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碩 士 論 文

志村曲線之定義方程式

Equations of Shimura Curves

研 究 生：施建宏

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中 華 民 國 一 百 零 二 年 七 月

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志村曲線之定義方程式

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摘　　要

論文中，我們將決定數個志村曲線的方程式。論文中最主要的想法是建構兩個合適且 weight 為 0 的 Borcherds form 以產生在$X_0^D(1)$模函數組成的體，藉由 Schofer 的方程式來計算不同 Borcherds form 在 CM 點上的值最後找出兩個 Borcherds form 的關聯性。

Equations of Shimura Curves

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ABSTRACT

In the thesis， we will determine the equations of Shimura curves $X_0^D(1)/W_D$ of genus one for several D， where W_D denotes the group of all Atkin-Lehner involutions on $X_0^D(1)$. The main idea is to construct two suitable Borcherds forms of weight 0 that generate the field of modular functions on $X_0^D(1)/W_D$ and use Schofer's formula for values of Borcherds forms at CM-points to find the relation between the two Borcherds forms.

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謹誌于交通大學

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目　　　　錄

1. Introduction

Let $X_0^D(N)$ denote the Shimura curve associated to the Eichler order of level N in an indefinite quaternion algebra of discriminant D. In [17], Shimura showed that $X_0^D(N)$ is the moduli space of principally polarized abelian surfaces with quaternionic multiplication by the Eichler order. Furthermore, he proved that $X_0^D(N)$ are algebraic curves defined over $\mathbb{Q}$. However, because of the lack of effective methods in constructing modular functions on Shimura curves, it has been very difficult to determine equations for Shimura curves and there are only a handful of Shimura curves whose equations are known [10, 11, 13, 14, 15]. All the existing methods rely on the Cerednik-Drinfeld theory [5, 6, 7] of p-adic uniformization of Shimura curves to some degree.

In [12], as an application of their explicit methods for Shimura curves, Guo, Lin, and Yang determine the equations of all hyperelliptic Shimura curves $X_0^D(N)$ whose hyperelliptic involutions are Atkin-Lehner involutions. (In particular, the genus of $X_0^D(N)/W_{D,N}$ is zero for the curves they consider, where $W_{D,N}$ denotes the group of all Atkin-Lehner involutions on $X_0^D(N)$.) The methods of [12] do not rely on the Cerednik-Drinfeld theory of p-adic uniformization of Shimura curves. Instead, the idea is simple in the sense that if we can find two modular functions x and y on the Shimura curves that generate the field of modular functions and suppose that we can determine the values of the modular functions x and y at sufficiently many points, then the relation between x and y can be easily determined. Now the construction of modular functions is done via the theory of Borcherds forms [3, 4] and Schofer's formula [16], together with the formula of Kudla, Rapoport, and Yang [], yield the values of these Borcherds forms at CM-points. Then equations of Shimura curves follow.

The idea of using Borcherds forms to study Shimura curves is not new. For instance, Errthum [9] used Borcherds forms and Schofer's formula to determine the values of Hauptmoduls of the Shimura curves $X_0^6(1)/W_6$ and $X_0^{10}(1)/W_{10}$ at CM-points, known as singular moduli, verifying Elkies' numerical computation [8]. Also, Schofer himself uses his formula to give a criterion for what primes can appear in the prime factorization of the norm of the differences between two singular moduli. This criterion is an analogue of the result of Gross and Zagier for the case of the classical modular curve $X_0(1)$. The main contribution in [12] is a systematic way to construct Borcherds forms.

In [12], the authors also determine the equations of Shimura curves $X_0^D(1)/W_D$ of positive genus for several D and discuss how to use Borcherds forms to compute Hecke operators and heights of CM-divisors on the Jacobian variety of $X_0^D(1)/W_D$. The main goal of this thesis is to give more examples of equations of Shimura curves.

2. Quaternion Algebras and Shimura Curves

The following contents are mostly from Bayer [1, 18].

A K-algebra B (associative and with unity) is a vector space over a field K with ring structure and with unity.

Definition 1. *A quaternion K-algebra B is a central simple K-algebra of dimension 4 over K. We denote by B^* its groups of units, $B^* = \{u \in H : \exists v \in H, uv = vu = 1\}$.*

Over a field K of characteristic different from 2, every quaternion algebra B has a K-basis $\{1, i, j, ij\}$ satisfying the relations $i^2 = a$, $j^2 = b$, and $ij = -ji$, for some $a, b \in K^*$. Conversely, a K-basis and relations such as the previous ones, plus the associative property, define a quaternion K-algebra. In this case, we denote by $\left(\frac{a,b}{K}\right)$ the quaternion algebra B and the basis $\{1, i, j, ij\}$ is called the canonical basis. Of course different couples may lead to isomorphic quaternion K-algebras.

Definition 2. *A quaternion $\omega = x + yi + zj + tij$ in B is called pure if $x = 0$.We denote by B_0 the K-vector space of pure quaternions.*

A quaternion K-algebra is either a skew field or an algebra isomorphic to the matrix algebra $M(2, K)$; in the first case it is called a division K-algebra, and in the second one, a matrix K-algebra. If K is algebraically closed, we only obtain matrix algebras. If K is a local field different form $\mathbb{C}$, there exists a unique division quaternion K-algebra up to isomorphism. If $K = \mathbb{R}$, the unique quaternion division algebra B is the Hamilton quaternion algebra.

Definition 3. *Every quaternion K-algebra $B = \left(\frac{a,b}{K}\right)$ is provided with a K-endomorphism which is an involutive antiautomorphism called conjugation; it is denoted by $w \mapsto \overline{w}$. If $w = x + yi + zj + tij$, with $x, y, z, t \in K$, then $\overline{w} = x - yi - zj - tij$. The reduced trace and the reduced norm are defined by $tr(w) = w + \overline{w}$ and $n(w) = w\overline{w}$, respectively. Thus, $tr(w) = 2x$ and $n(w) = x - ay^2 - bz^2 + abt^2$. Note $w \in B_0$ if and only if $\overline{w} = -w$ in fact, B_0 is the set of quaternions of reduced trace equal to O. The elements in B^* are the elements of nonzero reduced norm.*

In the following, we assume that K is a number field.

