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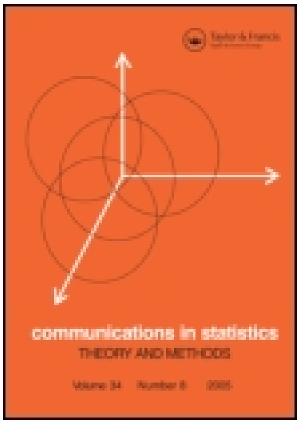

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Distributional and inferential properties of the process accuracy and process precision indices

W.L. Pearn [a] , G.H. Lin [b] & K.S. Chen [c]

[a] Department of Industrial Engineering & Management , National Chiao Tung University , Hsinchu , Taiwan

[b] Department of Industrial Engineering & Management , National Chiao Tung University , Hsinchu , Taiwan

[c] Department of Industrial Engineering & Management , National Chin-Yi Institute of Technology , Taichung , Taiwan

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DISTRIBUTIONAL AND INFERENTIAL PROPERTIES OF THE PROCESS ACCURACY AND PROCESS PRECISION INDICES

W. L. PEARN
Department of Industrial Engineering & Management
National Chiao Tung University
Hsinchu, Taiwan ROC

G. H. LIN
Department of Industrial Engineering & Management
National Chiao Tung University
Hsinchu, Taiwan ROC

K. S. CHEN
Department of Industrial Engineering & Management
National Chin-Yi Institute of Technology
Taichung, Taiwan ROC

Keywords and Phrases: process accuracy index; process precision index; process yield; process mean; process standard deviation.

ABSTRACT

Process capability indices such as C_p, k, and C_{pk}, have been widely used in manufacturing industry to provide numerical measures on process potential and performance. While C_p measures overall process variation, k measures the degree of process departure. In this paper, we consider the index C_p and a transformation of k defined as $C_a = 1 - k$ which measures the degree of process centering. We refer to C_p as the process precision index, and C_a as the process accuracy index. We consider the estimators of C_p and C_a, and investigate their statistical

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properties. For C_p, we obtain the UMVUE and the MLE. We show that this UMVUE is consistent, and asymptotically efficient. For C_a, we investigate its natural estimator. We obtain the first two moments of this estimator, and show that the natural estimator is the MLE, which is asymptotically unbiased and asymptotically efficient. We also propose an efficient test based on the UMVUE of C_p. We show that the proposed test is the UMP test.

1. INTRODUCTION

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications (including the target value and specification limits), have been the focus of recent research in quality assurance and process capability (quality) analysis. Those capability indices, quantifying process potential and performance, are important for any successful quality improvement activities and quality program implementation. Several indices widely used in manufacturing industry providing numerical measures on whether a process meets the preset quality requirement, include C_p, k, and C_{pk} which are defined as the following (see Kane (1986)):

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$k = \frac{|\mu - m|}{d},$$

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

where USL and LSL are the upper and the lower specification limits preset by the process engineers or product designers, μ is the process mean, σ is the process standard deviation, m is the mid-point between the upper and the lower specification limits (m = (USL + LSL)/2), and d is half length of the specification interval (d = (USL - LSL)/2). We have assumed the target value T = m (which is quite common in practical situations) for simplicity of our discussions.

The index C_p was designed to measure the magnitude of the overall process variation. For processes with two-sided specification limits, the percentage of nonconforming items (%NC) can be calculated as 1 - F(USL) + F(LSL), where F(·) is the cumulative distribution function of the process characteristic X. On the assumption of normality, %NC can be expressed as:

$$\%NC = 1 - \Phi\left\{\frac{USL - \mu}{\sigma}\right\} + \Phi\left\{\frac{LSL - \mu}{\sigma}\right\},$$

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where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution. If the process is perfectly centered, then %NC can be expressed alternatively as %NC = $2 - 2\Phi(3C_p)$. For example, $C_p = 1.00$ corresponds to %NC = 2700 ppm, and $C_p = 1.33$ corresponds to %NC = 63 ppm. Thus, the index C_p provides an exact measure of the actual process yield. Since C_p measures the magnitude of process variation, C_p may be viewed as a *process precision* index.

