國立交通大學

應用數學系

博　　士　　論　　文

志村曲線上的自守型式

Automorphic Forms on Shimura Curves

研 究 生：涂芳婷

指導老師：楊一帆　教授

中 華 民 國 一 Ｏ 二 年 九 月

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研 究 生：涂芳婷　　Student：Fang-Ting Tu

指導教授：楊一帆　　Advisor：Yifan Yang

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博 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science,

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

PhD

In

Applied Mathematics
September 2013
Hsinchu, Taiwan, Republic of China

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研究生：涂芳婷　　　　指導教授：楊一帆 教授

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摘　要

在上個世紀，模型式和模曲線在數論的發展上佔了很重要地位。志村的曲線是模曲線的一個推廣，因此自守型式和志村曲線的算術性質在近代數論的發展也是舉足輕重。我們的主要目標是研究自守型式的算術性質。這篇論文的工作是研究自守型式算術性質的一個起點。

根據楊一帆教授最近的結果，我們可以用 Schwarzian 微分方程的解來描述虧格為零的志村曲線上的自守型式，這提供了我們一個明確的方法來對自守型式作計算並幫助我們瞭解自守型式的算術性質。因此，如何找到的相關的 Schwarzian 微分方程就成為我們現在最重要的問題。

在這篇論文中，我們決定了大部分虧格為零志村曲線的Schwarzian微分方程。另外，在學習自守型式的算術性質時，我們有個有趣的發現： ${}_2F_1$-超幾何函數的代數變換。這主要的概念是把志村曲線上的自守型式用超幾何函數來表示，並利用自守型式之間的相等關係，我們就可以看到這些有趣的代數變換。

中華民國一０二年九月

Automorphic Forms on Shimura Curves

Student: Fang-Ting Tu Advisor: Yifan Yang

Department (Institute) of Applied Mathematics

National Chiao Tung University

Abstract

During the last century, modular forms and modular curves played important roles in the developments of number theory. Shimura curves are natural generalizations of classical modular curves. The arithmetic properties of automorphic forms and Shimura curves are particularly important in modern number theory. Our aim is to study the arithmetic properties of automorphic forms and automorphic functions on Shimura curves. The work in this dissertation is a starting point.

Due to the recent work of Yifan Yang, if the Shimura curve is of genus zero, then one can express its automorphic forms in terms of the solutions of the associated Schwarzian differential equation. This provides a concrete space of automorphic forms. We then can do explicit computation on the spaces to study the arithmetic properties of automorphic forms and functions. Therefore, the main question is how to find the Schwarzian differential equations.

In this thesis, we determine the Schwarzian differential equations for certain Shimura curves of genus zero. As a byproduct of study on automorphic forms on Shimura curves, we also obtain several algebraic transformations of $_2F_1$-Hypergeometric functions. This discovery is achieved by interpreting Hypergeometric functions as automorphic forms on Shimura curves.

Acknowledgement

First of all, I would like to express my deepest appreciation to my advisor Professor Yifan Yang for his enthusiasm, insight, and patience throughout years of guidance in my graduate school career. Much contents and the results of the dissertation herein are discussed with him. Without his constant encouragement, patient instruction, and enthusiastic supports, it is impossible for me today to be in this position. Many thanks are due to Professor Hashimoto of Waseda University for providing me the opportunity to be a research student under his supervision. He gives me several enlightening advices during the periods while I stay there. Part of the results in Chapter 3 included in this thesis was written at that time.

I would also like to show my gratitude to Professor Ming-Lun Hsieh, Professor Ming-Hsuan Kang, Professor Wen-Ching Winnie Li, and Professor Jing Yu for reviewing my dissertation and serving on the committee of my thesis defence.

For my student life, I felt an immense gratitude to Professor Winnie Li, Professor Jing Yu, as well as the government of Republic of China. Because of their organizations and supports, I have the chances to attend conferences, workshops, and study aboard. Also, I would like to thank my friends, especially best friends, and those in Japan. Thank you for doing me favors generously when I got into troubles. Thank you guys for making my student life colorful and smooth. There are too many persons to whom I want to express my gratitude, so please forgive me if I do not list all of you.

Finally, I am particularly indebted to my family for their love and support in my entire life.

TU, Fang-Ting

Department of Applied Mathematics

College of Science

National Chiao Tung University

2013. 9. 12

Contents

Chapter 1

Introduction

During the last century, modular forms and modular curves played important roles in the developments of number theory. A reason of this fact is because of the connection with the moduli space of elliptic curves, and that the elliptic curves, being algebraic curves of the smallest positive genus, are related with many non-trivial Diophantine problems in number theory. For example, the arithmetic properties of elliptic curves are essential in Andrew Wiles' proof of Fermat's Last Theorem. Shimura curves are natural generalizations of classical modular curves. Similar to the classical modular curves, Shimura curves are moduli spaces of certain abelian surfaces with quaternionic multiplication. The arithmetic properties of Shimura curves are particularly important in modern number theory. Our aim is to study the arithmetic of automorphic forms and automorphic functions on Shimura curves. The work in this dissertation is a starting point.

A Shimura curve is a quotient space of the upper half plane $\mathfrak{h} = \{\tau : \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ obtained by certain quaternion order. More precisely, we let K be a totally real number of degree n and B be a quaternion algebra over K that splits exactly at one infinite place, that is,

$$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M(2, \mathbb{R}) \times \mathbb{H}^{n-1},$$

where $M(2, \mathbb{R})$ is the algebra of 2 by 2 matrices over $\mathbb{R}$ and $\mathbb{H}$ is Hamilton's quaternion algebra. Up to conjugation, there is a unique embedding ι_∞ from B into $M(2, \mathbb{R})$. Given an order $\mathcal{O}$ of B, we let $\mathcal{O}^1$ be the group of the elements of reduced norm 1 of $\mathcal{O}$. Then the image $\Gamma(O) = \iota_\infty(\mathcal{O}^1)$ under the embedding ι_∞ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and hence there is a group action of $\Gamma(\mathcal{O})$ on $\mathfrak{h}$ by the usual fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathcal{O}).$$

When $B \neq M(2, \mathbb{Q})$, we denote by $X(\mathcal{O})$ the Riemann surface $\Gamma(\mathcal{O})\backslash\mathfrak{h}$. This is the so-called Shimura curve associated to $\mathcal{O}$. In the case of $B = M(2, \mathbb{Q})$, the compactified curve $\Gamma(\mathcal{O})\backslash(\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}))$ by adjoining cusps is the classical modular curve.

When $B \neq M(2, \mathbb{Q})$, an automorphic form of weight k on $\Gamma(\mathcal{O})$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),\ \forall\, \tau \in \mathfrak{h},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O}).$$

For the classical modular forms, i.e., the case of $B = M(2, \mathbb{Q})$, we need additional conditions on cusps.

Even though it is true that many theoretical aspects of classical modular curves can be extended to the case of Shimura curves, to the best knowledge of the author, it is not true for explicit methods. In the case of classical modular curves, many problems about modular curves can be answered using Fourier expansions of modular forms or modular functions involved, and there are many explicit methods for constructing modular functions, modular forms and computing their Fourier expansions. In fact, because the Fourier coefficients of a normalized Hecke eigenform on congruence subgroups are identical with the eigenvalues of Hecke operators, one can compute the expansions of Hecke eigenforms without actually constructing them. However, unlike their classical counterpart, Shimura curves do not have cusps and hence automorphic forms or automorphic functions on Shimura curves do not have Fourier expansions. Because of this, as far as we know, there have been very few explicit methods to construct automorphic forms and automorphic functions on Shimura curves. Also, any method for classical modular curves that uses Fourier expansions can not possibly be extended to the case of Shimura curves. Therefore, the question is how to construct automorphic forms on Shimura curves with Taylor series at a CM-point.

Recently, Yang [33] had a breakthrough for constructing automorphic forms on Shimura curves. In the work of Yang [33], he proposed a new method to study automorphic forms on Shimura curves of genus zero, in which automorphic forms are expressed in terms of solutions of Schwarzian differential equations. He then demonstrated how to compute Hecke operators explicitly on these automorphic forms. Moreover, since Schwarzian differential equations that with exactly 3 singularities are essentially hypergeometric, this approach leads to many identities among hypergeometric functions by interpreting the hypergeometric functions as automorphic forms on Shimura curves. This was the main theme of my joint paper with Yang [24], the author [22] also gave more examples of algebraic transformations of hypergeometric functions to illustrate the role Shimura curves play in proving these identities.

Due to the results of Yang [33], once the Schwarzian differential equation for a Shimura curve of genus zero is determined, we can study the arithmetic properties of the automorphic forms on this Shimura curve as t-series, where t is a generator of the field of functions on the Shimura curve of genus zero. Because of the importance of Schwarzian differential equations in explicit methods for Shimura curves, one of the main goals is to determine Schwarzian differential equations for as many Shimura curves as possible. Especially, we are most interested in the Shimura curves attached to Eichler orders of the indefinite quaternion algebras over $\mathbb{Q}$ and their quotients by Atkin-Lehener involutions.

We denote by $X_0^D(N)$ the Shimura curve obtained by an Eichler order of level N in an indefinite quaternion algebra defined over $\mathbb{Q}$ of discriminant D. (When $D = 1$, the curve $X_0^1(N)$ is the classical modular curve $X_0(N)$.) Let $W_{D,N}$ be the group of all the Atkin-Lehner involutions w_m of $X_0^D(N)$. In this dissertation, let us focus on the Shimura curves $X_0^D(N)/G$, quotient by some subgroup G of $W_{D,N}$, $D > 1$. We will determine the Schwarzian differential equations for certain Shimura curves $X_0^D(N)/W_{D,N}$ of genus zero.

In order to determine the Schwarzian differential equation for a given Shimura curve, we will first compute the defining equations of Shimura curves over $\mathbb{Q}$, and then construct coverings, we can find the coverings between Shimura curves. These relations will help us determine the Schwarzian differential equations. The key ingredients for determination of the equations of Shimura curves are the Čerednik-Drinfeld theory of p-adic uniformization for Shimura curves, and the Jacquet-Langlands correspondence. The Jacquet-Langlands correspondence gives a bijection from automorphic representations on $X_0^D(N)$ and certain modular representations on $X_0(DN)$. This tells us the isogeny class of a given Shimura curve which is an elliptic curve defined over $\mathbb{Q}$. The Čerednik-Drinfeld theory gives us the information of the bad reductions of Shimura curves, and then we can determine the isomorphism class of the given Shimura curve.

For the rest of this dissertation, we will first say a few words about quaternion algebras, Shimura curves and then introduce my recent work of automorphic forms on Shimura curves. In Chapter 2, we introduce quaternion algebras, quaternion orders, Shimura curves, automorphic forms and automorphic functions on Shimura curves. In Chapter 3, we briefly recall some basic and useful properties of the Eichler orders of level (D, N), the Shimura curves $X_0^D(N)$, automorphic forms on $X_0^D(N)$, the Čerednik-Drinfeld theory of p-adic uniformization for Shimura curves, and the Jacquet-Langlands correspondence.

In Chapter 4, we provide the connection between the automorphic forms on Shimura curves and the Schwarzian differential equations. Also, we will work out Schwarzian differential equations for certain Shimura curves $X_0^D(N)/W_{D,1}$ of genus zero. As applications of the arithmetic of automorphic forms on Shimura curves of genus zero, in Chapter 5, we compute Hecke operators T_p with prime p on $X_0^{14}(1)/W_{14,1}$ and use numerical computation to obtain Ramanujan-type series for the curve $X_0^{14}(1)/W_{14,1}$. This gives a numerical evidence to Yang's conjecture in [32].

Finally, in Chapter 6, as a byproduct of the study on arithmetic properties of automorphic forms, we obtain some algebraic transformations of ${}_2F_1$-hypergeometric functions.

For the future studies on the arithmetic of automorphic forms on Shimura curves of genus zero, we plan to determine the coordinates of CM-points on Shimura curves. The CM-points on Shimura curves correspond to abelian surfaces with endomorphism algebra equal to a matrix algebra of degree 2 over an imaginary quadratic number field. Another application is related to the Ramanujan-type formulae for Shimura curves. Moreover, a main future work is to generalize Yang's result. One restriction of Yang's approach is that the genus of the Shimura curve has to be zero. That is, it is not known how to express automorphic forms on Shimura curves using solutions of Schwarzian differential equations when the genus is positive. We will try to extend Yang's method to higher genus cases. Elkies [6], Greenberg, Voight [11, 28, 29, 30] also introduced many methods to do computations on the arithmetic of the Shimura curves $X_0^D(N)$, $X_0(\mathfrak{N})$ which is associated to a quaternion algebra defined over a totallyreal number field F, or the Shimura curves arising the arithmetic triangle groups. For instances, they compute CM-points on the Shimura curves, determine the system of Hecke eigenvalues by using the Jacquet-Langlands correspondences. Another furure work is to generalize their results.

Chapter 2

Quaternion algebras and Shimura curves

In this chapter, we will briefly recall some basic definitions and properties of quaternion algebras, especially quaternion algebras over a local field or number field. Then we will define the Shimura curves. Most of the materials are taken from the references [1, 26]. From now on, we let K be a field with characteristic not 2.

2.1 Quaternion algebras

2.1.1 Quaternion algebras and quadratic forms

A **quaternion algebra** B over a field K is a central simple algebra of dimension 4 over K, or equivalently, there exist $i, j \in B$ and $a, b \in K^*$ so that

$$B = K + Ki + Kj + Kij, \quad i^2 = a,\ j^2 = b,\ ij = -ji.$$

In such case, we denote by $\left(\frac{a,b}{K}\right)$ the quaternion algebra B, which has canonical K-basis $\{1, i, j, ij\}$. Familiar examples are Hamilton's quaternions $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ and the matrix algebra $M(2, K) \cong \left(\frac{1,1}{K}\right)$.

Theorem 2.1.1. *If a quaternion algebra B over K has a zero divisor, then it is isomorphic to $M(2, K)$.*

According to Theorem 2.1.1, if a has a square root α in K then the quaternion algebra B has a zero divisor $h = \alpha - i$, and B is isomorphic to the 2-by-2 matrix algebra. Hence, if K is an algebraically closed field, then the only structure of K-quaternion algebra is the matrix algebra.

Notice that an element h in a quaternion algebra satisfies a monic polynomial over K of degree less than 2. Therefore, any quaternion algebra B is provided with a unique K-linear anti-involution $^-\colon B \longrightarrow B$,

$$\bar{h} = a_0 - a_1 i - a_2 j - a_3 ij, \text{ if } h = a_0 + a_1 i + a_2 j + a_3 ij \in \left(\frac{a,b}{K}\right).$$

This map is called the conjugation. The **reduced trace**, and **reduced norm** on B are defined by

$$\mathrm{tr}(h) = h + \bar{h}, \quad \text{and} \quad n(h) = h\bar{h},$$

respectively. We remark that $\mathrm{tr}(h) = 2h$ and $n(h) = h^2$, if h lies in the center K. If $B = M(2, K)$ then the reduced trace and reduced norm of an element $h \in B$ are the trace and the determinant of h. These maps tr and n lead to a nondegenerate symmetric K-bilinear form on B, which is given by $\mathrm{tr}(x\bar{y})$. In other words, the quaternion algebra B is a quadratic space with the quadratic form given by the reduced norm of B.

Recall that a quadratic space with a quadratic form Q is said to be isotropic if there is a non-zero element x so that $Q(x) = 0$. We have the following facts.

Theorem 2.1.2. *For a quaternion algebra $B = \left(\frac{a,b}{K}\right)$ over K, the following are equivalent.*

(1) B is isomorphic to $M(2, K)$.

(2) B is not a division quaternion algebra.

(3) B is isotropic as a quadratic space with the reduce norm.

(4) The quadratic form $ax^2 + by^2$ represents 1.

(5) If $F = K(\sqrt{b})$, then a is an element of $N_{F/K}(F)$.

Denote B_0 by the pure quaternion space, $B_0 = \{x \in B : \mathrm{tr}(x) = 0\}$.

Theorem 2.1.3. *Let B and B' be two quaternion algebras over K. Then B is isometric to B' if and only if B_0 and B'_0 are isomorphic. Equivalently, the quaternion algebras $\left(\frac{a,b}{K}\right)$, $\left(\frac{a',b'}{K}\right)$ are isomorphic if and only if the quadratic forms*

$$ax^2 + by^2 - abz^2 \text{ and } a'x^2 + b'y^2 - a'b'z^2$$

are equivalent over K.

2.1.2 Automorphism theorem

Theorem 2.1.4. *(***Noether-Skolem Theorem***)*
Let L, L' be two commutative K-algebras over K contained in a quaternion algebra B over K. Then all K-isomorphism from L to L' can be extended to an inner automorphism of B. The K-automorphisms of B are all inner automorphisms.

Remark 2.1.5. *An inner automorphism of B is an automorphism given by $k \mapsto hkh^{-1}$, for some invertible element h of B. Therefore, according to the Theorem 2.1.4, the automorphism group of the quaternion algebra B, $\mathrm{Aut}_K(B)$, is isomorphic to B^*/K^*.*

Corollary 2.1.6. *For all separable quadratic algebras F over K contained in B, there exists an element $\theta \in K^\times$ such that*

$$B = F + Fu, \quad u^2 = \theta \text{ and } um = \sigma(m)u,$$

where σ denotes the non-trivial K-automorphism of F. In this case, we use the symbol $\{F, \theta\}$ to denote the quaternion algebra B.

Remark 2.1.7. *Let $\sigma : F \longrightarrow L$ be a nontrivial K-automorphism of L. Then there exist $u \in B^*$ so that $umu^{-1} = \sigma(m)$, for all $m \in F$. The fact $t(u) = 0$ implies that $u^2 = \theta \in K$. In this way, we realize B as $B = \{F, \theta\}$, moreover, $B = \left(\frac{a,b}{K}\right) = \{K(i), b\}$.*

2.2 Orders and Ideals

As the fractional ideals in a number field, there is a similar theory for ideals in a quaternion algebra. Let R be a Dedekind domain and K be its field of fractions. An **R-lattice** of a K-vector space V is a finitely generated R-module contained in V. A **complete R-lattice** Λ of V is an **R-lattice** Λ of V such that $K \otimes_R \Lambda \simeq V$.

Example 2.2.1. *We consider the cases in the quaternion algebras and quadratic number fields.*

1. *Let $\Lambda_1 = R + Ri$ and $\Lambda_2 = R + Ri + Rj + Rij$. Then they are both R-lattice of H and Λ_2 is complete.*
2. *Given $R = \mathbb{Z}$, $K = \mathbb{Q}$. Let $V = \mathbb{Q}(\sqrt{m})$ and Λ be its number ring, where m is a square-free integer. Then Λ is a complete lattice.*

Definition 2.2.1. *An* **ideal** *of a quaternion algebra B is a complete R-lattice in B. If an ideal of B is also a ring with unity, it is called an* **order**. *Moveover, we say that I is a* **left ideal** *of $\mathcal{O}$ if $\mathcal{O}I \subset I$; I is a* **right ideal** *of $\mathcal{O}$ if $I\mathcal{O} \subset I$.*

Definition 2.2.2. *A* **maximal order** *of B is an order that is not properly contained in another order of B. An intersection of two maximal orders of B is called an* **Eichler order**.

Now if an ideal I is given, we can define two orders associated to I, **the left order of** I,

$$\mathcal{O}_\ell(I) = \{h \in B : hI \subseteq I\},$$

and **the right order of** I,

$$\mathcal{O}_r(I) = \{h \in B : Ih \subseteq I\}.$$

Definition 2.2.3. *An ideal I is said to be* **two-sided** *if $\mathcal{O}_\ell(I) = \mathcal{O}_r(I)$, said to be* **integral** *if I is contained in both $\mathcal{O}_\ell(I)$ and $\mathcal{O}_r(I)$. If $\mathcal{O}_\ell(I)$ and $\mathcal{O}_r(I)$ are maximal orders, then I is called a* **normal ideal**.

An element x of a quaternion algebra B is called to be **integral over** R if $R[x]$ is a R-lattice of B. For instance, the element i in the classical quaternion algebra $H = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$ is an integral element but $i/2$ is not. Actually, we have a useful criterion to determine whether if an element is integral or not.

Lemma 2.2.2. *An element of a quaternion algebra B is integral if and only if its reduced trace and norm are in the ring R.*

Also, we have an equivalently definition of an order of a quaternion algebra.

Proposition 2.2.3. *Let B be a quaternion algebra over K.*

1. *$\mathcal{O}$ is an order of B if and only if $\mathcal{O}$ is a ring of integral elements in B which contains R and K-basis for B.*

2. *Every order is contained in a maximal order.*

The second proposition is followed from the first one and Zorn's Lemma. From this proposition, we can see that an integral ideal is an ideal whose elements are all integral elements.

There are also the analogue of the norm of an ideal, and the discriminant of an order as in the algebraic number theory. The **inverse of** I is defined to be

$$I^{-1} = \{h \in A : IhI \subset I\},$$

which is also an ideal. The **norm of** I, $n(I)$, is the R-fractional ideal generated by $\{n(x) : x \in I\}$. The **dual** I^* of I is

$$I^* = \{h \in A : \mathrm{tr}(hI) \subset R\}.$$

The **discriminant of an order** $\mathcal{O}$ is $D_{\mathcal{O}} = n(\mathcal{O}^*)^{-1}$. If I is a left ideal of $\mathcal{O}$, then the **discriminant of** I is given by $D_I = n(I^*)^{-1}n(I)$.

Proposition 2.2.4. *We have the following properties:*

(1) $II^{-1} \subseteq \mathcal{O}_\ell(I)$ *and* $I^{-1}I \subseteq \mathcal{O}_r(I)$.

(2) The square of discriminant of $\mathcal{O}$, $D^2_{\mathcal{O}}$, is equal to the ideal over R generated by

$$\{\det(\mathrm{tr}(x_ix_j)) : 1 \le i, j \le 4, x_i, x_j \in \mathcal{O}\}.$$

In particular, if $\mathcal{O}$ has free basis $\{e_1, e_2, e_3, e_4\}$ over R, then $D^2_{\mathcal{O}}$ is the principal R-ideal $\det(\mathrm{tr}(e_ie_j))R$.

(3) If an order $\mathcal{O}'$ is contained in the other order $\mathcal{O}$, then $D_{\mathcal{O}}$ divides $D_{\mathcal{O}'}$. Therefore, $D_{\mathcal{O}} = D_{\mathcal{O}'}$ is and only if $\mathcal{O} = \mathcal{O}'$.

(3) If I is a left ideal of an order $\mathcal{O}$, then $D_I = n(I)^2 D_{\mathcal{O}}$ and

$$D^2_I = \{\det(\mathrm{tr}(x_ix_j)) : 1 \le i, j \le 4, x_i, x_j \in I\}.$$

Example 2.2.5. *(1) The discriminant of the order $M(2, R)$ is R.*

(2) Consider the two orders

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij$$

and

$$\mathcal{O}' = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1 + i + j + ij}{2}$$

in the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$. It obvious that $\mathcal{O} \subset \mathcal{O}'$ and

$$D^2_{\mathcal{O}'} = 4\mathbb{Z} \supset 16\mathbb{Z} = D_{\mathcal{O}}.$$

In the case of the quaternion algebra $B = M(2, K)$. One can identify B with the endomorphsim ring of some vector space over K. To be more precise, let V be a vector space over K with basis $\{e_1, e_2\}$. Then with respect to this basis, $M(2, K)$ is viewed as End(V). Given a complete R-lattice Λ in V, we can see that

$$\text{End}(\Lambda) = \{\alpha \in \text{End}(V) : \alpha\Lambda \subset \Lambda\}$$

is a maximal order in End(V). Conversely, for a given order $\mathcal{O}$ in End(V), we can associate an R-module

$$\Lambda = \{\alpha e_i : \alpha \in \mathcal{O}, i = 1, 2\},$$

which is a complete R-lattice, to the order $\mathcal{O}$ contained in End(Λ).

Proposition 2.2.6. *If R is a principal ideal domain, then each maximal order in $M(2, K)$ is conjugate to the maximal order $M(2, R)$.*

2.3 Quaternion Algebras over Local Fields

For a local field K, there are at most 2 non-isomorphic structures of quaternion algebras over K. If $K = \mathbb{C}$, there is only one $\mathbb{C}$-quaternion algebra, namely, the matrix algebra $M(2, \mathbb{C})$. For the Archimedean local field $\mathbb{R}$, a quaternion algebra over $\mathbb{R}$ is either isomorphic to $M(2, \mathbb{R})$ or the quaternions of Hamilton $\mathbb{H}$. If K is non-Archimedean, then a quaternion algebra over K is isomorphic to exactly one of $M(2, K)$ or the unique division quaternion algebra over K.

Theorem 2.3.1. *(**Frobenius Theorem**)*
Let D be a division ring containing $\mathbb{R}$ in its center of finite dimension over $\mathbb{R}$. Then D is isomorphic to $\mathbb{H}$, the Hamiltonian quaternion.

Hence, Frobenius' Theorem tells us that a quaternion algebra is either isomorphic to $M(2, \mathbb{R})$ or $\mathbb{H}$.

2.3.1 Quaternion algebra over non-Archimedean local fields

For a non-Archimedean local field K, we let R be its ring of integers and π be a fixed uniformizer with respect to the valuation ν.

Theorem 2.3.2. *There is a unique division quaternion algebra over K and it is isomorphic to $\left(\frac{\pi,e}{K}\right)$, where $K(\sqrt{e})$ is the unique unramified quadratic extension of K.*

While $h \neq 0$ in $\left(\frac{\pi,e}{K}\right)$, the map ω given by $\omega(h) = \frac{1}{2}\nu(N(h))$ defines a discrete valuation on the division algebra $\left(\frac{\pi,e}{K}\right)$.

We define the **Hasse invariant** of the quaternion algebra B by

$$\varepsilon(B) = \begin{cases} 1, & \text{if } B \cong M(2,K), \\ -1, & \text{otherwise.} \end{cases}$$

In the case of $K = \mathbb{Q}_p$, the Hasse invariant of $B = \left(\frac{a,b}{\mathbb{Q}_p}\right)$ coincides with the **Hilbert Symbol** $(a,b)_p$, which is given by

$$(a,b)_p = \begin{cases} 1, & \text{if } ax^2 + by^2 \text{ reprents } 1, \\ -1, & \text{otherwise.} \end{cases}$$

Remark 2.3.3. *From the Theorem 2.3.2, for $p > 2$, we have a simple description for the Hilbert symbol $(a,b)_p$ with $p \nmid a$,*

$$(a,b)_p = \begin{cases} 1, & \text{if } p \nmid a, b, \\ \left(\frac{a}{p}\right), & p \nmid a,\ p \mid b, \end{cases}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

2.3.2 Orders in $B = \left(\frac{\pi,e}{K}\right)$

For the unique division quaternion algebra $B = \left(\frac{\pi,e}{K}\right)$, it is known that there is a unique maximal order in B, which is the associated valuation ring

$$\mathcal{O} = \{h \in B : w(h) \geq 0\} = \{h \in B : N(h) \in R\}$$

with respective to the valuation w. The ring

$$P = \{h \in B : w(h) > 0\}$$

is a two-sided prime ideal of $\mathcal{O}$.

Theorem 2.3.4. *Let $B = \left(\frac{e,\pi}{K}\right)$, $F = K(\sqrt{e})$, and $\mathcal{O}$ be the unique maximal order in B. Then we have*

1. *$P = \mathcal{O}j$ is a prime ideal of $\mathcal{O}$ and $P^2 = \mathcal{O}\pi$.*
2. *$\mathcal{O} = R_F + R_F j$, where R_F is the ring of integers of F.*
3. *The discriminant of $\mathcal{O}$ is $D_{\mathcal{O}} = \pi^2 R$.*

2.3.3 Orders in $M(2, K)$

If B is isomorphic to $M(2, K)$, then as the consideration in the end of the last section, each maximal order in B is then isomorphic to the maximal order $M(2, R)$. We now let $B = M(2, K)$.

Theorem 2.3.5. *1. A maximal order of $M(2, K)$ is conjugate to $M(2, R)$ by an element of $\mathrm{GL}(2, K)$.*

2. The set of all maximal orders is in one-to-one correspondence with the cosets

$$K^*\mathrm{GL}(2, R)\backslash\mathrm{GL}(2, K).$$

The standard coset representatives of $K^*\mathrm{GL}(2, R)\backslash\mathrm{GL}(2, K)$ are

$$\begin{pmatrix} \pi^a & c \\ 0 & \pi^b \end{pmatrix},$$

where a and b are nonnegative integers and c are from $R/(\pi)^b$, subject to the condition that $v(c) = 0$ if $a, b > 0$. Therefore, we can classify all maximal orders of $M(2, K)$ as

$$\begin{pmatrix} \pi^a & c \\ 0 & \pi^b \end{pmatrix}^{-1} M(2, R) \begin{pmatrix} \pi^a & c \\ 0 & \pi^b \end{pmatrix}, \quad a, b \geq 0, \ c \bmod \pi^b,$$

and $c \notin \pi R$ if $a, b > 0$.

Also, we can classify the Eichler order of $M(2, K)$.

Proposition 2.3.6. (Hijikata)
If $\mathcal{O}$ is an order in $M(2, K)$, then the followings are equivalent.

1. *$\mathcal{O}$ is an Eichler order.*
2. *There exists a unique pair of maximal orders $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$.*
3. *There exists $n \in \mathbb{Z}_{>0}$ such that $\mathcal{O}$ is conjugate to $\left(\begin{smallmatrix} R & R \\ \pi^n R & R \end{smallmatrix}\right)$.*
4. *The order $\mathcal{O}$ contains $R \oplus R$ as a subring.*

We say that an Eichler order in $M(2, K)$ is **of level** $\pi^n R$, if it is conjugate to $\left(\begin{smallmatrix} R & R \\ \pi^n R & R \end{smallmatrix}\right)$.

We now introduce the graph of maximal orders of $M(2, K)$. First, let us define the distance between the maximal orders. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be two maximal orders in $M(2, K)$. If the the Eichler order $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$ is of index q^n in $\mathcal{O}_1$, then the **distance** between $\mathcal{O}_1$ and $\mathcal{O}_2$ is $d(\mathcal{O}_1, \mathcal{O}_2) = n$, where q is the cardinality of the residue field $R/\pi R$. Equivalently, the Eichler order $\mathcal{O}_1 \cap \mathcal{O}_2$ is of level $\pi^n R$.

Now we define a graph X of maximal orders as follows. The vertices of X are the maximal orders and two vertices are connected by a simple edge if the two corresponding maximal orders has distance 1.

Example 2.3.7. *Let* $\mathcal{O}_0 = M(2, R)$*,*

$$\mathcal{O}_1 = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathcal{O}_0 \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & \pi^{-1}R \\ \pi R & R \end{pmatrix},$$

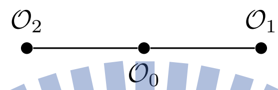
and

$$\mathcal{O}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}^{-1} \mathcal{O}_0 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} R & \pi R \\ \pi^{-1}R & R \end{pmatrix}.$$

We have $\mathcal{O}_0 \cap \mathcal{O}_1 = \begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}$*,*

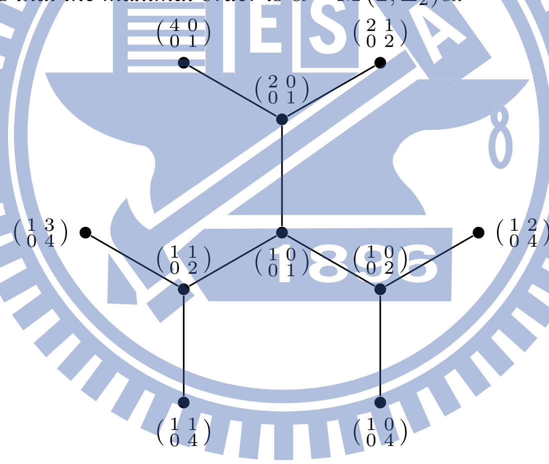
$$\mathcal{O}_0 \cap \mathcal{O}_2 = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix} \cong \begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}, \quad \mathcal{O}_1 \cap \mathcal{O}_2 = \begin{pmatrix} R & \pi R \\ \pi R & R \end{pmatrix} \cong \begin{pmatrix} R & R \\ \pi^2 R & R \end{pmatrix}.$$

Thus, $d(\mathcal{O}_0, \mathcal{O}_1) = d(\mathcal{O}_0, \mathcal{O}_2) = 1$ *and* $d(\mathcal{O}_1, \mathcal{O}_2) = 2$*. The subgraph of these maximal orders is*



Proposition 2.3.8. *The graph* X *is a* $(q+1)$*-regular tree, i.e., a connected graph without cycles, and every vertex has precisely* $q+1$ *edges connecting to it.*

Example 2.3.9. *Here is a subtree of maximal orders of* $M(2, \mathbb{Q}_2)$*. The matrix* α *next to a vertex means that the maximal order is* $\alpha^{-1} M(2, \mathbb{Z}_2)\alpha$*.*



Remark 2.3.10. *We remark that the group* $\mathrm{PGL}(2, K)$ *acts on the coset* $K^*\mathrm{GL}(2, R)\backslash\mathrm{GL}(2, K)$*, and hence acts by conjugation on the tree of maximal orders in* $M(2, K)$*. In particular,* $\mathrm{PGL}(2, K)$ *acts on the set*

$$\mathfrak{L}^{(n)} = \{(\mathcal{O}_1, \mathcal{O}_2) : d(\mathcal{O}_0, \mathcal{O}_2) = n\}$$

double transitively.

2.4 Quaternion Algebras over Number Fields

We now recall the classification of quaternion algebras over a number field. Let K be a number field, and R be its ring of integers. Let K_v be the local field with respect to the place v of K.

2.4.1 Classification of quaternion algebras over number fields

A quaternion algebra B over a number field K is said to be **ramified at** v if $B_v = B \otimes K_v$ is a division algebra. Otherwise, B is **unramified or split at** v.

Theorem 2.4.1. *(***Hasse-Minkowski Theoerm***)*
The quaternion algebra B is isomorphic to $M(2, K)$ if and only if B splits over K_v for all places v.

Let $\mathrm{Ram}(B)$ denote the set of ramified places of B. The reduced discriminant of quaternion algebra B is the integral ideal of R defined by

$$D_B = \prod_{v \in \mathrm{Ram}(B)} v.$$

In the case that R is a principal ideal domain, we identify the ideal D_B with its generator, up to units. That is, $D_B R = \prod_{v \in \mathrm{Ram}(B)} v$; for a quaternion algebra over $\mathbb{Q}$, its discriminant is an integer.

The structure of the quaternion algebra B is uniquely determined by the reduced discriminant.

Theorem 2.4.2. *(1) The cardinality of Ram(B) is finite and even.*

(2) Two quaternion algebras B and B' over K are isomorphic if and only if Ram$(B) =$ Ram(B').

(3) Given a finite set S of noncomplex places of K such that $|S|$ is even, there exists a quaternion algebra B over K such that Ram$(B) = S$.

Therefore, if an even number of noncomplex places of K is given, then there exists one and only one K-quaternion algebra that ramifies exactly at these places.

Example 2.4.3. *(1) A quaternion algebra over a number field K is isomorphic to $M(2, K)$ if and only if $D_B = R$.*

(2) The discriminant of the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is 2, since the values of the Hilbert symbols are

$$(-1, -1)_p = \begin{cases} -1, & \text{if } p = \infty, 2, \\ 1, & \text{if } p > 2. \end{cases}$$

For any field F, if B is a quaternion algebra over F and L is a field extension of F. We say that L **splits** B if $L \otimes_F B$ is isomorphic to $M(2, L)$. We now address the conditions that when a K-quaternion algebra B splits over a quadratic extension field F of K. In particular, one has the conditions for which quadratic fields can be embedded into B. Let L be a finite extension field over K, and w be a place of L.

Proposition 2.4.4. *Let B be a quaternion algebra over K. Then B splits over L if and only if B_v splits over L_w for any place $w|v$ of L. In particular, if L is a quadratic field over K, then followings are equivalent:*

(1) The field L is a splitting field for B.

(2) The field L is K-isomorphic to a maximal subfield of B containing K.

(3) There exists an embedding over K form L into B.

(4) Each place v in K that ramifies in B is not totally split in L.

For a totally real number field K, if a quaternion algebra over K is ramified at all the real infinite places, we say that the quaternion algebra is **definite**; otherwise, it is **indefinite**. We remark that a quaternion algebra B is definite if and only if the quadratic form given by $< x, y >= \mathrm{tr}(x\bar{y})$ on B is positive definite.

2.4.2 Orders in a quaternion algebra over a number field

Let I be an ideal in a quaternion algebra B over a number field K. Denote R_v the ring of integers of the localization K_v. Then the localization $I_v = I \otimes_{\mathbb{Z}} R_v$ is an ideal in the quaternion algebra B_v and $I = B \cap (\prod_v I_v)$. As the Hasse-Minkowshi theorem for quaternion algebras, being a maximal order or an Eichler order satisfied the local-global correspondence.

Proposition 2.4.5. *Let Λ be a lattice in a quaternion algebra B over K. For any finite place v in K, we consider a local lattice L_v in B_v. Assume that $L_v = \Lambda_v$ for almost all v. Then there exists a lattice Λ' in B such that $\Lambda'_v = L_v$ for all finite places v.*

This gives us the existence of a global lattice.

Note that if $\mathcal{O}$ is a maximal order of B, it is clear that $\mathcal{O}_v$ is again an order in B_v and $(D_{\mathcal{O}})_v = D_{\mathcal{O}_v}$. We have a criterion for global maximal orders from the information of the discriminants.

Proposition 2.4.6. *An order $\mathcal{O}$ is maximal in the quaternion algebra B if and only if its discriminant is equal to the discriminant of B, i.e, $D_{\mathcal{O}} = D_B$.*

Example 2.4.7. *In the quaternion algebra $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, the order $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij$ is a maximal order with $D_{\mathcal{O}} = 2 = D_B$.*

Definition 2.4.1. *The* **level** *of a global Eichler order is the unique integral ideal $N_{\mathcal{O}}$ in R so that $N_{\mathcal{O}_v}$ is the level of each $\mathcal{O}_v$ at each finite place of K. That is, $N_{\mathcal{O}} = \prod_v N_{\mathcal{O}_v}$. If R is a PID, we identify the ideal $N_{\mathcal{O}}$ with its generator, up to units.*

Unlike the case of maximal orders, we have no explicit classification of Eichler orders in terms of the discriminant.

Proposition 2.4.8. *If $\mathcal{O}$ is an Eichler order of level N, then the discriminant of $\mathcal{O}$ is $D_{\mathcal{O}} = D_B N$.*

Lemma 2.4.9. *Let I be an ideal in B and its right order $\mathcal{O} = \mathcal{O}_r(I)$ is a maximal order. Then there exists an element $h_v \in B_v^*$ so that $I_v = h_v \mathcal{O}_v$.*

Corollary 2.4.10. *For an ideal I in B, the right order of I, $\mathcal{O}_r(I)$, is maximal if and only if the left order of I, $\mathcal{O}_\ell(I)$, is maximal.*

Corollary 2.4.11. *If I is a normal ideal in B, then $I^{-1}I = \mathcal{O}_r(I)$ and $II^{-1} = \mathcal{O}_\ell(I)$.*

2.5 Shimura Curves

We are now in a position to introduce Shimura curves. In this section, we will focus on the indefinite quaternion algebras over totally real number fields, especially the rational field.