Definition 4. *Let B be a quaternion K-algebra. For each place v of K, $B_v := K_v \otimes B$ is a quaternion K_v-algebra. If B_v is a division algebra, we say that B is ramified at v; otherwise, we say that B is unramified at v.*

Definition 5. *Let B be a quaternion K-algebra and F a field extension of K. The field F splits B if $B_F := F \otimes_K H \simeq M(2, F)$.*

Theorem 1 ([1])**.**
1. *A quaternion K-algebra B is ramified at a finite even number of places.*
2. *Two quaternion K-algebras are isomorphic if and only if they are ramified at the same places.*
3. *Given an even number of noncomplex places of K, there exists a quaternion K-algebra that ramifies exactly at these places.*

Definition 6. *The reduced discriminant D_B of a quaternion K-algebra B is the integral ideal of R equal to the product of prime ideals of R that ramify in B.*

Definition 7. *Two quaternion K-algebras are isomorphic if and only if they have the same reduced discriminant. In particular, a quaternion K-algebra B is a matrix K-algebra if and only if $D_B = R$.*

Definition 8. *Let B be a quaternion K-algebra and F a field extension of K. The field F splits B if $H_F := F \otimes_K B \simeq M(2, F)$.*

Let $F = K(\alpha)$ be a quadratic field that splits a quaternion K-algebra B. An embedding $\phi : F \hookrightarrow H$ is characterized by an element $\phi(\alpha) \in B$ such that $n(\phi(\alpha)) = n(\alpha)$ and $tr(\phi(\alpha)) = tr(\alpha)$.

Lemma 1 ([1]). *Let $B = \left(\frac{p,q}{\mathbb{Q}}\right)$, p, q primes and $\left(\frac{\cdot}{\cdot}\right)$ be the Legendre symbol. Then, there always exists an embedding $B \otimes M(2, \mathbb{R})$. Moreover, B is a matrix algebra if and only if one of the following conditions are satisfied: $p = q - 2$; $p \equiv q \equiv 1 \mod 4$; $q = 2$ and $p \equiv \pm 1 \mod 8$; $p \neq q$, $p \neq 2$, $q \neq 2$, $\left(\frac{q}{p}\right) = 1$, and either p or q is congruent to 1 modulo 4.*

Lemma 2 ([1]). *Let $B = \left(\frac{p,q}{\mathbb{Q}}\right)$, p, q primes, $p \equiv q \equiv 3 \mod 4$ and $\left(\frac{q}{p}\right) \neq 1$. Then $D_H = 2p$. If $q = 2$, $p \equiv 3 \mod 8$, then $D_B = pq = 2p$. If $p \neq q$, p or q congruent to 1 modulo 4 and $\left(\frac{q}{p}\right) = -1$, then $D_H = pq$.*

From above statements, we could get the following theorem.

Theorem 2 ([1]). *Let $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a quaternion algebra.*

(1) *If $D_B = 1$, then $B \simeq M(2, \mathbb{Q}) \simeq \left(\frac{1,-1}{\mathbb{Q}}\right)$.*
(2) *If $D_B = 2p$, p prime and $p \equiv 3 \mod 4$, then $B \simeq \left(\frac{p,-1}{\mathbb{Q}}\right)$.*
(3) *If $D_B = pq$, p, q primes, $q \equiv 1 \mod 4$ and $\left(\frac{p}{q}\right) = -1$, then $B \simeq \left(\frac{p,q}{\mathbb{Q}}\right)$.*

If a and b are prime numbers, the algebra B satisfies one, and only one, of the three previous statements.

From the above theorem, we could get the following proposition.

Proposition 1 ([1])**.** *Given p and q two different prime numbers, let B be a quaternion Q-algebra of discriminant $D_B = pq$.*

(1) *If $p \equiv 3 \mod 4$ and $q = 2$, then $B \simeq \left(\frac{p,-1}{\mathbb{Q}}\right)$.*
(2) *If $p \equiv 5 \mod 8$ and $q = 2$, then $B \simeq \left(\frac{p,2}{\mathbb{Q}}\right)$.*
(3) *If $p \equiv -1 \mod 8$ and $q = 2$, then $H \simeq \left(\frac{2p,-r}{\mathbb{Q}}\right)$, where r is a prime number such that $\left(\frac{r}{p}\right) = \left(\frac{r}{2}\right) = -1$.*
(4) *If p or q is congruent to 1 modulo 4 and $\left(\frac{q}{p}\right) \neq 1$, then $B \simeq \left(\frac{p,q}{\mathbb{Q}}\right)$.*
(5) *If p or q is congruent to 1 modulo 4 and $\left(\frac{q}{p}\right) = 1$, then $B \simeq \left(\frac{pq,-r}{\mathbb{Q}}\right)$, where r is a prime number such that $\frac{r}{s} = \pm 1$ according to $s \equiv \mp 1 \mod 4$, respectively, for $s = p, q$; moreover, if p or q is congruent to 3 modulo 4, then necessarily $r \equiv 3 \mod 4$.*
(6) *If $p \equiv q \equiv 3 \mod 4$, then $H \simeq \left(\frac{pq,-1}{\mathbb{Q}}\right)$.*

Definition 9. *An element $\alpha \in B$ is said to be integral over K if $n(\alpha)$ and $tr(\alpha)$ are in R. In general, the set of integral elements in a quaternion algebra is not a ring.*

Definition 10. *An R-lattice Λ of B is a finitely generated R-torsion-free Rmodule contained in H. An R-ideal I of B is an R-lattice such that $K \otimes_R \Lambda \simeq H$. The inverse of an ideal I is the R-ideal $I^{-1} = \{ h \in H \mid IhI \subseteq I \}$. An R-ideal is said to be integral if all its elements are integral.*

Definition 11. *A subset $\mathcal{O}$ of H is called an R-order of H if it satisfies the following conditions.*

(1) *$\mathcal{O}$ is a ring whose elements are integral and $R \otimes O = H$.*
(2) *$\mathcal{O}$ is an R-ideal that is a ring.*

Definition 12. *An Eichler R-order in a quaternion algebra H is the intersection of two maximal R-orders of H.*

Theorem 3 ([1])**.** *Let H be a quaternion $\mathbb{Q}$-algebra of discriminant D. Then, for each integer N such that $\gcd(D, N) = 1$, there exist Eichler orders of level N.*

Definition 13. *The Poincaré half-plane is the complex upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.*

The definition of proper and discontinuous action is equivalent to the fact that Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Definition 14. *The action of Γ in $\mathcal{H}$ gives an equivalence relation between the points: two points z, $z' \in \mathcal{H}$ are called equivalent with respect to Γ if and only if $z' = \gamma(z)$ for some $\gamma \in \Lambda$.*

Definition 15. *A point $x \in \mathbb{R} \cup \{\infty\}$ is called parabolic (hyperbolic, respectively) with respect to Γ if there exists a transformation $\gamma \in \Gamma$ which is parabolic (hyperbolic, respectively) and such that $\gamma(x) = x$. A point $z \in \mathcal{H}$ is called elliptic with respect to Γ if there exists an elliptic transformation $\gamma \in \Gamma, \gamma \neq \pm Id$, such that $\gamma(z) = z$. The isotropy group of a point z with respect to Γ is the group $\Gamma_z = \{\gamma \in \Gamma | \gamma(z) = z\}$.*

In the following, we want to introduce Shimura curves associated to quaternion algebras. Let D, $N \in \mathbb{N}$ and $\gcd(D,N) = 1$. Let B be an indefinite quaternion $\mathbb{Q}$-algebra of discriminant D. Choose an Eichler order $\mathcal{O}_0^D(N)$ of level N in B, and fix a monomorphism $\Phi : B \hookrightarrow M(2,\mathbb{R})$.