While the precision index C_p measures the magnitude of process variation, the index k measures the departure of process mean, μ, from the center-point m. Therefore, the transformation of k defined as $C_a = 1 - k$ measures the degree of process centering (the ability to cluster around the center), which can be regarded as a *process accuracy* index. For example, $C_a = 1$ indicates that the process is perfectly centered ($\mu = m$), $C_a > 1/2$ indicates that μ is within half of the specification interval, and $C_a = 0$ indicates that μ is on the specification limits (μ = USL, or μ = LSL). On the other hand, if $C_a < 0$ then it indicates that μ falls outside the specification limits (μ > USL, or μ < LSL). Obviously, the process is severely off-center and it needs an immediate troubleshooting.

2. ESTIMATION OF C_p

To estimate the precision index C_p, we consider the natural estimator $\hat{C}_p$ defined as the following, where $S = [\sum_{i=1}^{n} (X_i - \overline{X})^2/(n-1)]^{1/2}$ is the conventional estimator of the process standard deviation σ, which may be obtained from a stable process.

$$\hat{C}_p = \frac{USL - LSL}{6S}.$$

The natural estimator $\hat{C}_p$ can be alternatively written as:

$$\hat{C}_p = (n-1)^{1/2}\left(\frac{USL - LSL}{6\sigma}\right)\left[\frac{(n-1)S^2}{\sigma^2}\right]^{-1/2} = (n-1)^{1/2} C_p \left[\frac{(n-1)S^2}{\sigma^2}\right]^{-1/2}.$$

On the assumption of normality, the statistic $(n-1)S^2/\sigma^2$ is distributed as χ^2_{n-1}, a chi-square with n - 1 degrees of freedom. Therefore, the probability density function of $\hat{C}_p$ can be expressed as (Chou and Owen (1989)):

$$f(x) = 2\frac{\left(\sqrt{(n-1)/2}\, C_p\right)^{n-1}}{\Gamma[(n-1)/2]}(x)^{-n}\exp[-(n-1)(C_p)^2(2x^2)^{-1}],$$

for $x > 0$. The r-th moment of $\hat{C}_p$, therefore can be calculated as the following:

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$$E\left(\left[\hat{C}_p\right]^r\right) = \frac{\Gamma\left[\frac{n-r-1}{2}\right]}{\Gamma\left[\frac{n-1}{2}\right]}\left[\frac{(n-1)C_p^2}{2}\right]^{r/2},$$

and the first two moments as well as the variance may be obtained as (see also Chou and Owen (1989), and Pearn, Kotz and Johnson (1992)):

$$E\left(\hat{C}_p\right) = \frac{\Gamma\left[\frac{n-2}{2}\right]}{\Gamma\left[\frac{n-1}{2}\right]}\sqrt{\frac{(n-1)}{2}}\, C_p\,,$$

$$E\left(\left[\hat{C}_p\right]^2\right) = \frac{\Gamma\left[\frac{n-3}{2}\right]}{\Gamma\left[\frac{n-1}{2}\right]}\frac{(n-1)C_p^2}{2} = \frac{n-1}{n-3}C_p^2\,,$$

$$\mathrm{Var}\left(\hat{C}_p\right) = \left\{\frac{n-1}{n-3} - \frac{n-1}{2}\Gamma\left[\frac{n-2}{2}\right]^2\Gamma\left[\frac{n-1}{2}\right]^{-2}\right\}C_p^2\,.$$

It can be shown that the coefficient of $E(\hat{C}_p)$, $[(n-1)/2]^{1/2}\Gamma[(n-2)/2]/\Gamma[(n-1)/2] > 1$ for all n. For $n \geq 15$, this coefficient can be accurately approximated by $(4n-4)/(4n-7)$. Therefore, the natural estimator $\hat{C}_p$ is biased, which over-estimates the actual value of C_p. Table 1 displays the values of $E(\hat{C}_p)$ under the condition $C_p = 1$ for various sample sizes n. For the percentage bias to be less than one percent ($|E(\hat{C}_p) - C_p|/C_p \leq 0.01$), it requires the sample size $n > 80$.