Assume that K is a totally real number field and take a quaternion algebra B over K that splits exactly at one infinite place among all infinite places. That is, $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M(2,\mathbb{R}) \times \mathbb{H}^{[K:\mathbb{Q}]-1}$,where $\mathbb{H}$ is Hamilton's quaternions. Notice that we have a natural embedding from B into $B \otimes_{\mathbb{Q}} \mathbb{R}$, we now let $i_\infty : B \hookrightarrow M(2,\mathbb{R})$ be the projection onto the first factor. Let $\mathcal{O}$ be an order of B,

$$\mathcal{O}^1 = \{\gamma \in \mathcal{O} : n(\gamma) = 1\}, \quad \text{and} \quad \Gamma(\mathcal{O}) = i_\infty(\mathcal{O}^1).$$

Then $\Gamma(\mathcal{O})$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and hence it acts on the upper half plane $\mathfrak{h} = \{\tau : \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ by the usual fractional linear transformations.

We denote $X(\mathcal{O})$ the quotient space $\Gamma(\mathcal{O})\backslash\mathfrak{h}$ (or $\Gamma(\mathcal{O})\backslash\mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ if $B = \mathrm{M}(2,\mathbb{Q})$), which has a complex structure as a compact Riemann surface. It is the so-called **Shimura curve** associated to $\mathcal{O}$. In the case of the classical modular curve, the associated quaternion algebra is the matrix algebra $B = \mathrm{M}(2,\mathbb{Q})$ with discriminant $D = 1$.

Example 2.5.1. *(1) Let $B = M(2,\mathbb{Q})$. If $\mathcal{O} = M(2,\mathbb{Z})$, then $\Gamma(\mathcal{O}) = \mathrm{SL}(2,\mathbb{Z})$ and $X(\mathcal{O})$ is the classical modular curve $X(1) = X_0(1)$. For the Eichler order $\mathcal{O} = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z}\end{smallmatrix}\right)$, $X(\mathcal{O})$ is the modular curve $X_0(N)$.*

(2) Let $\mathcal{O}$ be the order $\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+ij}{2}$ in the quaternion algebra $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$. The quaternion algebra is ramified at the finite places 2 and 3. An embedding $i_\infty : B \to M(2,\mathbb{R})$ is given by

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$$

and

$$i_\infty(\mathcal{O}^1) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha\overline{\alpha} + \beta\overline{\beta} = 1,\ \alpha,\beta \in \mathbb{Z}[\sqrt{3}] \right\}.$$

2.6 Signatures of Shimura curves

Recall that a nonidentity element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}(2, \mathbb{R})$ is called **parabolic**, **hyperbolic**, or **elliptic** if γ has one fixed point, 2 distinct points of $\mathbb{P}^1(\mathbb{R})$, or a pair of conjugate complex numbers, respectively. The points τ fixed by γ are the roots of

$$c\tau^2 + (d - a)\tau - b = 0.$$

Hence, it can be simplified that γ is parabolic, elliptic, or hyperbolic, corresponding to whether $|\mathrm{tr}(\gamma)| = 2$, $|\mathrm{tr}(\gamma)| < 2$, or $|\mathrm{tr}(\gamma)| > 2$.

Definition 2.6.1. *Let γ be an element of $\Gamma(\mathcal{O})$.*

1. *The fixed point of a parabolic element is called a* **cusp**. *We let $e = \infty$.*
2. *The point τ in the upper half-plane fixed by an elliptic element is called an* **elliptic point** *of order e, where e is the number of elements in $\Gamma(\mathcal{O})/\pm 1$ that fixes τ. In other words, e is the order of the isotropy subgroup of τ in $\Gamma(\mathcal{O})/\pm 1$.*

Note that cusps can only appear when the quaternion algebra is $\mathrm{M}(2, \mathbb{Q})$. Therefore, if $B \neq \mathrm{M}(2, \mathbb{Q})$, the quotient space $\Gamma(\mathcal{O}) \setminus \mathfrak{h}$ is a compact Riemann surface; if $B = \mathrm{M}(2, \mathbb{Q})$, we compactify the Riemann surface $\Gamma(\mathcal{O}) \setminus \mathfrak{h}$ by adjoining cusps.

Proposition 2.6.1. *If $\Gamma(\mathcal{O})$ has a parabolic element, then the related quaternion algebra must be $\mathrm{M}(2, \mathbb{Q})$.*

Proof. Let $\gamma \in \Gamma(\mathcal{O})$ be a parabolic element and h be the associated element in $\mathcal{O}^1$. Then $\mathrm{tr}(h) = 2$ or -2, and $N(h) = 1$. Note that ± 1 are elements of $\mathcal{O}^1$ and hence $\pm 1 - h$ belong to $\mathcal{O}$. Without loss generality, we may assume that $\mathrm{tr}(h) = 2$. Then $1 - h$ is an element has reduced trace 0 and reduced norm 0. This means that the quaternion algebra has a zero divisor element $1 - h$ and hence it is isomorphic to the 2-by-2 matrix algebra over a totally real number field. The only possibility is the $\mathbb{Q}$-quaternion algebra $\mathrm{M}(2, \mathbb{Q})$, for which splits at exactly one real place. □

For the curve $X(\mathcal{O})$ with genus g, it is well-known that there exist hyperbolic elements a_1, …, a_g, b_1, …, b_g, and elliptic or parabolic elements c_1, …, c_r that generate $\Gamma(\mathcal{O})/\pm 1$ with relations

$$[a_1, b_1] \ldots [a_g, b_g] c_1 \ldots c_r = 1, \text{ where } [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}.$$

We let $(g; e_1, \ldots, e_r)$ be the **signature** of the curve $X(\mathcal{O})$. The number e_j runs over all $\Gamma(\mathcal{O})$-inequivalent cusps and elliptic points. In particular, if a Shimura curve $X(\mathcal{O})$ has signature $(0; e_1, e_2, e_3)$, we say that $\Gamma(\mathcal{O})$ is an **arithmetic triangle group**.

2.7 Automorphic Forms on Shimura Curves

Let $X(\mathcal{O}) = \Gamma(\mathcal{O}) \setminus \mathfrak{h}$ be the Shimura curve associated to the order $\mathcal{O}$ in an indefinite quaternion algebra B. In this section, we let k be a non-negative even integer.

Definition 2.7.1. *An* **automorphic form** *of weight k on $\Gamma(\mathcal{O})$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\tau \in \mathfrak{h}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O})$.

If f is meromorphic and $k = 0$, then f is called an **automorphic function***. Moreover, if the Shimura curve is of genus 0, an automorphic function is said to be a* **Hauptmodul** *if it generates the field of automorphic functions on $\Gamma(\mathcal{O})$.*

Remark 2.7.1. *For the quaternion algebra $B = M(2, \mathbb{Q})$, we also need additional conditions at cusps. However, we do not consider the classical modular curves here, so we need not to consider the cusps. The curves mentioned in the following discussions are always concerned to be the quotient space related the quaternion algebra $B \neq \mathrm{M}(2, \mathbb{Q})$ (if not be pointed out).*

The automorphic forms of a given weight k form a complex vector space. We denote it by $\mathrm{S}_k(\Gamma(\mathcal{O}))$ or $S_k(X(\mathcal{O}))$. It is easy to see that the weight 0 automorphic forms on $\Gamma(\mathcal{O})$ are exactly the constant functions. Using the Riemann-Roch Theorem, one can figure out the dimension formula of $\mathrm{S}_k(\Gamma(\mathcal{O}))$.

Proposition 2.7.2. *If the signature of $X(\mathcal{O})$ is $(g; e_1, \ldots, e_r)$, then the dimension of the space of automorphic forms of weight k on $\Gamma(\mathcal{O})$ is*

$$\dim \mathrm{S}_k(\Gamma(\mathcal{O})) = \begin{cases} 1, & \text{if } k = 0, \\ g, & \text{if } k = 2, \\ (g-1)(k-1) + \sum_j \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor, & \text{if } k \geq 4. \end{cases}$$

Chapter 3

The Shimura Curves $X_0^D(N)$

In this chapter, we will review some facts about the Shimura curves $X_0^D(N)$, which is obtained by the Eichler order $\mathcal{O}(D, N)$ of level N in an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant D. Most of the materials are coming from [1, 4, 5, 15].

3.1 Eichler orders $\mathcal{O}(D, N)$ and Shimura curves $X_0^D(N)$

Let B be a quaternion algebra over $\mathbb{Q}$ of discriminant D. According to the proposition 2.4.5, for each positive integer N with $\gcd(D, N) = 1$, there exists an Eichler order of level N. We now give a characterizations of Eichler orders in a quaternion algebra over $\mathbb{Q}$.

Proposition 3.1.1. *Let $\mathcal{O}$ be an order in a $\mathbb{Q}$-quaternion algebra B of discriminant D. Let N be a positive integer relatively prime to D. Then the following conditions are equivalent:*

(1) $\mathcal{O}$ is an Eichler order of level N.

(2) For each prime number, the localization $\mathcal{O}_p$ is maximal if $p \nmid N$, and is isomorphic to the order $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ if $p \mid N$.

(3) For each prime number, the localization $\mathcal{O}_p$ is maximal if $p \mid D$, and is isomorphic to the order $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ if $p \nmid D$.

Proposition 3.1.2. *Let $\mathcal{O}$ be an order in a $\mathbb{Q}$-quaternion algebra B of discriminant D.*

(1) If $\mathcal{O}$ is an Eichler order with norm $N_{\mathcal{O}}$ with $\gcd(D, N_{\mathcal{O}}) = 1$, then its discriminant $D_{\mathcal{O}}$ is equal to $DN_{\mathcal{O}}$.

(2) If $D_{\mathcal{O}} = DN$ is a square-free integer, then $\mathcal{O}$ is an Eichler order of level N.

(3) Let $\mathcal{O}$ and $\mathcal{O}'$ be orders in B and they are conjugate. Then $\mathcal{O}$ is an Eichler order of level N if and only if $\mathcal{O}'$ is an Eichler order of level N.

Theorem 3.1.3. *In an indefinite quaternion algebra over $\mathbb{Q}$, there is only one Eichler order of a given level N, up to conjugation. Moreover, such an Eichler order contains a unit of norm -1.*

We use the notation $\mathcal{O} = \mathcal{O}(D, N)$ to indicate the Eichler order of level N in an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D, where D, N are coprime positive integers. In literature, sometimes, the order $\mathcal{O}(D, N)$ is said to be the Eichler order of level (D, N). We remark that when $N = 1$, the order $\mathcal{O}(D, N)$ is a maximal order.

3.1.1 The Shimura curves $X_0^D(N)$

Note that Theorem 3.1.3 implies that the Shimura curve $X(\mathcal{O})$ attached to the Fuchsian group defined from $\mathcal{O} = \mathcal{O}(D, N)$ is only dependent on the discriminant D and the level N. The curve $X(\mathcal{O})$ has a canonical model as a projective curve defined over $\mathbb{Q}$ (Shimura [19]). Here, we use the notation $X_0^D(N)$ to denote the corresponding Shimura curve.

Theorem 3.1.4. *Let $\mathcal{O}$ be the Eichler order of level N in an indefinite $\mathbb{Q}$-quaternion algebra B with discriminant D. There is a projective algebraic curve $X(\mathcal{O})$ over $\mathbb{Q}$ such that there exists an open immersion of Riemann surfaces*

$$\Gamma(\mathcal{O}) \setminus \mathfrak{h} \hookrightarrow X(\mathcal{O})(\mathbb{C}).$$

When $D \neq 1$, this map is a biregular isomorphism.

Therefore, the curve $X(\mathcal{O}(D, N))$ has a canonical model over $\mathbb{Q}$, we denote it by $X_0^D(N)$. The notion of such Shimura curves generalizes that of the classical modular curves $X_0^1(N) = X_0(N)$.

3.1.2 The Atkin-Lehner involutions on $X_0^D(N)$

Like the theory of the classical modular curve, we can define the Aktin-Lehner group of the curves $X_0^D(N)$.

For a compact Riemann surface X uniformized by a Fuchsian group Γ, the quotient group of the normalizer of Γ in $\mathrm{GL}(2, \mathbb{R})^+$ by Γ acts as automorphisms on X. Here we let $\mathcal{O} = \mathcal{O}(D, N)$ and take $\Gamma = \Gamma(\mathcal{O})$, for convenience. To obtain such automorphisms, we pullback to the order $\mathcal{O}$ in the $\mathbb{Q}$-quaternion algebra B.

For an integer $m \mid DN$ with $\gcd(m, DN/m) = 1$, we then have an ideal $I = x_m\mathcal{O} = \mathcal{O}x_m$ with $I^2 = m\mathcal{O}$, for some $x_m \in \mathcal{O}$ with $n(x_m) = m$. Since $\mathcal{O}$ has a unit of reduced norm -1, the norm 1 group $\mathcal{O}^1$ is equal to the conjugation $x_m\mathcal{O}^1x_m^{-1}$. Hence, x_m gives an automorphism w_m of $X_0(D, N)$ with $w_m^2 = id$. This is called **Atkin-Lehner involution** associated to m.

The **Atkin-Lehner group**

$$W_{D,N} = \{w_m : m \mid DN, \gcd(m, DN/m) = 1\}$$

is an automorphism group of $X_0^D(N)$ associated to the the group $N_{B^+}(\mathcal{O}^1)/\mathbb{Q}^*\mathcal{O}^1$, where $N_{B^+}(\mathcal{O}^1) = \{h \in B^* : h\mathcal{O}^1h^{-1} = \mathcal{O}^1, n(h) > 0\}$ is the normalizer of $\mathcal{O}$ in the subgroup of B^* collecting the positive reduced norm elements. The elements w_m of $W_{D,N}$ can be taken to be any generator of the only 2-sided ideal of reduced norm m of $\mathcal{O}$ when $m \neq 1$. Hence the group $W_{D,N}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of prime factors of DN.

3.2 Optimal Embeddings

Let $\mathcal{O}$ be an order of a quaternion algebra B over the field K. Let F be a quadratic extension over K, and $\mathcal{O}_F$ be its ring of integers. For a given order Λ of $\mathcal{O}_F$, an **embedding of Λ in $\mathcal{O}$** is an embedding from F into B such that $\phi(\Lambda) \subseteq \mathcal{O}$; an **optimal embedding of Λ in $\mathcal{O}$** is an embedding from F into B such that

$$\phi(F) \cap \mathcal{O} = \phi(\Lambda).$$

We let $\mathcal{E}(\mathcal{O}, \Lambda) = \mathcal{E}_K(\mathcal{O}, \Lambda)$ be the set of all optimal embeddings of the given quadratic order Λ into the order $\mathcal{O}$.

In the following discussion, we are going to consider the optimal embeddings of quadratic orders into a given Eichler order $\mathcal{O} = \mathcal{O}(D, N)$.

3.2.1 Optimal embeddings of quadratic orders into $\mathbb{Q}$-quaternion algebras

We first consider the case when B is a quaternion algebra over $\mathbb{Q}$ of discriminant D and $F = \mathbb{Q}(\sqrt{d_F})$ is a quadratic extension field over $\mathbb{Q}$ of discriminant d_F. We recall that there is an embedding from F into B if and only if for any prime p in $\mathbb{Q}$ so that $\mathbb{Q}_p \otimes B \ncong \mathrm{M}(2, \mathbb{Q}_p)$, the prime number p does not completely split in F. In other words, we have an embedding $F \hookrightarrow B$ defined over $\mathbb{Q}$ if and only if the Legendre symbol $\left(\frac{d_F}{p}\right) \neq 1$ if $p \nmid D$. Naturally, we have an action of B^* on the set $\{\phi : F \hookrightarrow B$ is an embedding defined over $\mathbb{Q}\}$ given by $\phi^h = h^{-1}\phi h$, for any element $h \in B^*$.

Proposition 3.2.1. *Let $\phi : F \hookrightarrow B$ be an embedding defined over $\mathbb{Q}$. For any element $h \in B^*$, one has $\phi \in \mathcal{E}(\mathcal{O}, \Lambda)$ if and only if $\phi^h \in \mathcal{E}(h^{-1}\mathcal{O}h, \Lambda)$.*

The following fact give conditions for the existence of optimal embeddings.

Lemma 3.2.2. *Let B_p be the division quaternion algebra over $\mathbb{Q}_p$, and $\mathcal{O}_p$ be the maximal order of B_p. If there exists an embedding from F_p into B_p, we consider an order Λ_p in F_p. Then $\mathcal{E}(\mathcal{O}_p, \Lambda_p)$ is nonempty if and only if Λ_p is a maximal order.*

Proposition 3.2.3. *Let $\mathcal{O}$ be an Eichler order of level N in the $\mathbb{Q}$-quaternion aglebra B. Let F be a quadratic number field such that there is an embedding from F into B, and Λ be an order of conductor m in F. Then*

(1) If $\mathcal{E}(\mathcal{O}, \Lambda)$ is non-empty, then $\gcd(D, m) = 1$.

(2) *If $N = 1$ and B is indefinite, then $\mathcal{E}(\mathcal{O}, \Lambda)$ is non-empty if and only if $\gcd(D, m) = 1$.*

Moreover, while B is an indefinite quaternion algebra, there is exactly one structure of an Eichler order $\mathcal{O} = \mathcal{O}(D, N)$ of level N with $gcd(D, N) = 1$. The action of B^* on field embeddings gives an action of the normalizer of $\mathcal{O}$ in B^* on $\mathcal{E}(\mathcal{O}, \Lambda)$.

Corollary 3.2.4. *Let $\mathcal{O}$ be an Eichler order in an indefinite $\mathbb{Q}$-quaternion algebra B. Let $N_{B^*}(\mathcal{O})$ be the normalizer of $\mathcal{O}$ in B^* and G be a subgroup of $N_{B^*}(\mathcal{O})$. Then the action of G on $\mathcal{E}(\mathcal{O}, \Lambda)$ is an equivalence relation. Here, $\phi, \phi' \in \mathcal{E}(\mathcal{O}, \Lambda)$ are G-equivalent if there is an element $h \in G$ such that $\phi' = h^{-1}\phi h$.*

3.2.2 Optimal embeddings of quadratic orders into $\mathcal{O}(D, N)$

In this subsection, we will count the the number of optimal embeddings of quadratic orders into the Eichler order $\mathcal{O}(D, N)$ of an indefinite $\mathbb{Q}$-quaternion algebra of discriminant D.

Let $\Lambda = \Lambda(d_F, m)$ be an order of conductor m in the field $F = \mathbb{Q}(\sqrt{d_F})$, where d_F is the discriminant of the quadratic field F. Denote by

$$\nu(D, N, d_F, m; \mathcal{O}^*) := \#\mathcal{E}(\mathcal{O}, \Lambda)/\mathcal{O}^*$$

the class number of $\mathcal{O}^*$-equivalent optimal embeddings of Λ in $\mathcal{O}$. In the local case, we let $\nu_p(D, N, d_F, m; \mathcal{O}^*) = \sharp\,\mathcal{E}(\mathcal{O}_p, \Lambda_p)/\mathcal{O}_p^*$ denote the corresponding class number, where $\mathcal{O}_p$, Λ_p are the localization of $\mathcal{O}$ and Λ at prime p, respectively.

Theorem 3.2.5. *Assume that there is an embedding of F into B and $\gcd(m, D) = 1$. Then*

$$\nu(D, N, d_F, m; \mathcal{O}^*) = h(d_F, m) \prod_{p|DN} \nu_p(D, N, d_F, m; \mathcal{O}^*),$$

where $h(d_F, m)$ is the ideal class number of the order $\Lambda = \Lambda(d_F, m)$, and the local class numbers are given by

(1) *If $p \mid D$, then $\nu_p(D, N, d_F, m; \mathcal{O}^*) = 1 - \left(\frac{d_F}{p}\right)$.*

(2) *If $p \mid N$ and $p^2 \nmid N$, then*

$$\nu_p(D, N, d_F, m; \mathcal{O}^*) = \begin{cases} 1 + \left(\frac{d_F}{p}\right), & \text{if } p \nmid m, \\ 2, & \text{if } p \mid m \end{cases}$$

(3) *Assume $N = p^r u_1$, with $p \nmid u_1$, $r \geq 2$. Write $m = p^k u_2$, with $p \nmid u_2$.*

   (a) *If $r < 2k$, then*

$$\nu_p(D, N, d_F, m; \mathcal{O}^*) = \begin{cases} p^{k/2} + p^{k/2-1}, & \text{if } k = 0 \bmod 2, \\ 2p^{(k-1)/2}, & \text{if } k = 1 \bmod 2. \end{cases}$$

(b) If $r = 2k$, then $\nu_p(D, N, d_F, m; \mathcal{O}^*) = p^{k-1}\left(1 + p + \left(\frac{d}{p}\right)\right)$.

(c) If $r = 2k+1$, then

$$\nu_p(D, N, d_F, m; \mathcal{O}^*) = \begin{cases} 2\psi_p(m), & \text{if } \left(\frac{d_F}{p}\right) = 1, \\ p^k, & \text{if } \left(\frac{d_F}{p}\right) = 0, \\ 0, & \text{if } \left(\frac{d_F}{p}\right) = -1. \end{cases}$$

(d) If $r > 2k+1$, then

$$\nu_p(D, N, d_F, m; \mathcal{O}^*) = \begin{cases} 2\psi_p(m), & \text{if } \left(\frac{d_F}{p}\right) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, the function ψ_p is a multiplicative function given by

$$\begin{cases} \psi_p(p^k) = p^{k-1}(p+1) \\ \psi_p(m) = 1, \text{ if } \gcd(m, p) = 1. \end{cases}$$

Corollary 3.2.6. *If the integer N is square-free, and assume that there exists an embedding of F into B, $\gcd(m, D) = 1$, then the class number of optimal embeddings of Λ into $\mathcal{O}$ can be expressed as*

$$\nu(D, N, d_F, m; \mathcal{O}^*) = \begin{cases} 0, & \text{if there exists } p \mid N, p \nmid m, \left(\frac{d_F}{p}\right) = -1, \\ h(d_F, m)2^{s+t}, & \text{otherwise,} \end{cases}$$

where s is the number of prime factors p of D so that p is inert in F and t is the number of prime factors of N that splits in F or divides m.

3.3 Complex Multiplication Points on $X_0^D(N)$

When F is an imaginary quadratic field, and assume that F embeds in the indefinite $\mathbb{Q}$-quaternion algebra B. Then, for any embedding $\phi : F \hookrightarrow B$, the image of F^* in $B^* \setminus \mathbb{Q}^*$ under ϕ has a unique fixed point on the upper half-plane $\mathfrak{h}$.

To be more precise, it is known that two elements $\gamma, \gamma' \in \mathrm{GL}(2, \mathbb{R})$ have the same fixed points if and only if there exist real constants $\lambda \neq 0$ and μ so that $\gamma' = \lambda\gamma + \mu \cdot 1$. Now, if i_∞ stands for the fixed embedding of the infinite $\mathbb{Q}$-quaternion algebra B into $\mathrm{M}(2, \mathbb{R})$ and ϕ is an embedding from F into B, we then have precisely one fixed point in $\mathfrak{h}$ under the action of the set $i_\infty(\phi(F^*))$. In this case, we denote τ_ϕ the fixed point in $\mathfrak{h}$. It is a **complex multiplication point** (briefly, **CM-point**) on the associated Shimura curve X.

Definition 3.3.1. *Let Λ be an order of discriminant $d_\Lambda = m^2 d_F$ in the imaginary quadratic field F. A point $\tau \in X_0^D(N)$ is said to be a* **CM-point by Λ** *or* **CM-point of discriminant d_Λ** *if it is fixed by $i_\infty(\phi)$, i.e. $\tau = \tau_\phi$ on $X_0^D(N)$, for an optimal embedding ϕ in $\mathcal{E}(\mathcal{O}(D, N), \Lambda)$.*

Remark 3.3.1. *A point on $X_0^D(N)$ is elliptic if and only if it is a* CM*-point by the ring of integers $\mathbb{Z}[\sqrt{-1}]$ or $\mathbb{Z}[(1+\sqrt{-3})/2]$.*

3.3.1 The set of CM-points by an order

It is clear that there are many CM-points on the curve $X_0^D(N)$. However, for a given order Λ, the number of CM-points by Λ is related to the number of non-equivalent optimal embeddings of Λ into the order $\mathcal{O}(D,N)$ and it is finite.

Proposition 3.3.2. *Let ϕ, $\phi' \in \mathcal{E}(\mathcal{O}(D,N),\Lambda)$. Then $\tau_\phi = \tau_{\phi'}$ under the action of $\Gamma(\mathcal{O}(D,N))$ if and only if ϕ is $\mathcal{O}(D,N)^1$-equivalent to ϕ' or $-\phi'$, where $-\phi$ is the embedding defined by $(-\phi)(\sqrt{d_F}) = -\phi(\sqrt{d_F})$.*

Note that $-\phi(F) = \phi(F)$ and $-\phi(\Lambda) = \phi(\Lambda)$, hence ϕ and $-\phi$ have the same fixed point in $\mathfrak{h}$. Also, they are either simultaneously optimal or not.

Proof. May assume that ϕ is equivalent to ϕ'. Suppose that $h \in \mathcal{O}(D,N)^1$ is the element such that $h^{-1}\phi(\alpha)h = \phi'(\alpha)$, for all $\alpha \in F^* \setminus \mathbb{Q}^*$. Fixing $\alpha \in F \setminus \mathbb{Q}$, let γ_h, γ, and γ' in $\Gamma(\mathcal{O}(D,N))$ be the corresponding elements to h, $\phi(\alpha)$, $\phi'(\alpha)$. Then $\gamma_h^{-1}\gamma\gamma_h = \gamma'$ and hence $\tau_{\phi'} = \gamma_h^{-1}\tau_\phi$, which is $\Gamma(\mathcal{O}(D,N))$-equivalent to the point τ_ϕ.

Conversely, suppose that there exists $\gamma_h \in \Gamma(\mathcal{O}(D,N))$ so that $\gamma_h^{-1}\tau_\phi = z_{\phi'}$. Write $h \in \mathcal{O}(D,N)^1$ as the associated element to γ_h. Now, we choose $\alpha \in F \setminus \mathbb{Q}$ with $\mathrm{tr}_{\mathbb{Q}}^F(\alpha) = 0$. Then both of $\phi'(\alpha)$ and $h^{-1}\phi(\alpha)h$ fix the point $\tau_{\phi'}$. Considering the elements $\gamma = i_\infty(\phi(\alpha))$ and $\gamma' = i_\infty(\alpha)$, one has the identity

$$\gamma_h^{-1}\gamma\gamma_h = \lambda\gamma' + \mu\cdot 1, \quad \lambda \neq 0, \mu \in \mathbb{R}.$$

By the assumption of $\mathrm{tr}_{\mathbb{Q}}^F(\alpha) = 0$, we can get that the constant μ must be 0, since the trace is $\mathbb{Q}$-linear and preserved by conjugation. The relation between determinants,

$$N_{\mathbb{Q}}^F(\alpha) = \det(\gamma) = \lambda^2\det(\gamma') = \lambda^2 N_{\mathbb{Q}}^F(\alpha) \quad \text{and} \quad N_{\mathbb{Q}}^F(\alpha) \neq 0,$$

implies that $\gamma_h^{-1}\gamma\gamma_h = \pm\gamma'$. That is, the embedding ϕ' is $\mathcal{O}(D,N)^1$-equivalent to ϕ or $-\phi$. □

Lemma 3.3.3. *If ϕ is an embedding from F into B, then ϕ is not $\mathcal{O}^1$-equivalent to $-\phi$, for any order $\mathcal{O}$ in B.*

Proof. Suppose that ϕ is $\mathcal{O}^1$-equivalent to $-\phi$. For a fixed $\alpha \in F - \mathbb{Q}$ with $\mathrm{tr}_{\mathbb{Q}}^F(\alpha) = 0$, there is an element $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\gamma^{-1}i_\infty(\phi(\alpha))\gamma = -i_\infty(\phi(\alpha)).$$

Note that if we choose the element α with trace not 0 then the lemma hold by the properties of trace. Now we consider the associated quadratic forms. Since $\det(i_\infty(\phi(\alpha))) = N_{\mathbb{Q}}^F(\alpha) > 0$, we will get a contradiction. □

From above results, to count the number of CM-points by the order Λ is equivalently to count the number of the non-equivalent class $\mathcal{E}(\mathcal{O}(D,N),\Lambda))$ under the action of $\mathcal{O}(D,N)^*$. We now let $\mathbf{CM}(\mathbf{d}_\Lambda)$ denote the set of CM-points of discriminant d_Λ, up to $\mathcal{O}(D,N)^*$-equivalence. Also, we use the same notation $\mathbf{CM}(\mathbf{d}_\Lambda)$ or $\mathbf{CM}(\Lambda)$ to indicate the set of the in-equivalent optimal embeddings of Λ into $\mathcal{O}(D,N)$. In the stance, the optimal embedding corresponding to a point τ, means that the $\mathcal{O}(D,N)$-equivalent optimal embedding which fixes the point $\tau \in \mathfrak{h}$.

Theorem 3.3.4. *Fix $\Lambda = \Lambda(d_\Lambda)$ an order of index m in the the imaginary quadratic field F which has discriminant d_F.*

$$\#\mathrm{CM}(d_\Lambda) = \#\mathrm{CM}(\Lambda) = \nu(D, N, d_F, m; \mathcal{O}(D,N)^*),$$

the class number of $\mathcal{O}(D,N)^*$-equivalent optimal embeddings of Λ in $\mathcal{O}(D,N)$ mentioned in Section 3.2.2.

3.3.2 Fixed points of Atkin-Lehner involutions

Regarding Atkin-Lehner involutions acting on $X_0^D(N)$ as optimal embeddings, CM-points arise in a natural way as fixed points of Atkin-Lehner involutions on $X_0^D(N)$.

For a given involution w_m, we let h be its corresponding element in the order $\mathcal{O} = \mathcal{O}(D,N)$ with $\mathcal{O}h = h\mathcal{O}$, $n(h) = m$. Assume that $P \in X_0^D(N)(\mathbb{C})$ is a fixed point of w_m on the curve $X_0^D(N)$ and $\tau \in \mathfrak{h}$ representing for P. Then we have $h\tau = u\tau$, for some $u \in \mathcal{O}^1$. (Here, we use the notation $h\tau$ to simplify the action of γ_h on $\tau \in \mathfrak{h}$ with $\gamma_h \in \mathrm{SL}(2,\mathbb{R})$.) Therefore, we may assume that $h\tau = \tau$ and $\mathrm{tr}(h) \geq 0$, by replacing $-h$ by h if necessary. Since h fixes a pair of conjugate complex numbers τ and $\bar{\tau}$, the field $\mathbb{Q}(h)$ containing h and $\mathbb{Q}$ is an imaginary quadratic field.

Observe that the conjugation $\bar{h}$ of h generated the same principal ideal $\mathcal{O}h = \mathcal{O}\bar{h}$, $n(h) = m$, and $\mathrm{tr}(h) \in Q$. One has that $\bar{h} = uh$, for some $u \in \mathcal{O}^1 \cap \mathbb{Q}(h)$. In particular,

$$u = \begin{cases} \zeta_4, & \text{if } m = 2, \\ \zeta_3, & \text{if } m = 3, \\ -1, & \text{else.} \end{cases}$$

Now let Λ be the quadratic order $\mathcal{O}(D,N) \cap \mathbb{Q}(h)$. It is clear that Λ contains the ring $\mathbb{Z}[h]$. Then for a given fixed point $P \in X_0^D(N)$ of w_m, we can associated 2 optimal embeddings of R into $\mathcal{O}(D,N)$, corresponding to h and $\bar{h}$. Consider an embedding $u = \gamma^{-1}h\gamma$, which is $\mathcal{O}(D,N)^*$-equivalent to h. If $n(\gamma) = 1$, then u fixes $\gamma\tau$, which represents the same point P; if $n(\gamma) = -1$, then u fixes the point $\gamma(\bar{\tau})$ associated to the point $\bar{P}$, the complex conjugate point on the Shimura curve $X_0^D(N)$. We can see that P is a real point (i.e. $P = \bar{P}$) if and only if h is $\mathcal{O}(D,N)^*$-equivalent to $\bar{h}$.

Proposition 3.3.5. *(Ogg [15]) Assume that $m > 1$ is a square-free exact divisor of DN. Then the set of the fixed points of an Atkin-Lehner involution w_m on $X_0^D(N)$ is*

$$\begin{cases} \mathrm{CM}(-4) \cup \mathrm{CM}(-8), & \text{if } m = 2, \\ \mathrm{CM}(-m) \cup \mathrm{CM}(-4m), & \text{if } m = 3 \bmod 4, \\ \mathrm{CM}(-4m), & \text{else.} \end{cases}$$

We remark that in the case m is not square-free, the description of the fixed points is complicated. In general, they will be a proper sunset of $\cup_{f^2|4m}\mathrm{CM}(-4m/f^2)$.

3.3.3 Fields of definition of CM-points

Let Λ be an order with discriminant d in the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-s})$. Set $I(\Lambda)$ be the group of the fractional invertible ideal classes of Λ, and H_Λ be the ring class field of Λ. By class field theory, we have the Artin isomorphism from $I(\Lambda)$ to H_Λ by $[\mathfrak{p}] \mapsto \mathrm{Frob}_\mathfrak{p}$, for all primes $\mathfrak{p}$ of F unramified in H_Λ. Denote $\mathbb{Q}(P)$ be the number field generated by the coordinates of the CM-point $P \in \mathrm{CM}(d)$ on $X_0^D(N)$. Then we have fundamental result due to Shimura, which is the so-called Shimura's reciprocity law.

Theorem 3.3.6. *[19](Shimura's reciprocity law) Let Φ be the natural uniformization map $\mathfrak{h} \to \Gamma(\mathcal{O}(D,N)) \setminus \mathfrak{h}$, $\tau \in \mathfrak{h}$ so that $\Phi(\tau) = P$ has* CM *by the order Λ. Then*

(1) $H_\Lambda = F \cdot \mathbb{Q}(P)$.

(2) Let ϕ be the embedding $\Lambda \hookrightarrow \mathcal{O}(D,N)$ corresponding to the point τ. Assume that $\mathfrak{a} \in I(\Lambda)$ and $\sigma_\mathfrak{a}$ is the Artin symbol attached to $\mathfrak{a}$. Then action of the Galois group $\mathrm{Gal}(H_\Lambda/F) \simeq \mathrm{Pic}(\Lambda)$ is given by

$$\sigma_\mathfrak{a}(P) = \Phi(\alpha^{-1}\tau),$$

where α is some element in $\mathcal{O}(D,N)$ with $n(\alpha) > 0$ satisfying the identity

$$\phi(\mathfrak{a})\mathcal{O}(D,N) = \alpha\mathcal{O}(D,N).$$

3.4 Signatures

Recall that the genus of a Shimura curve X is given by

$$g(X) = 1 + \frac{\mathrm{Vol}(X)}{2} - \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right),$$

where the sum runs through all elliptic points with e_i being their respective orders. Considering a normalization $\int\int dxdy/y^2\pi$ for the hyperbolic area, from [17], the formulae for the area (volume) and the genus of $X_0^D(N)$ are

$$\mathrm{Vol}\left(X_0^D(N)\right) = \frac{DN}{6}\prod_{p|D}\left(1 - \frac{1}{p}\right)\prod_{p|N}\left(1 + \frac{1}{p}\right)$$

and

$$g(X_0^D(N)) = 1 + \frac{\mathrm{Vol}(X_0^D(N))}{2} - \frac{1}{2}\sum_{e_i}\left(1 - \frac{1}{e_i}\right).$$

In particular, the total number of elliptic points of order 2 and 3, say v_2 and v_3, are given by

$$v_2 = \begin{cases} \prod_{p|D}\left(1 - \left(\frac{-4}{p}\right)\right)\prod_{p|N}\left(1 + \left(\frac{-4}{p}\right)\right), & \text{if } 4 \nmid N, \\ 0, & \text{if } 4 \mid N, \end{cases}$$

and

$$v_3 = \begin{cases} \prod_{p|D}\left(1-\left(\frac{-3}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right), & \text{if } 9 \nmid N, \\ 0, & \text{if } 9 \mid N. \end{cases}$$

These can be obtained equivalently by counting the number of optimal embeddings from the maximal order in the fields $\mathbb{Q}(\sqrt{-4})$ and $\mathbb{Q}(\sqrt{-3})$ into the quaternion order $\mathcal{O}(D,N)$.

Note that the ramification points of this covering $X_0^D(N) \longrightarrow X_0^D(N)/\langle w_m\rangle$ are the exact fixed points of w_m on the curve $X_0^D(N)$. Therefore, from the Riemann-Hurwitz formula, we can deduce that the genus of the quotient curve $X_0^D(N)/\langle w_m\rangle$ is equal to $(g+1)/2 - B_m/2$, where g is the genus of $X_0^D(N)$, and B_m is the number of the fixed points of w_m on $X_0^D(N)$.

From Proposition 3.3.5, it is easy to determine the number of elliptic points on $X_0^D(N)/G$ for any subgroup G of $W_{D,N}$ such that m is squarefree for any w_m in G.

Lemma 3.4.1. *[23] Let G be a nontrivial subgroup of the group $W_{D,N}$ of Atkin-Lehner involutions on $X_0^D(N)$ such that m is squarefree for any $w_m \in G$. Then the only possible orders of elliptic points on $X_0^D(N)/G$ are 2, 3, 4, and 6.*

1. *If $w_2 \in G$, then the number of elliptic points of order 2 on $X_0^D(N)/G$ is*

$$\frac{2}{|G|}\begin{cases} \sum_{w_m\in G, m\neq 1} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) - \#\mathrm{CM}(-3) & , \text{ if } w_3 \in G, \\ \sum_{w_m\in G, m\neq 1} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) & , \text{ if } w_3 \notin G. \end{cases}$$

   If $w_2 \notin G$, then the number is $(\#\mathrm{CM}(-4)+2A)/|G|$, where A is

$$\begin{cases} \sum_{w_m\in G, m\neq 1} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) - \#\mathrm{CM}(-3) & , \text{ if } w_3 \in G, \\ \sum_{w_m\in G, m\neq 1} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) & , \text{ if } w_3 \notin G. \end{cases}$$

   (If $-m$ is not a discriminant, we simply set $\#\mathrm{CM}(-m)=0$.)

2. *If $w_3 \in G$, then there are no elliptic points of order 3 on $X_0^D(N)/G$. If $w_3 \notin G$, then the number of elliptic points of order 3 is $\#\mathrm{CM}(-3)/|G|$.*

3. *If $w_2 \notin G$, then there are no elliptic points of order 4 on $X_0^D(N)/G$. If $w_2 \in G$, then the number of elliptic points of order 4 is $2\#\mathrm{CM}(-4)/|G|$.*

4. *If $w_3 \notin G$, then there are no elliptic points of order 6 on $X_0^D(N)/G$. If $w_3 \in G$, then the number of elliptic points of order 6 is $2\#\mathrm{CM}(-3)/|G|$.*

Proof. The fact that only 2, 3, 4, and 6 can be the orders of elliptic points on $X_0^D(N)/G$ is well-known.