The group $\Gamma_0^D(N) := \{\phi(\alpha) : \alpha \in \mathcal{O}(D,N),\ n(\alpha) = 1\} \subseteq \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian group of the first kind acting on the Poincaré half-plane. The quotient $\Gamma_0^D(N) \setminus \mathcal{H}$ is a Riemann surface.

The theory of Shimura [17] gives a canonical model $X_0^D(N)$ over $\mathbb{Q}$ for $\Gamma_0^D(N) \setminus \mathcal{H}$ and a modular interpretation.

The Shimura curves have the following properties ([17]):

(1) $X_0^D(N)$ is a projective curve defined over $\mathbb{Q}$.

(2) There exists a mapping $j_{D,N} : \mathcal{H} \to X_0^D(N)(\mathbb{C})$ that factorizes in an isomorphism between the analytic space $\Gamma_0^D(N)\backslash\mathcal{H}$ and a Zariski open set in $X_0^D(N)(\mathbb{C})$.

(3) Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field splitting the algebra H. Let ϕ be an embedding of F into H, and $z \in \mathcal{H}$ the unique common fixed point of all the elements in $\Phi(\phi(F^*))$. Then the coordinates of the point $j_{D,N}(Z)$ are algebraic,

more specifically, $j_{D,N}(Z) \in X_0^D(N)(F_{ab})$, where $F_{ab} \subseteq \mathbb{C}$ denotes the maximal abelian extension of F.

Definition 16. *The canonical model $X_0^D(N)$ is called the Shimura curve associated with the subgroup $\Gamma_0^D(N)$.*

Definition 17. *Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field splitting the algebra. Let $\phi : F \hookrightarrow B$ be an imbedding of F into B. Let R be the quadratic order in F such that $\phi(F) \cap \mathcal{O}(D,N) = \phi(R)$. Then the unique fixed point of $\Phi(\phi(F^*)$ in $\mathcal{H}$ is called a CM-point of discriminant* disc R*, where* disc R *denotes the discriminant of R.*

The case $D = 1$ corresponds to a nonramified quaternion algebra $B \simeq M(2,\mathbb{Q})$. If $D > 1$, the quaternion algebra B is ramified; In this case, the Riemann surface $\Gamma_0^D(N)\backslash\mathcal{H}$ is already compact.

Lemma 3 ([2]). *Assume that m is a squarefree divisor of DN such that $(m, \frac{DN}{m})$ = 1. Then the set of the fixed points of an Atkin-Lehner involution ω_m, $m > 1$, on $X^D(N)$ is*

$$\begin{cases} \mathrm{CM}(-4) \cup \mathrm{CM}(-8) & \text{if } m = 2, \\ \mathrm{CM}(m) \cup \mathrm{CM}(-4m) & \text{if } m \equiv 3 \mod 4, \\ \mathrm{CM}(-4m) & \text{else.} \end{cases}$$

3. Borcherds and Computations of Singular Moduli

§ 3.1 Modular forms on orthogonal groups

Definition 18. *Let k be an integer. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k if*

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad for \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \quad and \quad \tau \in \mathcal{H}$$

Definition 19. *Let k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k if*

(1) *f is holomorphic on $\mathcal{H}$,*
(2) *f is weakly modular of weight k,*
(3) *f is holomorphic at ∞.*

The set of modular forms of weight k is denoted $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Definition 20. *Let V be a finite dimensional vector space over a field F and ϕ be a bilinear form on V. If ϕ is symmetric and nondegenerate, then the automorphisms of ϕ is called the orthogonal group of ϕ.*

Let $L \subset \mathbb{R}^n$ be a lattice of dimension n with the associated quadratic form Q and the bilinear form $\langle \cdot, \cdot \rangle$. Assume that L is nonsingular and let (b^+, b^-) be its signature. For $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ let $V(k) = L \otimes k$ and extend the definition $\langle \cdot, \cdot \rangle$ to $V(k)$ bilinearly. Let $O_V(\mathbb{R})$ be the orthogonal group for V and $O_V^+(\mathbb{R})$ be the subgroup of elements whose spinor norm has the same sign as the discriminant. Let also $O_L = \{\sigma \in O_V(\mathbb{R}) : \sigma(L) = L\}$ and $O_L^+ = O_L \cap O_V(\mathbb{R})^+$. In order to define modular forms on a subgroup Γ of O_L^+, we assume that the signature of L is $(b, 2)$ and consider the set

$$K = \{z \in V(\mathbb{C}) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle > 0\}/\mathbb{C}^\times.$$

The set K has two connected components. Pick one of them and designate it to be K^+, called the *tubedomain* for L. The orthogonal group $O_V^+(\mathbb{R})$ acts transitively on K^+. That is, K^+ is a symmetric domain for $O_V^+(\mathbb{R})$. The space K^+ can be identified with the Grassmanian $\mathrm{Gr}(V) := \{W \subset V(\mathbb{R}) : \dim W = 2, \langle \cdot, \cdot \rangle \mid_W < 0\}$, the set of oriented negative 2-planes of V. Namely, for $W = \mathbb{R}x + \mathbb{R}y \subset V(\mathbb{R})$ with $\langle x, x \rangle = \langle y, y \rangle = -1$, $\langle x, y \rangle = 0$ and suitable orientation, we have $x + iy \in K^+$. Let $\widetilde{K}^+ = \{\, \omega \in V(\mathbb{C}) \setminus \{0\} : [\omega] \in K^+\}$.

Definition 21. *Let $L^\vee = \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z}$ for all $y \in L\}$ be the dual lattice of L and $\{e_\gamma : \gamma \in L^\vee/L\}$ be the standard basis for the space $\mathbb{C}[L^\vee/L]$. The metapletic group*

$$\widetilde{\mathrm{SL}} = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \quad \mathrm{SL}(2, \mathbb{Z}) \right\}$$

is generated by

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

Then the Weil representation ρ_L *of* $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ *associated to* L *is defined by*

$$\rho_L(T)e_\gamma = e^{2\pi i \langle \gamma, \gamma \rangle /2} e_\gamma$$

$$\rho_L(S)e_\gamma = \frac{e^{2\pi i \langle b^- - b^+ \rangle /8}}{\sqrt{\mid L^\vee / L \mid}} \sum_{\delta \in L^\vee / L} e^{2\pi i \langle \gamma, \delta \rangle} e_\delta$$

We say a holomorphic function $f : \mathcal{H} \to \mathbb{C}[L^\vee / L]$ *is a modular form of weight* k *and type* ρ_L *if*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho_L \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) f(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Such a modular form admits a Fourier Expansion

$$f(\tau) = \sum_{\gamma \in L^\vee / L} f_\gamma(\tau) e_\gamma = \sum_{\gamma \in L^\vee / L} \sum_{m \in \mathbb{Q}} c_\gamma(\tau) q^m r_\gamma$$

We say f is *weakly holomorphic* if only a finite number of $c_\gamma(m)$ with $m < 0$ are nonzero.