By setting

$$b_f = \frac{\Gamma\left[\frac{n-1}{2}\right]}{\Gamma\left[\frac{n-2}{2}\right]}\left(\sqrt{\frac{(n-1)}{2}}\right)^{-1},$$

we may obtain an unbiased estimator $\tilde{C}_p = b_f\hat{C}_p$. That is, $E(\tilde{C}_p) = C_p$. Since $b_f < 1$, then the variance of $\tilde{C}_p$ is smaller than that of the natural estimator $\hat{C}_p$. That is, $\mathrm{Var}(\tilde{C}_p) < \mathrm{Var}(\hat{C}_p)$. In the following, we investigate the statistical properties of $\tilde{C}_p$. We show that $\tilde{C}_p$ is the UMVUE of C_p, which is consistent and asymptotically efficient.

An estimator $\hat{\theta}_n$ of θ is said to be consistent if for all $\varepsilon > 0$, $p(|\hat{\theta}_n - \theta| > \varepsilon) \to 0$ as $n \to \infty$ for all θ. A sufficient condition for the consistency is that $E(\hat{\theta}_n) \to \theta$ and $\mathrm{Var}(\hat{\theta}_n) \to 0$. Under regular conditions, the estimator $\hat{\theta}_n$ is said to be asymptotically efficient if $\hat{\theta}_n$ is asymptotically normal, $n^{1/2}[\hat{\theta}_n - E(\hat{\theta}_n)] \to 0$, and $\{n\,\mathrm{Var}[\hat{\theta}_n - E(\hat{\theta}_n)]\}$ converges to the Cramer-Rao Bound.

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Table 1. Values of $E(\hat{C}_p)$ corresponding to $C_p = 1$ for various sample sizes n.

Sample size	$E(\hat{C}_p)$	Sample size	$E(\hat{C}_p)$
10	1.094	100	1.008
20	1.042	110	1.007
30	1.027	120	1.006
40	1.020	140	1.005
50	1.016	170	1.004
60	1.013	220	1.003
70	1.011	310	1.002
80	1.010	510	1.001
90	1.009	1690	1.000

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Theorem 1. If the process characteristic follows normal distribution, then

(a) $\tilde{C}_p$ is the UMVUE of C_p.
(b) $\tilde{C}_p$ is consistent.
(c) $n^{1/2}(\tilde{C}_p - C_p)$ converges to $N(0, [C_p]^2/2)$ in distribution.
(d) $\tilde{C}_p$ is asymptotically efficient.

Proof: (a) We first note that the statistic $(\overline{X}, S^2)$ is sufficient and complete for (μ, σ^2). Since $E(\tilde{C}_p) = C_p$, and $\tilde{C}_p$ is a function of $(\overline{X}, S^2)$ only, then by Lehmann-Scheffe' Theorem (Arnold (1990)) $\tilde{C}_p$ is the UMVUE of C_p.

(b) For all $\varepsilon > 0$, $p(|\tilde{C}_p - C_p| > \varepsilon) < E(\tilde{C}_p - C_p)^2/\varepsilon^2$. Now, $E(\tilde{C}_p - C_p)^2 = Var(\tilde{C}_p) = E(\tilde{C}_p)^2 - C_p^2$. By Stirling's formula, we can show that $E(\tilde{C}_p)^2$ converges to C_p^2. Hence, $E(\tilde{C}_p - C_p)^2$ converges to zero. Therefore, $\tilde{C}_p$ converges to C_p in probability and $\tilde{C}_p$ must be consistent.