Let $w_m \in G$. By Proposition 3.3.5, the fixed points of w_m consist of $\mathrm{CM}(-4)$, $\mathrm{CM}(-m)$, or $\mathrm{CM}(-4m)$, depending on m. If $m \neq 1, 3$, then points in $\mathrm{CM}(-4m)$

or $\mathrm{CM}(-m)$ are fixed only by w_m and every other Atkin-Lehner involution other than w_1 permutes them. Thus, there are totally $|G|/2$ points in $\mathrm{CM}(-4m)$ or $\mathrm{CM}(-m)$ that are mapped to the same point in the covering $X_0^D(N) \to X_0^D(N)/G$. For points in $\mathrm{CM}(-4)$, which constitute elliptic points of order 2 on $X_0^D(N)$, they are also fixed by w_2. Thus, if $w_2 \in G$, then there are $2\#\mathrm{CM}(-4)/|G|$ elliptic points of order 4 on $X_0^D(N)/G$. If $w_2 \notin G$, points in $\mathrm{CM}(-4)$ contribute another $\#\mathrm{CM}(-4)/|G|$ elliptic points of order 2 on $X_0^D(N)/G$. For points in $\mathrm{CM}(-3)$, which are elliptic points of order 3 on $X_0^D(N)$, they are also fixed by w_3. If $w_3 \in G$, then they become elliptic points of order 6 on $X_0^D(N)/G$ and there are $2\#\mathrm{CM}(-3)/|G|$ such points. If $w_3 \notin G$, then they remain elliptic points of order 3. There are $\#\mathrm{CM}(-3)/|G|$ such points. Summarizing, we get the lemma. □

3.5 Čerednik-Drinfeld Theory

In this section, we will review the Čerednik-Drinfeld theory of the p-adic uniformization for Shimura curves, which gives a description of the bad reduction of Shimura curves $X_0^D(N)$. In the following, for fixed integers D and N, we will use X to denote the Shimura curve $X_0^D(N)$.

Due to the moduli interpretation of Shimura curves, the curve X admit a canonical model over $\mathbb{Q}$. Following from the work of Morita, Čerednik, and Drinfeld, there exists a proper integral model $M = M(D,N)/\mathbb{Z}$ of X which extends the moduli interpretation to arbitrary schemes over $\mathbb{Z}$ and it is smooth over $\mathbb{Z}[\frac{1}{DN}]$. It is known that the curve X has good reduction only at the prime numbers p with $p \nmid DN$. For a prime divisor p of D, the curve $X/\mathbb{Q}_p$ defined over $\mathbb{Q}_p$ is a Mumford curve. By Mumford's theory, the curve X has a p-adic uniformization expressing it as a quotient of the p-adic upper half plane $\mathfrak{h}_p$ by the action of a discrete subgroup Γ of $\mathrm{PGL}(2,\mathbb{Q}_p)$. The theory of Čerednik-Drinfeld provides an explicit description of this p-adic uniformization. It describes $X \times \mathbb{Q}_p$ as a quadratic twist of $\Gamma \setminus \mathfrak{h}_p$ over $\mathbb{Q}_p$.

In the following, we will also describe the connection between Brandt matrices and the bad reductions of X from the theory of Čerednik-Drinfeld. Let us fix the notations K_p, K_p^{unr}, and $\mathbb{Z}_p^{\mathrm{unr}}$, as the unramified quadratic extension of $\mathbb{Q}_p$, the maximal unramified extension of $\mathbb{Q}_p$, and the ring of integers of K_p^{unr}, respectively.

3.5.1 The Čerednik-Drinfeld theory

Let p be a prime with $p \mid D$, and $\mathcal{O} = \mathcal{O}(D/p, N)$ be an Eichler order of level N in a definite quaternion algebra B' defined over $\mathbb{Q}$ of discriminant D/p. Let $\mathbb{Z}^{(p)}$ be the set $\mathbb{Z}[\frac{1}{p}]$ and $\mathcal{O}^{(p)} = \mathcal{O} \otimes \mathbb{Z}^{(p)}$. Define $\widetilde{\Gamma}_0 = \mathcal{O}^{(p)^*}$ and

$$\widetilde{\Gamma}_+ = \left\{ x \in \widetilde{\Gamma}_0 : \mathrm{Ord}_p(n(x)) \equiv 0 \bmod 2 \right\}.$$

Also, we let $\Gamma_0 = \widetilde{\Gamma}_0/\mathbb{Z}^{(p)^*}$ and define

$$\Gamma_+ = \widetilde{\Gamma}_+/\mathbb{Z}^{(p)^*}.$$

Identifying the quaternion algebra $B' \otimes \mathbb{Q}_p$ with the quaternion algebra $M(2, \mathbb{Q}_p)$, the groups $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_+$ can be considered as discrete compact subgroups of $\mathrm{GL}(2, \mathbb{Q}_p)$ containing the element $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, and Γ_0 and Γ_+ can be viewed as discrete compact subgroups of $\mathrm{PGL}(2, \mathbb{Q}_p)$. Then the quotients $\Gamma_0 \setminus \mathfrak{h}_p$ and $\Gamma_+ \setminus \mathfrak{h}_p$ exist. Moreover, let $\Gamma = \Gamma_0$ or Γ_+, there exists a unique scheme $\mathcal{P}_\Gamma$ proper over $\mathbb{Z}_p$ such that the formal completion of $\mathcal{P}_\Gamma$ along its closed fibre is canonically the quotient $\Gamma \setminus \mathfrak{h}_p$ over $\mathbb{Z}_p$. Note that the scheme $\mathcal{P}_\Gamma$ is projective over $\mathbb{Z}_p$, and its generic fibre X_Γ is a smooth curve defined over $\mathbb{Q}_p$.

Theorem 3.5.1. *(Čerednik-Drinfeld) There is an isomorphism over $\mathbb{Z}_p$ such that*

$$X \times \mathbb{Q}_{p^2} \simeq X_{\Gamma_+}^{\chi},$$

where χ is the character $\chi :$ Gal$(K_p/\mathbb{Q}_p) \longrightarrow$ Aut$(X_{\Gamma_+} \otimes K_p)$ defined by $Frob \mapsto w_p$, and $X_{\Gamma_+}^{\chi}$ is the quadratic twist of X_{Γ_+} by χ.

3.5.2 Dual graph and bad reduction

Let Δ be the Burhat-Tits tree of $\mathrm{SL}(2, \mathbb{Q}_p)$, ie., $\mathrm{PGL}(2, \mathbb{Q}_p)/\mathrm{PGL}(2, \mathbb{Z}_p)$, on which $\mathrm{PGL}(2, \mathbb{Q}_p)$ acts in the usual manner. According to the Čerednik and Drinfeld's result, the special fiber of $M \otimes \mathbb{Z}_p$ is determined by a quadratic twist by the finite graph $G = \Gamma_+ \setminus \Delta$ with lengths. Geometrically, a vertex v of the graph G is corresponding to the irreducible rational component C_v of M_p, where M_p is the closed fiber of M at the prime p. An edge e of length $\ell(e)$ connecting vertices v and v' is corresponding to an intersection point x of the component C_v and $C_{v'}$ locally at which

$$M_x \times \widehat{\mathbb{Z}_p}^{\mathrm{unr}} \simeq \mathrm{Spec}\left(\widehat{\mathbb{Z}_p}^{\mathrm{unr}}[X, Y]/\left(XY - p^{\ell(e)}\right)\right).$$

Now, let us see some properties of the graph G.

We first consider the finite graph $G_0 = \Gamma_0 \setminus \Delta$ with lengths. Let $I_1, I_2, \ldots, I_h$ be a completely representatives of the left ideals of $\mathcal{O}$, and let $\mathcal{O}_i$ be the right order of I_i, $i = 1 \ldots h$. The vertices of the graph G_0 form the set $\mathrm{Ver}(G_0) = V$, where V collects the right orders $\mathcal{O}_i$. The vertices v_1 and v_2 are linked by an edge if and only if the intersection of the corresponding orders $\mathcal{O}_1$ and $\mathcal{O}_2$ is an Eichler order $\mathcal{O}(D/p, Np)$, up to conjugation. Observe that the group Γ_+ is a subgroup of index 2 of the group Γ_0, and the quotient group Γ_0/Γ_+ is generated by $\gamma_p \Gamma_+$, where γ_p is corresponding to an element of $\mathcal{O}$ with reduced norm p. We can construct the graph G with lengths from the graph G_0.

The vertices of the graph G are the set $\mathrm{Ver}(G) = V \cup V'$, where $V' = \gamma_p V$ with $v' = \gamma_p v$. There are no edges in G connecting 2 vertices from the same set V or V'. Let $\ell(v)$ be the weight of the a vertex v, and $\ell(e)$ be the length of an edge e. One has the following facts.

Proposition 3.5.2. *For a given vertex $v \in V$, let $v' = \gamma_p v \in V'$.*

1. *The weight $\ell(v_i)$ of the vertex v_i is equal to the half of the number of the units in the corresponding order $\mathcal{O}_i$. That is ,*

$$\ell(v_i) = \frac{\#\mathcal{O}_i^*}{2}.$$

Furthermore, we have the equality $\ell(v) = \ell(v')$.

2. *The number of the edges e_α with lengths $\ell(e_\alpha)$ joining v_i and v'_j coincides with that of v'_i and v_j.*

3. *For all edges connecting to a vertex v, we have $\ell(e) \mid \ell(v)$ and*

$$\sum_{e \in \mathit{Star}(v)} \frac{\ell(v)}{\ell(e)} = p + 1,$$

if we let $\mathit{Star}(v)$ be the set of all the edges connecting to the vertex v.

On the other hand, we can get the information of the graph G from the theory of Brandt matrices. Let $A = (a_{i,j}) \in M(h, \mathbb{Z})$ be the Brandt matrix attached to the order $\mathcal{O}$. The entry $a_{k,\ell}$ is the number of the $\mathcal{O}_k$-left ideals of reduced norm p which are equivalent to the the ideal $I_k^{-1} I_\ell$, and the equality $a_{k,\ell}/\#\mathcal{O}_k^* = a_{\ell,k}/\#\mathcal{O}_\ell^*$ holds for every ℓ, k.

Proposition 3.5.3. 1. *The number of the edges e with given lengths $\ell(e)$ joining v_i and v'_j is the number $a_{i,j}$, and*

$$a_{i,j} = \sum_{v_i \xrightarrow{e} v'_j} \frac{\ell(v_i)}{\ell(e)}.$$

In particular, it always holds that

$$a_{i,j}/\ell(v_i) = a_{j,i}/\ell(v_j).$$

2. *For each row, we have*

$$\sum_{j=1}^{h} a_{i,j} = p + 1.$$

These give us the information of the finite graph G with lengths and thus we can determine the special fibre M_p when $p \mid D$.

When $p \mid N$, we have the simpler result to determine the fibre M_p. In summary, when $p \mid N$, we let $I_1, I_2, \ldots, I_h$ be a completely representatives of the left ideals of $\mathcal{O}(Dp, N/p)$, and let $\mathcal{O}_i$ be the right order of I_i, $i = 1 \ldots h$. Then the irreducible components of M_p meet at h points with thinkness $\frac{\#\mathcal{O}_i^*}{2}$, $i = 1 \ldots h$.

3.6 The Jacquet-Langlands correspondence

From the Jacquet-Langlands correspondence, we can see a connection between the space of cusp forms on classical modular curves and the space of automorphic forms on Shimura curves $X_0^D(N)$.

The definition of Hecke operators on the space of automoprhic forms on Shimura curves $X_0^D(N)$ are the same as that of the classical modular forms. We assume that $\mathcal{O} = \mathcal{O}(D, N)$ is an Eichler order of level N in an indefinite quaternion algebra of discriminant D. Now fix an imbedding $\iota : B \longrightarrow M(2, \mathbb{R})$.

Definition 3.6.1. *Let p be a prime with $p \nmid DN$, and $\alpha \in \mathcal{O}$ be such $N(\alpha) = p$. Then for an automorphic form $f(\tau)$ of even weight k on $\Gamma = \Gamma(\mathcal{O})$, the action of* **Hecke operator** *T_p on $f(\tau)$ is defined by*

$$T_p : f(\tau) \mapsto p^{k/2-1} \sum_{\gamma \in \Gamma \backslash \Gamma \iota(\alpha) \Gamma} \frac{(\det \gamma)^{k/2}}{(c\tau + d)^k} f(\gamma\tau),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Hecke operators T_n for general n with $\gcd(n, DN) = 1$ are more complicated. As in the case of classical modular curves, there exists a basis of $S_k(\mathcal{O})$ consisting of simultaneous eigenforms for all T_n, with $(n, DN) = 1$. The Jacquet-Langlands correspondence gives an isomorphism of Hecke modules from $S_k(\mathcal{O}(D, N))$ to the space of cusp forms of weight k and level N which are new at all primes dividing D.

Now let $S_k(D, N)$ stand for the space of automorphic forms of weight k on $\Gamma(\mathcal{O}(D, N))$ and simply $S_k(M) = S_k(1, M)$, the space of cusp forms of weight k on $\Gamma_0(M)$. Denote by $w_m = w_m(D, N)$ the Atkin-Lehner involution in $\mathcal{O}(D, N)$. Then the Jacquet-Langlands correspondence in our case can be stated as follows.

Proposition 3.6.1 ([12, 18, 33])**.** *We have*

$$S_k(D, N) \simeq S_k^{D-new}(DN) := \bigoplus_{d|N} \bigoplus_{m|\frac{N}{d}} S_k^{new}(dD)^{[m]}$$

as Hecke modules, where

$$S_k^{new}(dD)^{[m]} = \left\{ f(m\tau) : f(\tau) \in S_k^{new}(dD) \right\},$$

and $S_k^{new}(M)$ is the subspace of newforms of $S_k(M)$. Moreover, for a prime $p \mid D$, if the action of the Atkin-Lehner involution $w_p(1, DN)$ on a normalized Hecke eigenform $f \in S_k^{D-new}(DN)$ is $w_p(1, DN)f = \varepsilon_p f$, then the action of w_p on the corresponding automorphic form $\widetilde{f} \in S_k(D, N)$ is

$$w_p \widetilde{f} = -\varepsilon_p \widetilde{f}.$$

According to the Jacquet-Langlands correspondence, we can see that each Hecke eigenform $\widetilde{f}$ for T_p is with the same eigenvalues as the cusp form f.

Chapter 4

Automorphic Forms in Terms of Solutions of Schwarzian Differential Equations

Let B be an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. For an Eichler order $\mathcal{O}$ of level N, $(D,N)=1$, in B, we let $X_0^D(N)$ denote the Shimura curve associated to $\mathcal{O}$. For each divisor m of DN with $(m, DN/m)=1$, we let w_m denote the Atkin-Lehner involution on $X_0^D(N)$ and $W_{D,N}$ be the group of all Atkin-Lehner involutions. We also let the subgroup of $W_{D,N}$ consisting of w_m, $m|D$, be denoted by W_D.

Many properties and theories about classical modular curves can be extended to the case of Shimura curves. In the classical case, many results are relying on the Fourier expansions of modular forms. However, because of the absence of cusps in the case of general Shimura curves ($D \neq 1$), it is not easy to determine Taylor coefficients of automorphic forms and functions. Therefore, there have been very few results on arithmetic of Shimura curves, and few methods to construct automorphic forms and functions on Shimura curves. One of the few methods uses differential equations satisfied by automorphic forms and automorphic functions. (See [2, 6, 33].) The idea is that even though it is difficult to explicitly construct automorphic functions that can be put into practical use, the Schwarzian differential equations associated to automorphic functions in the case of Shimura curves of genus zero can often be determined using analytic information of the automorphic functions and coverings between Shimura curves. Then one can use the solutions of the Schwarzian differential equations in place of automorphic forms to study properties of automorphic forms.

From the result of Yang [33], every automorphic form on a Shimur curve X which is of genus zero can be expressed by the solutions of Schwarzian differential equation associated to X. In view of the significance of Schwarzian differential equations, it is important to determine the Schwarzian differential equation for each of the Shimura curves $X_0^D(N)/G$, $G < W_{D,N}$, of genus zero. In [6], Elkies worked out the Schwarzian equation on $X_0^{10}(1)/W_{10}$, $X_0^{14}(1)/W_{14}$, and $X_0^{15}(1)/W_{15}$. Bayer and

Travesa [2] computed all the Schwarzian differential equations for the Shimura curves $X_0^6(1)/G$ with $G < W_6$. In [33], Yang also gave Schwarzian differential equation on $X_0^6(1)/W_6$ and $X_0^{10}(1)/W_{10}$ from the properties of the automorphic derivatives.

In this chapter, we will consider the cases $X_0^D(N)/W_D$ when there exists a square-free integer $M > 1$ such that $X_0^D(M)/W_D$ has genus zero. The reason for this restriction is that we need additional information from coverings between Shimura curves of genus zero in order to completely determine the differential equations. (Note that in [33], a covering between Shimura curves of different levels is also needed in order to compute Hecke operators.) In the process, we also need work out equations for some Shimura curves of genus one and hyperelliptic Shimura curves, which are useful in determining the covering maps between Shimura curves. As a byproduct of our computation of coverings $X_0^D(N)/W_D \to X_0^D(1)/W_D$, we can also determine the values of Hauptmoduls at several CM-points.

In this chapter, we will describe a way to construct automorphic forms on Shimura curves in Section 4.1 . The rest of this chapter is organized as follows. In Section 4.2, we determine all Shimura curves $X_0^D(N)/W_D$ of genus 0, $N > 1$. In Section 4.3, we will find explicit coverings of $X_0^D(N)/W_D \to X_0^D(1)/W_D$. The equations for Shimura curves and the methods to obtain them given in [8, 9, 14] are important here. The explicit coverings will be used later. In Section 4.4, we will list the Schwarzian differential equations for the selected Shimura curves. These results is mainly following the preprint [23].

4.1 Automorphic Forms on Shimura Curves and Schwarzian Differential Equations

Let $t(\tau)$ be a non-constant automorphic function on a Shimura curve X. It is straightforward to verify that $t'(\tau)$ is a meromorphic automorphic form of weight 2 on X and that the **Schwarzian derivative**

$$\{t,\tau\} := \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2$$

is a meromorphic automorphic form of weight 4 on X. Thus, the ratio of $\{t,\tau\}$ and $t'(\tau)^2$ is an automorphic function on X. In particular, if X has genus zero and $t(\tau)$ is a Hauptmodul, i.e., the function t generates the field of automorphic functions on X, then

$$Q(t) := -\frac{\{t,\tau\}}{2t'(\tau)^2}$$

is a rational function of t. In literature [2], given a thrice-differentiable function f of z, the function

$$D(f,z) := -\frac{\{f,z\}}{2f'(z)^2}$$

is called the **automorphic derivative** associated to f.

Now the relation $2Q(t)t'(\tau)^2 + \{t, \tau\} = 0$ can also be written as

$$\frac{d^2}{dt(\tau)^2} t'(\tau)^{1/2} + Q(t) t'(\tau)^{1/2} = 0.$$

In other words, if we consider $t'(\tau)^{1/2}$ as a function of t, then $t'(\tau)^{1/2}$ is a solution of the differential equation

$$\frac{d^2}{dt^2} f + Q(t) f = 0.$$

Definition 4.1.1. *The differential equation $d^2 f/dt^2 + Q(t) f = 0$ is called the* **Schwarzian differential equation** *associated to $t(\tau)$.*

This differential equation is a Fuchsian differential equation. For each singularity, there is a basis of local solutions of the form

$$x^e (1 + a_1 x + a_2 x + \cdots),$$

where e is the local exponent at the singular point. We also remark that this differential equation can be regarded as a normal form for all atomorphic differential equation associated to the group Γ with $X = \Gamma \setminus \mathfrak{h}$, because it depends only on the chosen of $t(\tau)$.

4.1.1 Automorphic forms on Shimura curves of genus zero

The significance of Schwarzian differential equations can be seen from the following result.

Proposition 4.1.1 ([33, Theorem 4])**.** *Assume that a Shimura curve X has genus zero with elliptic points $\tau_1, \ldots, \tau_r$ of orders $e_1, \ldots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \ldots, r$. For a positive even integer $k \geq 4$, then a basis for $S_k(X)$ is*

$$t'(\tau)^{k/2} t(\tau)^j \prod_{j=1, a_j \neq \infty}^{r} (t(\tau) - a_j)^{-\lfloor k(1-1/e_j)/2 \rfloor}, \quad j = 0, \ldots, d_k - 1,$$

where $d_k = \dim S_k(X)$ and it is equal to $1 - k + \sum_j^r \lfloor \frac{k}{2} \left(1 - \frac{1}{e_j}\right) \rfloor$.

Moreover, the automorphic derivative $Q(t)$ satisfies some conditions.

Proposition 4.1.2. *Assume that X has genus zero with elliptic points $\tau_1, \ldots, \tau_r$ of order $e_1, \ldots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \ldots, r$. Then the automorphic derivative $Q(t) = D(t, \tau)$ is equal to*

$$Q(t) = \frac{1}{4} \sum_{j=1, a_j \neq \infty}^{r} \frac{1 - 1/e_j^2}{(t - a_j)^2} + \sum_{j=1, a_j \neq \infty}^{r} \frac{B_j}{t - a_j}$$

for some constants B_j. Moreover, if $a_j \neq \infty$ for all j, then the constants B_j satisfy

$$\sum_{j=1}^{r} B_j = \sum_{j=1}^{r}\left(a_jB_j + \frac{1}{4}(1-1/e_j^2)\right) = \sum_{j=1}^{r}\left(a_j^2B_j + \frac{1}{2}a_j(1-1/e_j^2)\right) = 0.$$

Also, if $a_r = \infty$, then B_j satisfy

$$\sum_{j=1}^{r-1} B_j = 0, \qquad \sum_{j=1}^{r-1}\left(a_jB_j + \frac{1}{4}(1-1/e_j^2)\right) = \frac{1}{4}(1-1/e_r^2).$$

In other words, if we can determine the Schwarzian differential equation associated to a Hauptmodul on a Shimura curve, then we can express automorphic forms of any even weight k on this Shimura curve in terms of solutions of the differential equation.

Corollary 4.1.3. *Let X be a Shimura curve of genus zero with elliptic points $\tau_1, \ldots, \tau_r$ of order $e_1, \ldots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$. Suppose that $\{g_1, g_2\}$ is a basis for the solution space of the Schwarzian differential equation associated to t,*

$$f'' + Q(t)f = 0.$$

Then a basis for $S_k(X)$ is given by

$$(g_1 + Cg_2)^k t(\tau)^j \prod_{i=1, a_i\neq\infty}^{r} (t(\tau) - a_i)^{-\lfloor \frac{k(1-1/e_i)}{2} \rfloor}, \quad j = 0, \ldots, d_k - 1,$$

for some constant $C \in \mathbb{C}$.

This provides a concrete space that we can use to study properties of automorphic forms. For example, in [33], Yang devised a method to determine Hecke eigenforms in the spaces of automorphic forms, expressed in terms of solutions of Schwarzian differential equations.

Now the upshot is that it is often possible to determine a Schwarzian differential equation without constructing a Hauptmodul first. This is especially true when a Shimura curve of genus zero has three elliptic points. This is due to the well-known fact that a second-order Fuchsian differential equation with precisely three singularities is uniquely determined its local exponents at the three points.

4.1.2 Hypergeometric functions as automorphic forms on Shimura curves

In the case that the Shimura curve of genus 0 has exactly 3 elliptic points, since the number of singularities of the differential equation is 3, the differential equation is essentially a hypergeometric differential equation. Then one can express the automorphic forms by using ${}_2F_1$-hypergeometric functions.

To be more precise, when a Shimura curve has signature $(0; e_1, e_2, e_3)$, we let τ_1, τ_2, τ_3 be the three elliptic points corresponding to e_1, e_2, e_3. Since X has genus 0, there exists a unique Hauptmodul t that takes values $0, 1, \infty$ at τ_1, τ_2, τ_3, respectively. According to Proposition 4.1.3, the functions $t'(\tau)^{1/2}$ and $\tau t'(\tau)^{1/2}$, as functions of t, satisfy the differential equation $f'' + Q(t)f = 0$, where

$$Q(t) = \frac{1}{4}\left(\frac{1-1/e_1^2}{t^2} + \frac{1-1/e_2^2}{(t-1)^2}\right) + \frac{B_1}{t} + \frac{B_2}{t-1}$$

with

$$B_2 = \frac{1}{4}\left(-1 + \frac{1}{e_1^2} + \frac{1}{e_2^2} - \frac{1}{e_3^2}\right), \qquad B_1 = -B_2.$$

The local exponents at $0, 1, \infty$ are $\{1/2-1/(2e_1), 1/2+1/(2e_1)\}$, $\{1/2-1/(2e_2), 1/2-1/(2e_2)\}$, and $\{-1/2-1/(2e_3), -1/2+1/(2e_3)\}$, respectively. Therefore, the function $t^{-1/2+1/(2e_1)}(1-t)^{-1/2+1/(2e_2)}t'(\tau)^{1/2}$, as a function of z, satisfies the hypergeometric differential equation

$$\theta(\theta+c-1)F - t(\theta+a)(\theta+b)F = 0, \qquad \theta = t\frac{d}{dt}$$

with

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

Combining this with Proposition 4.1.3, we see that every automorphic form on X can be expressed in terms of hypergeometric functions.

Proposition 4.1.1 ([33, Theorem 9]). *Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $t(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1, e_2, and e_3, respectively. Let $k \geq 4$ be an even integer. Then a basis for the space of automorphic forms of weight k on X is given by*

$$t^{\{k(1-1/e_1)/2\}}(1-t)^{\{k(1-1/e_2)/2\}}t^j\left({}_2F_1(a,b;c;t) + Ct^{1/e_1}{}_2F_1(a',b',c';t)\right)^k,$$

$j = 0, \ldots, \lfloor k(1-1/e_1)/2\rfloor + \lfloor k(1-1/e_2)/2\rfloor + \lfloor k(1-1/e_3)/2\rfloor - k$, for some constant C, where for a rational number x, we let $\{x\}$ denote the fractional part of x,

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

and

$$a' = a + \frac{1}{e_1}, \qquad b' = b + \frac{1}{e_1}, \qquad c' = c + \frac{2}{e_1}.$$

In [24], Yang and the author of the present paper obtained several new algebraic transformation of ${}_2F_1$-hypergeometric functions by interpreting identities among hypergeometric functions as identities among automorphic forms on different Shimura curves. In chapter 6, we will introduce how we obtain algebraic transformations of ${}_2F_1$-Hypergeometric functions.

4.1.3 Transformation laws of automorphic derivatives

For general Shimura curves, the following properties of Schwarzian differential equations and automorphic derivatives are very useful in determining the differential equations.

Proposition 4.1.4. *[33] Automorphic derivatives have the following properties.*

1. $D((az+b)/(cz+d), z) = 0$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$.
2. $D(g \circ f, z) = D(g, f(z)) + D(f, z)/(dg/df)^2$.

Proposition 4.1.5. *[33] Let $t(\tau)$ be a Hauptmodul for a Shimura curve X of genus 0. Let $R(x) \in \mathbb{C}(x)$ be the rational function such that the automorphic derivative $Q(t) = D(t,\tau)$ is equal to $R(t)$. Assume that γ is an element of $\mathrm{GL}(2,\mathbb{R})$ normalizing the order $\mathcal{O}$ associated to X and let σ be the automorphism of X induced by γ. If $\sigma : t \mapsto (at+b)/(ct+d)$, then $R(x)$ satisfies*

$$\frac{(ad-bc)^2}{(cx+d)^4} R\left(\frac{ax+b}{cx+d}\right) = R(x).$$

Proof. We shall compute $D(t(\gamma\tau), \tau)$ in two ways. By Proposition 4.1.4, we have

$$D(t(\gamma\tau),\tau) = D\left(\frac{at(\tau)+b}{ct(\tau)+d}, t(\tau)\right) + \frac{D(t(\tau),\tau)}{(dt(\gamma\tau)/dt(\tau))^2} = 0 + \frac{(ct+d)^4 R(t)}{(ad-bc)^2}.$$

On the other hand, by the same proposition, we also have

$$D(t(\gamma\tau),\tau) = D(t(\gamma\tau),\gamma\tau) + \frac{D(\gamma\tau,\tau)}{(dt(\gamma\tau)/d\gamma\tau)^2} = R(t(\gamma\tau)) = R\left(\frac{at+b}{ct+d}\right).$$

Comparing the two expressions, we get the formula. □

4.2 Shimura Curves of Genus Zero

From now on, let us consider the Shimura curves $X_0^D(N)$ and fix the notation $W_D = W_{D,1}$. In this section, we will determine all pairs of integers (D,N), $D, N > 1$, such that $X_0^D(N)/W_D$ has genus 0, where N is a squarefree integer. We will need explicit coverings $X_0^D(N)/W_D \to X_0^D(1)/W_D$ in order to determine Schwarzian differential equations.

A formula for the genus of $X_0^D(N)/G$, $G < W_{D,N}$, will involve the numbers of CM points of certain discriminants. For the goal of this section, we only need to know the number of CM-points associated to $K = \mathbb{Q}(\sqrt{-m})$ with $m|D$ of discriminant -3, d_K, or $4d_K$ in the case $d_K \equiv 1 \bmod 4$.

Lemma 4.2.1 ([15], or Section 3.3 and Section 3.2.2)**.** *For $m|D$ or $m = 3$, let d_K denote the discriminant of the field $K = \mathbb{Q}(\sqrt{-m})$. We have*

$$\#\mathrm{CM}(d_K) = h(d_K)\begin{cases} 0, & \text{if } p^2|N \text{ for some } p|d_K, \\ \prod_{p|D}\left(1-\left(\frac{d_K}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{d_K}{p}\right)\right), & \text{if } p^2 \nmid N \text{for any } p|d_K. \end{cases}$$

Also, for $m|D$ with $m \equiv 3 \bmod 4$, we have

$$\#\mathrm{CM}(4d_K) = \delta h(4d_K) \begin{cases} 0, & \text{if } 2|D, \\ \prod_{p|D}\left(1-\left(\frac{4d_K}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{4d_K}{p}\right)\right), & \text{if } 2\nmid D, \end{cases}$$

where when $m \equiv 7 \bmod 8$,

$$\delta = \begin{cases} 6, & \text{if } 8|N, \\ 4, & \text{if } 4\|N, \\ 2, & \text{if } 2\|N, \\ 1, & \text{if } 2\nmid N, \end{cases}$$

and when $m \equiv 3 \bmod 8$,

$$\delta = \begin{cases} 0, & \text{if } 8|N, \\ 2, & \text{if } 2|N \text{ or } 4|N, \\ 1, & \text{if } 2\nmid N. \end{cases}$$

Here $h(d)$ is the class number of the imaginary quadratic order of discriminant d.

Lemma 4.2.2. *The complete list of integers (D, N) with $D, N > 1$ such that the Shimura curve $X_0^D(N)/W_D$ has genus zero, is*

$$(6,5),\ (6,7),\ (6,13),\ (10,3),\ (10,7),\ (14,3),\ (14,5),$$
$$(15,2),\ (15,4),\ (21,2),\ (26,3),\ (35,2),\ (39,2).$$

Proof. Let Γ be a congruence Fuchsian subgroup of $\mathrm{SL}(2,\mathbb{R})$. (See [13] for the definition of a congruence Fuchsian subgroup. The groups considered here are all congruence Fuchsian subgroups.) A famous result of Selberg [16] stated that if Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then the first eigenvalue λ_1 of the Laplace operator on the space of square-integrable function on $\Gamma\backslash\mathfrak{h}$ is not less than $3/16$. By combining this result with the Jacquet-Langlands correspondence, Vignéras [27] showed that the same inequality also holds for congruence Fuchsian subgroups coming from indefinite quaternion algebras over $\mathbb{Q}$ of discriminant not equal to 1.

On the other hand, Zograf [34] showed that if the area $\mathrm{Vol}(\Gamma\backslash\mathfrak{h})$ is at least $16(g(\Gamma)+1)$, then $\lambda_1 < 4(g(\Gamma)+1)/\mathrm{Vol}(\Gamma\backslash\mathfrak{h})$. Here $g(\Gamma)$ denotes the genus of Γ and the area is normalized such that $A(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}) = 1/6$. Combining Selberg's inequality and Zograf's result, one sees that if a congruence Fuchsian subgroup has genus 0, then the area must be less than $64/3$.

Now recall that the area of $X_0^D(N)$ is given by

$$\frac{DN}{6}\prod_{p|D}\left(1-\frac{1}{p}\right)\prod_{p|N}\left(1+\frac{1}{p}\right).$$

This immediately shows that if the number of prime factors of D is at least 6, then the genus of $X_0^D(N)/W_D$ cannot be 0 for any $N \geq 2$. Also, if $D = pq$ is a product of two primes such that $(p-1)(q-1) > 512/3$, then $X_0^D(N)/W_D$ must have a positive genus for any $N \geq 2$. A similar bounds exists for the case D has 4 prime factors. This leaves finitely many cases to check.

Note that the genus of a Shimura X is given by

$$g(X) = 1 + \frac{\text{Vol}(X)}{2} - \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right),$$

where the sum runs through all elliptic points with e_i being their respective orders. For $X = X_0^D(N)/W_D$, by Lemma 3.4.1, we have

$$\begin{aligned} g(X) = 1 + \frac{\text{Vol}(X)}{2} - \frac{1}{4}\sum_{m|D, m\neq 1,3} \frac{1}{2^{r-1}}(\#\text{CM}(-4m) + \#\text{CM}(-m)) \\ - \begin{cases} \frac{1}{4\cdot 2^r}\#\text{CM}(-4), & \text{if } 2\nmid D, \\ \frac{3}{8\cdot 2^{r-1}}\#\text{CM}(-4), & \text{if } 2|D \end{cases} \\ - \begin{cases} \frac{1}{3\cdot 2^r}\#\text{CM}(-3), & \text{if } 3\nmid D, \\ \left(\frac{1}{4\cdot 2^{r-1}}\#\text{CM}(-12) + \frac{5}{12\cdot 2^{r-1}}\#\text{CM}(-3)\right), & \text{if } 3|D, \end{cases} \end{aligned}$$

where r is the number of prime divisors of D. (Of course, if d is not a discriminant, then we simply let $\text{CM}(d)$ be the empty set.)

Using the Selberg-Zograf bound, the genus formula in the paragraph above and Lemma 4.2.1, we check case by case that the pairs of integers given in the lemma are the only cases where $X_0^D(N)/W_D$, $N > 1$, has genus zero. □

We now tabulate all Shimura curves $X_0^D(M)/W_D$ of genus 0 for integers D that appear in the lemma. We will also give a description of their elliptic points. These Shimura curves are the curves that we wish to determine their Schwarzian differential equations. Here v_j denotes the number of elliptic points of order j on $X_0^D(M)/W_D$. Here we also let $\text{CM}(-m)$ denote the set of points on $X_0^D(N)/W_D$ that are the image of CM points of discriminants $-m$ under the covering $X_0^D(N) \to X_0^D(N)/W_D$. The number n in $\text{CM}(-m)^{\times n}$ means the number of elements in $\text{CM}(-m)$ is n. If $n = 1$, we omit this annotation.

D,N	v_2,v_3,v_4,v_6	elliptic points
$6,1$	$1,0,1,1$	$\mathrm{CM}(-3),\ \mathrm{CM}(-4),\ \mathrm{CM}(-24)$
$6,5$	$2,0,2,0$	$\mathrm{CM}(-4)^{\times 2},\ \mathrm{CM}(-24)^{\times 2}$
$6,7$	$2,0,0,2$	$\mathrm{CM}(-3)^{\times 2}, \mathrm{CM}(-24)^{\times 2}$
$6,13$	$0,0,2,2$	$\mathrm{CM}(-3)^{\times 2}, \mathrm{CM}(-4)^{\times 2}$
$10,1$	$3,1,0,0$	$\mathrm{CM}(-3),\ \mathrm{CM}(-8),\ \mathrm{CM}(-20),\ \mathrm{CM}(-40)$
$10,3$	$4,1,0,0$	$\mathrm{CM}(-3),\ \mathrm{CM}(-8)^{\times 2},\ \mathrm{CM}(-20)^{\times 2}$
$10,7$	$4,2,0,0$	$\mathrm{CM}(-3)^{\times 2},\ \mathrm{CM}(-20)^{\times 2},\ \mathrm{CM}(-40)^{\times 2}$
$14,1$	$3,0,1,0$	$\mathrm{CM}(-4),\ \mathrm{CM}(-8),\ \mathrm{CM}(-56)^{\times 2}$
$14,3$	$6,0,0,0$	$\mathrm{CM}(-8)^{\times 2},\ \mathrm{CM}(-56)^{\times 4}$
$14,5$	$4,0,2,0$	$\mathrm{CM}(-4)^{\times 2},\ \mathrm{CM}(-56)^{\times 4}$
$15,1$	$3,0,0,1$	$\mathrm{CM}(-3),\ \mathrm{CM}(-12),\ \mathrm{CM}(-15),\ \mathrm{CM}(-60)$
$15,2$	$6,0,0,0$	$\mathrm{CM}(-12)^{\times 2},\ \mathrm{CM}(-15)^{\times 2}, \mathrm{CM}(-60)^{\times 2}$
$15,4$	$8,0,0,0$	$\mathrm{CM}(-12)^{\times 2},\ \mathrm{CM}(-15)^{\times 2},\ \mathrm{CM}(-60)^{\times 4}$
$21,1$	$5,0,0,0$	$\mathrm{CM}(-4),\ \mathrm{CM}(-7),\ \mathrm{CM}(-28), \mathrm{CM}(-84)^{\times 2}$
$21,2$	$7,0,0,0$	$\mathrm{CM}(-4),\ \mathrm{CM}(-7)^{\times 2},\ \mathrm{CM}(-28)^{\times 2},\ \mathrm{CM}(-84)^{\times 2}$
$26,1$	$5,0,0,0$	$\mathrm{CM}(-8),\ \mathrm{CM}(-52),\ \mathrm{CM}(-104)^{\times 3}$
$26,3$	$8,0,0,0$	$\mathrm{CM}(-8)^{\times 2}, \mathrm{CM}(-104)^{\times 6}$
$35,1$	$6,0,0,0$	$\mathrm{CM}(-7),\ \mathrm{CM}(-28),\ \mathrm{CM}(-35),\ \mathrm{CM}(-140)^{\times 3}$
$35,2$	$10,0,0,0$	$\mathrm{CM}(-7)^{\times 2},\ \mathrm{CM}(-28)^{\times 2},\ \mathrm{CM}(-140)^{\times 6}$
$39,1$	$6,0,0,0$	$\mathrm{CM}(-52)^{\times 2},\ \mathrm{CM}(-39)^{\times 2},\ \mathrm{CM}(-156)^{\times 2}$
$39,2$	$10,0,0,0$	$\mathrm{CM}(-52)^{\times 2},\ \mathrm{CM}(-39)^{\times 4},\ \mathrm{CM}(-156)^{\times 4}$

4.3 Coverings of Shimura Curves

The goal of this section is to obtain explicit coverings of $X_0^D(N)/W_D \to X_0^D(1)/W_D$ for pairs of D and N given in Lemma 4.2.2. That is, we wish to find a Hauptmodul t_1 of $X_0^D(1)/W_D$, a Hauptmodul t_N of $X_0^D(N)/W_D$, and the relation between them. Of course, there are infinitely many choice for t_1 and t_N. For $X_0^D(N)/W_D$, we will choose t_N such that the Atkin-Lehner involution w_N acts by $w_N : t_N \mapsto -t_N$. This will make the determination of Schwarzian differential equation simpler.

Case $D = 6$ In the case $D = 6$, all the coverings $X_0^6(N)/W_6 \to X_0^6(1)/W_6$, $N = 5, 7, 13$, are already given in [6]. Here we just modify the t_N in [6] such that the new t_N satisfies $w_N : t_N \mapsto -t_N$.

Lemma 4.3.1 ([6]). *1. There is a Hauptmodul t_1 for $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -24, -4, and -3, respectively.*

2. There is a Hauptmodul $t = t_5$ for $X_0^6(5)/W_6$ that takes values $\pm i/8$ and $\pm\sqrt{-6}/3$ at the CM-points of discriminants -4 and -24, respectively. The relation between t_1 and t is

$$t_1 = \frac{(2+3t^2)(34-117t+1824t^2)^2}{125(1+6t)^6} = 1 + \frac{27(1+64t^2)(3-7t)^4}{125(1+6t)^6}.$$

The Atkin-Lehner involution w_5 acts by $w_5 : t \mapsto -t$.