Theorem 4 ([3, 4])**.** *Let* f *be a weakly holomorphic vector-valued modular form of weight* $1 - b/2$ *and type* ρ_L *with Fourier expansion*

$$f(\tau) = \sum_{\lambda \in L^\vee / L} f_\lambda e_\lambda = \sum_{\lambda \in L^\vee / L} \sum_{m \in \mathbb{Q}} c_\lambda(m) q^m e_\lambda.$$

Assume that $c_\lambda(m) \in \mathbb{Z}$ *for* $m \le 0$*. Then there exists a meromorphic modular form* Ψ_f *on the orthogonal group*

$$O^+_{L,f} := \{\sigma \in O^+_L : \ f_{\sigma\lambda} = f_\lambda \text{ for all } \lambda \in L^\vee / L\} \subseteq O^+_L$$

with the following properties.

(1) *The weight of* Ψ_f *is* $c_0(0)/2$.

(2) *The divisor of* Ψ_f *is given by*

$$\operatorname{div}(\Psi_f) = \sum_{\lambda} \sum_{m<0} c_\lambda(m) Z(-m, \lambda),$$

where the $Z(-m, \lambda)$ *are rational quadratic divisors.*

Definition 22. *The modular form* Ψ_f *in the above theorem is called the Borcherds form associated to* f.

A relation between f and Ψ_f could be described as follows. Let $\theta_L(\tau, v)$, $\tau \in \mathcal{H}$, $v \in \mathrm{Gr}(V)$, be the Siegel theta function. For vector-valued modular form f of weight $1 - b/2$ and type ρ_L, we define

$$\Phi_f(v) = \int_{\mathrm{SL}(2,\mathbb{Z})/\mathcal{H}}^{\cdot} \bar{\theta}(\tau, v) f(\tau) \frac{dxdy}{y}, \quad \tau = x + iy.$$

Then $\Phi_f(v) : \mathrm{Gr}(V) \to \mathbb{C}$ is an automorphic function invariant under O_L^+. When z is not in the divisor of Ψ_f, we have

$$\Phi_f(v) = -2 \log \|\Psi_f(v)\|^2.$$

§ 3.2 Borcherds forms as modular forms on Shimura Curves

Let $\mathcal{O} = \mathcal{O}(D, N)$, $(D, N) = 1$, be an Eichler order of level N in an indefinite quaternion algebra B of discriminant D. Consider the set

$$L = \{\alpha \in \mathcal{O} : tr(\alpha) = 0\}.$$

Define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on L by

$$\langle \alpha, \beta \rangle = tr(\alpha \overline{\beta}).$$

Then L becomes a lattice of rank 3 and signature $(1, 2)$. If we choose a representative for B to be $\left(\frac{a,b}{\mathbb{Q}}\right)$ with $a, b > 0$ and fix the embedding $\iota : B \hookrightarrow M(2, \mathbb{R})$ to be the one determined by

$$\iota(i) = \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, \qquad \iota(j) = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix},$$

then the symmetric space K can be identified with $\mathcal{H}^{\pm}$, the union of the upper half-plane and the lower half-plane, by

$$\tau \in \mathcal{H}^{\pm} \longmapsto z(\tau) = \frac{1+\tau^2}{2\sqrt{a}} i + \frac{\tau}{\sqrt{b}} j + \frac{1-\tau^2}{2\sqrt{ab}} ij.$$

We let K^+ be the piece corresponding to $\mathcal{H}^+$.

It is clear that if α is an element of B normalizing $\mathcal{O}$, then the function $\sigma_\alpha : B \to B$ defined by

$$\sigma_\alpha(\eta) = \alpha\eta\alpha^{-1}$$

is an automorphism of the lattice L. In fact, it follows from the Noether-Skolem theorem that, up to ± 1, all automorphisms of L arise this way. That is, we have

$$O_L = \{\sigma_\alpha : \ \alpha \in N_B(\mathcal{O})\} \times \{\pm 1\},$$

where $N_B(\mathcal{O})$ denotes the normalizer of $\mathcal{O}$ in B. Furthermore, we can verify that

$$O_L^+ = \{\sigma_\alpha : \ \alpha \in N_B(\mathcal{O}), \ n(\alpha) > 0\} \times \{\pm 1\}.$$

Now the group $N_B^+(\mathcal{O})$ acts on both $\mathcal{H}^+$ and K^+. (The action on $\mathcal{H}^+$ is linear fractional transformation through $\iota(\alpha)$ and the action on K^+ is conjugation.) We can check that the actions are compatible. That is, the diagram

$$\begin{array}{ccc} \mathcal{H}^+ & \longrightarrow & K^+ \\ {\scriptstyle \iota(\alpha)} \downarrow & & \downarrow {\scriptstyle \sigma_\alpha} \\ \mathcal{H}^+ & \longrightarrow & K^+ \end{array}$$

commutes. Thus, if $f = \sum f_\eta e_\eta$ is a weakly holomorphic modular form of weight $1/2$ and type ρ_L such that

$$\{\sigma \in O_L^+ : \ f_{\sigma\eta} = f_\eta \text{ for all } \eta \in L^\vee/L\},$$

then $\psi_f(\tau) = \Psi_f(z(\tau))$ is a meromorphic modular form on $N_B^+(\mathcal{O})$. In other words, ψ_f becomes a modular form on the Shimura curve $X_0^D(N)/W_{D,N}$.

§ 3.3 Construction of Borcherds forms

Definition 23. *For $k \in \frac{1}{2}\mathbb{Z}$, let the slash operator of weight k of an element $\gamma \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ be defined by*

$$f \mid_{\gamma^{\pm}}^{k} (\tau) = (\pm\sqrt{c\tau + d})^{-2k} f(\gamma\tau).$$

The slash operator satisfies

$$f \mid_{\gamma_1\gamma_2}^{k} = (f \mid_{\gamma_1}^{k}) \mid_{\gamma_2}^{k}$$

Lemma 4 ([3])**.** *Let N be the level of the lattice Suppose f is a scalar-valued weight k modular form on $\widetilde{\Gamma_0}(N)$ with character χ_L. Then the function $F_f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$ defined by*

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma_0(\mathbb{N})} \backslash \widetilde{SL_2}(\mathbb{Z})} f|_{\gamma}^{k}(\tau)\rho_{\Lambda_L}(\gamma^{-1})e_0.$$

is a weakly holomorphic vector-valued modular form of weight k and type ρ_L. Moreover, if $\langle \eta, \eta \rangle = \langle \eta', \eta' \rangle$, then the η-component and the η'-component of F_f are equal.

