1$, then the transformation, and thus the equivalence of the NME and the SE, fail.

Define $Y = A_2^\star X$, the NME (3.7) has the form

$$A_2 Y^{-\star} Y - A_2 Y^{-1} A_2^\star - Y^\star - A_1 = 0. \tag{3.13}$$

Without the first term, (3.13) looks very similar to (3.5). Similar to (3.6), we have the corresponding factorization

$$P_3(\lambda) = (\lambda Y^\star - A_2) Y^{-\star} (\lambda Y + Y^\star) Y^{-1} (\lambda A_2^\star - Y) \tag{3.14}$$

The structure-preserving solution of the PEVP (3.2) then involves the following steps: Solve (3.7), or (3.13) for a solved X or associated Y. Apply the QR algorithm to $\lambda I - X$, or we can use the QZ algorithm to $\lambda A_2^\star - Y$. Using a structure-preserving algorithm to solve the palindromic linearization $\lambda Y + Y^\star$.

3.3 Palindromic Factorization for $d = 5$

Rewrite the palindromic matrix polynomial of degree $d = 2r + 1$ in the following symmetric form:

$$S_r(\lambda) \equiv \sum_{i=0}^{r} \left(M_i^\star \lambda^{i+1/2} + M_i \lambda^{-i-1/2} \right), \tag{3.15}$$

where $A_i = M_{i-r-1}^\star$ $(i > r)$ or $A_i = A_{d-i}^\star$ $(i \leq r)$, with $[x]$ denoting the largest integer less than x and $i = 1, \cdots, d$. Motivated by the $d = 3$ case, we are seeking the following structured factorization:

$$S_r(\lambda) = \left(I - \lambda^{-1} X^{-\star} \right) \left[\sum_{i=0}^{r-1} \left(B_i^\star \lambda^{i+1/2} + X^\star B_i X^{-1} \lambda^{-i-1/2} \right) \right] (\lambda I - X) \tag{3.16}$$

Note that the middle factor has the form

$$\sum_{i=0}^{r-1} \left(B_i^\star \lambda^{i+1/2} + X^\star B_i X^{-1} \lambda^{-i-1/2} \right) = S_{r-1}(\lambda) X^{-1}$$

where the symmetric

$$S_{r-1}(\lambda) \equiv \sum_{i=0}^{r-1} \left(\widetilde{B}_i^\star \lambda^{i+1/2} + \widetilde{B}_i \lambda^{-i-1/2} \right), \quad \widetilde{B}_i \equiv X^\star B_i$$

is a symmetric representation of a palindromic matrix polynomial of degree $d = 2(r-1)$. Consequently, in general, $P_d(\lambda)$ in (3.3) or its symmetric representation $S_r(\lambda)$ in (3.16) may be recursively factorized until only linear factors exist, if all the associated SEs can be solved. Note also that even if X is not exact, the palindromic structure in the factored form of $S_r(\lambda)$ is preserved.

Expanding (3.16) and matching the coefficient matrices for $d = 5$ and $r = 2$, we obtain

$$M_2 = B_1, \tag{3.17}$$

$$M_1 = -X^\star B_1 + B_0 - B_1 X^{-1}, \tag{3.18}$$

$$M_0 = X^\star B_1 X^{-1} + X^{-\star} B_0^\star X - X^\star B_0 - B_0 X^{-1}. \tag{3.19}$$

Substitute B_0 and B_1 from (3.17) and (3.18) into (3.19), we obtain the NME for X (for $d = 5$):

$$R(X) \equiv M_0 + X^\star M_1 + (X^\star)^2 M_2 + M_1 X^{-1} + M_2 X^{-2} \tag{3.20}$$

$$+X^\star M_2 X^{-1} - X^{-\star} M_1^\star X - X^{-\star} M_2^\star X^2 - (X^{-\star})^2 M_2^\star X = 0. \tag{3.21}$$