(c) If the process characteristic is normally distributed, then it is clear that the statistic $n^{1/2}(S^2 - \sigma^2)$ converges to $N(0, 2\sigma^4)$ in distribution. We apply Cramer-δ Theorem (Arnold (1990)) with g(t) defined as $g(t) = d/(3t^{1/2})$. Since $g'(t) = -d/(6t^{3/2})$, and $[g'(\sigma^2)]^2 = (C_p)^2/(4\sigma^4)$, then $n^{1/2}[g(S^2) - g(\sigma^2)] = n^{1/2}[d/(3S) - d/(3\sigma)] = n^{1/2}(\hat{C}_p - C_p)$ converges to $N(0, 2\sigma^4[g'(\sigma^2)]^2)$, or $N(0, [C_p]^2/2)$ in distribution. By (b) $\tilde{C}_p$ converges to C_p in probability, then by Slutzky's Theorem (Arnold (1990)) $n^{1/2}(\tilde{C}_p - \hat{C}_p)$ converges to $N(0, [C_p]^2/2)$, and so $n^{1/2}(\tilde{C}_p - C_p)$ converges to $N(0, [C_p]^2/2)$ in distribution.

If the knowledge on whether $p(\mu \geq m) = 1$, or 0 is available, then we can consider the estimator $\tilde{C}_a = 1 - [(\bar{X} - m)\,\mathrm{sgn}(\mu - m)]/d$, where $\mathrm{sgn}(\mu - m) = 1$ if $\mu - m \geq 0$, and $\mathrm{sgn}(\mu - m) = -1$ if $\mu - m < 0$. Thus, $\tilde{C}_a = 1 - (\bar{X} - m)/d$ if $\mu \geq m$, and $\tilde{C}_a = 1 - (m - \bar{X})/d$ if $\mu < m$. We can show that the estimator $\tilde{C}_a$ is the MLE, and the UMVUE of C_a. We can also show that the estimator $\tilde{C}_a$ is consistent, and efficient.

Theorem 3. If the process characteristic follows normal distribution, then

(a) $\tilde{C}_a$ is the MLE of C_a.
(b) $\tilde{C}_a$ is the UMVUE of C_a.
(c) $\tilde{C}_a$ is consistent.
(d) $n^{1/2}(\tilde{C}_a - C_a)$ converges to $N(0, 1/[3C_p]^2)$ in distribution.

Proof: (a) We first note that the statistic $(\bar{X}, [(n-1)/n]S^2)$ is the MLE of (μ, σ^2). By the invariance property of the MLE, $\hat{k}$ is the MLE of k, and $\tilde{C}_a$ is the MLE of C_a.

(b) From Theorem 2(d), the Cramer-Rao bound $= 1/[9n\,C_p^2]$. Since $\tilde{C}_a$ is distributed as $N(C_a, 1/[9nC_p^2])$, then $\tilde{C}_a$ is efficient for C_a, and is the UMVUE of C_a.

(c) For all $\varepsilon > 0$, $p(|\tilde{C}_a - C_a| > \varepsilon) < E(\tilde{C}_a - C_a)^2/\varepsilon^2$. Now, $E(\tilde{C}_a - C_a)^2 = \mathrm{Var}(\tilde{C}_a) = 1/[9nC_p^2]$ converges to zero, and so $\tilde{C}_a$ must be consistent.

(d) From (c), $\tilde{C}_a$ converges to C_a in probability. From Theorem 2(c), $n^{1/2}(\hat{C}_a - C_a)$ converges to $N(0, 1/[3C_p]^2)$ in distribution. By Slutzky's Theorem (Arnold (1990)), $n^{1/2}(\tilde{C}_a - C_a)$ converges to $N(0, 1/[3C_p]^2)$ in distribution..