Now recall the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k),$$

which is a weight $1/2$ modular form. The η-function satisfies

$$\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau), \qquad \eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\eta(\tau).$$

Lemma 5. *Let N be the level of a lattice L. Assume that r_d, $d|N$, are integers such that*

(1) $|L^\vee/L| \prod_{d|N} d^{r_d}$ *is the square of a rational number,*
(2) $\sum_{d|N} r_d d \equiv 0 \mod 24$,
(3) $\sum_{d|N} r_d (N/d) \equiv 0 \mod 24$.

Then the function $\prod_{d|N} \eta(d\tau)^{r_d}$ is a modular form of weight $\sum_{d|N}(r_d/2)$ and character χ_L.

§ 3.4 Borcherds Forms at CM Points

In the following, we will discuss some results about Borcherds forms at CM points([9, 16]).

As before, let $L = \{\alpha \in \mathcal{O} : \ tr(\alpha) = 0\}$. Let $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. For $x \in V$ with positive norm, let

$$U = \{y \in V : \langle x, y\rangle = 0\}, \quad L_+ = \mathbb{Q}x \cap L, \qquad L_- = U \cap L.$$

Then U is a negative definite 2-plane and thus can be identified with a point in the symmetric space K for the orthogonal group O_L. In our setting, the corresponding point in K is a CM-point of discriminant $-c^2\langle x, x\rangle$ for some $c \in \mathbb{Q}^\times$. In this section, we recall Schofer's formula for values of a Borcherds form at such a CM-point.

Definition 24. *For $\mu \in L/L_-$ and $\phi_\mu = char(\mu + L_-)$, let $E(\tau, s; \phi_{mu}, +1)$ be the incoherent Eisenstein series of weight 1 with Fourier Expansion*

$$E(\tau, s; \phi_{mu}, +1) = \sum_{m \in \mathbb{Q}} A_\mu(s, m, v) q^m$$

with the Fourier coefficients having Laurent expansions $A_\mu = b_\mu(m, v)s + O(s^2)$ at $s = 0$.

For $\eta \in L^\vee/L$ and $m \in \mathbb{Q}$ define,

$$k_\eta(m) = \sum_{\lambda \in L/L_+ + L_-} \ \sum_{x \in \eta_+ + \lambda_+ + L_+} k^-_{\eta_- + \lambda_-}(m - Q(x)),$$

where

$$k^-_\mu(m') = \begin{cases} \lim_{v\to\infty} b_\mu(m', v), & \text{if } m' > 0, \\ k_0(0)\phi_\mu(0), & \text{if } m' = 0, \\ 0, & \text{if } m' < 0. \end{cases}$$

Here

$$k_0(0) = \log\left(|\Delta| \mid + 2\frac{\Lambda'(1, \chi_\Delta)}{\Lambda(1, \chi_\Delta)}\right),$$

where Δ is the discriminant of the imaginary quadratic number field $\mathbb{Q}(-\sqrt{x, x})$ and $\Lambda(s, \chi_\Delta)$ is the complete L-function $\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})L(s, \chi_\Delta)$ associated with the Dirichlet character χ_Δ.

Theorem 5 (Corollary 3.4 of [16]). *Assume that $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$, $c_0(0) = 0$, and that the CM-point corresponding to x is a CM-point of discriminant d. Assume that the set $\mathrm{CM}(d)$ of CM-points of discriminant d does not meet the divisor of the Borcherds form ψ_f*

of weight 0. *Then Then*

$$\frac{1}{|\mathrm{CM}(d)|}\sum_{\tau\in\mathrm{CM}(d)}\log|\psi_f(\tau)| = -\frac{1}{4}\sum_{\eta}\sum_{m\geq 0} c_\eta(m)k_\eta(m).$$

From [3] we could know that $x_D : X_D \to \mathbb{P}^1$ is a Borcherds form. Thus, we could use the above results to compute the singular moduli.

4. Equations of Shimura Curves

Example 1. *Consider the Shimura Curve* $X_0^{158}(1)/W_{158,1}$. *By finding suitable eta-products, we construct 4 modular forms* f_1, f_2, f_3, f_4 *of weight* $1/2$ *on* $\Gamma_0(316)$ *with Fourier expansions*

$$f_1 = q^{-45} - q^{-36} + q^{-31} - q^{-18} - q^{-11} + q^{-9} - q^{-5} + q^{-4} + q^{-2} - q^{-1} + \cdots$$

$$f_2 = q^{-97} + q^{-76} + 2q^{-64} - q^{-52} - 2q^{-40} + 2q^{-32} - 2q^{-23} - q^{-20} - q^{-19} - q^{-18} - 2q^{-16} + 4q^{-15} + 4q^{-14} + q^{-13} - q^{-11} + 2q^{-10} - q^{-8} + 4q^{-6} + q^{-5} + 4q^{-3} + \cdots$$

$$f_3 = -q^{-1} - q^{-2} - 3q^{-3} + q^{-4} - 2q^{-5} - 3q^{-6} + q^{-8} + q^{-9} - 2q^{-10} - q^{-13} - 3q^{-14} - 3q^{-15} + 2q^{-16} + q^{-19} + q^{-20} + q^{-23} + q^{-31} - 2q^{-32} - q^{-36} + 2q^{-40} + q^{-42} + q^{-52} - 2q^{-64} - q^{-76} + \cdots$$

$$f_4 = -2q^{-1} - q^{-2} - 2q^{-3} + q^{-4} - 2q^{-5} - 2q^{-6} + q^{-8} + q^{-9} - 2q^{-10} - q^{-13} - 2q^{-14} - 2q^{-15} + 2q^{-16} + q^{-19} + q^{-20} + q^{-23} + q^{-31} - 2q^{-32} - q^{-36} + 2q^{-40} + q^{-49} + q^{-52} - 2 * q^{-64} - q^{-76} + \cdots$$

so the Borcherds forms have divisors

$$\operatorname{div}\Phi_{f_1} = 2P_{-180} - P_{-11} - P_{-72}, \quad \operatorname{div}\Phi_{f_2} = 2P_{-388} - P_{-11} - P_{-72}$$

$$\operatorname{div}\Phi_{f_3} = 2P_{-168} - P_{-8} - P_{-20}, \quad \operatorname{div}\Phi_{f_4} = 2P_{-196} - P_{-8} - P_{-20}$$

Thus, we could know that Φ_{f_2} *is a polynomial of degree 1 in* Φ_{f_1} *and* Φ_{f_4} *is a polynomial of* Φ_{f_3}. *There are modular functions* x *and* y *on* $X_0^{158}(1)/W_{158,1}$ *and let* $\operatorname{div} x = \operatorname{div}\Phi_{f_1}$ *and* $\operatorname{div} y = \operatorname{div}\Phi_{f_3}$ *such that x has poles at* P_{-11} *and* P_{-72} *and y has poles at* P_{-8} *and* P_{-20}.