With nine terms (and in general $(r+1)^2$ terms), we don't like to solve such a long equation like this. However, similar to Section 2, we can prove that (3.20) is equivalent to (3.8) when a solved X is sought. Similar to (3.10), we have

$$\begin{aligned}
S(R(X)) &= R(X) + R(X)^\star X \\
&= M_0 + X^\star M_1 + (X^\star)^2 M_2 + M_1 X^{-1} + M_2 X^{-2} \\
&\quad +X^\star M_2 X^{-1} - X^{-\star} M_1^\star X - X^{-\star} M_2^\star X^2 - (X^{-\star})^2 M_2^\star X \\
&\quad +M_0^\star X + M_1^\star X^2 + M_2^\star X^3 + X^{-\star} M_1^\star X + (X^{-\star})^2 M_2^\star X \\
&\quad +X^{-\star} M_2^\star X^2 - X^\star M_1 - (X^\star)^2 M_2 - X^\star M_2 X^{-1} \\
&= M_2^\star X^3 + M_1^\star X^2 + M_0^\star X + M_0 + M_1 X^{-1} + M_2 X^{-2} \\
&= P_5(X) X^{-2} = 0.
\end{aligned}$$

With the same argument using the property of the $\star$-Sylevester operator $S(\cdot)$, we have shown a similar result as Theorem 2.1 that the NME (3.20) (or $R(X)=0$) and the SE (3.8) (or $P_5(X)=0$) are equivalent for a soluble X.

The challenge is to generalize the tedious argument for $d=3,5$ to the general case for all odd values of d.

3.4 Palindromic Factorization for Odd Degree d

We shall proof a general version of Theorem 2.1 by generalizing the results in Section 3.3 to a recursive argument, without writing down the tedious NMEs.

Theorem 3.2. *For odd values of $d=2r+1$, a soluble X satisfies the nonlinear matrix equation, obtained by matching the coefficient matrices of $S_r(\lambda)$ in (3.15) and its factorization in (3.16), if and only if it satisfies the solvent equation (3.8): $P_d(X)=0$.*

That is to say, any palindromic matrix polynomial of odd degree can be factorized in a structure-preserving way as in (3.16) by solving the corresponding solvent equation (3.8) for a solved X.

Proof. We shall follow the steps in Section 3.3 for the $d=5$ case. Expanding (3.16) and matching the coefficient matrices for a general $d=2r+1$, we obtain

$$M_r = B_{r-1}, \tag{3.22}$$

$$M_{r-1} = -X^\star B_{r-1} + B_{r-2} - B_{r-1}X^{-1}, \tag{3.23}$$

$$M_i = -X^\star B_i + B_{i-1} + X^\star B_{i+1}X^{-1} - B_iX^{-1} \quad (i=r-2,\cdots,2), \tag{3.24}$$

$$M_1 = -X^\star B_1 + B_0 + X^\star B_2X^{-1} - B_1X^{-1}, \tag{3.25}$$

$$M_0 = X^\star B_1X^{-1} + X^{-\star}B_0^\star X - X^\star B_0 - B_0X^{-1}. \tag{3.26}$$

The first d equations of (3.22)–(3.25) lead to all the values of B_i $(i = 0, \cdots, r-1)$, in terms of M_i $(i = 0, \cdots, r)$ and X. By substituting into the last equation (3.26), we obtain the NME, which is very tedious and we shall apply a recursive argument to avoid writing it down explicitly. Note that (3.22) and (3.23) are just special cases of (3.24) with $i = r, r-1$, in which those B_i with $i \geq r$ degenerate to zero.

From the last equation (3.26) and similar to (3.10) and (3.11), we obtain the NME:

$$R(X) \equiv M_0 + X^\star B_0 + B_0 X^{-1} - X^{-\star} B_0^\star X - X^\star B_1 X^{-1} = 0 \tag{3.27}$$

and the $\star$-Sylvester operator

$$S(R(X)) = R(X) + R(X)^\star X. \tag{3.28}$$

From (3.27) and (3.28), we obtain

$$\begin{aligned} S(R(X)) &= (M_0 + X^\star B_0 + B_0 X^{-1} - X^{-\star} B_0^\star X - X^\star B_1 X^{-1}) \\ &\quad + (M_0^\star X + B_0^\star X^2 + X^{-\star} B_0^\star X - X^\star B_0 - X^{-\star} B_1^\star X^2) \\ &= M_0^\star X + B_0^\star X^2 - X^{-\star} B_1^\star X^2 + M_0 + B_0 X^{-1} - X^\star B_1 X^{-1} \\ &= 0 \end{aligned} \tag{3.29}$$