In fact, since $\bar{X}$ and S^2 are mutually independent, then $\tilde{C}_a$ and $\tilde{C}_p$ are also mutually independent. Therefore, since $Z = 3\sqrt{n}\,C_p(\tilde{C}_a - C_a)$ is distributed as N(0, 1) and $W = (n-1)(b_f C_p)^2/[\tilde{C}_p]^2$ is distributed as χ^2_{n-1}, then $3n^{1/2}\tilde{C}_p(\tilde{C}_a - C_a)/b_f = Z/[W/(n-1)]^{1/2}$ is distributed as t_{n-1}, a t distribution with n - 1 degrees of freedom. Therefore, the α-level confidence interval for C_a can be established as:

$$\left[\tilde{C}_a - \frac{b_f\, t_{n-1,\alpha/2}}{3\sqrt{n}\,\tilde{C}_p},\ \tilde{C}_a + \frac{b_f\, t_{n-1,\alpha/2}}{3\sqrt{n}\,\tilde{C}_p}\right].$$

where $t_{n-1,\alpha}$ is the upper α-th quantile of the t_{n-1} distribution. The length, l', of the confidence interval, therefore is

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Table 4. $E(l')$ for C_a with $C_p = 1$ and $\alpha = 0.05$.

Sample Size	E(l')	Sample Size	E(l')
10	0.464	120	0.120
20	0.308	140	0.111
30	0.247	160	0.104
40	0.212	180	0.100
50	0.189	200	0.093
60	0.171	220	0.088
70	0.158	240	0.085
80	0.148	260	0.081
90	0.139	280	0.078
100	0.132	300	0.076

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$$l' = 2 \times \frac{b_f\, t_{n-1,\alpha/2}}{3\sqrt{n}\,\tilde{C}_p}.$$

The expected value $E(l')$ and the variance $\mathrm{Var}(l')$ of the length of the confidence interval l', can be found as:

$$E(l') = 2\left(\frac{t_{n-1,\alpha/2}}{3\sqrt{n}}\right)\sqrt{\frac{2}{n-1}}\,\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\,\frac{1}{C_p},$$

$$\mathrm{Var}(l') = 4\left(\frac{t_{n-1,\alpha/2}}{3\sqrt{n}}\right)^2\left\{1-\left[1-\sqrt{\frac{2}{n-1}}\,\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}\right]^2\right\}\frac{1}{(C_p)^2},$$

for given sample size n. Table 4 displays the expected lengths, $E(l')$, of the α-level confidence intervals for the accuracy index C_a with $C_p = 1$ and $\alpha = 0.05$ for various sample sizes n.

4. TESTS FOR PROCESS CAPABILITY

To judge whether a given process meets the preset capability requirement and runs under the desired quality condition. We can consider the following statistical testing hypothesis for C_p: H_0: $C_p \le C$, and H_1: $C_p > C$. Process fails to meet the capability (quality) requirement if $C_p \le C$, and meets the capability requirement if $C_p > C$. We define the test $\phi^*(x)$ as: $\phi^*(x) = 1$ if $\tilde{C}_p > c_0$, and $\phi^*(x) = 0$ otherwise. Thus, the test ϕ^* rejects the null hypothesis H_0 ($C_p \le C$) if $\tilde{C}_p > c_0$,

Table 5. Critical values c_0 for C = 1.00 with n = 10(10)100, and α = 0.01, 0.025, 0.05.