Computing singular moduli, we could find

	-4	-8	-11	-19	-20	-40	-52	-67	-88
$\mid x \mid$	1	1/3	∞	1	3	1	1	13/7	11/5
$\mid a+bx \mid$	2	8/3	∞	3	4	2	3	11/7	18/5
$\mid y \mid$	1	∞	4	6	∞	5	2	2	5
$\mid c+dy \mid$	2	∞	8/7	4/7	∞	6/7	12/7	12/7	6/7

From P_{-4} and P_{-19}, we could set up the following equations.

Since the value of $\mid a+bx \mid$ for P_{-4} and P_{-19} are different, thus the they couldn't have the same value of x. By considering Cramer's formula, we only need to change the sign of $\mid a+bx \mid$ for P_{-4} and P_{-19} and fixing the value of x.

(1) *If the x value of P_{-4} is 1 and the x value of P_{-19} is -1, and assume $\mid a+bx \mid$ for P_{-4} is 2, and $\mid a+bx \mid$ for P_{-19} is 3 we could get*

$$\begin{cases} a+b=2 \\ a-b=3 \end{cases}$$

Then we could solve $(a,b) = (\frac{5}{2}, -\frac{1}{2})$.

(2) *If the x value of P_{-4} is 1 and the x value of P_{-19} is -1, and assume $\mid a+bx \mid$ for P_{-4} is 2, and $\mid a+bx \mid$ for P_{-19} is -3 we could get*

$$\begin{cases} a+b=-2 \\ a-b=3 \end{cases}$$

Then we could solve $(a,b) = (\frac{1}{2}, -\frac{5}{2})$.

(3) *If the x value of P_{-4} is 1 and the x value of P_{-19} is -1, and assume $\mid a+bx \mid$ for P_{-4} is -2, and $\mid a+bx \mid$ for P_{-19} is -3 we could get*

$$\begin{cases} a+b=-2 \\ a-b=-3 \end{cases}$$

Then we could solve $(a,b) = (-\frac{5}{2}, -\frac{1}{2})$.

(4) *If the x value of P_{-4} is 1 and the x value of P_{-19} is -1, and assume $\mid a+bx \mid$ for P_{-4} is 2, and $\mid a+bx \mid$ for P_{-19} is -3 we could get*

$$\begin{cases} a+b=2 \\ a-b=-3 \end{cases}$$

We could solve $(a,b)=(-\frac{1}{2},\frac{5}{2})$. By examing the values of P_{-8} and P_{-40}, we could get$(\frac{5}{2},-\frac{1}{2})$ is the correct answer. Thus, we could get value of x for each point.
By the similar argument as above, we could get $(c,d)=(\frac{16}{7},-\frac{2}{7})$ and get the values of y. Then we could the following results.

$$\mid x \mid = \mid \Phi_{f_1} \mid, \quad \mid \frac{5}{2}-\frac{1}{2}x \mid = \mid \Phi_{f_2} \mid /(2\cdot 79^2)$$

$$\mid y \mid = \mid 2^2 79^{\frac{3}{2}} \Phi_{f_3} \mid, \quad \mid \frac{16}{7}-\frac{7}{2}y \mid = \mid 2^3 79 \Phi_{f_4} \mid /7$$

and the values of x and y at various CM-points are

	-4	-8	-11	-19	-20	-40	-52	-67	-88
x	1	1/3	∞	-1	-3	1	-1	13/7	-11/5
y	1	∞	4	6	∞	5	2	2	5

For the coordinates, we see that the relation between x and y is

$$a(3x+1)(x+3)y^2+(bx^2+cx+d)y+(x^2+fx+g)=0.$$

From the information at the other CM-points $P_{-4}, P_{-19}, P_{-40}, P_{-52}, P_{-67}, P_{-88}$ we could get

$$a=\frac{5}{400}, \quad b=-\frac{146}{400}, \quad c=-\frac{320}{400}, \quad d=0, \quad f=-\frac{640}{400}, \quad g=\frac{256}{400}.$$

Therefore, the relation between x and y is

$$(3x+1)(x+3)y^2+(-32x^2-64x)y+(80x^2+64x-64)=0.$$

Thus, we could get the Weierstrass model is

$$y^2=x^3-139x+454.$$

By letting

$$x_1=\frac{1}{4}x+\frac{1}{4}.$$

$$y_1 = \frac{1}{8}y - \frac{1}{8}x - \frac{5}{8}.$$

The minimal model is

$$y_1^2 + x_1 y_1 + y_1 = x_1^3 - x_1^2 - 9x_1 + 9,$$

Example 2. *Consider the Shimura Curve $X_0^{214}(1)/W_{214,1}$. By finding suitable eta-products, we construct 4 modular forms f_1, f_2, f_3, f_4 of weight $1/2$ of type ∞ on $\Gamma_0(428)$ with Fourier expansions*

$$\begin{aligned} f_1 = {} & q^{-190} - q^{-92} - q^{-76} + 2 * q^{-64} + 2q^{-56} + 2q^{-52} - 3q^{-48} - 3q^{-47} - q^{-44} - 3q^{-43} + \\ & q^{-40} + q^{-39} - q^{-37} + 2q^{-36} + 2q^{-23} - 11q^{-22} - 11q^{-21} + q^{-19} - 3q^{-18} - \\ & 3q^{-17} - 2q^{-14} - 2q^{-13} + 2q^{-12} + q^{-11} - q^{-10} - 3q^{-9} + 3q^{-8} - 3q^{-7} - \\ & 11q^{-5} - 4q^{-4} + q^{-3} - 14q^{-2} + 3q^{-1} + \cdots \end{aligned}$$

$$\begin{aligned} f_2 = {} & q^{-147} - q^{-92} - q^{-76} + q^{-56} + 2q^{-52} - 3q^{-48} - q^{-47} - q^{-44} - 5q^{-43} + q^{-39} - \\ & q^{-37} + 2q^{-36} + 2q^{-23} + 6q^{-22} + 6q^{-21} + q^{-19} - 5q^{-18} - 5q^{-17} - \\ & q^{-14} - 2q^{-13} + 2q^{-12} + q^{-11} - 3q^{-9} + 5q^{-8} - 5q^{-7} + 6q^{-5} - 2q^{-4} + \\ & q^{-2} + 3q^{-1} + \cdots \end{aligned}$$