3. *There is a Hauptmodul $t = t_7$ for $X_0^6(7)/W_6$ that takes values $\pm\sqrt{-3}/9$ and $\pm\sqrt{-6}/8$ at the CM-points of discriminants -3 and -24, respectively. The relation between t_1 and t is*

$$t_1 = -\frac{(3+32t^2)(78-396t+1963t^2-12312t^3)^2}{4(1+27t^2)(3+10t)^6}$$

The Atkin-Lehner involution w_7 acts by $w_7 : t \mapsto -t$.

4. *There is a Hauptmodul $t = t_{13}$ for $X_0^6(13)/W_6$ that takes values $\pm 4\sqrt{-3}/9$ and $\pm 3i/4$ at the CM-points of discriminants -3 and -4, respectively. The relation between t_1 and t is*

$$t_1 = 1 - \frac{27(9+16t^2)(144-98t+246t^2-161t^3)^4}{16(16+27t^2)(30+3t+55t^2)^6}.$$

The Atkin-Lehner involution w_{13} acts by $w_{13} : t \mapsto -t$.

Proof. In [6], Elkies already showed that explicit coverings of $X_0^6(N)/W_6 \to X_0^6(1)/W_6$, $N = 5, 7, 13$, are given by

$$t_1 = 1 + 135s^4 + 324s^5 + 540s^6, \qquad w_5 : s \longmapsto \frac{42-55s}{55+300s},$$

$$t_1 = -\frac{(4s^2+4s+25)(2s^3-3s^2+12s-2)^2}{108(7s^2-8s+37)}, \qquad w_7 : s \longmapsto \frac{116-9s}{9+20s},$$

and

$$t_1 = \frac{(s^7-50s^6+63s^5-5040s^4+783s^3-168426s^2-6831s-1864404)^2}{4(7s^2+2s+247)(s^2+39)^6}$$

with

$$w_{13} : s \longmapsto \frac{5s+72}{2s-5},$$

respectively. Choosing t such that

$$s = \frac{7t-3}{30t+5}, \qquad s = \frac{-29t+6}{10t+3}, \qquad s = \frac{-8t+9}{2t+1},$$

respectively, we get the lemma. □

Case $D = 10$ The covering $X_0^{10}(3)/W_{10} \to X_0^{10}(1)/W_{10}$ has also been given in [6]. Here we mainly work on the case $N = 7$.

Lemma 4.3.2. *1. There is a Hauptmodul t_1 for $X_0^{10}(1)/W_{10}$ that takes values 0, ∞, 2, and 27 at the CM-points of discriminants -3, -8, -20, and -40, respectively.*

2. *There is a Hauptmodul $t = t_3$ for $X_0^{10}(3)/W_{10}$ that takes values 0, $\pm 1/4\sqrt{-2}$, $\pm 1/\sqrt{-5}$ at the CM-points of discriminants -3, -8, and -20, respectively. The relation between t_1 and t is*

$$t_1 = \frac{108t(1-2t)^3}{(1+32t^2)(1+7t)^2} = 2 - \frac{2(1+5t^2)(1-20t)^2}{(1+32t^2)(1+7t)^2}.$$

   The Atkin-Lehner involution w_3 acts by $w_3 : t \mapsto -t$.

3. *There is a Hauptmodul $t = t_7$ for $X_0^{10}(7)/W_{10}$ that takes values $\pm 1/3\sqrt{-3}$, $\pm 1/2\sqrt{-5}$, and $\pm\sqrt{-10}/16$ at the CM-points of discriminants -3, -20, and -40, respectively. The relation between t_1 and t is*

$$t_1 = \frac{8(1+27t^2)(2-3t+44t^2)^3}{7(1+4t+55t^2+102t^3+736t^4)^2}.$$

   The Atkin-Lehner involution w_7 acts by $w_7 : t \mapsto -t$.

Proof. In [6], it is shown that an explicit covering $X_0^{10}(3)/W_{10} \to X_0^{10}(1)/W_{10}$ is given by

$$t_1 = \frac{216(s-1)^3}{(s+1)^2(9s^2-10s+17)}$$

with $w_3 : s \mapsto 10/9 - s$. Let t be the Hauptmodul of $X_0^{10}(1)/W_{10}$ with

$$s = \frac{2}{9t} + \frac{5}{9}.$$

Then the relation of t_1 and t and the action of w_3 are given as in the lemma.

We next consider the case $N = 7$. According to Theorem 3.4 of [9], an equation for $X_0^{10}(7)$ is given by

$$y^2 = -27x^4 - 40x^3 + 6x^2 + 40x - 27. \tag{4.1}$$

The actions of the Atkin-Lehner involutions on this model of $X_0^{10}(7)$ are given by

$$w_{70} : (x,y) \longmapsto (x,-y), \qquad w_5 : (x,y) \longmapsto \left(-\frac{1}{x}, -\frac{y}{x^2}\right),$$

and

$$w_{10} : (x,y) \longmapsto \left(\frac{2x+1}{x-2}, \frac{5y}{(x-2)^2}\right).$$

Since CM(-20) are fixed points of w_5, their coordinates on (4.1) are $(i, \pm 2\sqrt{5}(1 + 2i))$ and $(-i, \pm 2\sqrt{5}(1-2i))$. Likewise, we find that CM(-40) have coordinates

$(2+\sqrt{5}, \pm 8\sqrt{-10}(2+\sqrt{5}))$ and $(2-\sqrt{5}, \pm 8\sqrt{-10}(2-\sqrt{5}))$. Furthermore, from the method of [9], we know that the two points at infinity are CM-points of discriminant -3. Thus, the coordinates of $\mathrm{CM}(-3)$ are ∞, $(0, \pm 3\sqrt{-3})$, $(2, \pm 15\sqrt{-3})$, and $(-1/2, \pm 15\sqrt{-3}/4)$.

From (4.1), we can obtain an equation $w^2+27z^2+40z+20=0$ for $X_0^{10}(7)/\langle w_{10}\rangle$, where the covering $X_0^{10}(7) \to X_0^{10}(7)/\langle w_{10}\rangle$ is given by

$$(x,y) \longmapsto (w,z) = \left(\frac{y}{x-2}, \frac{x^2+1}{x-2}\right).$$

On this equation for $X_0^{(10)}(7)/\langle w_{10}\rangle$, the actions of the Atkin-Lehner involutions are given by

$$w_{70} = w_7 : (w,z) \longmapsto (-w,z), \qquad w_2 = w_5 : (w,z) \longmapsto \left(\frac{w}{2z+1}, \frac{-z}{2z+1}\right).$$

The coordinates of $\mathrm{CM}(-3)$ are the two points at ∞ and $(\pm 3\sqrt{-3}/2, -1/2)$. Also, the coordinates of $\mathrm{CM}(-20)$ are $(\pm 2\sqrt{-5}, 0)$, and the coordinates of $\mathrm{CM}(-40)$ are $(\pm 8\sqrt{-2}(2+\sqrt{5}), 4+2\sqrt{5})$ and $(\pm 8\sqrt{-2}(2-\sqrt{5}), 4-2\sqrt{5})$.

Now set $t=(z+1)/w$. We can check that t is invariant under w_2 and that $(w,z) \mapsto t=(z+1)/w$ is 2-to-1. Thus, t is a Hauptmodul of $X_0^{10}(7)/W_{10}$. The coordinates of CM-points of discriminants -3, -20, and -40 are $\pm 1/3\sqrt{-3}$, $\pm 1/2\sqrt{-5}$, and $\pm\sqrt{-10}/16$, respectively. It follows that the relation between t_1 and t is

$$t_1 = \frac{A(1+27t^2)(1+a_1t+a_2t^2)^3}{(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2}$$

with

$$\begin{aligned} &A(1+27t^2)(1+a_1t+a_2t^2)^3 - 2(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2 \\ &\quad = B(1+20t^2)(1+c_1t+c_2t^2+c_3t^3)^2, \\ &A(1+27t^2)(1+a_1t+a_2t^2)^3 - 27(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2 \\ &\quad = C(1+128t^2/5)(1+d_1t+d_2t^2+d_3t^3)^2 \end{aligned}$$

for some constants A, B, C, a_j, b_j, c_j, and d_j. Comparing the coefficients, we get

$$t_1 = \frac{8(1+27t^2)(2-3t+44t^2)^3}{7(1+4t+55t^2+102t^3+736t^4)^2}$$

(or the same expression with t replaced by $-t$). This proves the lemma. □

Case $D=14$ The case $D=14$ is also worked out in [6]. Here we only need to make a change of variable so that w_N acts by $w_N : t_N \to -t_N$.

Lemma 4.3.3 ([6]). *1. There is a Hauptmodul t_1 for $X_0^{14}(1)/W_{14}$ that takes values ∞, 0, and $(-13\pm 7\sqrt{-7})/32$ at CM-points of discriminants -4, -8, and -56, respectively.*

2. *There is a Hauptmodul $t = t_3$ for $X_0^{14}(3)/W_{14}$ that takes values $\pm 1/\sqrt{-2}$ and $(\pm 9\sqrt{-7} \pm 4\sqrt{-14})/49$ at CM-points of discriminants -8 and -56, respectively. The relation between t_1 and t is*

$$t_1 = \frac{4(1+2t^2)(1-5t)^2}{9(1+t)^4}.$$

The Atkin-Lehner involution w_3 acts by $w_3 : t \mapsto -t$.

3. *There is a Hauptmodul $t = t_5$ for $X_0^{14}(5)/W_{14}$ that takes values $\pm i/4$ and $(\pm 5\sqrt{-7} \pm 4\sqrt{-14})/7$ at CM-points of discriminants -4 and -56, respectively. The relation between t_1 and t is*

$$t_1 = -\frac{5(1-t+17t^2-13t^3)^2}{(1+16t^2)(1+3t)^4}.$$

The Atkin-Lehner involution w_5 acts by $w_5 : t \mapsto -t$.

Proof. In [6], it is shown that explicit coverings $X_0^{14}(N)/W_{14} \to X_0^{14}(1)/W_{14}$ can be given by

$$t_1 = \frac{1}{27}(s^4 + 2s^3 + 9s^2), \qquad w_3 : s \longmapsto \frac{5-2s}{2+s}$$

and

$$t_1 = -\frac{(256s^3 + 224s^2 + 232s + 217)^2}{50000(s^2+1)}, \qquad w_5 : s \longmapsto \frac{24-7s}{7+24s},$$

respectively. Choosing t with

$$s = \frac{1-5t}{1+t}, \qquad s = \frac{3-16t}{4+12t},$$

respectively, we get the lemma. □

Case $D = 15$ An explicit covering $X_0^{15}(2)/W_{15} \to X_0^{15}(1)/W_{15}$ is given in [6]. Here we only need to make a change of variable so that w_N acts by $w_N : t_N \to -t_N$.

Lemma 4.3.4. 1. *There is a Hauptmodul for $X_0^{15}(1)/W_{15}$ that takes values ∞, 0, 81, and 1 at CM-points of discriminants -3, -12, -15, and -60, respectively.*

2. *There is a Hauptmodul t_2 for $X_0^{15}(2)/W_{15}$ that takes values ± 1, $\pm\sqrt{-15}/3$, and $\pm 1/5$ at CM-points of discriminant -12, -15, and -60, respectively. The relation between t_1 and t_2 is*

$$t_1 = \frac{27(1-t_2)(1-3t_2)^2}{2(1+t_2)^3} = 1 + \frac{(1-5t_2)(5-7t_2)^2}{2(1+t_2)^3} = 81 - \frac{27(1+5t_2)(5+3t_2^2)}{2(1+t_2)^3}.$$

The Atkin-Lehner involution w_2 acts by $w_2 : t_2 \mapsto -t_2$.

Proof. In [6], an explicit covering $X_0^{15}(2)/W_{15} \to X_0^{15}(1)/W_{15}$ is given by

$$t_1 = \frac{1}{4}s(s-3)^2, \qquad w_2 : s \longmapsto \frac{36}{s}.$$

Choosing a Hauptmodul t for $X_0^{15}(2)/W_{15}$ with

$$s = \frac{6-6t}{1+t},$$

we establish the claim about $X_0^{15}(2)/W_{15}$. □

Case $D = 21$ We will need an equation for some Atkin-Lehner quotient of $X_0^{21}(2)$ in order to determine the coordinates of elliptic points on $X_0^{21}(2)$.

Lemma 4.3.5. *An equation for $X_0^{21}(2)/\langle w_{21}\rangle$ is $y^2 = (x+12)(x^2-7x+28)$. Moreover, the action of the Atkin-Lehner involution $w_3 = w_7$ on this curve is given by $(x,y) \mapsto (x,-y)$. Also, the two rational points ∞ and $(-12,0)$ are the CM-points of discriminants -28, and the other two 2-torsion points $((7 \pm 3\sqrt{-7})/2, 0)$ are the CM-points of discriminant -7.*

Proof. We follow the methods of [9]. The Shimura curve $X_0^{21}(2)/\langle w_{21}\rangle$ has genus 1. By [9, Lemma 5.10], the two CM-points of discriminant -28 are $\mathbb{Q}$-rational points on this curve. Thus, $X_0^{21}(2)/\langle w_{21}\rangle$ is an elliptic curve over $\mathbb{Q}$. Now in the space $S_2(\Gamma_0(42))^{21\text{-new}}$ the unique Hecke eigenform with $+$-eigenvalue for w_{21} is coming from the newform space of $S_2(\Gamma_0(42))$. Therefore, the elliptic curve $X_0^{21}(2)/\langle w_{21}\rangle$ has conductor 42. Using the Cerednik-Drinfeld theory of p-adic uniformization of Shimura curves, we find that the types of singular fibers at primes of bad reduction of $X_0^{21}(2)/\langle w_{21}\rangle$ agree with those of the elliptic curve 42A1, in Cremona's notation. The global minimal model of the elliptic curve 42A1 is $y^2 + xy + y = x^3 + x^2 - 4x + 5$. With a simple change of variables, we write it as $y^2 = (x+12)(x^2-7x+28)$.

Now the covering $X_0^{21}(2)/\langle w_{21}\rangle \to X_0^{21}(2)/W_{21}$ is ramified at the two CM-points of discriminant -7 and the two CM-points of discriminant -28. If we let one of the CM-points of discriminant -28 be the point at infinity, then an equation for $X_0^{21}(2)/\langle w_{21}\rangle$ is of the form $y^2 = f(x)$ for some polynomial $f(x) = x^3 + \cdots$ of degree 3 in $\mathbb{Q}[x]$ with the Atkin-Lehner involution w_3 acting by $(x,y) \mapsto (x,-y)$. Up to a transformation of the form $x \mapsto ax+b$, this polynomial $f(x)$ must be the polynomial $(x+12)(x^2-7x+28)$. This proves the lemma. □

Remark 4.3.6. *According to Cremona's table of elliptic curves [3], the elliptic curve 42A1 has 8 rational points. Thus, $X_0^{21}(2)/\langle w_{21}\rangle$ also has 8 $\mathbb{Q}$-rational points. Two of them are the CM-points of discriminant -28 mentioned above. The rest of $\mathbb{Q}$-rational points consist of two CM-points of discriminant -4 and four CM-points of discriminant -16.*

Lemma 4.3.7. *There is a Hauptmodul t_1 for $X_0^{21}(1)/W_{21}$ that takes values 49, 0, ∞, and $(47 \pm 8\sqrt{-3})/7$ at CM-points of discriminants -4, -7, -28, and -84, respectively.*

Also, there is a Hauptmodul $t = t_2$ for $X_0^{21}(2)/W_{21}$ that takes values 0, $\pm 1/3\sqrt{-7}$, ± 1, and $\pm 1/3\sqrt{-3}$ at CM-points of discriminants -4, -7, -28, and -84, respectively. The relation between t_1 and t is

$$t_1 = \frac{49(1+t)(1+63t^2)}{(1-t)(1-15t)^2} = 49 + \frac{1568t(1-3t)^2}{(1-t)(1-15t)^2}.$$

The Atkin-Lehner involution w_2 acts by $w_2 : t \mapsto -t$.

Proof. According to [9], an equation for $X_0^{21}(1)$ is given by $y^2 = -7x^4 + 94x^2 - 343$ with the actions of the Atkin-Lehner involutions given by

$$w_3 : (x, y) \longmapsto (-x, -y), \qquad w_7 : (x, y) \longmapsto (-x, y), \qquad w_{21} : (x, y) \longmapsto (x, -y).$$

The Atkin-Lehner involution w_7 fixes the two points at ∞ and $(0, \pm 7\sqrt{-7})$. Since the equation has a symmetry $(x, y) \longmapsto (7/x, 7y/x^2)$, we might as well assume that the two points $(0, \pm 7\sqrt{-7})$ are the CM-points of discriminant -7 and the two points at infinity are the CM-points of discriminant -28. Moreover, the four points with $y = 0$ correspond to the four CM-points of discriminant -84.

Since w_3 acts by $(x, y) \to (-x, -y)$, an equation for $X_0^{21}(1)/\langle w_3 \rangle$ is $y^2 = -7x^3 + 94x^2 - 343x$, where the covering $X_0^{21}(1) \to X_0^{21}(1)/\langle w_3 \rangle$ is given by $(x, y) \mapsto (x^2, xy)$. Then $t_1 = x$ generates the function field of X_0^{21}/W_{21}. The values of t_1 at the CM-points of discriminants -7, -28, and -84 are 0, ∞, and $(47 \pm 8\sqrt{-3})/7$, respectively. The value of t_1 at the CM-point of discriminant -4 will be determined later.

By Lemma 4.3.5, an equation $X_0^{21}(2)/\langle w_{21} \rangle$ is $y^2 = (x + 12)(x^2 - 7x + 28)$ with the Atkin-Lehner involution $w_3 = w_7$ acting by $(x, y) \to (x, -y)$. Thus, $s = x$ generates the function field of $X_0^{21}(2)/W_{21}$. According to the lemma, the values of s at the CM-points of discriminant -7 are $(7 \pm 3\sqrt{-7})/2$ and those at CM-points of discriminant -28 are -12 and ∞. The Atkin-Lehner involution w_2 switches the two CM-points of discriminant -28. It also switches the two CM-points of discriminant -7. (Note that in general, w_2 can send a CM-point of discriminant $-d$ on $X_0^D(N)/G$ to a CM-point of discriminant $-4d$ and vice versa. Here because w_2 is defined over $\mathbb{Q}$, it must send a $\mathbb{Q}$-rational point to another $\mathbb{Q}$-rational point.) These informations suffice to determine w_2 in terms of s. We find

$$w_2 : s \longmapsto \frac{-12s + 112}{s + 12}.$$

Choosing a new Hauptmodul

$$t = \frac{4 - s}{28 + s},$$

we have $w_2 : t \mapsto -t$. The new coordinates of CM-points of discriminants -7 and -28 are $\pm 1/3\sqrt{-7}$ and ± 1, respectively. Also, since w_2 fixes the unique CM-point of discriminant -4, we find that the CM-point of discriminant -4 has coordinate 0. We now determine the relation between t_1 and t.

Replacing t by $-t$ if necessary, we may assume that the CM-point of discriminant -28 of $X_0^{21}(2)/W_{21}$ that lies above the CM-point of discriminant -7 of $X_0^{21}(1)/W_{21}$

is -1. Then

$$t_1 = \frac{A(1+t)(1+63t^2)}{(1-t)(1-at)^2}$$

for some constants A and a. Since $X_0^{21}(2)/W_{21} \to X_0^{21}(1)/W_{21}$ is also ramified at the CM-points of discriminant -84, the discriminant of the polynomial

$$A(1+t)(1+63t^2) - B(1-t)(1-at)^2$$

in t must be divisible by the polynomial $7B^2-94B+343$. This gives us two conditions on A and a. Solving them for A and a, we find that the only legitimate values for A and a are $A = 49$ and $a = 15$. Because t has value 0 at the CM-point of discriminant -4 on $X_0^{21}(2)/W_{21}$, the CM-point of -4 on $X_0^{21}(1)/W_{21}$ has coordinate 49. This proves the lemma. □

Case $D = 26$ We first recall a lemma of González and Rotger [8].

Lemma 4.3.8 ([8, Proposition 2.1])**.** *Let C be a hyperelliptic curve of genus 2 defined over a field k of characteristic not equal to 2 or 3 and let w be its hyperelliptic involution. Assume that the group of automorphisms of C over k contains a subgroup $\langle u_1, u_2 = u_1 \cdot w\rangle$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and denote by C_i the elliptic quotient $C/\langle u_i\rangle$. If the two elliptic curves*

$$E_1 : y^2 = x^3 + A_1x + B_1, \qquad E_2 : y^2 = x^3 + A_2x + B_2$$

are isomorphic to C_1 and C_2 over k, respectively. Then C admits a hyperelliptic equation of the form $y^2 = ax^6 + bx^4 + cx^2 + d$, where $a \in k^*$, $b \in k$ are solutions of

$$\begin{aligned} 27a^3B_2 &= 2A_1^3 + 27B_1^2 + 9A_1B_1b + 2A_1^2b^2 - B_1b^3, \\ 9a^2A_2 &= -3A_1^2 + 9B_1b + A_1b^2, \end{aligned}$$

$c = (3A_1 + b^2)/(3a)$, $d = (27B_1 + 9A_1b + b^3)/(27a^2)$, and the involution u_1 on C is given by $(x, y) \mapsto (-x, y)$.

Lemma 4.3.9. *The Shimura curves $X_1 : X_0^{26}(3)/\langle w_2, w_3\rangle$, $X_2 : X_0^{26}(3)/\langle w_2, w_{39}\rangle$, and $X_3 : X_0^{26}(3)/\langle w_6, w_{13}\rangle$ are elliptic curves over $\mathbb{Q}$ with defining equations*

$$\begin{aligned} X_1 &: y^2 = x^3 - 3403x - 83834, \\ X_2 &: y^2 = x^3 - 43x + 166, \\ X_3 &: y^2 = x^3 + 621x + 9774. \end{aligned}$$

Moreover, on the equation for X_1, the point at ∞ is the CM-point of discriminant -312, and the involution $(x, y) \mapsto (x, -y)$ is the Atkin-Lehner involution $w_{13} = w_{26} = w_{39} = w_{78}$. On the equation for X_2, the point at ∞ is the CM-point of discriminant -24 and the involution $(x, y) \mapsto (x, -y)$ is the Atkin-Lehner involution

$w_3 = w_6 = w_{13} = w_{26}$. *On the equation for* X_3*, the point at* ∞ *is the CM-point of discriminant* -8 *and the involution* $(x,y) \mapsto (x,-y)$ *is the Atkin-Lehner involution* $w_2 = w_3 = w_{26} = w_{39}$. *In all three cases, the* 2*-torsion points are the CM-points of discriminant* -104 *on their respective curves.*

Proof. The fact that the three curves in the lemma have genus one can be verified either by using the genus formula, together with Proposition 3.3.5, Lemmas 3.4.1, and 4.2.1, or by counting the dimensions of subspaces of $S_2(\Gamma_0(78))^{26\text{-new}}$ with appropriate eigenvalues for the Atkin-Lehner involutions. We omit the details.

On X_1, there is a unique CM-point of discriminant -312, which must be a $\mathbb{Q}$-rational point. Thus, X_1 is an elliptic curve over $\mathbb{Q}$. Likewise, X_2 and X_3 have unique CM-points of discriminants -24 and -8, respectively. They are also elliptic curve over $\mathbb{Q}$.

Observe that all cusp forms in $S_2(\Gamma_0(78))^{26\text{-new}}$ having -1 eigenvalue for w_2 are from the cusp form of level 26 corresponding to the isogeny class 26B of elliptic curves, in Cremona's notation. Thus, X_1 and X_2 are isomorphic to either 26B1 or 26B2. Similarly, we find that the one-dimensional subspace of $S_2(\Gamma_0(78))^{26\text{-new}}$ that has eigenvalue $+1$ for both w_6 and w_{13} is coming from the cusp form associated to 26A. Using the Cerednik-Drinfeld theory to compute the types of singular fibers at primes 2 and 13, we see that X_1 is isomorphic to the elliptic curve 26B2, X_2 is isomorphic to 26B1, and X_3 is isomorphic to 26A3. If we put the CM-point of discriminant -312 on X_1, that of discriminant -24 on X_2, and that of discriminant -8 on X_3 at ∞, respectively, and require that the Atkin-Lehner involutions w_{13}, w_3, and w_2 act by $(x,y) \to (x,-y)$ on the three curves, respectively, we get the equations for the three curves. □

Lemma 4.3.10. *1. An equation for the curve* $X_0^{26}(3)/\langle w_2 \rangle$ *is*

$$y^2 = -\frac{2197}{3}x^6 - 362x^4 - 55x^2 - \frac{8}{3}$$

with the actions of the Atkin-Lehner involutions given by

$$w_3 : (x,y) \longmapsto (-x,y), \qquad w_{13} : (x,y) \longmapsto (x,-y).$$

On this model, the two CM-points of discriminant -312 *are the two points at infinity, and the two CM-points of discriminant* -24 *are* $(0, \pm 2\sqrt{-6}/3)$.

2. *An equation for the curve* $X_0^{26}(3)/\langle w_6 \rangle$ *is*

$$y^2 = \frac{2197}{72}x^6 - \frac{699}{8}x^4 - \frac{225}{8}x^2 - \frac{81}{8}$$

with the actions of the Atkin-Lehner involutions given by

$$w_2 : (x,y) \longmapsto (-x,y), \qquad w_{26} : (x,y) \longmapsto (x,-y).$$

On this model, the two CM-points of discriminant -312 *are the two points at infinity, and the two CM-points of discriminant* -8 *are* $(0, \pm 9\sqrt{-2}/4)$.

3. An equation for $X_0^{26}(3)/\langle w_{39}\rangle$ is

$$y^2 = \frac{8}{9}x^6 + 9x^4 - 18x^2 + 81$$

with the actions of the Atkin-Lehner involutions given by

$$w_2 : (x, y) \longmapsto (-x, y), \qquad w_6 : (x, y) \longmapsto (x, -y).$$

On this model, the two CM-points of discriminant -24 are the two points at infinity, and the two CM-points of discriminant -8 are $(0, \pm 9)$.

Moreover, on each of these three curves, there are six CM-points of discriminant -104. Their coordinates are $(\alpha_j, 0)$, $j = 1, \ldots, 6$, where α_j are the zeros of their respective polynomials of degree 6.

Proof. We apply Proposition 2.1 of [8], cited as Lemma 4.3.8 above) with $C = X_0^{26}(3)/\langle w_2\rangle$, w_{13}, $u_1 = w_3$, $u_2 = w_{39}$, $A_1 = -3403$, $B_1 = -83834$, $A_2 = -43$, and $B_2 = 166$. We find an equation for $X_0^{26}(3)/\langle w_2\rangle$ is

$$y^2 = -\frac{2197}{3}x^6 - 362x^4 - 55x^2 - \frac{8}{3}$$

with the Atkin-Lehner involutions given by

$$w_3 : (x, y) \longmapsto (-x, y), \qquad w_{13} : (x, y) \longmapsto (x, -y).$$

Since CM-points of discriminant -24 are fixed points of the involution $w_6 = w_3 : (x, y) \to (-x, y)$, we see that their coordinates are $(0, \pm 2\sqrt{-6}/3)$. Likewise, CM-points of discriminant -312 are the fixed points of $w_{78} = w_{39} : (x, y) \mapsto (-x, -y)$, so they are the two points at infinity. Also, CM-points of discriminant -104 are the fixed point of $w_{26} = w_{13} : (x, y) \mapsto (x, -y)$. Their coordinates are $(\alpha_j, 0)$, $j = 1, \ldots, 6$, where α_j are the zeros of $-2197x^6/3 - 362x^4 - 55x^2 - 8/3$.

The equations of the other two curves are obtained in the same way. □

Lemma 4.3.11. *Let $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ be the equation for $X_0^{26}(3)/\langle w_2\rangle$ given in the previous lemma. Then the coordinates of the four CM-points of discriminant -8 are $(\pm 1/2\sqrt{-2}, \pm 3/16\sqrt{-2})$.*

Proof. By Lemma 4.3.10, an equation for $X_0^{26}(3)/\langle w_2\rangle$ is $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ with $w_3 : (x, y) \mapsto (-x, y)$ and $w_{13} : (x, y) \mapsto (x, -y)$. Thus, if we let $t_1 = x^2$, then t_1 is a Hauptmodul for $X_0^{26}(3)/W_{26,3}$. Likewise, if we let t_2 be the function x^2 in the equation $y^2 = 2197x^6/72 - 699x^4/8 - 225x^2/8 - 81/8$ for $X_0^{26}(3)/\langle w_6\rangle$, then t_2 is also a Hauptmodul for $X_0^{26}(3)/W_{26,3}$. It follows that $t_1 = (at_2 + b)/(ct_2 + d)$ for some a, b, c, d.

Now observe that the values of t_1 and t_2 at the CM-point of discriminant -312 are both ∞. Thus, $t_1 = at_2 + b$ for some a and b. Moreover, the values of t_1 and t_2 at the CM-points of discriminant -104 are the zeros of $f_1(z) = -2197z^3/3 - 362z^2 - 55z - 8/3$ and the zeros of $f_2(z) = 2197z^3/72 - 699z^2/8 - 225z/8 - 81/8$, respectively. Therefore, the constants a and b must satisfy $f_1(az + b) = Af_2(z)$ for some constant A. Comparing the coefficients, we find $A = 1/576$, $a = -1/24$ and $b = -1/8$. Since the value of t_2 at the CM-point of discriminant -8 is 0, the value of t_1 at the same point is $-1/8$, which implies that the four CM-points of discriminant -8 on $X_0^{26}(3)/\langle w_2\rangle$ has coordinates $(\pm 1/(2\sqrt{-2}), \pm 3/(16\sqrt{-2}))$ on the equation $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ for $X_0^{26}(3)/\langle w_2\rangle$. □

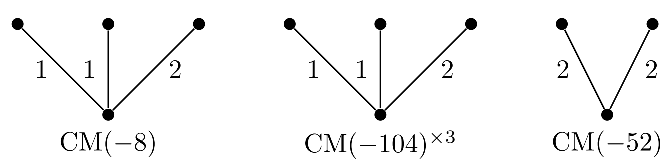
Lemma 4.3.12. *There is a Hauptmodul t_1 for $X_0^{26}(1)/W_{26}$ that takes values ∞, 0, and the three zeros of $-2x^3 + 19x^2 - 24x - 169$ at the CM-point of discriminant -8, the CM-point of discriminant -52, and three CM-points of discriminant -104, respectively. Also, there is a Hauptmodul $t = t_3$ for $X_0^{26}(3)/W_{26}$ that takes values $\pm 1/(2\sqrt{-2})$ and the six zeros of $-2197x^6/3 - 362x^4 - 55x^2 - 8/3$ at the two CM-points of discriminant -8 and the six CM-points of discriminant -104, respectively. Moreover, the relation between t_1 and t and the action of w_3 on t are given by*

$$t_1 = -\frac{3(1+t+10t^2)^2}{(1+8t^2)(1-t)^2}, \qquad w_3 : t \longmapsto -t.$$

Proof. According to Theorem 3.1 of [8], an equation for $X_0^{26}(1)$ is $y^2 = -2x^6 + 19x^4 - 24x^2 - 169$. In fact, the method used in [8] to deduce this equation also shows that the Atkin-Lehner involutions act by $w_{13} : (x,y) \mapsto (-x,y)$ and $w_{26} : (x,y) \mapsto (x,-y)$. Then the two points $(0, \pm 13\sqrt{-1})$ are the CM-points of discriminant -52, the two points at infinity are the fixed points of $w_2 : (x,y) \longmapsto (-x,-y)$, i.e., the two CM-points of discriminant -8, and the six points $(\alpha_j, 0)$, $j = 1, \ldots, 6$, are the six CM-points of discriminant -104, where α_j are the zeros of $-2x^6+19x^4-24x^2-169$. Thus, $t_1 = x^2$ is a Hauptmodul of $X_0^{26}(1)/W_{26}$ with values ∞, 0, the zeros of $-2x^3 + 19x^2 - 24x - 169$ at the CM-point of discriminant -8, the CM-point of discriminant -52, and the three CM-points of discriminant -104 on $X_0^{26}(1)/W_{26}$.

On the other hand, Lemmas 4.3.10 and 4.3.11 show that if we let t be the x in the equation $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ for $X_0^{26}(3)/\langle w_2\rangle$, then t is a Hauptmodul for $X_0^{26}(3)/W_{26}$ that takes values $\pm 1/(2\sqrt{-2})$ at the two CM-points of discriminant -8 and β_j, $j = 1, \ldots, 6$, at the six CM-points of discriminant -104, where β_j are the six zeros of $-2197x^6/3 - 362x^4 - 55x^2 - 8/3$. It is clear that w_3 acts on t by $w_3 : t \mapsto -t$.

The relation between t_1 and t is simple to determine. From the table at the end of Section 4.2, we know that the covering $X_0^{26}(3)/W_{26} \to X_0^{26}(1)/W_{26}$ is ramified precisely at the CM-points of discriminants -8, -52, and -104 of $X_0^{26}(1)/W_{26}$ with ramification types given by



It follows that

$$t_1 = \frac{A(1+a_1t+a_2t^2)^2}{(1+8t^2)(1+bt)^2}$$

for some constants A, a_1, a_2, and b such that

$$\begin{aligned} & -2f^3 + 19f^2g - 24fg^2 - 169g^3 \\ & \qquad = B(-2197t^6/3 - 362t^4 - 55t^2 - 8/3)(1 + c_1t + c_2t^2 + c_3t^3)^2 \end{aligned}$$

for some constants B, c_1, c_2, and c_3, where $f = A(1+8t^2)(1+at)^2$ and $g = (1+b_1t+b_2t^2)^2$. Comparing the coefficients, we find

$$t_1 = -\frac{3(1+t+10t^2)^2}{(1+8t^2)(1-t)^2} \quad \text{or} \quad t_1 = -\frac{3(1-t+10t^2)^2}{(1+8t^2)(1+t)^2}.$$

Both are valid, since the action of w_3 sends one to the other. This gives us the lemma. □

Case $D = 35$

Lemma 4.3.13. *An equation for* $X_0^{35}(1)/\langle w_5 \rangle$ *is*

$$y^2 = -(x+12)(7x+4)(x^3+4x^2+144x+80)$$

with the action $w_7 = w_{35}$ *given by* $w_7 : (x,y) \mapsto (x,-y)$*. The coordinates of CM-points of discriminants* -7, -28, -35, *and* -140 *are* $(-12,0)$, $(-4/7,0)$, ∞, *and* $(\alpha_j, 0)$, *respectively, where* α_j *are the three roots of* $x^3+4x^2+144x+80$.

An equation for $X_0^{35}(2)/\langle w_7 \rangle$ *is*

$$-2y^2 = (x^3+3x^2+11x+25)(x^3-3x^2+11x-25)$$

with the actions of $w_2 = w_{14}$ *and* $w_5 = w_{35}$ *given by* $w_2 : (x,y) \mapsto (-x,-y)$ *and* $w_5 : (x,y) \mapsto (x,-y)$*. The coordinates of CM-points of discriminants* -7, -8, -140, *and* -280 *are* $(\pm\sqrt{-7}, \pm 8)$, *two points at* ∞, $(\beta_j, 0)$, $j = 1, \ldots, 6$, *and* $(0, \pm 25/\sqrt{-2})$, *respectively, where* β_j *are the six roots of* $(x^3+3x^2+11x+25)(x^3-3x^2+11x-25)$.

Proof. In Section 10.4 of [14], Molina showed that an equation for $X_0^{35}(1)/\langle w_5 \rangle$ is

$$y^2 = -x(9x+4)(4x+1)(172x^3+176x^2+60x+7),$$

where $w_7 : (x,y) \mapsto (x,-y)$ and the points $(0,0)$, $(-4/9,0)$, $(-1/4,0)$, and $(\gamma_j, 0)$, $j = 1, \ldots, 3$, are the CM-points of discriminant -7, -28, -35, and -140, respectively. Here γ_j are the zeros of $172x^3+176x^2+60x+7$. Setting

$$(x,y) = \left(-\frac{x'+12}{4x'+28}, \frac{5y'}{16(x'+7)^3}\right),$$

we get the equation in our lemma. The reason for this change of variable is the following. The Shimura curve $X_0^{35}(1)/\langle w_7 \rangle$ has genus 1 and the unique CM-point of discriminant -35 is a $\mathbb{Q}$-rational point. Thus, it is an elliptic curve over $\mathbb{Q}$. Computing the singular fibers at primes of bad reduction, we find that it is isomorphic to the elliptic curve 35A1, which, after a change of variables, has an equation $y^2 = x^3+4x^2+144x+80$.

If we choose a Weierstrass equation for $X_0^{35}(1)/\langle w_7\rangle$ by requiring that the CM-point of discriminant -35 is the point at infinity and that w_5 acts by $(x,y)\to(x,-y)$, then up to a transformation of the form $x\to ax+b$, this Weierstrass equation must be $y^2=x^3+4x^2+144x+80$ and the three 2-torsion points $(\alpha_j,0)$ must be the three CM-points of discriminant -140. In view of this equation for $X_0^{35}(1)/\langle w_7\rangle$, we make the above change of variables for $X_0^{35}(1)/\langle w_5\rangle$.

We now consider the Shimura curve $X_0^{35}(2)/\langle w_7\rangle$. It is bielliptic with elliptic quotients $C_1 : X_0^{35}(2)/\langle w_7, w_{10}\rangle$ and $C_2 : X_0^{35}(2)/\langle w_2, w_7\rangle$. Here C_1 is an elliptic curve over $\mathbb{Q}$ because it has a unique CM-point of discriminant -8 and another two $\mathbb{Q}$-rational point coming from $\mathrm{CM}(-7)$. Likewise, C_2 is an elliptic curve over $\mathbb{Q}$ because C_2 has a unique CM-point of discriminant -280. By considering the eigenvalues of the Atkin-Lehner involutions associated to the eigenforms in $S_2(\Gamma_0(70))^{35\text{-new}}$, we find that both C_1 and C_2 fall in the isogeny class 35A, in Cremona's notation. Furthermore, by considering its singular fibers at primes of bad reduction using the Cerednik-Drinfeld theory, we find that C_1 is isomorphic to the elliptic curve 35A3 and C_2 is isomorphic to 35A2. We take $y^2=x^3-1728x+30672$ and $y^2=x^3-170208x-28273968$ to be (non-minimal) equations for 35A3 and 35A2, respectively.

Now if we choose a Weierstrass equation for C_1 by requiring that the CM-point of discriminant -8 is the infinity point and that the Atkin-Lehner involution w_2 acts by $(x,y)\mapsto(x,-y)$, then by a suitable transformation $x\mapsto ax+b$, the equation must be $y^2=x^3-1728x+30672$. Similarly, if we put the CM-point of discriminant -280 at infinity and require that w_5 acts by $(x,y)\mapsto(x,-y)$, then an equation for C_2 is $y^2=x^3-170208x-28273968$. Applying Lemma 4.3.8, we find an equation for $X_0^{35}(2)/\langle w_7\rangle$ is

$$\begin{aligned} y^2 &= -\frac{9}{2}(x^6+13x^4-29x^2-625) \\ &= -\frac{9}{2}(x^3+3x^2+11x+25)(x^3-3x^2+11x-25). \end{aligned}$$

Replacing y by $3y$, we get the equation

$$-2y^2=(x^3+3x^2+11x+25)(x^3-3x^2+11x-25) \tag{4.2}$$

as claimed in the lemma. According to Lemma 4.3.8, the Atkin-Lehner involutions act by

$$w_{10}:(x,y)\mapsto(-x,y), \qquad w_5:(x,y)\mapsto(x,-y), \qquad w_2:(x,y)\mapsto(-x,-y).$$

Since CM-points of discriminant -8, -140, and -280 on $X_0^{35}(2)/\langle w_7\rangle$ are fixed points of w_2, w_5, and w_{10}, respectively, we find that their coordinates are the two points at infinity, $(\beta_j,0)$, $j=1,\dots,6$, and $(0,\pm 25/\sqrt{-2})$, respectively, where β_j are the zeros of the polynomial on the right-hand side of (4.2).