For the initial step, substitute B_0 from (3.25) into (3.29), we have

$$\begin{aligned} S(R(X)) &= M_0^\star X + (M_1^\star + B_1^\star X + X^{-\star} B_1^\star - X^{-\star} B_2^\star X) X^2 - X^{-\star} B_1^\star X^2 + M_0 \\ &\quad + (M_1 + X^\star B_1 + B_1 X^{-1} - X^\star B_2 X^{-1}) X^{-1} - X^\star B_1 X^{-1} \\ &= M_0^\star X + M_1^\star X^2 + B_1^\star X^3 - X^{-\star} B_2^\star X^3 + M_0 + M_1 X^{-1} \\ &\quad + B_1 X^{-2} - X^\star B_2 X^{-2} = 0. \end{aligned}$$

It is then easy to see that the substitution of B_1 (and subsequently B_{i-1} for $i = 3, \cdots, r$) from (3.22)–(3.24), proves that

$$\begin{aligned} S(R(X)) &= M_0^\star X + M_0 + M_1^\star X^2 + M_1 X^{-1} + \cdots + M_r^\star X^{r+1} + M_r X^{-r} \\ &= S_r(X) X^{1/2} = P_d(X) X^{-r} = 0, \end{aligned}$$

which is equivalent to the SE (3.8).

With $S(\cdot)$ invertible for a soluble X, we have shown $S(R(X)) = 0 \Leftrightarrow R(X) = 0$ or the equivalence of the NME and the SE.

4

Conclusions and Future Work

In this thesis, we consider two themes related to palindromic matrix polynomials for solving nonlinear eigenvalue problems.

In the first topic (Chapter 2), we propose a palindromic quadratization to transform the $(\star, \varepsilon)$-palindromic matrix polynomial of even degree with $(\star, \varepsilon) \neq (T, -1)$ to a $(\star, \varepsilon)$-palindromic quadratic pencil which is emerged in solving higher order systems of ordinary or partial differential equations. Figure 2.1 illustrate the relationship among various structured polynomial eigenvalue problems. T-even, T-odd, T-anti-palindromic and T-palindromic polynomial eigenvalue problems of even degree can be P-quadratized to T-PQEPs. Consequently, the structure-preserving algorithm for solving palindromic quadratic eigenvalue problem based on $(\mathcal{S} + \mathcal{S}^{-1})$-transform and Patel's algorithm can be applied to slove the $(\star, \varepsilon)$-palindromic quadratic pencil($\cdot^{\top}$ case in [24] and we present $\cdot^{H}$ case). By the structure-preserving backward error analysis [47], we find the quadratization also caused considerable impact on our backward error. However, we provide a balancing technique that improve our backward error and easily implement in our original algorithm. Numerical experiments show that relative residuals of approximate eigenpairs for the palindromic polynomial eigenvalue problem computed by the PQ_SPA are better than those by the PL_SPA and `polyeig` in MATLAB. Furthermore, the computational cost for PQ_SPA is much cheaper than that for PL_SPA.

In chapter 3, we consider the structured factorization of a palindromic matrix polynomials of odd degree, instead of the orthodox palindromic linearization approach. We develop the theoretical aspects of structured factorization of palindromic matrix polynomials. To be in face of such factorizations, there are some difficult nonlinear matrix equations that have to be solved. However, we point out that these equations are equivalent to the well known solvent equation, when the solution X is solved. Without writing down the dreary NMEs, we provide a general version from the palindromic matrix polynomials of odd degree to a structure-preserving factorization by solving the solvent equation.

We have reported the numerical experiments which relative residuals of approximate eigenpairs for the palindromic polynomial eigenvalue problem with small size matrix. However, it is urgent and important to apply our quadratization to solve the large sparse matrix. In [24], W.-W. Lin propose Generalized $\top$-skew-Hamiltonian Arnoldi method to cope with $\top$-palindromic quadratic pencil. This will be a big step forward if we can use this technology to solve our $(\star, \varepsilon)$-palindromic quadratic pencil with quadratization transform.

Other polynomial eigenvalue problems of higher degree than two arise when discretizing linear eigenproblems by dynamic elements [46, 61] or by least squares elements [48], so it is essentially for us to deal with palindromic matrix polynomials of odd degree. Nevertheless, we face some difficult nonlinear matrix equations that have to be solved in section 3. The theoretical aspects of factorization of palindromic matrix polynomials cause another problem and it also cost numerous time for computer to solve the nonlinear matrix equations. Many pioneers do a lot of contributions in this field. For instance, V. Mehrmann [40] propose a "Good Linearizations" to settle structured factorization of a palindromic matrix polynomials. We also look forward to the palindromic matrix polynomials of odd degree can be decomposition well with different methods.

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