$$\begin{aligned} f_3 = {} & -q^{-108} + q^{-64} - q^{-56} - q^{-48} - q^{-43} - 2q^{-39} - q^{-36} + q^{-27} - 2q^{-23} - 6q^{-22} - \\ & 6q^{-21} + q^{-19} - q^{-18} - q^{-17} - 2q^{-16} + q^{-14} + q^{-13} + q^{-8} - q^{-7} - \\ & 6q^{-5} + q^{-4} - 2q^{-3} - 7q^{-2} + q^{-1} + \cdots \end{aligned}$$

$$\begin{aligned} f_4 = {} & q^{-163} - q^{-108} - q^{-76} + q^{-64} - q^{-56} - 2q^{-48} - 5q^{-43} - 2q^{-39} + q^{-27} + q^{-25} - \\ & q^{-23} + q^{-22} + q^{-21} + q^{-19} - 5q^{-18} - 5q^{-17} - 2q^{-16} + q^{-14} + q^{-12} - \\ & q^{-9} + 5q^{-8} - 5q^{-7} + q^{-5} - 2q^{-3} - 4q^{-2} + q^{-1} + \cdots \end{aligned}$$

so the Borcherds forms have divisors

$$\operatorname{div} \Phi_{f_1} = 2P_{-760} - P_{-36} - P_{-148}, \quad \operatorname{div} \Phi_{f_2} = 2P_{-147} - P_{-36} - P_{-148}$$

$$\operatorname{div}\Phi_{f_3} = P_{-19} + P_{-52} - P_{-3} - P_{-36}, \quad \operatorname{div}\Phi_{f_4} = P_{-100} + P_{-163} - P_{-3} - P_{-36}$$

Thus, we could know that Φ_{f_2} is a polynomial of degree 1 in Φ_{f_1} and Φ_{f_4} is a polynomial of Φ_{f_3}. There are modular functions x and y on $X_0^{214}(1)/W_{214,1}$ and let $\operatorname{div} x = \operatorname{div}\Phi_{f_1}$ and $\operatorname{div} y = \operatorname{div}\Phi_{f_3}$ such that x has poles at P_{-36} and P_{-148} and y has poles at P_{-3} and P_{-36} and zeros at P_{-19} and P_{-52}.

Computing singular moduli, we find

$$|\, x \,| = |\, 107^7\Phi_{f_1} \,|, \quad |\, \frac{6}{7} + \frac{2}{7}x \,| = |\, 2\Phi_{f_2} \,| /(107^{\frac{1}{2}})$$

$$|\, y \,| = |\, 2^5 107^{\frac{7}{2}}\Phi_{f_3} \,|, \quad |\, \frac{3}{2} + \frac{1}{2}y \,| = |\, 2^5 107^2\Phi_{f_4} \,|$$

and the values of x and y at various CM-points are

	-3	-4	-11	-19	-40	-52	-163
x	1/16	-16/3	1/2	9/4	17/6	23/4	276/26
y	∞	1/3	1	0	1	0	-3

For the coordinates, we see that the relation between x and y is

$$a(16x-1)y^2 + (bx^2 + cx + d)y + (4x-9)(4x-23) = 0.$$

From the information at the other CM-points $P_{-4}, P_{-11}, P_{-40}, P_{-163}$, we could get

$$a = -7, \quad b = -64, \quad c = 400, \quad d = -282.$$

Thus, the relation between x and y is

$$-7(16x-1)y^2 + (-64x^2 + 400x - 282)y + (4x-9)(4x-23) = 0.$$

So we could get the Weierstrass model

$$y^2 = x^3 - \frac{354656512}{3}x + \frac{14156973350912}{27}.$$

By letting

$$x_1 = \frac{1}{3136}x - \frac{1}{12}.$$

$$y_1 = \frac{1}{175616}y - \frac{1}{6272}x + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - 12x_1 + 16.$$

Example 3. *Consider the Shimura Curve $X_0^{218}(1)/W_{218,1}$. By finding suitable eta-products, we construct 4 modular forms f_1, f_2, f_3, f_4 of weight $1/2$ of type ∞ on $\Gamma_0(426)$ with Fourier expansions*

$$f_1 = -q^{-295} + 2q^{-236} + q^{-96} + 2q^{-76} - 2q^{-59} + q^{-55} - q^{-40} - 2q^{-39} - 3q^{-32} + q^{-30} - 2q^{-23} - 23q^{-21} - 23q^{-20} - 2q^{-19} - 23q^{-12} - q^{-11} + q^{-10} - 23q^{-9} - q^{-6} + q^{-2} + \cdots$$

$$f_2 = -q^{-295} + 2q^{-236} + q^{-133} + q^{-96} + 2q^{-76} - 2q^{-59} + q^{-55} - q^{-40} - 2q^{-39} - 3q^{-32} - 2q^{-23} - 25q^{-21} - 25q^{-20} - 2q^{-19} - 25q^{-12} - q^{-11} + q^{-10} - 25q^{-9} - q^{-6} + q^{-2} + \cdots$$

$$f_3 = q^{-295} - q^{-236} - q^{-76} + q^{-59} - q^{-44} + q^{-40} + q^{-32} + q^{-23} + 16q^{-21} + 16q^{-20} + q^{-19} + q^{-14} - q^{-13} + 16q^{-12} - q^{-10} + 16q^{-9} - q^{-2} + \cdots$$

$$f_4 = q^{-295} - q^{-236} + q^{-85} - q^{-76} + q^{-59} - q^{-44} + q^{-40} + q^{-32} + q^{-23} + 15q^{-21} + 15q^{-20} + q^{-19} - q^{-13} + 15q^{-12} - q^{-10} + 15q^{-9} - q^{-2} + \cdots$$

so the Borcherds forms have divisors

$$\operatorname{div}\Phi_{f_1} = 2P_{-120} - P_{-8} - P_{-11}, \quad \operatorname{div}\Phi_{f_2} = 2P_{-532} - P_{-8} - P_{-11}$$

$$\operatorname{div}\Phi_{f_3} = 2P_{-56} - P_{-11} - P_{-52}\ \operatorname{div}\Phi_{f_4} = 2P_{-340} - P_{-11} - P_{-52}$$

Thus, we could know that Φ_{f_2} is a polynomial of degree 1 in Φ_{f_1} and Φ_{f_4} is a polynomial of Φ_{f_3}. There are modular functions x and y on $X_0^{218}(1)/W_{218,1}$ and let $\operatorname{div} x = \operatorname{div}\Phi_{f_1}$ and $\operatorname{div} y = \operatorname{div}\Phi_{f_3}$ such that x has poles at P_{-8} and P_{-11} and y has poles at P_{-11} and P_{-52}.