To determine the coordinates of the four CM-points of discriminant -7, we observe that the curve $C_1 : X_0^{35}(2)/\langle w_7, w_{10}\rangle$ has exactly three $\mathbb{Q}$-rational points because it is isomorphic to the elliptic curve 35A3, which has precisely 3 $\mathbb{Q}$-rational points. Since we already know that C_1 has three $\mathbb{Q}$-rational points consisting of $\mathrm{CM}(-8)$

and $\mathrm{CM}(-7)$, any $\mathbb{Q}$-rational point of C_1 that is the CM-point of discriminant -8 will be a CM-point of discriminant -7. Now from the model $-2y^2 = x^6 + 13x^4 - 29x^2 - 625$ for $X_0^{35}(2)/\langle w_7\rangle$, we see that $-2y^2 = x^3 + 13x^2 - 29x - 625$ is also an equation for $X_0^{35}/\langle w_7, w_{10}\rangle$. On this model, the point at infinity is the CM-point of discriminant -8. Thus, the 3-torsion points $(-7, \pm 8)$ are the coordinates of CM-points of discriminant -7 on $X_0^{35}(2)/\langle w_7, w_{10}\rangle$. This in turn implies that the four CM-points of discriminant -7 on $X_0^{35}(2)/\langle w_7\rangle$ have coordinates $(\pm\sqrt{-7}, \pm 8)$. This completes the proof of the lemma. □

Lemma 4.3.14. *There is a Hauptmodul t_1 for $X_0^{35}(1)/W_{35}$ that takes values -12, $-4/7$, ∞, and the three zeros of $x^3 + 4x^2 + 144x + 80$ at the CM-points of discriminants -7, -28, -35, and -140, respectively. Also, there is also a Hauptmodul t for $X_0^{35}(2)/W_{35}$ that takes values $\pm\sqrt{-7}$, ± 5, the six zeros of $(x^3+3x^2+11x+25)(x^3-3x^2+11x-25)$, and 0 at the CM-points of discriminants -7, -8, -140, and -280, respectively. Moreover, the relation between t_1 and t is*

$$t_1 = -\frac{2(t-1)(t^2-6t+25)}{t^3+3t^2+11t+25}$$

and the Atkin-Lehner involution w_2 on t is given by $w_2 : t \mapsto -t$.

Proof. The existence of Hauptmoduls with the described values at CM-points follows immediately from Lemma 4.3.13. The fact that w_2 acts on t by $w_2 : t \mapsto -t$ also follows from the same lemma. We now determine the relation between Hauptmoduls.

The CM-point of discriminant -35 on $X_0^{35}(1)/W_{35}$ splits completely in the covering $X_0^{35}(2)/W_{35} \to X_0^{35}(1)/W_{35}$ and the three points lying above it are CM-points of discriminant -140 on $X_0^{35}(2)/W_{35}$. Replacing t by $-t$ if necessary, we may assume that the coordinates of these three points are the three zeros of $x^3 + 3x^2 + 11x + 25$. Considering CM-points of discriminant -7, we have

$$t_1 + 12 = \frac{A(t^2+7)(t-a)}{t^3+3t^2+11t+25} \tag{4.3}$$

for some constants A and a. The point $t = a$ is a CM-point of discriminant -28. Thus, the point $t = -a$ is the other CM-point of discriminant -28 and this point lies above the CM-point of discriminant -28 on $X_0^{35}(1)/W_{35}$. Therefore, we have

$$t_1 + \frac{4}{7} = \frac{B(t+a)(t-b)^2}{t^3+3t^2+11t+25} \tag{4.4}$$

for some constants B and b. Comparing (4.3) and (4.4), we find $A = 10$, $B = -10/7$, $a = -5$, and $b = 3$. It follows that

$$t_1 = -\frac{2(t-1)(t^2-6t+25)}{t^3+3t^2+11t+25}.$$

To check the correctness, we observe that the point t with $t^3 - 3t^2 + 11t - 25$ are lying above CM-points of discriminant -140 on $X_0^{35}(1)/W_{35}$. Thus, if we write $t_1^3 +$

$4t_1^2 + 144t_1 + 80$ as a rational function of t, then $t^3 - 3t^2 + 11t - 25$ should divide its numerator. Indeed, we find

$$t_1^3 + 4t_1^2 + 144t_1 + 80 = -\frac{200(t^3 - t^2 + 11t - 25)(t^3 - t^2 - 5t - 35)^2}{(t^3 + 3t^2 + 11t + 25)^3}$$

as expected. This proves the lemma. $\square$

Case $D = 39$

Lemma 4.3.15. *An equation for* $X_0^{39}(1)/\langle w_{13}\rangle$ *is*

$$y^2 = -(7x^2 + 23x + 19)(x^2 + x + 1)$$

with $w_3 = w_{39} : (x, y) \mapsto (x, -y)$*. Moreover, the coordinates of CM-points of discriminants* -52*,* -39*, and* -156 *are* $\left(\pm 2i, \pm\sqrt{13}(3 + 2i)\right)$*,* $((-1 \pm \sqrt{-3})/2, 0)$*, and* $((-23 \pm \sqrt{-3})/14, 0)$*, respectively.*

Proof. By [14], an equation for $X_0^{39}(1)$ is

$$y^2 = -(7x^4 + 79x^3 + 311x^2 + 497x + 277)(x^4 + 9x^3 + 29x^2 + 39x + 19)$$

with $w_{39} : (x, y) \mapsto (x, -y)$. Moreover, the coordinates of CM-points of discriminants -39 and -156 are $(\alpha_j, 0)$ and $(\beta_j, 0)$, $j = 1, \ldots, 4$, respectively, where α_j are the zeros of $x^4 + 9x^3 + 29x^2 + 39x + 19$ and β_j are the zeros of $7x^4 + 79x^3 + 311x^2 + 497x + 277$. Substituting x by $x - 2$, we obtain an equation

$$y^2 = -(7x^4 + 23x^3 + 5x^2 - 23x + 7)(x^4 + x^3 - x^2 - x + 1) \qquad (4.5)$$

with smaller coefficients. This hyperelliptic curve has an obvious automorphism $(x, y) \mapsto (-1/x, y/x^4)$. We will show that this is the Atkin-Lehner involution w_{13}.

The Atkin-Lehner w_{13} permutes the CM-points of discriminant -39. It also permutes the CM-points of discriminant -156. Thus, if w_{13} maps (x, y) to $((ax+b)/(cx+d), Cy/(cx + d)^4)$, then the constants a, b, c, and d must satisfy

$$(cx + d)^4 f_j\left(\frac{ax + b}{cx + d}\right) = C_j f_j(x)$$

for $f_1(x) = 7x^4 + 23x^3 + 5x^2 - 23x + 7$ and $f_2(x) = x^4 + x^3 - x^2 - x - 1$. We find w_{13} maps (x, y) to either $(-1/x, y/x^4)$ or $(-1/x, -y/x^4)$. The latter has no fixed points, so we conclude that w_{13} maps (x, y) to $(-1/x, y/x^4)$.

Now it is easy to show that $Y = y/x^2$ and $X = x - 1/x$ generate the function field of $X_0^{39}(1)/\langle w_{13}\rangle$. The relation between X and Y is also easy to find. It is

$$Y^2 = -(7X^2 + 23X + 19)(X^2 + X + 1), \qquad (4.6)$$

which gives us an equation for $X_0^{39}(1)/\langle w_{13}\rangle$. The coordinates of CM-points of discriminants -39 and -156 on $X_0^{39}(1)/\langle w_{13}\rangle$ are $\left((-1 \pm \sqrt{-3})/2, 0\right)$ and $\left((-23 \pm \sqrt{-3})/14, 0\right)$, respectively.

To determine the coordinates of CM-points of discriminant -52 on $X_0^{39}(1)/\langle w_{13}\rangle$, we first consider the CM-points of the same discriminant on $X_0^{39}(1)$. Since these points on $X_0^{39}(1)$ are the fixed points of w_{13} and on (4.5), the Atkin-Lehner involution w_{13} acts by $(x,y) \mapsto (-1/x, y/x^4)$, we find that the coordinates of CM-points of discriminant -52 on (4.5) are $\big(\pm i, \pm\sqrt{13}(3+2i)\big)$. This implies that the CM-points of discriminant -52 on $X_0^{39}(1)/\langle 13\rangle$ are $\big(\pm 2i, \pm\sqrt{13}(3+2i)\big)$. The proof of the lemma is complete. □

Lemma 4.3.16. *There is a Hauptmodul t_1 on $X_0^{39}(1)/W_{39}$ that takes values*

$$\pm 2i, \qquad \frac{-1\pm\sqrt{-3}}{2}, \qquad \frac{-23\pm\sqrt{-3}}{14}$$

at the CM-points of discriminants -52, -39, and -156, respectively. Also, there is a Hauptmodul t on $X_0^{39}(2)/W_{39}$ that takes values

$$\pm 3i, \qquad \frac{\pm 2\sqrt{-3}\pm\sqrt{-39}}{3}, \qquad \pm 1 \pm 2\sqrt{-3}$$

at the CM-points of discriminants -52, -39, and -156, respectively. Moreover, the relation between t_1 and t is

$$t_1 = -\frac{2(t^3+t^2+11t+3)}{(t^2+7)(t+3)}$$

and the Atkin-Lehner involution w_2 on t is $w_2 : t \mapsto -t$.

Proof. The existence of t_1 with the described properties follows from the previous lemma. Now let $s_1 = (t_1-2i)/(t_1+2i)$ so that s_1 takes values 0 and ∞ at the two CM-points of discriminant -52. Then the values of s_1 at the two CM-points of discriminant -156 are the zeros of

$$(9+46i)x^2 + 94x + (9-46i). \tag{4.7}$$

The covering $X_0^{39}(2)/W_{39} \to X_0^{39}(1)/W_{39}$ is ramified at $\mathrm{CM}(-52)\cup\mathrm{CM}(-156)$ of $X_0^{39}(1)/W_{39}$. There is a Hauptmodul s of $X_0^{39}(2)/W_{39}$ such that

$$s_1 = \frac{As(1-s)^2}{(1-as)^2}$$

for some complex numbers A and a. That is, s is determined by the property that it takes values 0 and 1 at the two points lying above the point $s_1 = 0$ with the point $s = 1$ having a ramification index 2 and value ∞ at the point lying above $s_1 = \infty$ with ramification index 1.

Now the condition that CM-points of discriminant -156 are ramified implies that the discriminant of

$$As(1-s)^2 - x(1-as)^2$$

as a polynomial in s must be divisible by the polynomial in (4.7). This gives two relations between A and a. Solving them for A and a, we find that the only legitimate choice is $A = 9 - 46i$ and $a = 13$. Then we have

$$t_1 = \frac{2i(s_1+1)}{-s_1+1} = \frac{4394is^3 + (-15548 - 5746i)s^2 + (2392 + 3926i)s - 92 + 18i}{(13s - 3 + 2i)(-169s^2 + (416 + 624i)s + 5 - 12i)}.$$

Let t be the Hauptmodul of $X_0^{39}(2)/W_{39}$ with

$$s = -\frac{3+2i}{13}\frac{(5+i)t + 3 - 15i}{(5-i)t + 3 + 15i}.$$

Then we have

$$t_1 = -\frac{2(t^3 + t^2 + 11t + 3)}{(t+3)(t^2+7)}.$$

The values of t at CM(-52), CM(-39), and CM(-156) can be read off from

$$t_1^2 + 4 = \frac{8(t^2+9)(t^2+2t+5)^2}{(t+3)^2(t^2+7)^2},$$

$$t_1^2 + t_1 + 1 = \frac{(t^2+2t+13)(3t^4+34t^2+27)}{(t+3)^2(t^2+7)^2},$$

and

$$7t_1^2 + 23t_1 + 19 = \frac{(t^2-2t+13)(t^2-6t+21)^2}{(t+3)^2(t^2+7)^2},$$

respectively. To determine the action of w_2 on t, we recall that w_2 switches the two points in CM(-52). It also exchanges the two zeros of $x^2 + 2x + 13$, corresponding to the two points in CM(-156) that lie above the CM-points of discriminant -39 on $X_0^{39}(1)/W_{39}$, with the two zeros of $x^2 - 2x + 13$, corresponding to the other two points in CM(-156) that lie above the CM-points of discriminant -156 on $X_0^{39}(1)/W_{39}$. From these informations, we can deduce that $w_2 : t \mapsto -t$. □

4.4 Schwarzian Differential Equations Associated to Shimura Curves of Genus Zero

Theorem 4.4.1. *Let $t = t_{D,N}$ the Hauptmoduls for $X_0^D(N)/W_D$ be chosen by Lemmas in Section 4.3. Then then automorphic derivatives $Q(t)$ associated to them are as follows. For $(D,N) = (6,1)$,*

$$Q(t) = \frac{108 - 113t + 140t^2}{576t^2(1-t)^2}.$$

For $(D,N) = (6,5)$,

$$Q(t) = -\frac{15(23 - 456t^2 + 1608t^4)}{2(2+3t^2)^2(1+64t^2)^2}.$$

For $(D,N)=(6,7)$,

$$Q(t)=-\frac{3(267+6480t^2+64352t^4)}{4(1+27t^2)^2(3+32t^2)^2}.$$

For $(D,N)=(6,13)$,

$$Q(t)=-\frac{3(12492+43272t^2+37541t^4)}{(9+16t^2)^2(16+27t^2)^2}.$$

For $(D,N)=(10,1)$,

$$Q(t)=\frac{3t^4-119t^3+3157t^2-7296t+10368}{16t^2(t-2)^2(t-27)^2}.$$

For $(D,N)=(10,3)$,

$$Q(t)=\frac{8-303t^2-1200t^4-95840t^6}{36t^2(1+32t^2)^2(1+5t^2)^2}.$$

For $(D,N)=(10,7)$,

$$Q(t)=-\frac{655+62410t^2+2237231t^4+35817920t^6+216522240t^8}{(1+27t^2)^2(1+20t^2)^2(5+128t^2)^2}.$$

For $(D,N)=(14,1)$,

$$Q(t)=\frac{192+440t+43t^2+1036t^3+960t^4}{16t^2(8+13t+16t^2)^2}.$$

For $(D,N)=(14,3)$,

$$Q(t)=-\frac{3(497-1988t^2+31494t^4+141436t^6+139601t^8)}{2(1+2t^2)^2(7+226t^2+343t^4)^2}.$$

For $(D,N)=(14,5)$,

$$Q(t)=-\frac{623+16772t^2+55178t^4-853468t^6+97503t^8}{(1+16t^2)^2(7+114t^2+7t^4)^2}.$$

For $(D,N)=(15,1)$,

$$Q(t)=\frac{177147-244944t+244242t^2-3680t^3+35t^4}{144t^2(1-t)^2(81-t)^2}.$$

For $(D,N)=(15,2)$,

$$Q(t)=\frac{3(385+5500t^2-2042t^4+35196t^6-2175t^8)}{4(1-t)^2(1+t)^2(1-5t)^2(1+5t)^2(5+3t^2)^2}.$$

For $(D,N)=(21,1)$,

$$Q(t)=\frac{21(40353607-17647350t+3561369t^2-477652t^3+31833t^4-630t^5+7t^6)}{16t^2(49-t)^2(343-94t+7t^2)^2}.$$

For $(D,N)=(21,2)$,

$$Q(t)=\frac{3(1-69t^2-4086t^4+23670t^6+6043653t^8+6781887t^{10})}{16t^2(1-t)^2(1+t)^2(1+27t^2)^2(1+63t^2)^2}.$$

For $(D,N)=(26,1)$,

$$Q(t)=\frac{85683+15210t+16694t^2-9480t^3+1363t^4-170t^5+12t^6}{16t^2(169+24t-19t^2+2t^3)^2}.$$

For $(D,N)=(26,3)$,

$$Q(t)=-\frac{6(85+3528t^2+60543t^4+552448t^6+2850579t^8+7990200t^{10}+9677785t^{12})}{(1+8t^2)^2(8+165t^2+1086t^4+2197t^6)^2}.$$

For $(D,N)=(35,1)$,

$$Q(t)=Q_1(t)/16(t+12)^2(7t+4)^2(t^3+4t^2+144t+80)^2,$$

where

$$\begin{aligned}Q_1(t)=&666427392t+1132800t^4+181420032-753984t^5+24576t^6+147t^8\\&+659096576t^2+85540864t^3+3808t^7.\end{aligned}$$

For $(D,N)=(35,2)$,

$$Q(t)=Q_1(t)/4(t^2+7)^2(t^2-25)^2(t^6+13t^4-29t^2-625)^2,$$

where

$$\begin{aligned}Q_1(t)=&2842805000t^2+91524600t^6-2082286t^8-217416t^{10}\\&+54644t^{12}+3784t^{14}+19t^{16}-992578125+1017474100t^4.\end{aligned}$$

For $(D,N)=(39,1)$,

$$Q(t)=\frac{-3Q_1(t)}{4(4+t^2)^2(1+t+t^2)^2(19+23t+7t^2)^2},$$

where

$$\begin{aligned}Q_1(t)=&2596+7104t+9692t^2+12348t^3+13149t^4+9522t^5\\&+4367t^6+1086t^7+97t^8.\end{aligned}$$

For $(D,N)=(39,2)$,

$$Q(t)=\frac{-9Q_1(t)}{4(9+t^2)^2(13+2t+t^2)^2(13-2t+t^2)^2(27+34t^2+3t^4)^2},$$

where

$$\begin{aligned}Q_1(t)=&419253003+119984328t^2+89200020t^4+43676088t^6+10194786t^8\\&+1272824t^{10}+87380t^{12}+3080t^{14}+43t^{16}.\end{aligned}$$

For these results, we take the Schwarzian differential equations associated to $X_0^{14}(1)/W_{14}$, $X_0^{14}(3)/W_{14}$, and $X_0^{14}(5)/W_{14}$ as examples for the proofs.

Proof. In Lemma 4.3.3, we see that there is a Hauptmodul t_1 on $X_0^{14}(1)/W_{14}$ with values ∞ at the elliptic point of order 4 and values 0, $(-13\pm 7\sqrt{-7})/32$ at the elliptic points of order 2. According to Proposition 4.1.2, the automorphic derivative $Q(t_1)$ associate to t_1 is

$$Q(t_1)=\frac{3}{16}-\frac{21+16B}{52t}+\frac{3(512t^2+416t-87)}{(16t^2+13t+8)^2}+\frac{4(21t+B(16t+13))}{13(16t^2+13t+8)},$$

for some constant B. We now use the covering $X_0^{14}(3)/W_{14}\longrightarrow X_0^{14}(1)/W_{14}$ to determine the constant B. More precisely, according to Proposition 4.1.4, we have the relation between $Q(t_1)$ and the automorphic derivative $Q(t)$ associative to a Hauptmodul t of $X_0^{14}(3)/W_{14}$,

$$Q(t)=D(t_1,t)+Q(t_1)/(dt_1/dt)^2.$$

Note that there is a Hauptmodul t for $X_0^{14}(3)/W_{14}$ that takes values $\pm 1/\sqrt{-2}$, $(\pm 9\sqrt{-7}\pm 4\sqrt{-14})/49$ at the 6 elliptic points of order 6. Thus, the automorphic derivative $Q(t)$ is

$$\begin{aligned}Q(t)=&\frac{3(2t^2-1)}{4(2t^2+1)^2}+\frac{3(18335t^2+38759t^4+117649t^6-791)}{4(7+226t^2+343t^4)^2}\\&+\frac{343(686C_4t^3+109C_3t^2+109C_4t+109C_5)}{436(7+226t^2+343t^4)}-\frac{1372C_4t+981+218C_3}{436(2t^2+1)},\end{aligned}$$

for some constants C_3, C_4, and C_5. Also, the action of the Atkin-Lehner involution w_3 on the Hauptmodul t is $w_3: t\mapsto -t$. Thus, by Proposition 4.1.5, the function $Q(t)$ satisfies

$$Q(t)=Q(-t),$$

and then we can get the value $C_4=0$.

Moreover, from the relation

$$t_1=\frac{4(1+2t^2)(1-5t)^2}{9(1+t)^4}$$

and Proposition 4.1.4,

$$Q(t)=D(t_1,t)+Q(t_1)/(dt_1/dt)^2,$$

we can find that

$$B = -\frac{373}{512},\ C_3 = -\frac{91}{9},\ \text{ and } C_5 = -\frac{1301}{3087}.$$

For the case of $X_0^{14}(5)/X_{14}$, the chosen Hauptmodul t takes values $\pm i/4$ at the elliptic points of order 4, $(\pm 5\sqrt{-7} \pm 4\sqrt{-14})/7$ at the elliptic points of order 2, and the action of Atkin-Lehner involution w_5 is $t \mapsto -t$. Therefore, the automorphic derivative associative to t is

$$\begin{aligned} Q(t) = & \frac{15(16t^2-1)}{2(16t^2+1)^2} + \frac{3(49t^6+399t^4+6351t^2-399)}{4(7t^4+114t^2+7)^2} \\ & - \frac{39+8B_1}{2(16t^2+1)} + \frac{7(B_1t^2+B_2)}{4(7t^4+114t^2+7)}, \end{aligned}$$

for some constants B_1 and B_2. From the relation

$$t_1 = -\frac{5(1-t+17t^2-13t^3)^2}{(1+16t^2)(1+3t)^4}$$

and Proposition 4.1.4, we can conclude that

$$Q(t) = -\frac{97503t^8 - 853468t^6 + 55178t^4 + 16772t^2 + 623}{(16t^2+1)^2(7t^4+114t^2+7)^2}.$$

□

Chapter 5

Applications of the Arithmetic of Automorphic Forms

From previous discussions, for a Shimura curve X having genus zero, we can use the solutions of the Schwarzian differential equations in place of automorphic forms (Chapter 4). Then we can do explicit computation on automorphic forms in terms of the solutions of the associated differential equations. This makes a powerful way to study the arithmetic properties of automorphic forms. For example, we can compute the Hecke operators on automorphic forms, modular equations for Shimura curves, determine the Hecke eigenforms and so on. In the paper [33], Yang computes Hecke operators on automorphic forms on Shimura curves $X_0^6(1)/W_6$ and on $X_0^{10}(1)/W_{10}$. He [31] also compute modular equations for Shimura curves.

A possible future work related to the arithmetic of automorphic forms on Shimura curves is Ramanujan-type series for Shimura curves. A typical example of Ramanujan-type identities for the classical modular curves is

$$\sum_{n=0}^{\infty} \frac{(6n+1)(1/2)_n^3}{(n!)^3} \left(\frac{1}{4}\right)^n = \frac{4}{\pi},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. It is known that such series is related to the Hecke theory of the classical modular curves and CM-theory. Natively, one expects that we can obtain Ramanujan-type series for Shimura curves. In the work of Yang [32], he gave several Ramanujan-type formulae for the Shimura curve $X_0^6(1)/W_6$. He also conjectures a general form for the Ramanujan-type identities for Shimura curves.

In this chapter, in support of his conjecture, we will numerically obtain Ramanujan-type identities for $X_0^{14}(1)/W_{14}$. However, we are not able to give a rigorous proof at present. In Section 5.1, we will compute Hecke operators on $X_0^{14}(1)/W_{14}$ and hence determine Hecke eigenforms. In Section 5.2, using the method developed in the previous chapter for obtaining bases of automorphic forms in terms of solutions of Schwarzian differential equations, we obtain Ramanujan-type identities for $X_0^{14}(1)/W_{14}$. This is mainly following the preprint [23].

5.1 Hecke Operators on $X_0^{14}(1)/W_{14}$

Assume that $\mathcal{O} = \mathcal{O}(D, N)$ is an Eichler order of level N in an indefinite quaternion algebra B of discriminant D. Fix an imbedding $\iota : B \longrightarrow M(2, \mathbb{R})$. Recall that for a given prime $p \nmid DN$ and $\alpha \in \mathcal{O}$ be such $N(\alpha) = p$,

$$T_p(f(\tau)) = p^{k/2-1} \sum_{\gamma \in \Gamma \backslash \Gamma \iota(\alpha) \Gamma} \frac{(\det \gamma)^{k/2}}{(c\tau + d)^k} f(\gamma\tau),$$

where $f(\tau)$ is an automorphic form of weight k on $\Gamma = \Gamma(\mathcal{O})$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Let t be the Hauptmodul of $X_0^{14}(1)/W_{14}$ with values ∞, $(-39 \pm 21\sqrt{-7})/16$ at the elliptic points of order 2, and with value 0 at the elliptic point of order 4. Let u be the Hauptmodul of $X_0^{14}(3)/W_{14}$ which is chosen so that it takes values $\pm 1/\sqrt{-2}$, and $\pm 9\sqrt{-7} \pm 4\sqrt{-14}/49$ at the CM-pints of discriminant -8, and -56, respectively. The relation between t and u is

$$t = f(u) = \frac{27(1+u)^4}{(1+2u^2)(1-5u)^2},$$

and the Atkin-Lehner involution w_3 sending u to $-u$. Then we can deduce that the Schwarzian differential equation associated to t is

$$\frac{d^2}{dt^2} f + \frac{3(64t^4 + 440t^3 + 129t^2 + 9324t + 25920)}{16t^2(8t^2 + 39t + 144)^2} f = 0.$$

Near the point P_4, the t-expansion of $t'(\tau)$ is the square of a linear combination of 2 solutions

$$g_1(t) = t^{3/8} \left(1 + \frac{131}{2304}t + \frac{21631}{3538944}t^2 - \frac{49745249}{29896998912}t^3 + \frac{16603576771}{91843580657664}t^4 + \cdots\right),$$

$$g_2(t) = t^{5/8} \left(1 + \frac{131}{3840}t + \frac{8923}{1966080}t^2 - \frac{257758957}{176664084480}t^3 + \frac{1646181570409}{9226105147883520}t^4 + \cdots\right)$$

of the Schwarzian differential equation associated to t.

Lemma 5.1.1. *Let g_1, g_2 be the functions given as above. We have*

$$\frac{\tau - P_4}{\tau - \overline{P_4}} = C\frac{g_2}{g_1} \quad \text{and} \quad t'(\tau) = -\frac{4\,(g_1 - Cg_2)^2}{C(P_4 - \overline{P_4})},$$

where P_4 denote the elliptic point of order 4 on the curve $X_0^{14}(1)/W_{14}$, and C is certain complex number.

Proof. Note that the functions $t'(\tau)^{1/2}$ and $\tau t'(\tau)^{1/2}$, as functions of t, satisfy the Schwarzian differential equation associated to t. Thus, there exist some constants a', b', c', d', so that

$$\tau = \frac{a'g_1 + b'g_2}{c'g_1 + d'g_2}$$

and hence

$$\frac{\tau - P_4}{\tau - \overline{P_4}} = \frac{ag_1 + bg_2}{cg_1 + dg_2}, \quad a, b, c, d \in \mathbb{C}.$$

On the other hand, we let γ denote a generator of the isotropy subgroup for P_4, then we have

$$t(\gamma\tau)^{1/4} = \zeta_4 t(\tau)^{1/4} \quad \text{and} \quad \frac{\gamma\tau - P_4}{\gamma\tau - \overline{P_4}} = \zeta_4 \frac{\tau - P_4}{\tau - \overline{P_4}},$$

for some primitive fourth root of unity ζ_4. Therefore, we can get that

$$\frac{\tau - P_4}{\tau - \overline{P_4}} = C\frac{g_2}{g_1}.$$

From this identity, we can get

$$\tau = (P_4 g_1 - \overline{P_4} C g_2)/(g1 - Cg2)$$

and

$$\frac{d\tau}{dt} = C(P_4 - \overline{P_4})\frac{g_1 dg_2/dt - g_2 dg_1/dt}{(g_1 - Cg_2)^2} = -\frac{C(P_4 - \overline{P_4})}{4(g_1 - Cg_2)^2}.$$

□

Then we can give a concrete basis for space $S_k(\Gamma)$. According to the Corollary 4.1.3, an automorphic form of weight $2k$ on $X_0^{14}(1)/W_{14}$ can be written as a linear combination of

$$t^{j-\lfloor 3k/4 \rfloor}\left(t^2 + \frac{39}{8}t + 18\right)^{-\lfloor k/2 \rfloor}(g_1(t) - Cg_2(t))^{2k} \tag{5.1}$$

with the constant C in Lemma 5.1.1, where $j = 0, \ldots, 1 - 2k + 3\lfloor k/2 \rfloor + \lfloor 3k/4 \rfloor$.

We now compute Hecke operators T_3 on the space $S_k(\Gamma)$ of automorphic forms on $X_0^{14}(1)/W_{14}$ relative to the basis given in 5.1. We first consider the case of T_3.

Let $B = \left(\frac{-1,7}{\mathbb{Q}}\right)$ be a quaternion algebra defined over $\mathbb{Q}$ of discriminant 14 which is generated by I and J with the relations $I^2 = -1$, $J^2 = 7$, $IJ = -JI$. Fix the maximal order to be $\mathcal{O} = \mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}(1 + I + J + IJ)/2$ and choose the embedding $\iota : B \longrightarrow M(2, \mathbb{R})$ to be $I \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $J \mapsto \begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix}$.

The curve $X_0^{14}(1)$ has 3 elliptic points of order 2 and an elliptic point of order 4. We choose the representatives of elliptic point of order 4 by $P_4 = i$ with the isotropy subgroup generated by $M_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Let A be the matrix $\frac{1}{2}\begin{pmatrix} 5+\sqrt{7} & -1+\sqrt{7} \\ 1+\sqrt{7} & 5-\sqrt{7} \end{pmatrix}$, which is the image of the element $2 + (1 + I + J + IJ)/2$ of reduced norm 3 in $\mathcal{O}$ under the embedding ι. A complete set of right coset representatives of $\Gamma \setminus \Gamma A \Gamma$ is given by

$$\gamma_0 = \frac{1}{2}\begin{pmatrix} 5 + \sqrt{7} & -1 + \sqrt{7} \\ 1 + \sqrt{7} & 5 - \sqrt{7} \end{pmatrix}, \quad \text{and} \quad \gamma_j = \gamma_0 M_4^j, \quad j = 1, 2, 3.$$

Then $\gamma_j P_4 = AP_4 = (5\sqrt{7} + 7 + 5i + \sqrt{-7})/12$. For these coset representatives, we can easily verify the following property.

Lemma 5.1.2. *Letting* $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$, *we have*

$$a_j = c_j P_4 + d_j = e^{2\pi ij/8}(5 + i + \sqrt{-7} - \sqrt{7})/2, \quad j = 0, 1, 2, 3.$$

To compute Hecke operator T_3 on the selected basis for $S_k(\Gamma)$, our goal is to determine the t-expansions of

$$t(\gamma_j\tau), \quad F(\gamma_j\tau), \quad \text{and } c_j\tau + d_j, \quad j = 0, \dots, 3,$$

where $F(t) = (g_1(t) - Cg_2(t))^2$.

Here, we will use the modular equation of level 3 to help us to decide the t-expansions of $t(\gamma_j\tau)$.

Lemma 5.1.3. *Let* γ_j, $j = 0, 1, 2, 3$, *be the coset representatives given above. In a neighborhood of* P_4, *the* t*-expansion of* $t(\gamma_j\tau)$ *is given by*

$$t(\gamma_j\tau) = \frac{9}{4} + (-i)^j\frac{33}{4}t^{1/4} + (-1)^j\frac{229}{16}t^{1/2} + i^j\frac{1897}{96}t^{3/4} + \frac{1791}{64}t + (-i)^j\frac{531689}{13824}t^{5/4} + \cdots$$

In particular, we have $t(\gamma_j P_4) = 9/4$.

Proof. At the beginning, let us consider the Hauptmoduls u and t we mentioned before. Note that the relation between t and u is

$$t = f(u) = \frac{27(1+u)^4}{(1+2u^2)(1-5u)^2},$$

and the action of w_3 is $u \mapsto -u$. Thus, we have

$$s = f(-u) = \frac{27(1-u)^4}{(1+2u^2)(1+5u)^2},$$

and the polynomial

$$\begin{aligned}\Phi_3(s,t) =& 924210st(s+t) - 2304(s^4+t^4) + 20736(s^3+t^3)\\ &- 8750s^3t^3 + 260415s^2t^2 + 193104st(s^2+t^2)\\ &- 5625s^2t^2(s^2+t^2) + 7200(st^4+s^4t) - 69984(s^2+t^2)\\ &- 10350(s^2t^2)(s+t) + 104976(s+t) + 1557954st - 59049.\end{aligned}$$

Solving the modular equation $\Phi_3(s,t) = 0$ for s, we find the 4 roots are

$$s_j = \frac{9}{4} + \zeta_4^j\frac{33}{4}t^{1/4} + \zeta_4^{2j}\frac{229}{16}t^{1/2} + \zeta_4^{3j}\frac{1897}{96}t^{3/4} + \frac{1791}{64}t + \zeta_4^j\frac{531689}{13824}t^{5/4} + \cdots$$

for $j = 0, 1, 2, 3$, where ζ_4 is a primitive fourth root of unity. The fourth root $t^{1/4} = t^{1/4}(\tau)$ of $t(\tau)$ is defined in a neighborhood of P_4 so that it becomes a holomorphic function of τ near P_4.

In view of $t(M_4\tau) = t(\tau)$, one has $t(M_4\tau)^{1/4} = \zeta t(\tau)^{1/4}$, for some fourth root of unity ζ. Note that the function $\tau \longrightarrow t(\tau)$ preserves orientation and is locally 4-to-1 at

P_4, hence the number ζ is actually $-i$. Without loss of generality, may assume that the expansion of $t(\gamma_0\tau)$ is s_0, and then we have

$$t(\gamma_j\tau) = t\left(\left(\gamma_0 M_4^j\right)\tau\right) = t\left(\gamma_0\left(M_4^j\tau\right)\right) = s_j$$

with $\zeta_4 = \zeta = -i$, for $j = 1, 2, 3$. □

Corollary 5.1.4. *We have the equality, for each j,*

$$\frac{33}{4}\zeta_4^j = \frac{12F\left(t(\gamma_j P_4)\right)}{(c_j P_4 + d_j)^2} = \frac{12F(9/4)}{(c_j P_4 + d_j)^2}.$$

Proof. Assume that the t-expansion of $t(\gamma_0\tau)$ is $A_0 + A_1 t^{1/4} + \cdots$. According to Lemma 5.1.1, we have

$$A_1 = \lim_{\tau\to P_4}\frac{t(\gamma_0\tau) - A_0}{g_2(t(\tau))/g_1(t(\tau))} = \lim_{\tau\to P_4}\frac{t(\gamma_0\tau) - A_0}{(\tau - P_4)/C(\tau - \overline{P_4})}.$$

By L'Hopital rule and Lemma 5.1.1, the equality becomes

$$A_1 = \frac{C(P_4 - \overline{P_4})(\det\gamma_0)t'(\gamma_0 P_4)}{(c_0 P_4 + d_0)^2} = \frac{12F(9/4)}{(c_0 P_4 + d_0)^2}.$$

From Lemma 5.1.3, we have

$$\frac{33}{4} = \frac{12F(9/4)}{(c_0 P_4 + d_0)^2}.$$

Using the same arguments, we can get the identity in this Corollary. □

As we see in the previous proof, if we suppose that the t-expansion of $F(\gamma_0\tau)$ is $\sum_{n=0}^{\infty} B_n t^{n/4}$, for some constants B_n, then we have

$$F(\gamma_j\tau) = F(t(\gamma_j\tau)) = \sum_{n=0}^{\infty} B_n(-i)^{nj}t^{n/4}, \quad j = 0, 1, 2, 3.$$

Also, from the above results, we can figure that the constant term B_0 is the value of $F(\gamma_j\tau)$ at $\tau = P_4$ for each coset representatives γ_j.

Corollary 5.1.5. *The constant term B_0 of the t-expansions of $F(\gamma_j\tau)$ is equal to $F(9/4)$, that is*

$$B_0 = \frac{66 - 33\sqrt{7} + (22\sqrt{7} - 11)i}{16}.$$

Proof. This can be easily verified form the Lemma 5.1.2 and Corollary 5.1.4. □

We then determine other coefficients B_n inductively. Denote by f_{2k} the automorphic form

$$t^{-\lfloor 3k/4\rfloor}\left(t^2+\frac{39}{8}t+18\right)^{-\lfloor k/2\rfloor}(g_1(t)-Cg_2(t))^k$$

of weight $2k$ in the equation (5.1). Observe that their expansions near P_4 are

$$\begin{aligned}
f_4 =& \frac{1}{18}t^{1/2}-\frac{2}{9}Ct^{3/4}+\frac{1}{3}C^2t-\frac{2}{9}C^3t^{5/4}+\left(\frac{1}{18}C^4-\frac{25}{10368}\right)t^{3/2}+\cdots,\\
f_8 =& \frac{1}{324}-\frac{2}{81}Ct^{1/4}+\frac{7}{81}C^2t^{1/2}-\frac{14}{81}C^3t^{3/4}+\left(\frac{35}{162}C^4-\frac{25}{93312}\right)t+\cdots,\\
f_{12} =& \frac{1}{5832}t^{1/2}-\frac{1}{486}Ct^{3/4}+\frac{11}{972}C^2t-\frac{55}{1458}C^3t^{5/4}-\frac{25}{1119744}t^{3/2}+\cdots,\\
f_{14} =& \frac{1}{5832}t^{1/4}-\frac{7}{2916}Ct^{1/2}+\frac{91}{5832}C^2t^{3/4}-\frac{91}{1458}C^3t+\cdots,\\
f_{18} =& \frac{1}{104976}t^{3/4}-\frac{1}{5832}Ct+\frac{17}{11664}C^2t^{5/4}-\frac{17}{2187}C^3t^{3/2}+\cdots,
\end{aligned}$$

we can use the basis of $S_k(\Gamma)$ described in equation (5.1) to get the coefficients B_n as the followings

$n \bmod 4$	0	1	2	3
k	8	14	$4, 12$	18

It is easier that if we use basis of $S_4(\Gamma)$ than if we use automorphic forms of weight 12 to compute the B_n with $n \equiv 2 \bmod 4$. Note that

$$\dim S_6(\Gamma)=\dim S_{10}(\Gamma)=0, \quad \dim S_{16}(\Gamma)=3,$$

and the t-expansion of f_{16} starts from a nonzero constant term, so we omit their expansions here.

For the purpose to determine the expansion $F(\gamma_0\tau)$, i.e. the number B_n, we first use Jacquet-Langlands correspondence to decide the representative matrix of T_3 on $S_k(\Gamma)$ with respect to the chosen basis.