Sample size	0.01	0.025	0.05
10	1.897	1.668	1.504
20	1.514	1.402	1.315
30	1.389	1.309	1.246
40	1.323	1.259	1.208
50	1.281	1.227	1.183
60	1.252	1.204	1.165
70	1.230	1.187	1.152
80	1.212	1.173	1.141
90	1.198	1.162	1.132
100	1.187	1.153	1.125

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with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly judging an incapable process ($C_p \le C$) as capable ($C_p > C$). The critical value, c_0, can be determined as:

$$p\{\tilde{C}_p > c_0 \mid C_p = C\} = \alpha$$

$$p\{[b_f \sqrt{n-1}\, C_p](K)^{-1/2} \ge c_0 \mid C_p = C\} = \alpha$$

$$p\{K \le (b_f)^2 (n-1)\left[\frac{C}{c_0}\right]^2\} = \alpha$$

Hence, we have

$$(b_f)^2 (n-1)\left[\frac{C}{c_0}\right]^2 = \chi^2_{n-1,\alpha}$$

where $\chi^2_{n-1,\alpha}$ is the lower α-th quantile of χ^2_{n-1} distribution, or,

$$c_0 = \frac{b_f \sqrt{n-1}\, C}{\sqrt{\chi^2_{n-1,\alpha}}}.$$

Therefore, if $\tilde{C}_p > c_0$, then $\phi^*(x) = 1$ and we reject the null hypothesis H_0 and conclude that the process meets the capability requirement ($C_p > C$). Otherwise, we can not conclude that the process meets the capability requirement. Tables 5 displays the critical values c_0 for C = 1.00 with sample sizes n = 10(10)100, and α-risk = 0.01, 0.025, 0.05 (the chance of incorrectly concluding a process with $C_p \le$ C as one with $C_p > C$).

Theorem 4. For the testing hypothesis H_0: $C_p \leq C$ and H_1: $C_p > C$, the test defined as $\phi^*(x) = 1$ if $\tilde{C}_p > c_0$, and $\phi^*(x) = 0$ otherwise, is the UMP test of level α, where c_0 is determined by $E_c[\phi^*(x)] = \alpha$..

Proof: For the test, the power function is:

$$\beta(C_p, \phi^*) = E_{C_p}[\phi^*(x)] = P_{C_p}\left[\chi^2_{n-1} < \frac{(n-1)\, b_f^2\, C_p^2}{c_0^2}\right].$$

For $\alpha(c_0) = \alpha$, $c_0 = \dfrac{C\, b_f \sqrt{n-1}}{\sqrt{\chi^2_{n-1,\,1-\alpha}}}$, where $\chi^2_{n-1,\,1-\alpha}$ satisfies

$P[\chi^2_{n-1} > \chi^2_{n-1,\,1-\alpha}] = 1 - \alpha.$

Since for $C_p' > C_p > 0$, $\dfrac{f_{\tilde{C}_p}(x', C_p')}{f_{\tilde{C}_p}(x', C_p)} > \dfrac{f_{\tilde{C}_p}(x, C_p')}{f_{\tilde{C}_p}(x, C_p)}$ if and only if $x' > x > 0$,

then $\{f_{\tilde{C}_p}(x, C_p) \mid C_p > 0\}$ has MLR (monotone likelihood ratio) property in $\tilde{C}_p$. Therefore, the test ϕ^* must be the UMP test.

5. CONCLUSIONS

Process capability indices such as C_p, k, and C_{pk}, have been widely used in manufacturing industry to provide numerical measures on process potential and performance. The index C_p measures the overall process variation, and the index k measures the degree of process departure. In this paper, we considered C_p and a transformation of k defined as $C_a = 1 - k$. We referred to C_p as the process precision index, and C_a as the process accuracy index which measures the degree of process centering.

We considered the estimators of C_p and C_a, and investigated their statistical properties. For C_p, we obtained the UMVUE and the MLE. We showed that this UMVUE is consistent, and asymptotically efficient. For C_a, we investigated its natural estimator. We showed that this natural estimator is the MLE, which is asymptotically unbiased and asymptotically efficient. In addition, we proposed an efficient test based on the UMVUE of C_p. Using this test, the practitioners can judge whether their processes meet the capability requirement preset in the factory. We showed that the proposed test is in fact the UMP test.