Computing singular moduli, we could find

$$|\ x\ |=|\ 109^{\frac{23}{2}}\Phi_{f_1}\ |/2^2, \quad |\ 4-2x\ |=|\ 109^{\frac{25}{2}}\Phi_{f_2}\ |/2$$

$$|\ y\ |=|\ 2\Phi_{f_3}\ |/109^8, \quad |\ \frac{8}{3} - \frac{2}{3}y\ |=|\ 2^2\Phi_{f_4}\ |/(3\cdot 109^{\frac{25}{2}})$$

and the values of x and y at various CM-points are

	-8	-11	-19	-24	-40	-52	-67	-148	-232
x	∞	∞	1	1/2	1/2	1	7/8	-1/2	-1/2
y	2	∞	1	2	1	∞	-2	-2	7/3

For the coordinates, we see that the relation between x and y is

$$a(x-1)y^2 + (bx^2 + cx + d)y + (x^2 + fx + g) = 0.$$

From the information at the other CM-points $P_{-4}, P_{-19}, P_{-40}, P_{-52}, P_{-67}, P_{-88}$, we could get

$$a = \frac{1}{8}, \quad b = -\frac{1}{2}, \quad c = \frac{1}{8}, \quad d = \frac{1}{4}, \quad f = -1, \quad g = \frac{1}{8}$$

Thus, the relation between x and y is

$$(x-1)y^2 + (-4x^2 + x + 2)y + (8x^2 - 8x + 1) = 0.$$

So we could get the Weierstrass model

$$y^2 = x^3 - \frac{97}{48}x + \frac{3601}{864}.$$

By letting

$$x_1 = x - \frac{1}{12}.$$

$$y_1 = y - \frac{1}{2}x + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - 2x_1 + 4.$$

Example 4. *In the following, we will give results for discrminant 226, 274, 278,298*

D=226,

The values of x and y at various CM-points are

	-3	-19	-20	-24	-27	-40	-43	-67	-148	-232
x	∞	0	1	0	∞	1	1/2	1/2	-3	-3
y	0	1	4	∞	∞	1	4	7/4	-17	0

The relation between x and y is

$$2xy^2 + (-9x^2 + 5x - 6)y + (2x + 6) = 0.$$

Setting

$$x_0 = -\frac{3(23x - 16y + 16)}{4x}.$$

$$y_0 = -\frac{18(3x^2 - 3xy - 3x + 4y - 4)}{x^2}.$$

We get a Weierstrass model

$$y_0^2 = x_0^3 - \frac{6507}{16}x_0 + \frac{33075}{32}.$$

By letting

$$x_1 = \frac{1}{9}x_0 - \frac{1}{12}.$$
$$y_1 = \frac{1}{27}y_0 - \frac{1}{18}x_0 + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - 5x_1 + 1.$$

Example 5. *D = 274,*

The values of x and y at various CM-points are

	-3	-20	-24	-27	-40	-43	-52	-67	-163
x	1	∞	-1	-2	-2	∞	-5/2	-1	-10/3
y	1	0	0	1	-1	∞	∞	-1/3	-1/3

The relation between x and y is

$$a(2x + 5)y^2 + (bx^2 + cx + d)y + (x + 1) = 0$$

From the information at the other CM-points $P_{-3}, P_{-40}, P_{-67}, P_{-163}$, we could get

$$a = 1, \quad b = -2, \quad c = -5, \quad d = -2$$

Thus, the relation between x and y is

$$(2x + 5)y^2 + (-2x^2 - 5x - 2)y + (x + 1) = 0.$$

Setting

$$x_0 = \frac{2x + 96y + 77}{12(2x + 5)}.$$
$$y_0 = -\frac{2(6x^2 + 8xy + 23x + 4y + 8)}{4x^2 + 20x + 25}.$$

We get a Weierstrass model

$$y_0^2 = x_0^3 - \frac{337}{48}x_0 + \frac{8281}{864}.$$

By letting

$$x_1 = x_0 - \frac{1}{12}$$
$$y_1 = y_0 - \frac{1}{2}x_0 + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - 7x_1 + 9.$$

Example 6. *D = 278,*

The values of x and y at various CM-points are

	-4	-11	-20	-24	-52	-67	-148	-163
x	1/3	1	1	∞	-3	3	3	1/3
y	1	3	1	∞	-1	∞	13/5	13/12

The relation between x and y is

$$a(x-3)y^2 + (bx^2 + cx + d)y + (x^2 + fx + g) = 0$$

From the information at the other CM-points $P_{-4}, P_{-11}, P_{-20}, P_{-52}, P_{-148}, P_{-163}$, we could get

$$a = -\frac{1}{2}, \quad b = \frac{1}{2}, \quad c = -\frac{5}{2}, \quad d = -2, \quad f = 1, \quad g = 1.$$

Thus, the relation between x and y is

$$-(x-3)y^2 + (x^2 - 5x - 4)y + (2x^2 + 2x + 2) = 0.$$

Setting

$$x_0 = \frac{97x - 120y + 21}{12(x-3)}.$$
$$y_0 = -\frac{23x^2 - 21xy + 15x - 37y + 8}{x^2 - 6x + 9}.$$

We get a Weierstrass model

$$y_0^2 = x_0^3 - \frac{49}{48}x_0 + \frac{7849}{864}.$$

By letting

$$x_1 = x_0 - \frac{1}{12}.$$
$$y_1 = y_0 - \frac{1}{2}x_0 + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - x_1 9.$$

Example 7. *D=298, The values of x and y at various CM-points are*

	-3	-8	-11	-40	-43	-52	-163	-232
x	1	3	3	3/2	∞	6	3/2	1
y	1	∞	1/3	∞	1/3	0	1	-5/3

The relation between x and y is

$$a(x-3)(2x-3)y^2 + (bx^2 + cx + d)y + (x-6) = 0$$

From the information at the other CM-points $P_{-3}, P_{-11}, P_{-163}, P_{-232}$, we could get

$$a = \frac{3}{2}, \quad b = -1, \quad c = \frac{15}{2}, \quad d = -\frac{9}{2}$$

Thus, the relation between x and y is

$$3(x-3)(2x-3)y^2 + (-2x^2 + 15x - 9)y + (2x - 12) = 0.$$

Setting

$$x_0 = \frac{3(72x^2y - 47x^2 - 324xy + 240x + 324y + 360)}{4(x^2 - 12x + 36)}.$$

$$y_0 = -\frac{27(90x^3y - 14x^3 - 981x^2y + 243x^2 + 2997xy - 1107x - 2592y + 864)}{2(x^3 - 18x^2 + 108x - 216)}.$$

We get a Weierstrass model

$$y_0^2 = x_0^3 - \frac{24651}{16}x_0 + \frac{806787}{32}.$$

By letting

$$x_1 = \frac{1}{9}x_0 - \frac{1}{12}.$$

$$y_1 = \frac{1}{27}y_0 - \frac{1}{18}x_0 + \frac{1}{24}.$$

The minimal model is

$$y_1^2 + x_1y_1 = x_1^3 - 19x_1 + 33.$$

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