Lemma 5.1.6. *For $k=4$, 8, 14, 18, let $F_{k,i}$, $k=1\dots d_k$ in (5.1) be the automorphic forms of weight k on Γ that spans the space $S_k(\Gamma)$. Then the representative matrices of T_3 with respect to $\{F_{k,i}\}_{i=1}^{d_k}$ are*

k	4	8	14	18
T_3	-2	$\begin{pmatrix} 48 & 50 \\ 108 & 22\end{pmatrix}$	-1026	4626

Proof. According to the Jacquet-Langlands correspondence,

$$S_k(\Gamma)\simeq S_k^{\text{new}}(\Gamma_0(14),-1,-1),$$

where $S_k^{\text{new}}(\Gamma_0(14),-1,-1)$ is the subspace of $S_k^{\text{new}}(\Gamma_0(14))$ with eigenvalues -1 for both w_2 and w_7. For $k=4$, the space $S_k^{\text{new}}(\Gamma_0(14))$ is of dimension 2, and the subspace $S_k^{\text{new}}(\Gamma_0(14),-1,-1)$ is spanned by the eigenform

$$f=q+2q^2-2q^3+4q^4-12q^5-4q^6+7q^7+8q^8-23q^9-24q^{10}+\cdots.$$

Thus, the eigenvalue of T_3 respect f is -2, the third coefficient of f. Here, we use the algebra computation system MAGMA to find the Hecke eigenforms. The eigenvalues of T_3 for the case $k = 14$, $k = 18$, can be determined in the same way.

For the case $k = 8$, the subspace $S_k^{\text{new}}(\Gamma_0(14), -1, -1)$ is 2 dimensional and spanned by

$$f = q+8q^2+aq^3+64q^4+(378-9a)q^5+8aq^6+343q^7+512q^8+(70a-1443)q^9+\cdots$$

with $a^2-70a = 744$ and its Galois conjugate. Therefore, the characteristic polynomial of the operator T_3 with respect to our basis for $S_k(\Gamma)$ is $x^2 - 70x - 744$. That is, the trace of the operator is 70, and its determinant is -744.

Note that the space $S_8(\Gamma)$ is spanned by

$$F_{8,1}(t) = \frac{1}{t^3(t^2+(39/8)t+18)^2}(g_1(t) - Cg_2(t))^4, \quad \text{and} \quad F_{8,2}(t) = tF_{8,1}(t).$$

The operator T_3 acts on $F_{8,1}$ and $F_{8,2}$ becoming

$$\begin{aligned} 3^7\sum_{j=0}^{3}\frac{F_{8,1}(t(\gamma_j\tau))}{(c_j\tau+d_j)^8} &= aF_{8,1}(t(\tau)) + bF_{8,2}(t(\tau)), \\ 3^7\sum_{j=0}^{3}\frac{F_{8,2}(t(\gamma_j\tau))}{(c_j\tau+d_j)^8} &= cF_{8,1}(t(\tau)) + dF_{8,2}(t(\tau)), \end{aligned}$$

the characteristic polynomial of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $x^2-70x-744$. Hence, the number d is equal to $70-a$.

Observe that the t-expansion of $F_{8,1}(t(\tau))$ is

$$\frac{1}{324} - \frac{2}{81}Ct^{1/4} + \frac{7}{81}C^2t^{1/2} - \frac{14}{81}C^3t^{3/4} - \left(\frac{25}{93312} - \frac{35}{162}C^4\right)t + \cdots.$$

If we evaluate the values at the point $\tau = P_4$, the Lemma 5.1.3 tells us that $t(\gamma_j P_4) = 9/4$, and we then have the equations

$$\begin{aligned} 3^7\sum_{j=0}^{3}\frac{F_{8,1}(9/4)}{(c_jP_4+d_j)^8} &= a/324 \\ 3^7\sum_{j=0}^{3}\frac{F_{8,2}(9/4)}{(c_jP_4+d_j)^8} &= \frac{9}{4}\left(3^7\sum_{j=0}^{3}\frac{f_{8,1}(9/4)}{(c_jP_4+d_j)^8}\right) = c/324. \end{aligned}$$

These imply that $c = 9a/4$. We now determine the value a. Since

$$3^7\sum_{j=0}^{3}\frac{F_{8,1}(9/4)}{(c_jP_4+d_j)^8} = \frac{2^{16}3^7}{3^{10}11^4}\sum_{j=0}^{3}\left(\frac{F(9/4)}{(c_jP_4+d_j)^2}\right)^4,$$

according to Corollary 5.1.4, we have

$$3^7\sum_{j=0}^{3}\frac{F_{8,1}(9/4)}{(c_jP_4+d_j)^8} = 4\frac{2^{16}3^7}{3^{10}11^4}\left(\frac{33}{4}\right)^4\frac{1}{12^4} = \frac{4}{27}.$$

In a word, we have the identity

$$\frac{4}{27} = 3^7 \sum_{j=0}^{3} \frac{F_{8,1}(9/4)}{(c_j P_4 + d_j)^8} = \frac{a}{324}.$$

Hence the number a must be 48, and $c = 108$. Together with the fact that the characteristic polynomial of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $x^2 - 70x - 744$, we can find that the representative matrix of T_3 with respect to the basis $\{F_{8,1}, F_{8,2}\}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 48 & 50 \\ 108 & 22 \end{pmatrix}.$$

□

For $k = 4,\ 14,\ 18$, let λ_k be the eigenvalue for T_3 given in previous Lemma, we have

$$3^{k-1} \sum_{j=0}^{3} \frac{f_k\left(t(\gamma_j \tau)\right)}{(c_j \tau + d_j)^k} = \lambda_k f_k(t(\tau));$$

for $k = 8$, we have the equality

$$3^7 \sum_{j=0}^{3} \frac{f_8\left(t(\gamma_j \tau)\right)}{(c_j \tau + d_j)^8} = (48 + 50t) f_8(t(\tau)).$$

Write

$$f_k(t) = d_k(t) F(t)^{k/2}, \quad d_k(t) = t^{-\lfloor 3k/4 \rfloor} \left(t^2 + \frac{39}{8} t + 18 \right)^{-\lfloor k/2 \rfloor}.$$

Now the t-expansions of τ and $t(\gamma_j \tau)$ are known in Lemma 5.1.1 and 5.1.3, so the part of $d_k(t(\gamma_j \tau))/(c_j \tau + d_j)^k$ can be work out. Also, we have the constant B_0 of the expansion of $F(\gamma_j \tau)$ near P_4 by Corollary 5.1.5. Thus, using these information, we can determine the other coefficients B_n of the expansion

$$F(\gamma_0 \tau) = B_0 + \sum_{n \geq 1} B_n t^{n/4}$$

inductively. Then the expansions of

$$F(\gamma_j \tau) = B_0 + \sum_{n \geq 1} B_n (-i)^{nj} t^{n/4}$$

near P_4 can be determined consequently. Here, let us list first 12 coefficients of the t-expansion of $F(\gamma_0 \tau)$. In the followings, we denote $M_1 = \left(6 - 3\sqrt{7} - i + 2\sqrt{-7}\right)$, $M_2 = \left(5 - \sqrt{7}\right) C$, and $M_3 = \left(6 - 3\sqrt{7} + i - 2\sqrt{-7}\right) C^2$.

$$
\begin{aligned}
B_0 &= \frac{11}{16} M_1, \\
B_1 &= \frac{229}{96} M_1 - \frac{11}{4} M_2, \\
B_2 &= \frac{1897}{384} M_1 - \frac{229}{24} M_2 + \frac{11}{16} M_3, \\
B_3 &= \frac{597}{64} M_1 - \frac{1897}{96} M_2 + \frac{229}{96} M_3, \\
B_4 &= \frac{1345607}{82944} M_1 - \frac{597}{16} M_2 + \frac{1897}{384} M_3, \\
B_5 &= \frac{8577605}{331776} M_1 - \frac{13443101}{207360} M_2 + \frac{597}{64} M_3, \\
B_6 &= \frac{427949389}{10948608} M_1 - \frac{10699507}{103680} M_2 + \frac{3357533}{207360} M_3,
\end{aligned}
$$

$$
\begin{aligned}
B_7 &= \frac{1249481879}{21897216} M_1 - \frac{8534385587}{54743040} M_2 + \frac{42708031}{1658880} M_3, \\
B_8 &= \frac{156151317775}{1926955008} M_1 - \frac{1556045479}{6842880} M_2 + \frac{4254891697}{109486080} M_3, \\
B_9 &= \frac{280396875558295}{2497333690368} M_1 - \frac{18652796644997}{57808650240} M_2 + \frac{6200954437}{109486080} M_3, \\
B_{10} &= \frac{3139891380163495}{20603002945536} M_1 - \frac{17432924774791}{39020838912} M_2 + \frac{5802349183013}{72260812800} M_3, \\
B_{11} &= \frac{111888301668967277}{54941341188096} M_1 - \frac{249723804965137451}{412060058910720} M_2 + \frac{6936494964167563}{62433342259200} M_3.
\end{aligned}
$$

This is enough to compute the Hecke operator T_3 for general automorphic forms on $X_0^{14}(1)/W_{14}$for general weights.

For computing Hecke operators T_p with prime $p \geq 5$, we can deduce the eigenvalues form T_3 and Jacquet-Langlands correspondence. For example, from the Jacquet-Langlands correspondence, the subspace $S_8^{\text{new}}(\Gamma_0(14), -1, -1)$ is 2 dimensional and spanned by

$$f = q + 8q^2 + aq^3 + 64q^4 + (378 - 9a)q^5 + 8aq^6 + 343q^7 + 512q^8 + (70a - 1443)q^9 + \cdots$$

with $a^2 - 70a = 744$ and its Galois conjugate. The eigenvalue for T_7 is 343. According to the Lemma 5.1.6, the matrix for T_5 relative to our basis of automorphic forms of weight 8 is

$$378 - 9\begin{pmatrix} 48 & 50 \\ 108 & 22 \end{pmatrix} = \begin{pmatrix} -54 & -450 \\ -972 & 180 \end{pmatrix}.$$

5.2 Ramanujan-type Formulae

Recall that if E is an elliptic curve defined over $\overline{\mathbb{Q}}$, which has CM by an imaginary quadratic field K of discriminant d, then up to an algebraic factor, the period of E can be expressed by

$$\Omega_d = \sqrt{\pi} \prod_{0<a<|d|} \Gamma\left(\frac{a}{|d|}\right)^{w_d\chi_d(a)/4h_d},$$

where w_d is the number of roots of unity in K, χ_d is the Kronecker character $\left(\frac{d}{\cdot}\right)$ associated to K, and h_d is the class number of K. In [32], Yang contributes many Ramamnujan-type series. For example,

$$\sum_{n=0}^{\infty}\left(74480n+\frac{6860}{3}\right)\frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!}\left(\frac{-7^4}{3375}\right)^n = 7^3\sqrt{5}\sqrt[4]{3375}\frac{4\pi}{\sqrt[4]{12}\Omega_{-4}^2},$$

which is related to the period of an elliptic curve with CM by $\mathbb{Q}(\sqrt{-1})$. The power series

$$\sum_{n=0}^{\infty}\frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!}t^n$$

mentioned above is the hypergeometric function

$${}_3F_2\left(\frac{1}{12},\frac{1}{4},\frac{5}{12};\frac{1}{2},\frac{3}{4};t\right) = {}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};t\right)^2.$$

Note that the function ${}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};t\right)$ is related to the Schwarzian differential equation associated to the Hauprmodul t of $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, and -3, respectively. Yang also gave other similar identities related to Ω_{-4}, and also the Ramanujan-type series related to Ω_{-3} for the curve $X_0^6(1)/W_6$.

In this article [32], he guess, in general, we can use the t-series expansion of a meromorphic form to obtain the Ramanujan-type identities, which are related to certain periods of elliptic curves with CM. That is, we may have

$$\sum_{n=0}^{\infty}(R_1 n + R_2)A_n t_0^n = R_3\frac{\pi}{\Omega_d^2},$$

where R_1, R_2, $R_3 \in \overline{\mathbb{Q}}$, $\sum_0^\infty A_n t^n$ is the expansion of a meromorphic automorphic form of weight 2 with respect to a Hauptmodul t of a Shimura curve of genus zero such that t takes value 0 at a CM-point of discriminant d, and t_0 is the value of t at some CM-point of discriminant $d' \neq d$. To be more precise, if g_1, g_2 are 2 linearly independent solutions of a given Schwarzian differential equation associated to a Shimura curve of genus 0. Write $g_1^2 = \sum_0^\infty A_n t^n$ and $g_2^2 = \sum_0^\infty B_n t^n$, then we expect that

$$\sum_{n=0}^{\infty}(R_1 n + R_2)\,A_n t_0^n = R_3\frac{\pi}{\Omega_d^2}$$
$$\sum_{n=0}^{\infty}(R_1 n + R_2 + R_1/a)\,B_n t_0^n = R_3\frac{\Omega_d^2}{\pi},$$

for certain positive integer a. We remark that the series also converge p-adically for the prime $p \mid M$ while $t_0 = M/N$. The p-adic numbers which they converge to should be related to the p-adic periods of certain elliptic curves with CM. It is natural to expect that those p-adic identities should be related to the p-adic periods of elliptic curves with CM. Yang also gave some numerical examples of the p-adic analogues for the Ramanujan-type series obtained from $X_0^6(1)/W_6$. Here, let us see some numerical examples coming from $X_0^{14}(1)/W_{14}$.

From the Lemma 4.3.3, we know that there is a Hauptmodul t for $X_0^{14}(1)/W_{14}$ that takes values ∞, 0, and $(-13 \pm 7\sqrt{-7})/32$ at CM-points of discriminants -4, -8, and -56, respectively. The t-series expansions of 2 linearly independent solutions of the Schwarzian differential equation associated to t (see Theorem 4.4.1),

$$\frac{d^2}{dt^2}f + Q(t)f = 0, \quad Q(t) = \frac{192 + 440t + 43t^2 + 1036t^3 + 960t^4}{16t^2(8 + 13t + 16t^2)^2},$$

are

$$g_1 = t^{1/4}\left(1 + \frac{23}{64}t + \frac{1867}{8192}t^2 - \frac{955937}{2621440}t^3 + \frac{157030847}{671088640}t^4 + \frac{3694251053}{42949672960}t^5 + \dots\right)$$
$$g_2 = t^{3/4}\left(1 + \frac{23}{192}t + \frac{3149}{24576}t^2 - \frac{434593}{1572864}t^3 + \frac{264972083}{1207959552}t^4 + \frac{39014127761}{850403524608}t^5 + \dots\right).$$

The Hauptmodule t takes values $t_0 = -13/81$ at the CM-points of discriminants -91 (This is given by Elkies [6]). We now let

$$\sum_{n=0}^{\infty} A_n = t^{-1/2}g_1^2, \quad \sum_{n=0}^{\infty} B_n = t^{-3/2}g_2^2,$$

and

$$C = \frac{81}{2548}\frac{\Gamma(5/8)\Gamma(7/8)}{\Gamma(1/8)\Gamma(3/8)} = \frac{81}{2548}\Omega_{-8}^2/\pi.$$

In this case, our numerical computation checked for 100-digits gives us that

$$\left(\sum_{n=0}^{\infty} R_1 n + R_2\right) A_n t_0^n = \frac{847}{18} 13^{3/4} 3C, \tag{5.2}$$

$$\left(\sum_{n=0} \infty R_1 n + R_1 + R_2\right) B_n t_0^n = \frac{847}{18} 13^{1/4} 27C^{-1}. \tag{5.3}$$

If we choose a Hauptmodule t that takes values 0, ∞, and $(-39 \pm 21\sqrt{-7})/16$ at CM-points of discriminants -4, -8, and -56, respectively. The Schwarzian differential equation associated to t is given by

$$\frac{d^2}{dt^2}f + Q(t)f = 0, \quad Q(t) = \frac{3(64t^4 + 440t^3 + 129t^2 + 9324t + 25920)}{16t^2(8t^2 + 39t + 144)^2},$$

and its 2 linearly independent solutions are

$$g_1 = t^{3/8}\left(1+\frac{131}{2304}t+\frac{21631}{3538944}t^2-\frac{49745249}{29896998912}t^3+\frac{16603576771}{91843580657664}t^4+\dots\right)$$
$$g_2 = t^{5/8}\left(1+\frac{131}{3840}t+\frac{8923}{1966080}t^2-\frac{257758957}{176664084480}t^3+\frac{646181570409}{9226105147883520}t^4+\dots\right).$$

The Hauptmodule t takes values $t_0 = 27/200$ at the CM-points of discriminants -168. Let

$$\sum_{n=0}^{\infty} C_n = t^{-3/4}g_1^2, \quad \sum_{n=0}^{\infty} D_n = t^{-5/4}g_2^2.$$

We have

$$\sum_{n=0}^{\infty}(R_1 n + R_2)\,C_n t_0^n = \frac{810000}{11^8}27^{3/4}200^{1/4}C,$$
$$\sum_{n=0}^{\infty}(R_1 n + R_2 + R_1/2)\,D_n t_0^n = \frac{810000}{11^8}27^{1/4}200^{3/4}C^{-1}$$

with $R_1 = 2904$, $R_2 = 12$, where

$$C = \frac{\Gamma(3/4)^2}{\Gamma(1/4)^2}\left(\frac{196}{3}\right)^{1/4} = \left(\frac{196}{3}\right)^{1/4}\Omega_{-4}^2/\pi.$$

Let $\Gamma_p(\cdot)$ stand for the p-adic Gamma function. The numerical results checked for 70 p-adic digits provide us that

$$\sum_{n=0}^{\infty}(R_1 n + R_2)\,C_n t_0^n = \frac{2^4\cdot 11^8}{9}\left(27^3 200\frac{98\Gamma_3(1/4)}{27\Gamma_3(3/4)}\right)^{1/4},$$
$$\sum_{n=0}^{\infty}(R_1 n + R_2 + R_1/2)\,D_n t_0^n = \frac{2^4\cdot 11^8}{9}\left(27\cdot 200^3\frac{27\Gamma_3(3/4)}{98\Gamma_3(1/4)}\right)^{1/4},$$

hold 3-adically with $R_1 = 29040$ and $R_2 = 120$.

For the numbers $\sum nA_n t_0^n$, $\sum A_n t_0^n$, $\sum nB_n t_0^n$, and $\sum B_n t_0^n$, after numerical computation, we can find that the equalities

$$\left(\sum_{n=0}^{\infty}(11011n + 7290)\,A_n t_0^n\right)^2 = 3^3\cdot 7\cdot 137\cdot 1571\frac{\Gamma_{13}(5/8)\Gamma_{13}(7/8)}{2\Gamma_{13}(1/8)\Gamma_{13}(3/8)},$$
$$\left(\sum_{n=0}^{\infty}(11011n + 75897)\,B_n t_0^n\right)^2 = 3^{12}\cdot 7\cdot 11^4\frac{\Gamma_{13}(1/8)\Gamma_{13}(3/8)}{8\Gamma_{13}(5/8)\Gamma_{13}(7/8)},$$

hold 13-adically.

Chapter 6

Algebraic Transformations of Hypergeometric Functions Arising from Theory of Shimura Curves

For real numbers a, b, c with $c \neq 0, -1, -2, \ldots$, the ${}_2F_1$-hypergeometric function (Gaussian hypergeometric function) is defined by the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

for $z \in \mathbb{C}$ with $|z| < 1$, where

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\ldots(a+n-1), & \text{if } n \geq 1, \end{cases}$$

is the Pochhammer symbol. The hypergeometric function ${}_2F_1(a, b; c; z)$ is a solution of the differential equation

$$\theta(\theta + c - 1)F - z(\theta + a)(\theta + b)F = 0, \qquad \theta = z\frac{d}{dz}.$$

This is a Fuchsian equation on the complex projective line with precisely 3 regular singular points at $z = 0, 1, \infty$ with local exponents $\{0, 1 - c\}$, $\{0, c - a - b\}$, and $\{a, b\}$, respectively.

Using the well-known fact in the classical analysis that a second-order linear ordinary differential equation with three regular singularities at 0, 1, ∞ is completely determined by the local exponents, one can easily deduce Euler's identity

$${}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right)$$

(among many other similar identities). Since the function $z/(z-1)$ is a rational function of degree 1 of z, we call this identity an algebraic transformation of degree 1 of hypergeometric functions. In this chapter, we are concerned with algebraic transformations of hypergeometric functions, that is, identities of the form

$$ {}_2F_1(a,b;c;z) = R(z)\,{}_2F_1\left(a',b';c';S(z)\right) \tag{6.1} $$

with suitable parameters a, b, c, a', b', c' and algebraic functions $R(z)$ and $S(z)$. If $w = R(z)$ is of degree m over the field $\mathbb{C}(z)$ or if z is of degree m over the field $\mathbb{C}(w)$, we say the algebraic transformation has *degree* m.

Beyond transformations of degree 1, one of the simplest examples is Kummer's quadratic transformation

$$ {}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4z(1-z)\right), \tag{6.2} $$

valid for any real numbers a, b with $a + b + 1/2 \neq 0, -1, -2, \ldots$. In [10], Goursat gave more than 100 algebraic transformations of degrees $2, 3, 4, 6$. One such example is

$$ {}_2F_1\left(a, a+\frac{1}{3}; \frac{1}{2}; \frac{z(9-8z)^2}{(4z-3)^3}\right) = \left(1+\frac{z}{3}\right)^{3a} {}_2F_1\left(3a, a+\frac{1}{6}; \frac{1}{2}; z\right) $$

of degree 3. (See Entry (96) on Page 132 of [10].) More recently, Vidūnas [25] gave dozens of new algebraic transformations of degrees $6, 8, 9, 10, 12$. For example, he showed that if we set $\beta = \pm\sqrt{-2}$,

$$ S(z) = \frac{4z(z-1)(8\beta z + 7 - 4\beta)^8}{(2048\beta z^3 - 3072\beta z^2 - 3264z^2 + 912\beta z + 3264z + 56\beta - 17)^3}, \tag{6.3} $$

and

$$ R(z) = \left(1 + \frac{16}{9}(4-17\beta)z - \frac{64}{243}(167-136\beta)z^2 + \frac{2048}{6561}(112-17\beta)z^3\right)^{-1/16}, $$

then

$$ {}_2F_1\left(\frac{5}{24}, \frac{13}{24}; \frac{7}{8}; z\right) = R(z)\,{}_2F_1\left(\frac{1}{48}, \frac{17}{48}; \frac{7}{8}; S(z)\right), $$

which is a transformation of degree 10. (See (32) of [10].) Vidūnas' examples usually involve Gröbner-basis computation. This is perhaps one of the reasons why Goursat could not find these transformations.

In a very recent paper, we [24] obtained many new algebraic transformations of hypergeometric functions. For example, one of our favorite identities is

$$ \begin{aligned} &{}_2F_1\left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; \frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ &\qquad = (1-z^2)^{1/20}(1+4z-z^2)^{1/4}\,{}_2F_1\left(\frac{3}{10}, \frac{2}{5}; \frac{9}{10}; z^2\right). \end{aligned} \tag{6.4} $$

The main novelty in [24] is the interpretation of hypergeometric functions as automorphic forms on Shimura curves. Then proving identities such as the one above amounts to showing two certain automorphic forms on two Shimura curves are equal. This point of view is especially useful in determining the function $R(z)$ in (6.1). As far as we know, this interpretation first appeared in [33].

In this chapter, we will present several new algebraic transformations and give examples of algebraic transformations of hypergeometric functions to illustrate the role Shimura curves play in proving these identities. Firstly, we will prove Kummer's quadratic transformation (6.2) in the cases when the hypergeometric functions are related to automorphic forms on Shimura curves, and obtain identities related to Class II in Takeuchi's classification of arithmetic triangle groups [20, 21]. We remark that these identities can also be deduced from the results in [24] and some classical algebraic transformations of hypergeometric functions. The purpose of proving these identities is to demonstrate the advantage of using Shimura curves in proving this kind of identities. We then prove identities related to Classes III and VI in Takeuchi's classification.

This chapter is mainly following the articles [22] and [24].

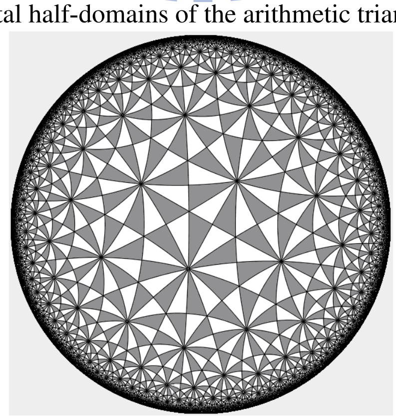
6.1 Preliminaries

In this section, we will review definitions arithmetic triangle groups, and their relations to hypergeometric functions.

6.1.1 Triangle groups

Suppose that a Shimura curve $X(\mathcal{O})$ has signature $(0; e_1, e_2, e_3)$. Then we say the group $\Gamma(\mathcal{O})$ is an **arithmetic triangle group**, and we denote it by $\Gamma(\mathcal{O}) = (e_1, e_2, e_3)$. The complete lists of all arithmetic triangle groups and their commensurability classes were determined by Takeuchi [20, 21].

If we cut each fundamental domain of an arithmetic triangle group $\Gamma(\mathcal{O})$ into 2 halves in a suitable way, then the fundamental half-domains give a tessellation of the upper half-plane $\mathfrak{h}$ by congruent triangles with internal angles π/e_1, π/e_2, and π/e_3. The following figure shows the tessellation of the unit disc, which is conformally equivalent to $\mathfrak{h}$, by fundamental half-domains of the arithmetic triangle group $(2, 3, 7)$.



Here each triangle represents a fundamental half-domain. Any combination of a grey triangle with a neighboring white triangle will be a fundamental domain for the triangle group $(2, 3, 7)$.

In general, for any discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$ such that $\Gamma\backslash\mathfrak{h}$ has finite volume, we can define its signature in the same way. If the signature is $(0; e_1, e_2, e_3)$, then we say Γ is a (hyperbolic) triangle group.

6.1.2 Automorphic forms on Shimura curves

We recall that if a Shimura curve X is of genus zero, Yang [33] shows that we can express the automomprhic forms on X by solutions of the Schwarzian differential equation associated to X. (Please see Section 4.1). In the case of arithmetic triangle groups, since the number of singularities of the differential equation is 3, the differential equation is essentially a hypergeometric differential equation. We then can use ${}_2F_1$-hypergeometric functions to express the automorphic forms (see Section 4.1.2).

Theorem 6.1.1. *Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $t(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1, e_2, and e_3, respectively. Let $k \geq 4$ be an even integer. Then a basis for the space of automorphic forms of weight k on X is given by*

$$t^{\{k(1-1/e_1)/2\}}(1-t)^{\{k(1-1/e_2)/2\}}t^j\left({}_2F_1(a, b; c; t) + Ct^{1/e_1}{}_2F_1(a', b', c'; t)\right)^k,$$

$j = 0, \ldots, \lfloor k(1-1/e_1)/2\rfloor + \lfloor k(1-1/e_2)/2\rfloor + \lfloor k(1-1/e_3)/2\rfloor - k$, for some constant C, where for a rational number x, we let $\{x\}$ denote the fractional part of x,

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

and

$$a' = a + \frac{1}{e_1}, \qquad b' = b + \frac{1}{e_1}, \qquad c' = c + \frac{2}{e_1}.$$

6.1.3 Algebraic transformations of hypergeometric functions

Consider the following situation. Suppose that $\Gamma_1 < \Gamma_2$ are two arithmetic triangle groups with Hauptmoduls z_1 and z_2, respectively. Since any automorphic function on Γ_2 is also an automorphic function on Γ_1, we have $z_2 = S(z_1)$ for some $S(x) \in \mathbb{C}(x)$. Likewise, if f_1 and f_2 are two automorphic forms of the same weight k on Γ_1 and Γ_2, respectively, then the ratio f_1/f_2 is an automorphic function on Γ_1 and hence is equal to $R(z_1)$ for some $R(x) \in \mathbb{C}(x)$. After taking the kth roots of the two sides of $f_1/f_2 = R(z_1)$, we obtain an algebraic transformation of hypergeometric function. This explains the existence of Kummer's, Goursat's and Vidūnas' transformations. (Of course, the triangle groups appearing in their transformations may not be arithmetic, but the argument above is still valid.)

More generally, if Γ_1 and Γ_2 are two commensurable arithmetic triangle groups such that the Shimura curve associated to $\Gamma = \Gamma_1 \cap \Gamma_2$ has genus 0. Let z be a

Hauptmodul on Γ. Then each of z_1 and z_2 is a rational function of z. Similarly, the ratio f_1/f_2 is also a rational function of z. In view of Theorem 6.1.1, we can obtain an algebraic transformation of the form

$$_2F_1\left(a_1, b_1; c_1; S_1(z)\right) = R(z)_2F_1\left(a_2, b_2; c_2; S_2(z)\right)$$

for some rational functions $S_1(z)$ and $S_2(z)$ and some algebraic function $R(z)$. This is the theory behind (6.4) and other algebraic transformations given in the paper.

Definition 6.1.1. *Let $S(z) \in \mathbb{C}(z)$ be a rational function. If the finite covering $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by $S : z \to S(z)$ is ramified at most at three points 0, 1, and ∞, then S is called a* **Belyi function**.

In practice, the Belyi functions $S_1(z)$ and $S_2(z)$ can be determined by the ramification data of the coverings of Shimura curves. The function $R(z)$ can be determined by Theorems 4.1.3 and 6.1.1.

6.2 Kummer's Quadratic Transformations and Automorphic Forms

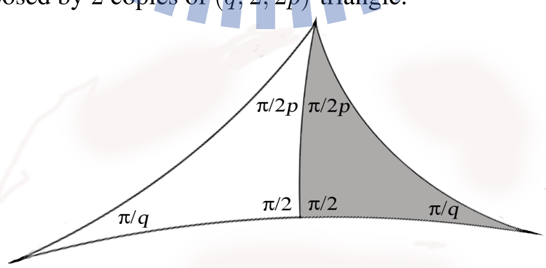
In this section, we will use our arguments to prove Kummer's quadratic transformation

$$_2F_1\left(2a, 2b; a+b+\frac{1}{2}; x\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4x(1-x)\right),$$

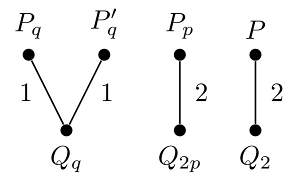
describe the automorphic forms on certain groups belong to Takeuchi's calss II, and obtain the related algebraic transformations.

6.2.1 Kummer's quadratic transformation

Note that the triangle group (q, q, p) is a subgroup of $(q, 2, 2p)$ of index 2. The (q, q, p)-triangle is decomposed by 2 copies of $(q, 2, 2p)$-triangle.



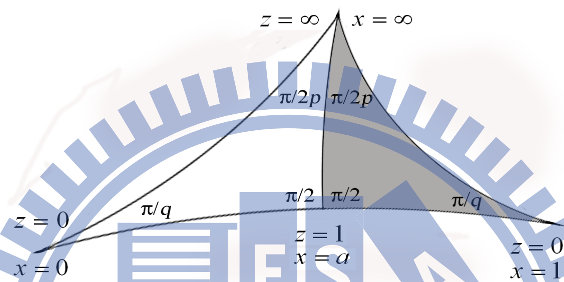
Let x be a Hauptmodul of $\Gamma_1 = (q, q, p)$ and z be a Hauptmodul of $\Gamma_2 = (q, 2, 2p)$. Label the elliptic points of $X_j = \Gamma_j \setminus \mathfrak{h}$ by P_q, P'_q, P_p for X_1 and Q_q, Q_2, Q_{2p} for X_2 such that the ramification data are given by



Here the numbers next to the lines are the ramification indices.

Assume that the values of x and z at these elliptic points are

$$x(P_q) = 0,\ x(P'_q) = 1,\ x(P_p) = \infty, \quad \text{and} \quad z(Q_q) = 0,\ z(Q_2) = 1,\ z(Q_{2p}) = \infty,$$



Then the corresponding hypergeometric functions are

$$ {}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; x\right), \quad \text{and} \quad {}_2F_1\left(\alpha, \beta; \alpha+\beta+\frac{1}{2}; z\right),$$

where

$$\alpha = \frac{1}{4} - \frac{1}{4p} - \frac{1}{2q}, \ \beta = \frac{1}{4} + \frac{1}{4p} - \frac{1}{2q}.$$

Also, the ramification data

z	0	1	∞
x	$0,\ 1$	$a,\ a$	$\infty,\ \infty$

at $z = 0, \infty$ implies $z = ux(1-x)$ for some constant u; the data at $z = 1$ implies $ux(1-x) = 1$ has a repeated root, which shows $u = 4$ and $a = 1/2$. Therefore, the relation between the Hauprmoduls z and x is $z = 4x(1-x)$, and thus the ratio between

$$ {}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; x\right), \quad \text{and} \quad {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4x(1-x)\right).$$

is an algebraic function of x. By considering the analytic behaviors, one can see that they are equal.

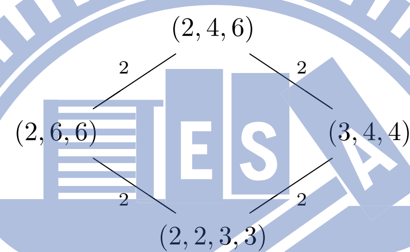
Remark. Here, we give another way to determine the value $\alpha = x(P)$. Let G be the group of all symmetries of the tessellation of the hyperbolic plane by the (q,q,p)-triangles and G_0 be the subgroup generated by the reflections across the edges of (q,q,p)-triangles. Then the factor group G/G_0 is of order 2. Since the group relation $\Gamma_1 < \Gamma_2$ admits the decomposition, the triangle group $\Gamma_2 = (q, 2, 2p)$ corresponds to the group G/G_0. Therefore, any element of Γ_2 not in Γ_1 induces an automorphim of order 2 on the curve X_2. Such an automorphism must fix the points P, P_p and permute the elliptic points P_q, P_q'. In terms of the Hauampmodul x, such an automorphism is given by

$$\sigma : x \mapsto 1 - x$$

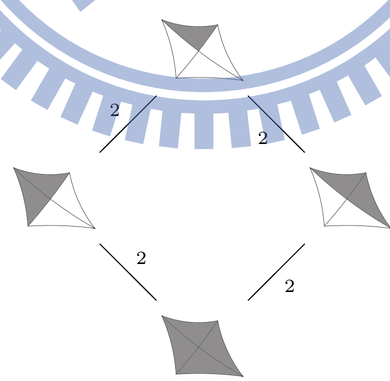
which implies that $x(P) = 1/2$.

6.2.2 Automorphic forms on arithmetic triangle groups in Takeuchi's class II and the associate algebraic transformations

Let us take Takeuchi's Class II of commensurable arithmetic triangle groups as an example, which comes from the quaternion algebra over $\mathbb{Q}$ with discriminant 6. This is a sub-diagram of the subgroup diagram of Calss II.



The node $(2, 2, 3, 3)$ in the diagram means that the related curve X, obtained by $\Gamma = (2, 6, 6) \cap (3, 4, 4)$, has signature $(0; 2, 2, 3, 3)$. The relations of these subgroups admit the Coxeter decompositions of a quadrilateral polygon that is symmetric with respect to both the diagonals as shown below



Associated to groups $(3, 4, 4)$, $(2, 6, 6)$ and Γ, we have the identities

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{3}{4}; \frac{z^2}{4(z-1)}\right) = (1-z))^{1/12}\, {}_2F_1\left(\frac{1}{12}, \frac{1}{4}; \frac{1}{2}; z(2-z)\right).$$

and

$$\sqrt{2}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4(z-1)}\right)=(1-z)^{1/3}(2-z)^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right).$$

Moreover, we can express all automorphic forms on Γ in terms of hypergeometric functions. (The algebraic transformation associated to the pair of groups $(2,4,6)$, $(3,4,4)$, and the pair of $(2,4,6)$, $(2,6,6)$ are Kummer's quadratic transformations, so we skip the associated transformations here.)

Let the Hauptmoduls be denoted by

$(2,3,3,2)$	$(4,4,3)$	$(2,6,6)$
z	u	t

where for (e_1,e_2,e_3), we choose the uniformizers in a way such that the values at the vertices e_1, e_2, e_3 are 0, 1, and ∞, respectively. For $(2,2,3,3)$, we assume that z takes values 0 at one of the elliptic point of order 2 and values 1 and ∞ the two elliptic points of order 3, respectively. Then from the ramification data, we have the relations

$$u=\frac{z^2}{4(z-1)}\quad\text{and}\quad t=z(2-z).$$

Note that the Hilbert-Poincaré series for X is

$$\sum_{k\geq 0}\dim S_k(\Gamma)x^k=1+x^4+x^6+x^8+x^{10}+3x^{12}+\cdots+5x^{24}+\cdots$$
$$=\frac{1+x^{12}}{(1-x^4)(1-x^6)}$$

which means that there are automorphic forms f_4, f_6, f_{12} of wight 4, 6, and 12 that generate the graded ring of automorphic forms. Moreover, there exists a linear relation among f_4^6, f_6^4, f_{12}^2, $f_4^3f_6^2$, $f_4^3f_{12}$, and $f_6^2f_{12}$.

On the other hand, according to the dimension formula, Proposition 2.7.2, we can find the dimensions of $S_k(\Gamma)$ on $\Gamma_1=(3,4,4)$ and $\Gamma_2=(2,6,6)$ are

$$\dim S_6(\Gamma_1)=1,\quad \dim S_8(\Gamma_1)=1,\quad \dim S_{12}(\Gamma_1)=1$$
$$\dim S_6(\Gamma_2)=0,\quad \dim S_8(\Gamma_2)=1,\quad \dim S_{12}(\Gamma_2)=2$$

Moreover, the space $S_6(\Gamma_1)$ can be spanned by

$$F_6(u)=u^{1/4}(1-u)^{1/4}\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right)+C_1u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^6,$$

for some constant C_1, the space $S_8(\Gamma_1)$ can be spanned by

$$F_8(u)=\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right)+C_1u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^8,$$

and $F_6(u)^2$ spans the automorphic forms of wight 12 on Γ_1. Similarly, on Γ_2, the sets $\{G_8(t)\}$ and $\{G_{12,1}(t),\, G_{12,2}(t)\}$ span the spaces of automorphic forms of weight 8 and 12, respectively, where

$$G_8(t) = (1-t)^{1/3}\left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^8,$$

$$G_{12,1}(t) = \left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^{12},$$

and

$$G_{12,1}(t) = t\left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^{12},$$

for some $C_2 \in \mathbb{C}$.

Substituting $u = z^2/4(z-1)$ and $t = z(2-z)$ into $F_6(u)$, $F_8(u)$, $F_6(u)^2$, $G_8(t)$, $G_{12,1}(t)$, and $G_{12,2}(t)$, they become automorphic forms on Γ. Also, the space $S_6(\Gamma)$ is equal to the space spanned by $F_6\left(z^2/(4z-4)\right)$, and the automorphic form

$$F_8\left(z^2/(4z-4)\right) = CG_8\left(z(2-z)\right), \quad C \in \mathbb{C}$$

is a basis of $S_8(\Gamma)$. Comparing the behaviors of these functions, we can find that the constant C is equal to 1, and $C_2 = (-1)^{1/4}C_1/2$. Thus, by taking 8th roots of the two sides, we can get the algebraic transformation

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4(z-1)}\right) = (1-z(2-z))^{1/24}\,{}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};z(2-z)\right).$$

$$\sqrt{2}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4(z-1)}\right) = (1-z)^{1/3}(2-z)^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right).$$

Observe that since the $\dim S_4(\Gamma) = \dim S_8(\Gamma) = 1$, if $S_4(\Gamma)$ is spanned by some automorphic form f_4 then f_4^2 spans $S_8(\Gamma)$, which can be also spanned by $F_8\left(z^2/(4z-4)\right)$. So we can choose

$$f_4 = \left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right) + C_1 u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^4,$$

and we can find

$$F_4^3\left(z^2/(4z-4)\right), \quad G_{12,1}\left(2z-z^2\right), \quad \text{and} F_6^2\left(z^2/(4z-4)\right)$$

form a basis of $S_{12}(\Gamma)$. (We remark that $4F_6^2\left(z^2/(4z-4)\right) = iG_{12,2}\left(2z-z^2\right)$.)