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Appendix 1

Theorem 1: The probability density function of $\hat{C}_a$ can be expressed as:

$$f(x) = 6\,C_p\sqrt{\frac{n}{2\pi}}\cosh\{9n[C_p]^2 k(1-x)\}\ \exp\left[\frac{-9n(C_p)^2[(1-x)^2+k^2]}{2}\right].$$

Proof: For $-\infty < x \leq 1$, the probability density function f(x) is

$$f(x) = \frac{d}{dx}p(\hat{C}_a \leq x) = \frac{d}{dx}p(1-\hat{k} \leq x) = \frac{d}{dx}p(\hat{k} \geq 1-x) = \frac{d}{dx}\{1 - p(\hat{k} \leq 1-x)\}$$

$= g(1-x)$, where $g(x)$ is the probability density function of $\hat{k}$.

Now, the statistic $Y = \frac{\bar{X}-m}{d}$ is distributed as $N\left(\frac{\mu - m}{d}, [9n(C_p)^2]^{-1}\right)$,
a normal distribution with mean $\mu_Y = (\mu - m)/d$ and variance $\sigma^2_Y = [9n(C_p)^2]^{-1}$.

Since $\hat{k} = |Y| = \left|\frac{\bar{X}-m}{d}\right|$ has a folded normal distribution, then the probability density function of $\hat{k}$ is:

$$g(y) = \phi(y) + \phi(-y)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y}\left\{\exp\left[-\frac{(y-\mu_Y)^2}{2(\sigma_Y)^2}\right] + \exp\left[-\frac{(y+\mu_Y)^2}{2(\sigma_Y)^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y}\exp\left[-\frac{y^2+(\mu_Y)^2}{2(\sigma_Y)^2}\right]\left\{\exp\left[\frac{y\times\mu_Y}{(\sigma_Y)^2}\right] + \exp\left[-\frac{y\times\mu_Y}{(\sigma_Y)^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y}\,2\exp\left[-\frac{y^2+(\mu_Y)^2}{2(\sigma_Y)^2}\right]\cosh\left(\frac{y\times\mu_Y}{(\sigma_Y)^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y}\,2\cosh\left(9nk(C_p)^2 y\right)\exp\left[\frac{-9n(C_p)^2[y^2+k^2]}{2}\right]$$

$$= 6\,C_p\sqrt{\frac{n}{2\pi}}\cosh\{9n[C_p]^2 k\,y\}\ \exp\left[\frac{-9n(C_p)^2[y^2+k^2]}{2}\right].$$

Therefore, the probability density function f(x) is

$$f(x) = 6\,C_p\sqrt{\frac{n}{2\pi}}\cosh\{9n[C_p]^2 k(1-x)\}\ \exp\left[\frac{-9n(C_p)^2[(1-x)^2+k^2]}{2}\right].$$

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Appendix 2

Theorem 2: The first two moments of $\hat{C}_a$ are:

$$E(\hat{C}_a) = C_a - \frac{1}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left[-\frac{\delta}{2}\right] + 2(1 - C_a)\,\Phi\{-3\sqrt{n}(C_p - C_{pk})\},$$

$$E[(\hat{C}_a)^2] = (C_a)^2 + \frac{1}{9n(C_p)^2} - \frac{2}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left[-\frac{\delta}{2}\right] +$$

$$4(1 - C_a)\,\Phi\{-3\sqrt{n}(C_p - C_{pk})\}, \text{ where } \delta = 9n(C_p - C_{pk})^2.$$

Proof: For simplicity of the derivation of the exact formulae for the moments, we assume that $\mu \geq m$. For the other case, $\mu < m$, the derivation and the result will be the same. From Theorem 1, the probability density function of $\hat{k}$ is

$$g(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y}\left\{\exp\left[-\frac{(y - \mu_Y)^2}{2(\sigma_Y)^2}\right] + \exp\left[-\frac{(y + \mu_Y)^2}{2(\sigma_Y)^2}\right]\right\}.$$