As a conclusion, the graded ring of automorphic forms on Γ can be generated by the following functions

$$f_4 = \left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4z-4}\right) + C_1\frac{z^2}{4z-4}^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4z-4)}\right)\right)^4,$$

$$f_6 = \left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4z-4}\right) + C_1\left(\frac{z^2}{4z-4)}\right)^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4z-4}\right)\right)^6,$$

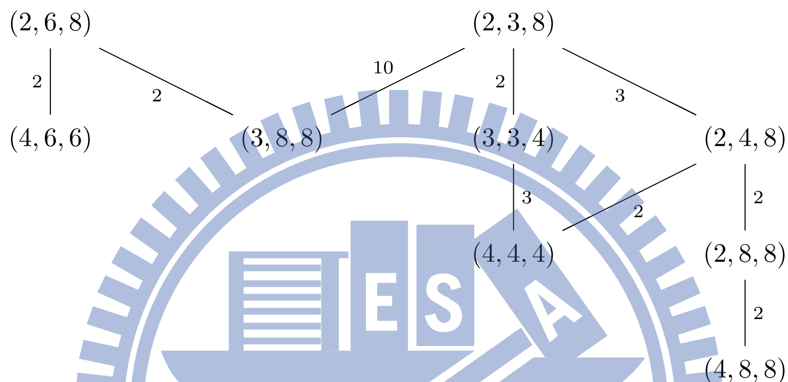
$$f_{12} = \left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};z(2-z)\right) + \frac{(-1)^{1/4}C_1}{2}(z(2-z))^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right)\right)^{12},$$

with the relation

$$f_4^6 - 4if_6^2 f_{12} - f_{12}^2 = 0.$$

6.3 Algebraic Transformations Associated to Class III

According to [21], Takeuchi's Class III of commensurable arithmetic triangle groups has the following subgroup diagram.



The main goal in this section is to prove an algebraic transformation associated to the pair of triangle groups $(4, 6, 6)$ and $(4, 4, 4)$.

Theorem 6.3.1. *Let α be a root of $x^2 + 3 = 0$ and β a root of $x^2 + 2 = 0$. We have*

$$\begin{aligned}&\frac{(1+z)^{1/8}(1-3z)^{1/8}}{(1+\alpha z)^{5/4}}\,{}_2F_1\left(\frac{5}{24}, \frac{3}{8}; \frac{3}{4}; \frac{12\alpha z(1-z^2)(1-9z^2)}{(1+\alpha z)^6}\right)\\ &\quad = \frac{1}{(1+(4+2\beta)z-(1+2\beta)z^2)^{1/2}}\,{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; \frac{3}{4}; R(z)\right),\end{aligned} \tag{6.5}$$

and

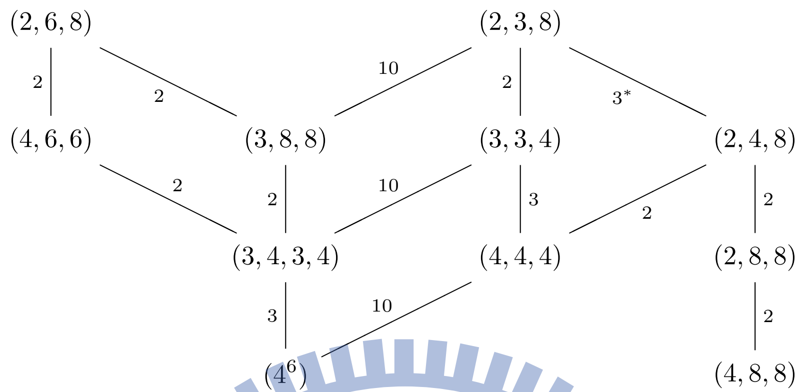
$$\begin{aligned}&\frac{(1-z)^{1/4}(1+z)^{5/8}(1-3z)^{1/4}(1+3z)^{5/8}}{(1+\alpha z)^{11/4}}\,{}_2F_1\left(\frac{11}{24}, \frac{5}{8}; \frac{5}{4}; \frac{12\alpha z(1-z^2)(1-9z^2)}{(1+\alpha z)^6}\right)\\ &\quad = \frac{(1+(-7+4\beta)z^2/3)}{(1+(4+2\beta)z-(1+2\beta)z^2)^{3/2}}\,{}_2F_1\left(\frac{3}{8}, \frac{5}{8}; \frac{5}{4}; R(z)\right)\end{aligned} \tag{6.6}$$

where

$$R(z) = -\frac{4(1+\beta)^4 z(1+(-7+4\beta)z^2/3)^4}{(1+z)(1-3z)(1+(4+2\beta)z-(1+2\beta)z^2)^4}.$$

We first determine the signatures of the intersections.

Lemma 6.3.2. *We have*



Moreover, the group of signature (4^6) *is a normal subgroup of the group of signature* $(3, 4, 3, 4)$. *(Here* (4^6) *is a shorthand for* $(4, 4, 4, 4, 4, 4)$.*)*

Proof. Let $\Gamma_1 = (3, 8, 8)$ and Γ_1' be its commutator subgroup. From the group presentation

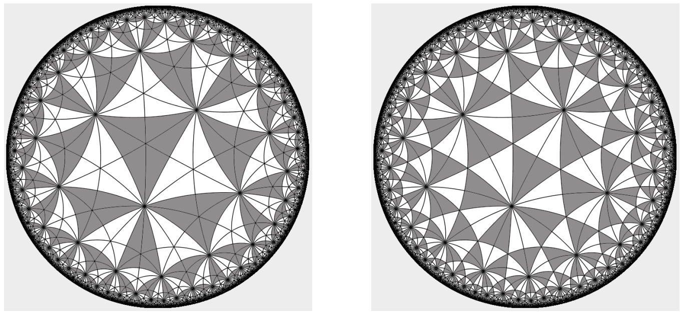
$$\Gamma_1 \simeq \langle \gamma_1, \gamma_2 : \gamma_1^3, \gamma_2^8, (\gamma_1\gamma_2)^8 = 1 \rangle$$

for Γ_1, we know that Γ_1/Γ_1' is cyclic of order 8. Thus, Γ_1 has exactly one subgroup of index 2, which must be the common intersection of the groups $(4, 6, 6)$, $(3, 8, 8)$ and $(3, 3, 4)$. The signature of this subgroup can be easily determined by observing that a covering of degree 2 from a Shimura curve to the Shimura curve associated to $(3, 8, 8)$ can only ramify at the two elliptic points of order 8. We find that the signature must be $(3, 4, 3, 4)$.

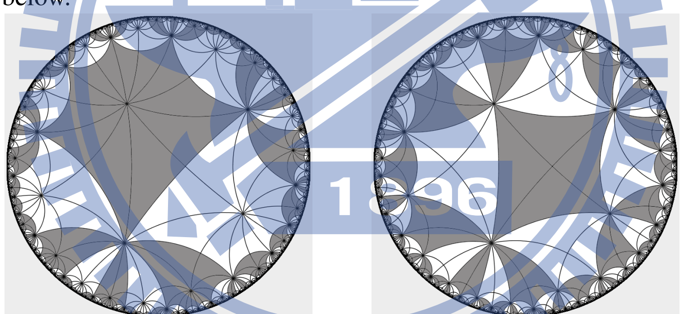
We next observe that the commutator subgroup Γ_2' of the group $\Gamma_2 = (3, 3, 4)$ is cyclic of order 3. Thus, Γ_2' is a normal subgroup of index 3 in Γ_2. This Γ_2' must be the same as the group of signature $(4, 4, 4)$. If $\Gamma_2' \neq (4, 4, 4)$, then $\Gamma_2' \cap (4, 4, 4)$ is a normal subgroup of $(4, 4, 4)$ of index 3, but the group $(4, 4, 4)$ cannot have a normal subgroup of index 3. We next determine the signature of the intersection of $\Gamma_3 = (3, 4, 3, 4)$ and $\Gamma_4 = (4, 4, 4)$.

Let X_j denote the Shimura curve associated to the group Γ_j. Since Γ_4 is a normal subgroup of Γ_2 of index 3, the intersection Γ_5 of Γ_3 and Γ_4 is a normal subgroup of index 3 in Γ_3, which implies that the two elliptic points of order 4 of X_3 must split completely on X_5. In view of the Riemann-Hurwitz formula, the two elliptic points of order 3 of X_3 must be totally ramified. We conclude that Γ_5 has signature (4^6).

In fact, the subgroup relations mentioned above can be visualized by the following figures.

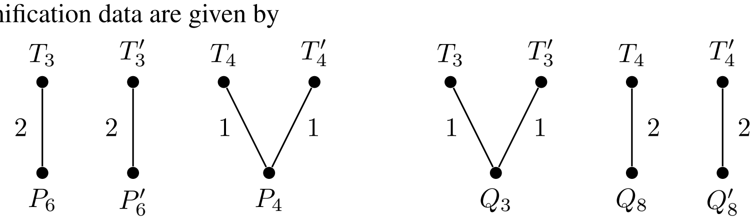


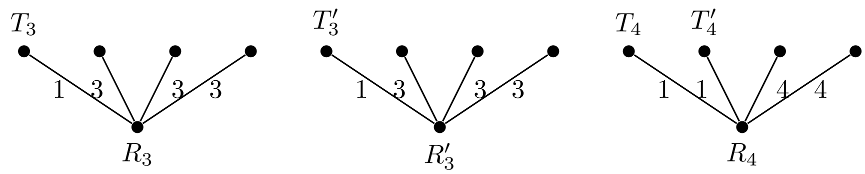
Here the small triangles are $(2,3,8)$-triangles. Let G be the group of all symmetries of the tessellation of the hyperbolic plane by the $(4,4,4)$-triangles and G_0 be the subgroup generated by the reflections across the edges of $(4,4,4)$-triangles. Then G/G_0 is isomorphic to D_3. The $(3,3,4)$-triangle group corresponds to the cyclic subgroup of order 3 in G/G_0, while the group $(2,3,8)$ corresponds the whole group G/G_0. Similarly, if we piece 12 copies of $(2,6,8)$-triangles around the vertex of inner angle $\pi/4$, we get a regular hexagon with inner angles $\pi/4$. Let H be the group of all symmetries of the tessellation by this regular hexagon and H_0 be the subgroup generated by the reflections across the edges of hexagons. Then H/H_0 is isomorphic to D_6. The unique cyclic subgroup of order 3 in H/H_0 corresponds to the group $(3,4,3,4)$. See the figures below.



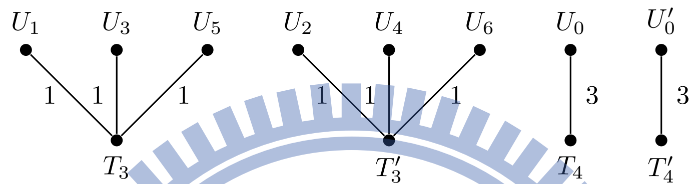
(The groups $(2,6,8)$, $(4,6,6)$, and $(3,8,8)$ correspond to the whole H/H_0, the cyclic subgroup of order 6 of H/H_0, and one of the D_3-subgroups, respectively.) $\square$

Now let $\Gamma_1 = (4,6,6)$, $\Gamma_2 = (3,8,8)$, $\Gamma_3 = (3,3,4)$, $\Gamma_4 = (4,4,4)$, $\Gamma_5 = (3,4,3,4)$, and $\Gamma_6 = (4^6)$. Let $X_j = X(\Gamma_j)$, $j = 1,\ldots,6$, be the corresponding Shimura curves. Label the elliptic points on X_1 by P_4, P_6, and P_6', those on X_2 by Q_3, Q_8, and Q_8', those on X_3 by R_3, R_3', and R_4, those on X_4 by S_4, S_4', S_4'', and those on X_5 by T_3, T_3', T_4, and T_4' (with the subscripts carrying the obvious meaning) such that the ramification data are given by



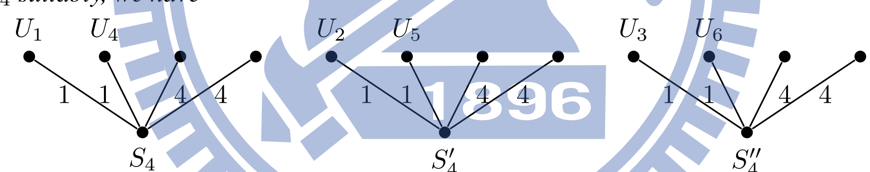


Label the elliptic points of X_6 by $U_1, \ldots, U_6$ such that the rotation around the center of the (4^6)-polygon by the angle $\pi/3$ permutes the six points cyclically. From the figures above, we know that if we label the points such that U_1 lies above T_3, then the ramification data for $X_6 \to X_5$ are



where U_0 and U_0' are the centers of the (4^6)-polygons. (The reader is reminded that each (4^6)-polygon represents only half of the fundamental domain for the Shimura curve X_6. Referring to the figure in the proof of the lemma above, a fundamental domain consists of a grey (4^6)-polygon and a neighboring white (4^6)-polygon.)

Lemma 6.3.3. *The two elliptic points of X_6 at the two ends of a diagonal of a (4^6)-polygon lie above the same elliptic point of X_4. That is, labeling the elliptic points of X_4 suitably, we have*



Moreover, if we choose Hauptmoduls $z_j(\tau)$ for X_j, $j = 1, \ldots, 6$, by requiring

$$\begin{aligned}
z_1(P_4) = 0, &\quad z_1(P_6) = 1, \quad z_1(P_6') = \infty, \\
z_2(Q_8) = 0, &\quad z_2(Q_3) = 1, \quad z_2(Q_8') = \infty, \\
z_3(R_4) = 0, &\quad z_3(R_3) = 1, \quad z_3(R_3') = \infty, \\
z_4(S_4) = 0, &\quad z_4(S_4') = 1, \quad z_4(S_4'') = \infty, \\
z_5(T_4) = 0, &\quad z_5(T_3) = 1, \quad z_5(T_4') = \infty, \\
z_6(U_1) = 0, &\quad z_6(U_3) = 1, \quad z_6(U_4) = \infty,
\end{aligned}$$

then we have

$$z_1 = \frac{4z_5}{(1+z_5)^2}, \qquad z_2 = z_5^2,$$

$$z_3 = \frac{3(\zeta-\zeta^2)z_4(1-z_4)}{(1+\zeta z_4)^3}, \qquad z_5 = \frac{3(\zeta-\zeta^2)z_6(1-z_6^2)}{1-9z_6^2},$$

$$z_3 = \frac{(28+16\beta)z_5(1+(-17+56\beta)z_5^2/81)^4}{(1+z_5)(1+(13+8\beta)z_5/3-(25+32\beta)z_5^2/9+(17-56\beta)z_5^3/81)^3},$$

and

$$z_4 = -\frac{4(1+\beta)^4 z_6(1+(-7+4\beta)z_6^2/3)^4}{(1+z_6)(1-3z_6)(1+(4+2\beta)z_6-(1+2\beta)z_6^2)^4},$$

where ζ is a 3rd root of unity and β is a root of $x^2+2=0$.

Proof. The ramification data for the covering $X_5 \to X_2$ and the assumption $z_2(Q_3) = z_5(T_3) = 1$ imply that $z_2 = z_5^2$ and

$$z(T_3') = -1.$$

The relation between z_1 and z_5 is easy to determine. We find $z_1 = 4z_5/(1+z_1)^2$.

To determine the relation between z_3 and z_4, we recall from Lemma 6.3.2 that Γ_4 is a normal subgroup Γ_3. Any element of Γ_3 not in Γ_4 induces an automorphism of order 3 on X_4. Such an automorphism must permute the three elliptic points S_4, S_4', and S_4'' cyclically. In term of the Hauptmodul z_4, such an automorphism is either

$$\sigma : z_4 \longmapsto \frac{-1}{z_4 - 1}$$

or its square. Moreover, the fixed points of such an automorphism are the ramified points in the covering $X_4 \to X_3$. That is, if we let S_0 and S_0' be the points lying above R_3 and R_3' respectively, then $z_4(S_0), z_4(S_0') \in \{-\zeta, -\zeta^2\}$, where ζ is a primitive 3rd root of unity. Then from the ramification data, we easily deduce that $z_3 = (\zeta - \zeta^2)z_4(1-z_4^2)/(1+\zeta z_4)^3$.

To determine the relation between z_5 and z_6, we argue similarly as above. The tessellation of the hyperbolic plane by Γ_6 has a D_6-symmetry, in addition to the symmetries arising from the reflections across the edges of the (4^6)-polygons. Thus, the automorphism group of X_6 is at least as large as D_6. This provides many useful informations. For example, if we let τ be the reflection across the diagonal joining U_1 and U_4, then τ induces an involution on X_6, which, in terms of z_6, is given by

$$\tau : z_6 \longmapsto -z_6,$$

which implies that

$$z_6(P_5) = -1.$$

Furthermore, let ρ denote the rotation by angle $\pi/3$ around the center of the hexagon. Then

$$\rho : z_6 \longmapsto \frac{cz_6+1}{-cz_6+c}$$

for some zero constant c since ρ maps 1 to ∞ and ∞ to -1. In light of $\rho^2 : 0 \to 1$, we conclude that $c = 3$ and

$$z_6(U_2) = 1/3, \qquad z_6(U_6) = -1/3.$$

It follows that $z_5 = Az_6(1-z_6^2)/(1-9z_6^2)$ for some A. This constant A has the property that $Ax(1-x^2)-(1-9x^2)$ has repeated roots. We find $A = \pm 3\sqrt{-3}$. The choice of the sign must be synchronized with the choice of the third root of unity in the relation between z_4 and z_5. This will be done later.

We now come to the more complicated part of the lemma. Let $\pi : X_6 \to X_4$ be the covering of the Shimura curves. Let γ be an element of Γ_5 not in Γ_6. Then γ normalizes both Γ_4 and Γ_6 and induces automorphisms ρ_1 and ρ_2 on X_4 and X_6, respectively. We may assume that $\rho_2 = \rho^2$, where ρ permutes $U_1, \ldots, U_6$ cyclically, as defined in the previous paragraph. It is easy to check that $\pi \circ \rho_1 = \rho_2 \circ \pi$. Thus, $\pi(U_1)$, $\pi(U_3)$, and $\pi(U_5)$ are three different elliptic points on X_4. We label them by S_4, S_4', and S_4'', respectively. Let V_1, V_2 be the two ramified points lying above S_4. Now there are three possibilities

$$\pi^{-1}(S_4) = \{U_1, U_2, V_1, V_2\},\ \pi^{-1}(S_4) = \{U_1, U_4, V_1, V_2\},\ \pi^{-1}(S_4) = \{U_1, U_6, V_1, V_2\}.$$

We will show that the correct one is $\{U_1, U_4, V_1, V_2\}$.

Let $V_j' = \rho_2(V_j)$ and $V_j'' = \rho_2^2(V_j)$ for $j = 1, 2$. If $\pi^{-1}(S_4) = \{U_1, U_2, V_1, V_2\}$, then we have

$$z_4 = \frac{Bz_6(1-3z_6)(z_6 - z_6(V_1))^4(z_6 - z_6(V_2))^4}{(1+z_6)(1+3z_6)(z_6 - z_6(V_1''))^4(z_6 - z_6(V_2''))^4}$$

for some constant B. The values of $z_6(V_1)$ and etc. must satisfy

$$\begin{aligned} &Bx(1-3x)(1-x/z_6(V_1))^4(1-x/z_6(V_2))^4 \\ &\qquad - (1+x)(1+3x)(1-x/z_6(V_1''))^4(1-x/z_6(V_2''))^4 \\ &= C(1-x)(1-x/z_6(V_1'))^4(1-x/z_6(V_2'))^4 \end{aligned} \tag{6.7}$$

for some constant C. Now if we let $p_1(x) = 1 + ax + bx^2 = (1 - x/z_6(V_1))(1 - x/z_6(V_2))$, then $(1 - x/z_6(V_1'))(1 - x/z_6(V_2'))$ and $(1 - x/z_6(V_1'')(1 - x/z_6(V_2''))$ are scalar multiples of

$$p_2(x) = (1+3x)^2 p_1\left(\frac{x-1}{3x+1}\right) = (1-a+b) + (6-2a-2b)x + (9+3a+b)x^2,$$

$$p_3(x) = (1-3x)^2 p_1\left(\frac{x+1}{1-3x}\right) = (1+a+b) + (-6-2a+2b)z + (9-3a+b)x^2,$$

respectively. Substituting these into (6.7) and equating the coefficients in the two sides, we find $A = B = 0$, $a = -2$, $b = -3$, but obviously this is invalid. This means that $\pi^{-1}(S_4) \neq \{U_1, U_2, V_1, V_2\}$. Likewise, $\pi^{-1}(S_4) \neq \{U_1, U_6, V_1, V_2\}$. Thus, we must have $\pi^{-1}(S_4) = \{U_1, U_4, V_1, V_2\}$. Now equating the coefficients in the two sides of

$$Bx(1+ax+bx^2)^4 - (1-x)(1+3x)p_2(x)^4 = C(1+x)(1-3x)p_3(x)^4$$

and excluding the invalid solutions, we get the claimed relation between z_4 and z_6. The relation between z_3 and z_5 can be determined by the known relation between z_3 and z_4, that between z_4 and z_6, and that between z_5 and z_6. This process also determines the choices of the third roots of unity in the relation between z_3 and z_4 and that between z_5 and z_6. We omit the details. □

Lemma 6.3.4. *The automorphic derivative $Q(z_6) = D(z_6, \tau)$ is equal to*

$$\begin{aligned} &\frac{15}{64}\left(\frac{1}{z_6^2} + \frac{1}{(1-z_6)^2} + \frac{1}{(1+z_6)^2} + \frac{1}{(z_6-1/3)^2} + \frac{1}{(z_6+1/3)^2}\right) \\ &\qquad + \frac{45}{128}\left(\frac{1}{1-z_6} + \frac{1}{1+z_6} + \frac{3}{1-3z_6} + \frac{3}{1+3z_6}\right). \end{aligned} \tag{6.8}$$

Proof. By Proposition 4.1.2, the rational function $R(x)$ such that automorphic $Q(z_6) = D(z_6, \tau)$ is equal to $R(z_6)$ is equal to

$$\begin{aligned} R(x) = &\frac{15}{64}\left(\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(x-1/3)^2} + \frac{1}{(x+1/3)^2}\right) \\ &+ \frac{B_1}{x} + \frac{B_2}{x-1} + \frac{B_3}{x+1} + \frac{B_4}{x-1/3} + \frac{B_5}{x+1/3} \end{aligned}$$

for some constants B_j satisfying

$$B_1 + B_2 + B_3 + B_4 + B_5 = 0, \quad B_2 - B_3 + \frac{1}{3}B_4 - \frac{1}{3}B_5 + \frac{15}{16} = 0. \tag{6.9}$$

Now the normalizer of Γ_6 in $\mathrm{SL}(2, \mathbb{R})$ contains at least the group of signature $(2, 6, 8)$. The factor group, in terms of the Hauptmodul z_6, is generated by $\sigma : z_6 \mapsto (3z_6 + 1)/(-3z_6 + 3)$ and $\tau : z_6 \mapsto -z_6$. By Proposition 4.1.5, $R(x)$ satisfies

$$R(-x) = R(x), \qquad \frac{144}{(-3x+3)^4} R\left(\frac{3x+1}{-3x+3}\right) = R(x).$$

Combining these informations with (6.9), we find

$$B_1 = 0, \quad B_2 = B_4 = -\frac{45}{128}, \quad B_3 = B_5 = \frac{45}{128}.$$

This gives us the formula. □

We now prove the theorem.

Proof of Theorem 6.3.1. By Proposition 2.7.2, we have

$$\dim S_6(\Gamma_1) = 1, \quad \dim S_6(\Gamma_4) = 1, \quad \dim S_6(\Gamma_6) = 7.$$

By Theorem 6.1.1, the one-dimensional spaces $S_6(\Gamma_1)$ and $S_6(\Gamma_4)$ are spanned by

$$F_1 = z_1^{1/4}(1-z_1)^{1/2}\left({}_2F_1\left(\frac{5}{24}, \frac{3}{8}; \frac{3}{4}; z_1\right) + C_1 z_1^{1/4} {}_2F_1\left(\frac{11}{24}, \frac{5}{8}; \frac{5}{4}; z_1\right)\right)^6 \tag{6.10}$$

and

$$F_2 = z_4^{1/4}(1-z_4)^{1/4}\left({}_2F_1\left(\frac{1}{8}, \frac{3}{8}; \frac{3}{4}; z_4\right) + C_2 z_4^{1/4} {}_2F_1\left(\frac{3}{8}, \frac{5}{8}; \frac{5}{4}; z_4\right)\right)^6 \tag{6.11}$$

for some complex numbers C_1 and C_2, respectively. Furthermore, by Theorem 4.1.3, if we let

$$f_1 = z_6^{3/8}\left(1 - \frac{15}{7}z_6^2 - \frac{111}{14}z_6^4 - \frac{2045}{46}z_6^6 - \frac{11355195}{39928}z_6^8 - \frac{77997477}{39928}z_6^{10} - \cdots\right)$$
$$f_2 = z_6^{5/8}\left(1 - \frac{5}{3}z_6^2 - \frac{245}{34}z_6^4 - \frac{7269}{170}z_6^6 - \frac{115223}{408}z_6^8 - \frac{55230121}{27880} - \cdots\right)$$

be a basis for the solution space of the Schwarzian differential equation $d^2f/dz_6^2 + Q(z_6)f = 0$, where $Q(z_6)$ is the rational function in (6.8), then a basis for $S_8(\Gamma_6)$ is

$$\{z_6^j g : j = 0, \ldots, 6\}, \qquad g = \frac{f_1 + C_3 z_6^{1/4}}{z_6^2(1-z_6^2)^2(1-9z_6^2)^2}.$$

Now from Lemma 6.3.3, we have

$$z_1 = \frac{12\alpha z_6(1-z_6^2)(1-9z_6^2)}{(1+\alpha z_6)^6}$$

and

$$z_4 = -\frac{4(1+\beta)^4 z_6(1+(-7+4\beta)z_6^2/3)^4}{(1+z_6)(1-3z_6)(1+(4+2\beta)z_6-(1+2\beta)z_6^2)^4},$$

where α is a root of $x^2+3=0$ and β is a root of $x^2+2=0$. Substituting these into (6.10) and (6.11) and comparing the coefficients, we find

$$F_1 = c_1(1+3z_6^2)^3 g$$

and

$$F_2 = c_2\left(1 + \frac{-7+4\beta}{3}z_6^2\right)\left(1+(4+2\beta)z_6-(1+2\beta)z_6^2\right) \times \left(1-(4+2\beta)z_6-(1+2\beta)z_6^2\right) g$$

for some constants c_1 and c_2. Taking the sixth roots of F_1 and F_2 and simplifying, we obtain the identities claimed in the theorem. □

Associated to this class, we also have the following identities.

Theorem 6.3.1. *(1) Corresponding to the pair of* $(4,6,6)$ *and* $(3,3,4)$ *are the following identities*

$$S^{1/8}{}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};\frac{4t}{(t+1)^2}\right) = (1+t)^{3/8}{}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right),$$
$$S^{7/8}{}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};\frac{4t}{(t+1)^2}\right) = R(1+t)^{5/8}{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right),$$

(2) *Corresponding to the pair of* $(3,8,8)$ *and* $(3,3,4)$ *are the following identities*

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right) &= S^{1/8}(1+t)^{1/24}{}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};t^2\right),\\
R{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right) &= S^{7/8}(1+t)^{7/24}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};t^2\right),
\end{aligned}
$$

(3) *Associated to the groups* $(3,8,8)$, $(4,6,6)$ *are the following identities*

$$
\begin{aligned}
{}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};\frac{4t}{(1+t)^2}\right) &= (1+t)^{5/12}{}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};t^2\right),\\
{}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};\frac{4t}{(1+t)^2}\right) &= (1+t)^{11/12}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};t^2\right),
\end{aligned}
$$

where

$$
R = 1+\frac{-17+56\beta}{81}t^2, \quad S = 1+\frac{13+8\beta}{3}t-\frac{25+32\beta}{9}t^2+\frac{17-56\beta}{81}t^3,
$$

and β *is a root of* x^2+2.

We remark here that these equalities can be deduced from the results described in Theorem 6.3.1. The purpose of the following proving these identities is to demonstrate the advantage of using Shimura curves in proving this kind of identities.

Proof. Let $\Gamma_1 = (4,6,6)$, $\Gamma_2 = (3,8,8)$, $\Gamma_3 = (3,3,4)$, $\Gamma = (3,4,3,4)$, and the Hauptmoduls be denoted by

$(4,6,6)$	$(8,3,8)$	$(4,3,3)$	$(4,3,4,3)$
z_1	z_2	z_3	t

where for (e_1,e_2,e_3), we choose the Hauptmoduls such that the values at the vertices e_1, e_2, e_3 are 0, 1, and ∞, respectively. For $(3,4,3,4)$, we assume that t takes value 1 at one of the elliptic point of order 3 and values 0 and ∞ the two elliptic points of order 4, respectively.

Then we can find that t takes value -1 at the other elliptic point of order 3, and the relations between these Hauptmoduls are

$$
z_1 = \frac{4t}{(1+t)^2}, \quad z_2 = t^2, \quad z_3 = \frac{4(7+4\beta)t\left(1+\frac{-17+56\beta}{81}t^2\right)^4}{(t+1)\left(1+\frac{13+8\beta}{3}t-\frac{25+32\beta}{9}t^2+\frac{17-56\beta}{81}t^3\right)^3}. \tag{6.12}
$$

By Proposition 2.7.2, we have

$$
\dim S_6(\Gamma_1) = \dim S_6(\Gamma_2) = \dim S_6(\Gamma_3) = 1 \quad \text{and} \quad \dim S_6(\Gamma) = 3.
$$

Therefore, the space $S_6(\Gamma_1)$ is spanned by

$$F_1 = z_1^{1/4}(1-z_1)^{1/2}\left({}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};z_1\right)+C_1 z_1^{1/4}{}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};z_1\right)\right)^6 \quad (6.13)$$

for some constant C_1; the space $S_6(\Gamma_2)$ is spanned by

$$F_2 = z_2^{5/8}\left({}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};z_2\right)+C_2 z_2^{1/8}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};z_2\right)\right)^6 \quad (6.14)$$

for some constant C_2, and the space $S_6(\Gamma_3)$ is spanned by

$$F_3 = z_3^{1/4}\left({}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};z_3\right)+C_3 z_3^{1/4}{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};z_3\right)\right)^6 \quad (6.15)$$

for some constant C_3.

By Theorem 4.1.3, a basis for the space $S_6(\Gamma)$ is

$$\{g,\ tg,\ t^2 g\},\qquad g=\frac{(f_1+Cf_2)^6}{t^2(1-t)^2(1+t)^2},$$

for some constant C, where $\{f_1, f_2\}$ is a basis for the solution space of the Schwarzian differential equation $d^2f/dt^2 + Q(t)f = 0$ associate to t.

Note that for any element γ of Γ_2 not Γ, we have the equality

$$t(\gamma\tau) = -t(\tau).$$

From the information and Theorem 4.1.3, we can get

$$Q(t)=\frac{15}{64t^2}+\frac{2}{9}\left(\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}-\frac{1}{t-1}+\frac{1}{t+1}\right).$$

Here, we choose a basis for the solution space of the Schwarzian differential equation $d^2f/dt^2 + Q(t)f = 0$ with t-series

$$f_1 = t^{5/8}\left(1-\frac{16}{81}t^2-\frac{1168}{12393}t^4-\frac{99568}{1673055}t^6-\frac{1922128}{45172485}t^8-\frac{32018768}{980508645}t^{10}-\cdots\right),$$
$$f_2 = t^{3/8}\left(1-\frac{16}{63}t^2-\frac{176}{1701}t^4-\frac{65008}{1056321}t^6-\frac{1792496}{42101937}t^8-\frac{254491952}{7957266093}t^{10}-\cdots\right).$$

After substituting (6.12) into (6.13), (6.14) and (6.15), one has

$$\begin{aligned} C^6F_1 &= \sqrt{2}(1-t^2)g,\\ C^6F_2 &= tg,\\ C^6F_3 &= \sqrt{2}(7+4\beta)^{1/4}\left(1+\frac{(-17+56\beta)}{81}t^2\right)g. \end{aligned} \quad (6.16)$$

Simplifying the relations

$$(7+4\beta)^{1/4}\left(1+\frac{-17+56\beta}{81}t^2\right)F_1=(1-t^2)F_3,$$

$$tF_3=\sqrt{2}(7+4\beta)^{1/4}\left(1+\frac{-17+56\beta}{81}t^2\right)F_2,$$

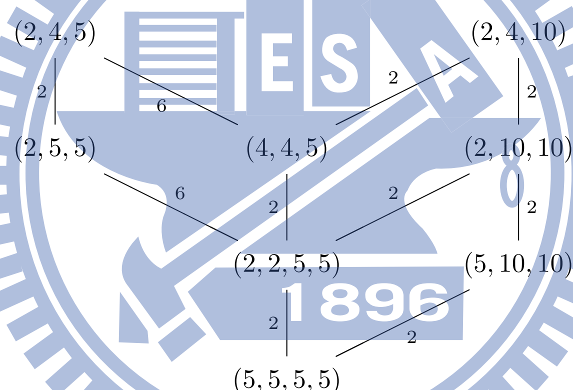
and

$$tF_1=\sqrt{2}(1-t^2)F_2,$$

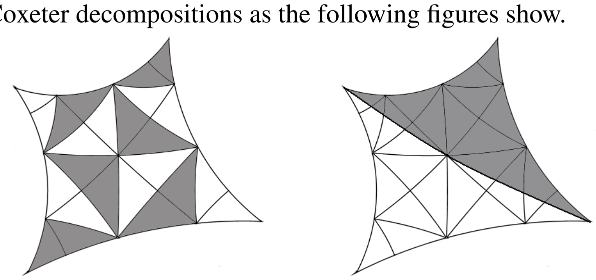
we can get the identities described in the theorems. □

6.4 Algebraic Transformations Associated to Class VI

According to Appendix A, the subgroup diagram for Takeuchi's Class VI is



Let $\Gamma_1=(2,5,5)$, $\Gamma_2=(5,10,10)$, $\Gamma_3=(5,5,5,5)$, and X_1, X_2, X_3 be the Shimura curves associated to these three groups. (The reader is reminded that the subgroup diagram should be read as "*there are* arithmetic Fuchsian subgroups of $\mathrm{SL}(2,\mathbb{R})$ such that their subgroup relations are given by the diagram".) The subgroups relations $\Gamma_3<\Gamma_1,\Gamma_2$ admit Coxeter decompositions as the following figures show.



Here the small triangles are $(2,4,5)$-triangles. Associated to this triplet of groups is the following identities.

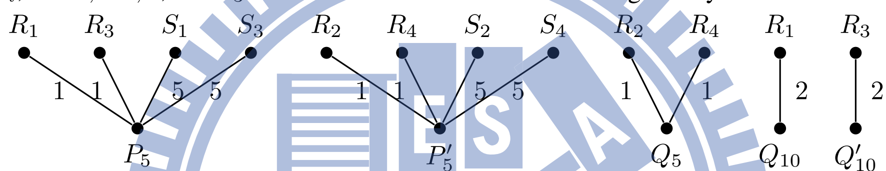
Theorem 6.4.1. *We have*

$$\begin{aligned} &{}_2F_1\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};\frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ &\qquad = (1-z^2)^{1/20}(1+4z-z^2)^{1/4}{}_2F_1\left(\frac{3}{10},\frac{2}{5};\frac{9}{10};z^2\right). \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} &(1-z-z^2){}_2F_1\left(\frac{1}{4},\frac{9}{20};\frac{6}{5};\frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ &\qquad = (1-z^2)^{1/4}(1+4z-z^2)^{5/4}{}_2F_1\left(\frac{2}{5},\frac{1}{2};\frac{11}{10};z^2\right). \end{aligned} \tag{6.18}$$

Proof. Label the elliptic points of X_j by P_2, P_5, P_5' for X_1, Q_5, Q_{10}, Q_{10}' for X_2, and R_i, $i=1,\dots,4$, for X_3 such that the ramifications data are given by



Here the numbers next to the lines are the ramification indices. We have omitted P_2 from the diagram. There are 6 points lying above P_2. Each has ramficiation index 2. Choose Hauptmoduls z_j for X_j by requiring

$$z_1(P_5)=0,\ z_1(P_2)=1,\ z_1(P_5')=\infty,\quad z_2(Q_{10})=0,\ z_2(Q_5)=1,\ z_2(Q_{10}')=\infty$$

and

$$z_3(R_1)=0,\ z_3(R_2)=1,\ z_3(R_3)=\infty.$$

The relation between z_2 and z_3 is easy to figure out. We have

$$z_2=z_3^2, \tag{6.19}$$

which implies that $z_3(R_4)=-1$. To determine the relation between z_1 and z_3, we observe that the tessellation of the hyperbolic plane by the $(5,5,5,5)$-polygons has extra symmetries by rotation by 90 degree around the center of any $(5,5,5,5)$-polygon. In terms of groups, this means that Γ_3 has a supergroup Γ normalizing Γ_3 such that Γ/Γ_3 is cyclic of order 4. (In fact, Γ is the $(4,4,5)$-triangle group in the subgroup diagram.) Therefore, the automorphism group of X_3 has an element σ of order 4 that permutes R_1,R_2,R_3,R_4 cyclically. In terms of the Hauptmodul, we have

$$\sigma : z_3 \longmapsto \frac{z_3+1}{z_3-1}.$$

Thus, if the value of z_3 at S_1 is a, then we have

$$z_3(S_1)=a,\quad z_3(S_2)=\frac{a-1}{a+1},\quad z_3(S_3)=-\frac{1}{a},\quad z_3(S_4)=-\frac{a+1}{a-1}.$$

Therefore, the relation between z_1 and z_3 is

$$z_1 = \frac{Bz_3(z_3-a)^5(z_3+1/a)^5}{(1-z_3^2)(z_3-(a-1)/(a+1))^5(z_3+(a+1)/(a-1))^5}$$

for some constant B. Moreover, the automorphism σ of X_3 rotates 4 of the six points lying above P_2 cyclically and fixes the other two. (The reader is reminded that each $(5,5,5,5)$-polygon represents only half of the fundamental domain for Γ_3. The two fixed of σ are the centers of the $(5,5,5,5)$-polygons.) In terms of the Hauptmodul z_3, this means that the values of z_3 at the two fixed points of σ are $\pm i$ and if the value of z_3 at one of the other 4 points above P_2 is b, then the values at the other 3 points are $-1/b$, $(b-1)/(b+1)$, and $-(b+1)/(b-1)$. Thus, we have

$$z_1 - 1 = \frac{C(1+z_3^2)^2(z_3-b)^2(z_3+1/b)^2(z_3-(b-1)/(b+1))^2(z_3+(b+1)/(b-1))^2}{(1-z_3^2)(z_3-(a-1)/(a+1))^5(z_3+(a+1)/(a-1))^5}$$

for some constant C. Comparing the two sides, we find $a=0,\pm1,\pm i$, $a^2+a-1=0$, or $a^2-a-1=0$. The first five solutions are invalid. The other two solutions give

$$z_1 = \frac{64z_3(1-z_3-z_3^2)^5}{(1-z_3^2)(1+4z_3-z_3^2)^5} \tag{6.20}$$

or

$$z_1 = -\frac{64z_3(1+z_3-z_3^2)^5}{(1-z_3^2)(1-4z_3-z_3^2)^5}. \tag{6.21}$$

Both are valid because of the following reason. Notice that Γ_2 normalizes Γ_3. If we take an element γ of Γ_2 not in Γ_3, then $\gamma^{-1}\Gamma_1\gamma$ is again a triangle of signature $(2,5,5)$ containing the same Γ_3. If the relation between the Hauptmoduls of Γ_1 and Γ_3 is (6.20), then the relation between the Hauptmoduls of $\gamma^{-1}\Gamma_1\gamma$ and Γ_3 will be (6.21).