Therefore,

$$E[\hat{k}] = \sqrt{\frac{2}{\pi}}\,\sigma_Y\exp\left[-\frac{1}{2}\left(\frac{\mu_Y}{\sigma_Y}\right)^2\right] + \mu_Y\left[1 - 2\,\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}\frac{1}{9n(C_p)^2}}\exp\left[-\frac{9n}{2}(C_p)^2k^2\right] + k\left[1 - 2\,\Phi(-3\sqrt{n}\,k\,C_p)\right]$$

$$= \sqrt{\frac{2}{n\pi}}\frac{1}{3C_p}\exp\left[-\frac{9n}{2}[C_p(1 - C_a)]^2\right] + k\left\{1 - 2\,\Phi[-3\sqrt{n}\,C_p(1 - C_a)]\right\}$$

$$= \sqrt{\frac{2}{n\pi}}\frac{1}{3C_p}\exp\left[-\frac{9n}{2}(C_p - C_{pk})^2\right] + k\left\{1 - 2\,\Phi[-3\sqrt{n}(C_p - C_{pk})]\right\}$$

$$= \sqrt{\frac{2}{n\pi}}\frac{1}{3C_p}\exp\left[-\frac{\delta}{2}\right] + k\left\{1 - 2\,\Phi[-3\sqrt{n}(C_p - C_{pk})]\right\}.$$

$$E\left(\hat{k}^2\right) = E(Y^2) = \mu_Y^2 + \sigma_Y^2 = k^2 + \frac{1}{9n(C_p)^2}$$

$$= (1 - C_a)^2 + \frac{1}{9n(C_p)^2} = \frac{9n(C_p)^2(1 - C_a)^2 + 1}{9n(C_p)^2} = \frac{9n(C_p - C_{pk})^2 + 1}{9n(C_p)^2}$$

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$$= \frac{\delta + 1}{9\,n\,(C_p)^2}\text{, where } \delta = 9\,n\,(C_p - C_{pk})^2.$$

Then, we may obtain $E[(\widehat{C}_a)$ and $E[(\widehat{C}_a)^2]$ as:

$$E[(\widehat{C}_a) = 1 - E(\widehat{k})$$

$$= 1 - \sqrt{\frac{2}{n\pi}}\,\frac{1}{3C_p}\,\exp\left[-\frac{\delta}{2}\right] - k\,\left\{1 - 2\,\Phi[-3\sqrt{n}\,(C_p - C_{pk})]\right\}$$

$$= (1 - k) - \sqrt{\frac{2}{n\pi}}\,\frac{1}{3C_p}\,\exp\left[-\frac{\delta}{2}\right] + 2k\,\left\{\Phi[-3\sqrt{n}\,(C_p - C_{pk})]\right\}$$

$$= C_a - \sqrt{\frac{2}{n\pi}}\,\frac{1}{3C_p}\,\exp\left[-\frac{\delta}{2}\right] + 2(1 - C_a)\,\left\{\Phi[-3\sqrt{n}\,(C_p - C_{pk})]\right\}.$$

$$E[(\widehat{C}_a)^2] = [E(\widehat{k})]^2 - 2\,E(\widehat{k}) + 1$$

$$= k^2 + \frac{1}{9n\,(C_p)^2} - 2\left\{\frac{1}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left[-\frac{\delta}{2}\right] + k - 2k\,\Phi[-3\sqrt{n}\,(C_p - C_{pk})]\right\} + 1$$

$$= (k^2 - 2k + 1) + \frac{1}{9n\,(C_p)^2} - \frac{2}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left[-\frac{\delta}{2}\right] + 4k\,\Phi\{-3\sqrt{n}\,(C_p - C_{pk})\}$$

$$= (C_a)^2 + \frac{1}{9n\,(C_p)^2} - \frac{2}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left[-\frac{\delta}{2}\right] + 4(1 - C_a)\,\Phi\{-3\sqrt{n}\,(C_p - C_{pk})\}.$$

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