By dimension formula and Theorem 6.1.1, we have

$$\dim S_8(\Gamma_1) = \dim S_8(\Gamma_2) = 1, \quad \dim S_8(\Gamma_3) = 7,$$

and the one-dimensional space $S_8(\Gamma_1)$ is spanned by

$$F_1 = z_1^{1/5}\left({}_2F_1\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};z_1\right) + C_1 z_1^{1/5}{}_2F_1\left(\frac{1}{4},\frac{9}{20};\frac{6}{5};z_1\right)\right)^8 \tag{6.22}$$

for some constant C_1, the function

$$F_2 = z_2^{3/5}(1-z_2)^{1/5}\left({}_2F_1\left(\frac{3}{10},\frac{2}{5};\frac{9}{10};z_2\right) + C_2 z_2^{1/10}{}_2F_1\left(\frac{2}{5};\frac{1}{2};\frac{11}{10};z_2\right)\right)^8 \tag{6.23}$$

is contained in $S_8(\Gamma_2)$ for some constant C_2. To get a basis for $S_8(\Gamma_3)$, we need to work out the Schwarzian differential equation associated to z_3. It is actually easy in this case.

Here we use we use the automorphism of X_3 coming from the normal subgroup relation $\Gamma_3 \lhd \Gamma_1$. Let γ be an element of Γ_2 not in Γ_3. We know that

$$z_3(\gamma\tau) = -z_3(\tau).$$

Now by Proposition 4.1.4 and Theorem 4.1.1, the function $z_3'(\tau)$, as a function of z_3, satisfies

$$\frac{d^2}{dz_3^2}f + Q(z_3)f = 0,$$

where

$$Q(z_3) = \frac{6}{25}\left(\frac{1}{z_3^2} + \frac{1}{(1-z_3)^2} + \frac{1}{(1+z_3)^2} + \frac{1}{1-z_3} + \frac{1}{1+z_3}\right).$$

Thus a basis for the solution space of the Schwarzian differential equation $d^2f/dz_3^2 + Q(z_3)f = 0$ is given by

$$f_1 = z_3^{2/5}\left(1 - \frac{4}{15}z_3^2 - \frac{52}{475}z_3^4 - \frac{13436}{206625}z_3^6 - \frac{46348}{1033125}z_3^8 - \frac{2024924}{60265625}z_3^{10} - \cdots\right)$$
$$f_2 = z_3^{3/5}\left(1 - \frac{12}{55}z_3^2 - \frac{28}{275}z_3^4 - \frac{2708}{42625}z_3^6 - \frac{393636}{8738125}z_3^8 - \frac{7503908}{218453125}z_3^{10} - \cdots\right).$$

By Corollary 4.1.3,

$$g,\ z_3g,\ z_3^2g,\ z_3^3g,\ z_3^4g, \qquad g = \frac{(f_1 + C_3f_2)^8}{z_3^3(1-z_3)^3(1+z_3)^3},$$

form a basis for $S_8(\Gamma_3)$ for some constant C_3. That is, after substituting (6.20) and (6.19) into (6.22) and (6.23), respectively, we have $F_1 = h_1(z_3)g$ and $F_2 = h_2(z_3)g$ for some polynomials $h_1(x)$ and $h_2(x)$ of degree ≤ 4. Indeed, by comparing the coefficients, we find

$$F_1 = 2^{6/5}(1 - z_3 - z_3^2)(1 + 4z_3 - z_3^2)g, \qquad F_2 = z_3g.$$

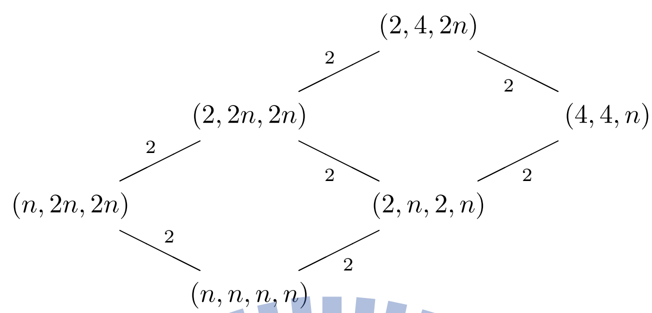
(It is easier if we take the 8th roots of the functions first.) Simplifying the relation $z_3F_1 = 2^{6/5}(1 - z_3 - z_3^2)(1 + 4z_3 - z_3^2)F_2$, we get the two identities in the theorem. This completes the proof. $\square$

6.5 Algebraic Transformations Associated to Other Classes

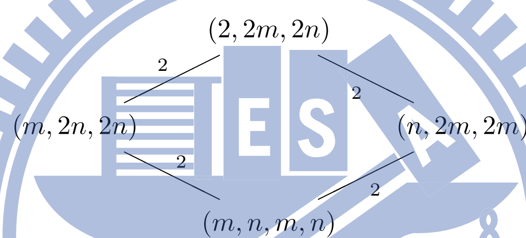
Note that the quaternion algebra in Class I is $M(2,\mathbb{Q})$, so the Shimura curves are just the classical modular curves. In this case, it is easier to use Fourier expansions of modular forms and modular functions. We will not discuss this case.

6.5.1 Classes II, V, and XII

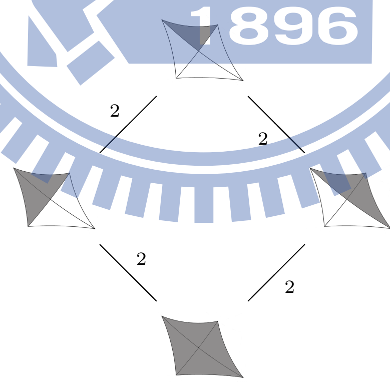
The subgroup diagrams of Class II, V, and XII are all of the form



The subgroup relation $(2,2n,2n) \cap (4,4,n) = (2,n,2,n)$ is a special case of



which arises from the Coxeter decompositions of a quadrilateral polygon that is symmetric with respect to both the diagonals as shown below



Associated to this family of subgroup relations is the following identity.

Theorem 6.5.1. *For real numbers a and b such that neither $b+3/4$ nor $2b+1/2$ is a nonpositive integer, we have*

$$(1+z)^{2a+2b}{}_2F_1\left(a+b,a+\frac{1}{4};b+\frac{3}{4};z^2\right) = {}_2F_1\left(a+b,b+\frac{1}{4};2b+\frac{1}{2};\frac{4z}{(1+z)^2}\right)$$

in a neighborhood of $z=0$.

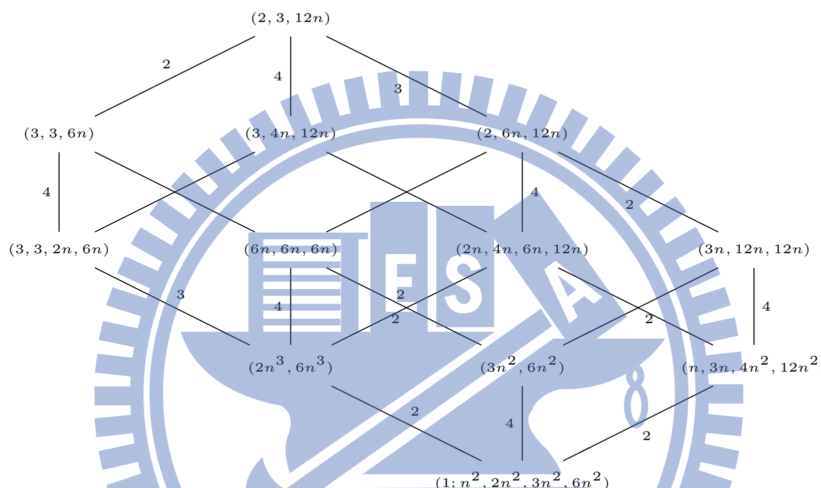
This identity can be easily proved using Kummer's quadratic transformation. Alternatively, one can verify that both sides are solutions of the differential equation

$$2z(1-z)(1+z)^2F''-(1+z)((3-4b)z^2+8(a+b)z-4b-1)F'-(a+b)(1+4b)(1-z)F=0.$$

and that the local behaviors at $z=0$ agree. We omit the details.

6.5.2 Classes IV, VIII, XI, XIII, XV, XVII

The subgroups diagrams of Classes IV, VIII, XI, XIII, XV, and XVIII are either of the form



or sub-diagram of it with Class XI having one extra node. There are two families of essentially new identities associated to these classes. One corresponds to the pair of $(3,3,6n)$ and $(3,4n,12n)$. (Theorem 6.5.2 below.) One corresponds to the pair of $(3,4n,12n)$ and $(2,6n,12n)$. (Theorem 6.5.3 below.)

Theorem 6.5.2. *For a real number a such that neither $3a+1$ nor $2a+1$ is a nonpositive integer, we have*

$$(1+z)^{a+1/6}(1-z/3)^{3a+1/2}{}_2F_1\left(2a+\frac{1}{3},a+\frac{1}{3};3a+1;z^2\right) = {}_2F_1\left(a+\frac{1}{6},a+\frac{1}{2};2a+1;\frac{16z^3}{(1+z)(3-z)^3}\right)$$

in a neighborhood of $z=0$.

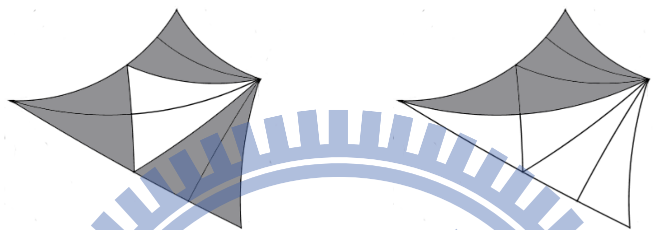
Theorem 6.5.3. *For a real number a such that neither $6a+1$ nor $4a+1$ is a nonpositive integer, we have*

$$(1-z)^{9a+3/4}{}_2F_1\left(4a+\frac{1}{3},2a+\frac{1}{3};6a+1;-\frac{27z^2(1-z)}{1-9z}\right) = (1-9z)^{a+1/12}{}_2F_1\left(3a+\frac{1}{4},a+\frac{1}{4};4a+1;-\frac{64z^3}{(1-z)^3(1-9z)}\right)$$

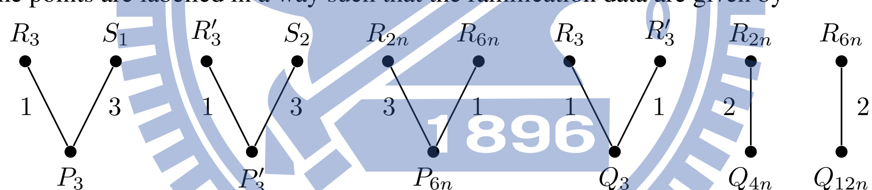
in a neighborhood of $z = 0$.

In principle, these two identities can be deduced from Kummer's and Goursat's transformations, once the related Belyi functions are determined. Here we briefly indicate how one can prove the theorems in the cases where the parameters correspond to discrete Fuchsian groups using theory of automorphic forms.

Proof of Theorem 6.5.2 in the cases of Shimura curves. For the pair of $(3, 3, 6n)$ and $(3, 4n, 12n)$, the subgroup relations admit Coxeter decompositions, as shown in the figures



Here the parameter n in the figures is 1 and the smaller triangles are $(2, 3, 12)$-triangles. Let $\Gamma_1 = (3, 3, 6n)$, $\Gamma_2 = (3, 4n, 12n)$, $\Gamma_3 = \Gamma_1 \cap \Gamma_2$, and let X_i, $i = 1, \ldots, 3$ be the associated Shimura curves. Denote by P_3, P_3', and P_{6n} the elliptic points of orders 3, 3, and $6n$ on X_1, by Q_3, Q_{4n}, and Q_{12n} the elliptic points of orders 3, $4n$, and $12n$ on X_2, and by R_3, R_3', R_{2n}, and R_{6n} the elliptic points of order 3, 3, $2n$, and $6n$ on X_3. The points are labelled in a way such that the ramification data are given by



Choose Hauptmoduls z_j on X_j, $j = 1, 2, 3$, by requiring

$$z_1(P_{6n}) = 0,\ z_1(P_3) = 1,\ z_1(P_3') = \infty, \quad z_2(Q_{4n}) = 0,\ z_2(Q_3) = 1,\ z_2(Q_{12n}) = \infty$$

and

$$z_3(R_{2n}) = 0,\ z_3(R_3) = 1,\ z_3(R_{6n}) = \infty.$$

It is easy to see from the ramification information that

$$z_2 = z_3^2, \tag{6.24}$$

which implies that $z_3(R_3') = -1$. For z_1, we have

$$z_1 = \frac{Az_3^3}{(1+z_3)(1-az_3)^3}$$

for some complex numbers A and a, where $1/a$ is the value of z_3 at S_1. These two numbers satisfy

$$1 - \frac{Az_3^3}{(1+z_3)(1-az_3)^3} = 1 - z_1 = \frac{(1-z_3)(1-bz_3)^3}{(1+z_3)(1-az_3)^3}, \tag{6.25}$$

where $1/b$ is the value of z_3 at S_2. Now observe that Γ_3 is a normal subgroup of Γ_2. Thus, an element of Γ_2 not in Γ_3 induces an automorphism on X_3. In terms of the Hauptmodul z_3, it is easy to see that this automorphism sends z_3 to $-z_3$. Since this automorphism maps S_1 to S_2, we find $b = -a$. Then comparing the two sides of (6.25), we get $A = 16/27$, $a = 1/3$, and

$$z_1 = \frac{16z_3^3}{(1+z_3)(3-z_3)^3}. \tag{6.26}$$

Now by Proposition 2.7.2, we have

$$\dim S_6(\Gamma_1) = \dim S_6(\Gamma_2) = 1, \quad \dim S_6(\Gamma_3) = \begin{cases} 2, & \text{if } n = 1, \\ 3, & \text{if } n \geq 2. \end{cases}$$

From now on, we assume that $n \geq 2$.

By Theorem 6.1.1, the one-dimensional spaces $S_6(\Gamma_1)$ and $S_6(\Gamma_2)$ are spanned by

$$\begin{aligned} F_1 = z_1^{1-1/2n}\Bigg(&{}_2F_1\left(\frac{1}{6}-\frac{1}{12n}, \frac{1}{2}-\frac{1}{12n}; 1-\frac{1}{6n}; z_1\right) \\ &+ C_1 z_1^{1/6n}\, {}_2F_1\left(\frac{1}{6}+\frac{1}{12n}, \frac{1}{2}+\frac{1}{12n}; 1+\frac{1}{6n}; z_1\right)\Bigg)^6 \end{aligned} \tag{6.27}$$

and

$$\begin{aligned} F_2 = z_2^{1-3/4n}\Bigg(&{}_2F_1\left(\frac{1}{3}-\frac{1}{6n}, \frac{1}{3}-\frac{1}{12n}; 1-\frac{1}{4n}; z_2\right) \\ &+ C_2 z_2^{1/4n}\, {}_2F_1\left(\frac{1}{3}+\frac{1}{12n}, \frac{1}{3}+\frac{1}{6n}; 1+\frac{1}{4n}; z_2\right)\Bigg)^6, \end{aligned} \tag{6.28}$$

respectively, for some constants C_1 and C_2. Also, if we let $f_1 = z_3^{1/2-1/4n}(1 + c_1 z + \cdots)$ and $f_2 = z_3^{1/2+1/4n}(1 + d_1 z + \cdots)$ be a basis of the solution space of the Schwarzian differential equation $d^2 f/dz_3^2 + Q(z_3)f = 0$ associated to z_3, then by Theorem 4.1.3, $S_6(\Gamma_3)$ is spanned by g, $z_3 g$, and $z_3^2 g$, where

$$g = \frac{(f_1 + C_3 f_2)^6}{z_3^2(1-z_3)^2(1+z_3)^2}$$

for some constant C_3. Now we substitute (6.26) and (6.24) into (6.27) and (6.24), respectively. We find

$$F_1 = a_1 z_3^{3-3/2n} + \cdots, \qquad F_2 = z_3^{2-3/2n} + \cdots,$$

where $a_1 = (16/27)^{1-1/2n}$, and thus

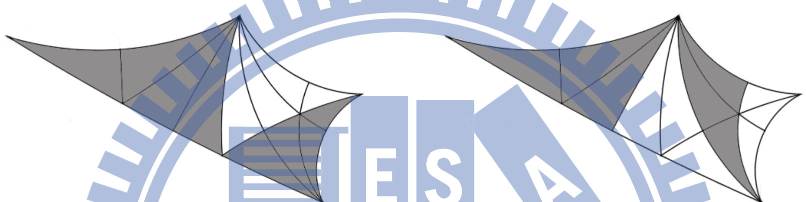
$$F_1 = a_1 z_3^2 g, \qquad F_2 = (z_3 + a_2 z_3^2) g$$

for some constant a_2. That is, $a_1 z F_2/F_1 = 1 + a_2 z_3$. We then take the 6th roots of the two sides and compare the coefficients of $z^{3/2-1/4n}$, we find that a_2 is actually 0. After simplifying, we arrive at

$$(1+z)^{1/6-1/12n}(1-z/3)^{1/2-1/4n}{}_2F_1\left(\frac{1}{3}-\frac{1}{6n},\frac{1}{3}-\frac{1}{12n};1-\frac{1}{4n};z^2\right)$$
$$= {}_2F_1\left(\frac{1}{6}-\frac{1}{12n},\frac{1}{2}-\frac{1}{12n};1-\frac{1}{6n};\frac{16z^3}{(1+z)(3-z)^3}\right).$$

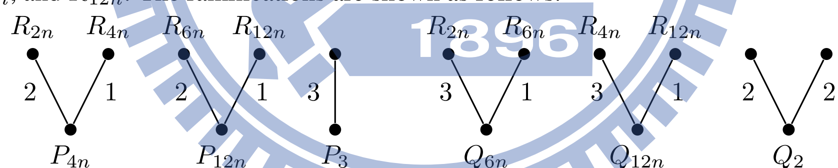
This proves Theorem 6.5.2 in the case the parameters correspond to arithmetic triangle groups. □

Proof of Theorem 6.5.3 in the cases of Shimura curves. The subgroups $(3,4n,12n)$, $(2,6n,12n)$ and their intersection admit Coxeter decompositions as the figures below show.



Here the parameter n in the figures is 1 and the small triangles are $(2,3,12)$-triangles.

Denote the groups $(3,4n,12n)$, $(2,6n,12n)$, and $(2n,4n,6n,12n)$ by Γ_1, Γ_2, and Γ_3, respectively. Label the elliptic points of $(3,4n,12n)$ by P_3, P_{4n}, and P_{12n}, those of $(2,6n,12n)$ by Q_2, Q_{6n}, and Q_{12n}, and those of $(2n,4n,6n,12n)$ by R_{2n}, R_{4n}, R_{6n}, and R_{12n}. The ramifications are shown as follows.



Choose Hauptmoduls z_j for Γ_j, $j = 1, \ldots, 3$, by requiring that

$$z_1(P_{4n}) = 0, \quad z_1(P_3) = 1, \quad z_1(P_{12n}) = \infty,$$
$$z_2(Q_{6n}) = 0, \quad z_2(Q_2) = 1, \quad z_2(Q_{12n}) = \infty,$$
$$z_3(R_{2n}) = 0, \quad z_3(R_{4n}) = 1, \quad z_3(R_{6n}) = \infty.$$

It is easy to work out the relation between z_1 and z_3 and that between z_2 and z_3. They are

$$z_1 = \frac{27z_3^2(1-z_3)}{1-9z_3}, \qquad z_2 = -\frac{64z_3^3}{(1-z_3)^3(1-9z_3)}. \tag{6.29}$$

Here $1/9$ is the value of z_3 at R_{12n}. We then follow the same arguments as before to obtain the claimed identities. We omit the details. □

APPENDIX A. LIST OF ARITHMETIC TRIANGLE GROUPS

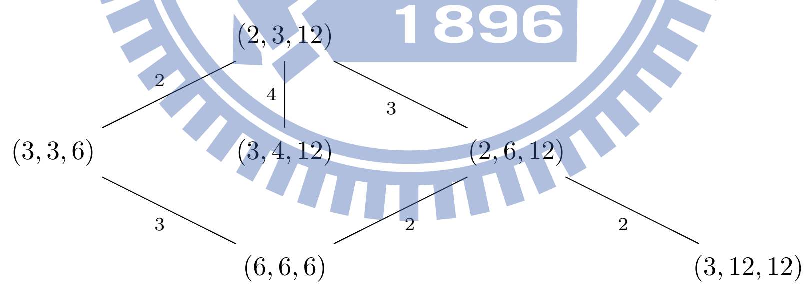
In this section, we determine the signatures of the intersections of commensurable triangle groups.

According to [21], there are totally 85 arithmetic triangle groups, falling in 19 different commensurability classes. Here we give the subgroup diagrams. Note that since most groups here have genus 0, we omit the genus information from the signature, unless the group has a positive genus. Also, to save space, the notation $(g; e_1^{n_1}, \ldots, e_r^{n_r})$ means that the Shimura curve has n_i elliptic points order e_i. Furthermore, for convenience, we will often call the groups by their signatures, even though this raises some ambiguity.

Remark 6.5.4. *There is some ambiguity when we say "the intersections of commensurable triangle groups" because there may be more than one orders whose norm-one groups have the same signature and the intersections of these groups with another group may have different signatures. For example, in the case* $B = M(2, \mathbb{Q})$*, the subgroups* $\Gamma_0(2)$ *and* $\Gamma^0(2)$ *of* $\mathrm{SL}(2, \mathbb{Z})$ *have the same signature* $(0; 2, \infty, \infty)$ *and the group* $\Gamma_0(4)$ *has signature* $(0; \infty, \infty, \infty)$*. The intersection of* $\Gamma_0(2)$ *and* $\Gamma_0(4)$ *is just* $\Gamma_0(4)$*, but the intersection of* $\Gamma^0(2)$ *and* $\Gamma_0(4)$ *has signature* $(0; \infty, \infty, \infty, \infty)$*. Thus, the subgroup diagrams described here should be read as* "there are *arithmetic groups whose subgroup relations are given by the subgroup diagrams"*.

Since it is not easy to describe explicitly the orders associated to arithmetic triangle groups, here we use group theory and properties of discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ to determine the signatures. We will work out the case of Class IV in [21] and omit the proof of the others.

According to [21], Class IV of arithmetic triangle groups has the following subgroup diagram.

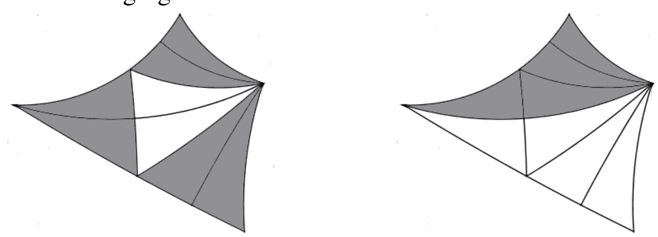


Here the numbers next to the lines are the indices. Set

$$\Gamma_1 = (2, 3, 12), \quad \Gamma_2 = (3, 3, 6), \quad \Gamma_3 = (3, 4, 12),$$
$$\Gamma_4 = (2, 6, 12), \quad \Gamma_5 = (6, 6, 6), \quad \Gamma_6 = (3, 12, 12),$$

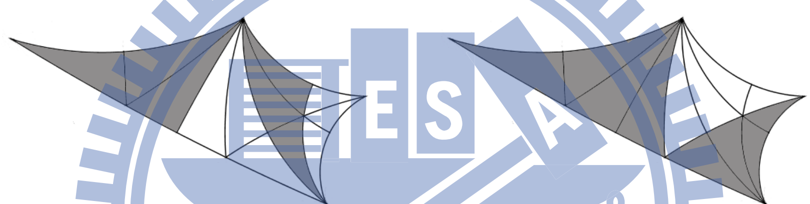
and let X_i, $i = 1, \ldots, 6$, denote the respective Shimura curves. To determine $\Gamma_2 \cap \Gamma_3$, we observe that Γ_2 is a normal subgroup of Γ_1 of index 2 and $\Gamma_1 = \Gamma_2\Gamma_3$. Thus, $\Gamma_2 \cap \Gamma_3$ is a normal subgroup of Γ_3 of index 2. Now the elliptic point of order 3 on X_3 must split into two points in $X(\Gamma_2 \cap \Gamma_3)$ because $2 \nmid 3$. Then from the Riemann-Hurwitz formula, we see that the elliptic points of order 4 and 12 must be ramified.

That is, the curve $X(\Gamma_2 \cap \Gamma_3)$ must have signature $(2,3,3,6)$. In fact, this can also be seen from the following figures.

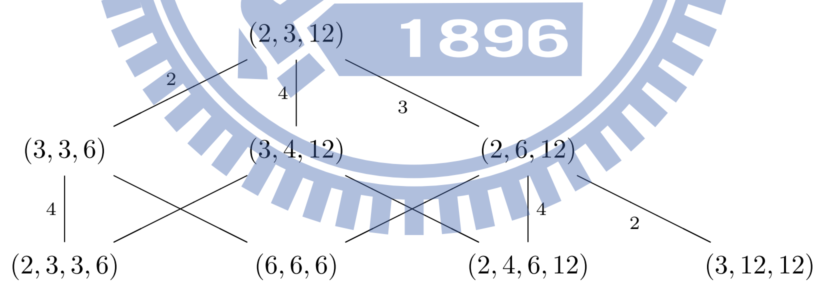


Here the smaller triangles are $(2,3,12)$-triangles. The figures show that the triangle group $(2,3,12)$ contains two subgroups of signatures $(3,3,6)$ and $(3,4,12)$, respectively, whose intersection has signature $(2,3,3,6)$. (In fact, the theoretical argument above shows that for any pair of subgroups of Γ_1 with signatures $(3,3,6)$ and $(3,4,12)$, respectively, the intersection must have signature $(2,3,3,6)$.)

Likewise, the figures



show that there are two subgroups of Γ_1 of signatures $(2,6,12)$ and $(3,4,12)$ such that there intersection has signature $(2,4,6,12)$. We have the following subgroup diagram.



Let $\Gamma_7 = (2,3,3,6)$ and $\Gamma_8 = (2,4,6,12)$ and X_7 and X_8 be their associated Shimura curves. Again, because Γ_5 is a normal subgroup of Γ_4 of index 2 and $\Gamma_5\Gamma_8 = \Gamma_4$, the intersection of Γ_5 and Γ_8 is a subgroup of index 2 of Γ_8. Now the group $(2,4,6,12)$ has many subgroups of index 2. (The structure of the quotient group of $(2,4,6,12)$ over its commutator subgroup is $C_2 \times C_4 \times C_6$.) To determine which of them is contained is the group $(6,6,6)$, we use the following properties.

1. If p is an elliptic point of order e on X_8, then its preimage in the covering $X(\Gamma_5 \cap \Gamma_8) \to X_8$ consists of either a single elliptic point of order $e/2$ or two elliptic points of order e.

2. The total branch number of any finite covering of compact Riemann surface is always even.

3. The volume of $X(\Gamma_5 \cap \Gamma_8)$ is twice of that of X_8. Thus, if $(g; e_1, \dots, e_r)$ is the signature of $X(\Gamma_5 \cap \Gamma_8)$, then we must have

$$2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{e_j}\right) = 2\left(2 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{12}\right) = 2.$$

From these informations, we find that possible signatures of a subgroup of index 2 of $(2, 4, 6, 12)$ are

$$\begin{gathered}(1; 2, 3, 6),\ (0; 2, 6^2, 12^2),\ (0; 3, 4^2, 12^2),\ (0; 4^2, 6^3) \\ (0; 2^3, 3, 12^2),\ (0; 2^3, 6^3),\ (0; 2^2, 3, 4^2, 6).\end{gathered} \tag{6.30}$$

Likewise, an elliptic point of order 6 on X_5 can

1. splits into 4 elliptic points of order 6, or

2. splits into 2 elliptic points of order 3, or

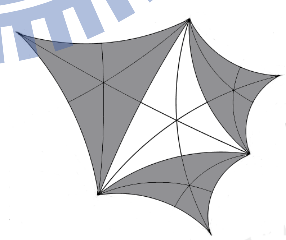
3. splits into 1 elliptic point of order 3 and 2 elliptic point of order 6, or

4. splits into 1 elliptic point of order 2 and 1 elliptic point of order 6,

in the covering $X(\Gamma_5 \cap \Gamma_8) \to X_5$ of degree 4. Also, the total branch number of $X(\Gamma_5 \cap \Gamma_8) \to X_5$ must be a positive even integer and the volume of $X(\Gamma \cap \Gamma_8)$ is 2. We find the possible signatures of a subgroup of index 4 of Γ_5 are

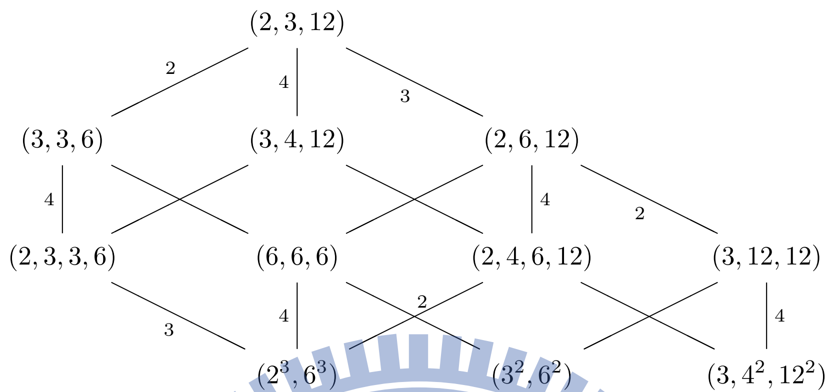
$$(0; 2^3, 6^3),\ (0; 2^2, 3^2, 6^2),\ (0; 2, 3^4, 6),\ (0; 3^6). \tag{6.31}$$

From (6.30) and (6.31), we conclude that the signature of $\Gamma_5 \cap \Gamma_8$ must be $(0; 2^3, 6^3)$. This can also be seen from the figures.



By the same argument, we can also show that the intersection of Γ_6 and Γ_8 must have signature $(0; 3, 4^2, 12^2)$ and the intersection of Γ_5 and Γ_6 has signature $(0; 3, 3, 6, 6)$.

The subgroup diagram becomes



Finally, we can show that the only possible signatures of subgroups of index 2 in $(2^3, 6^3)$ are

$$(0; 2^6, 3^2, 6^2),\ (0; 2^4, 3, 6^4),\ (0; 2^2, 6^6),\ (1; 2^4, 3^3),\ (1; 2^2, 3^2, 6^2),\ (1; 3, 6^4),\ (2; 3^3),$$

while the only possible signatures of subgroups of index 2 in $(3, 4^2, 12^2)$ are

$$(0; 3^2, 4^4, 6^2),\ (0; 2, 3^2, 4^2, 6, 12^2),\ (0; 2^2, 3^2, 12^4), (1; 2^2, 3^2, 6^2).$$

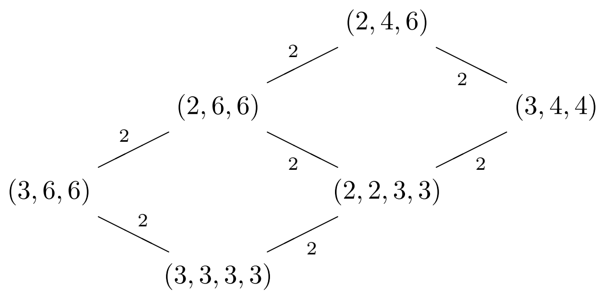
From these, we see that the common intersection of $(2^3, 6^3)$, $(3, 4^2, 12^2)$, and $(3^2, 6^2)$ has signature $(1; 2^2, 3^2, 6^2)$. This completes the proof of the case of Class IV.

Remark 6.5.5. *In literature [7], the decompositions of hyperbolic polygons shown in the figures above are called* Coxeter decompositions. *In general, a Coxeter decomposition is a decomposition of a polygon into finitely many Coxeter polygons such that if two Coxeter polygons share a common side, then they are symmetric with respect to the common side.*

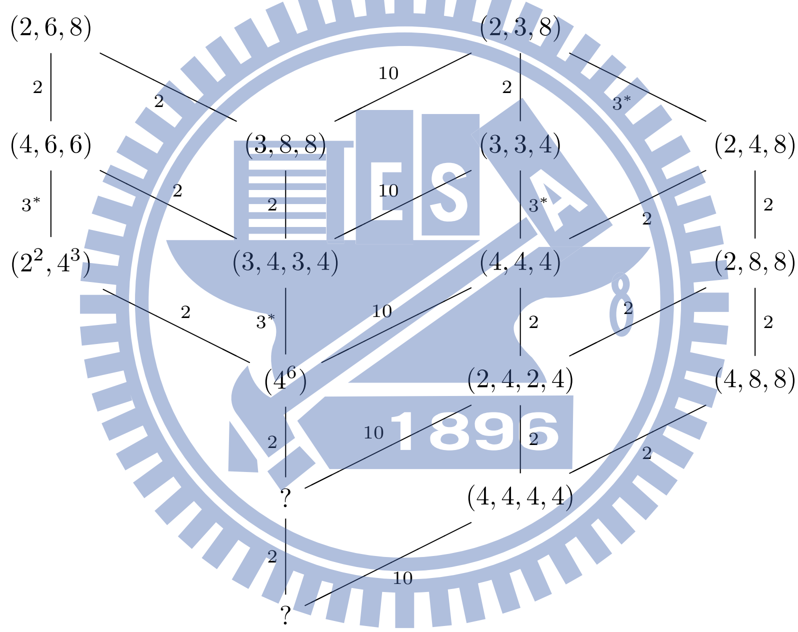
Note that not all subgroup relations given in Appendix A admit Coxeter decomposition. For example, in Class III, the group $(2, 4, 8)$ *is a subgroup of index* 3 *of the group* $(2, 3, 8)$*, but there is no way one can decompose a* $(2, 4, 8)$*-triangle into a union of three* $(2, 3, 8)$*-triangles. In the case of Class IV discussed above, the subgroup relation* $(2^3, 6^3) < (2, 3, 3, 6)$ *does not admit a Coxeter decomposition either.*

Now we give the subgroup diagrams for arithmetic triangle groups.

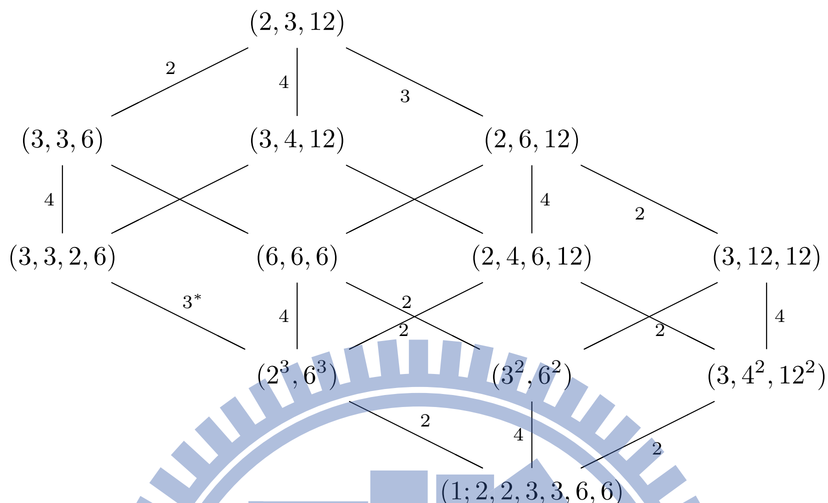
Class II



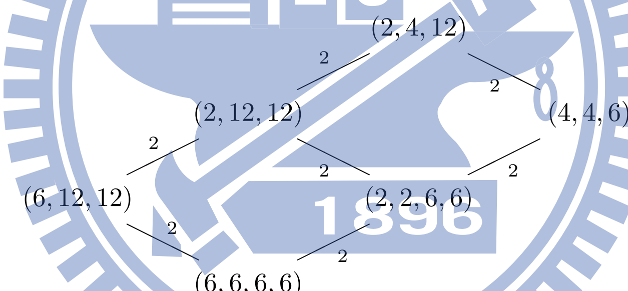
Class III



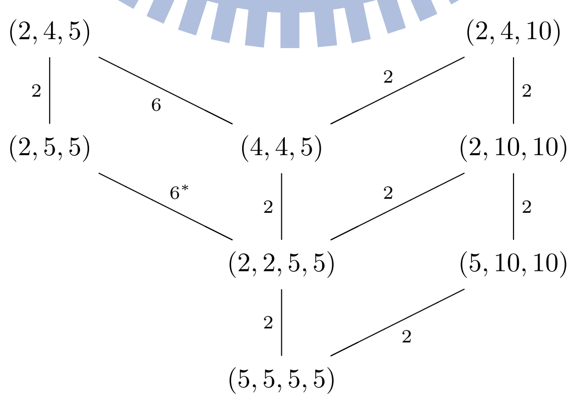
Class IV



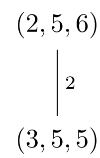
Class V



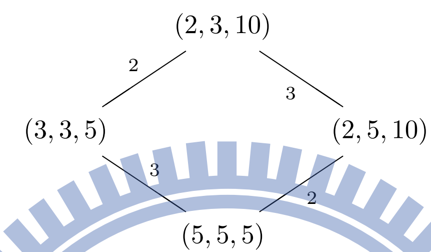
Class VI



Class VII



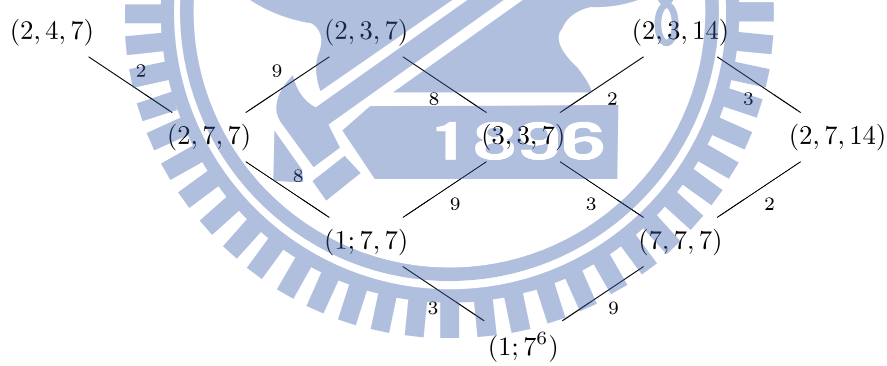
Class VIII



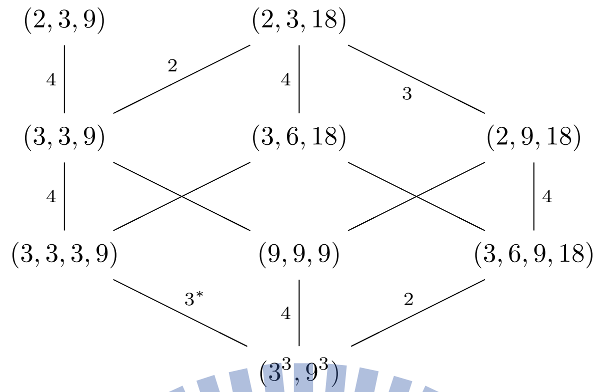
Class IX

$(3, 4, 6)$

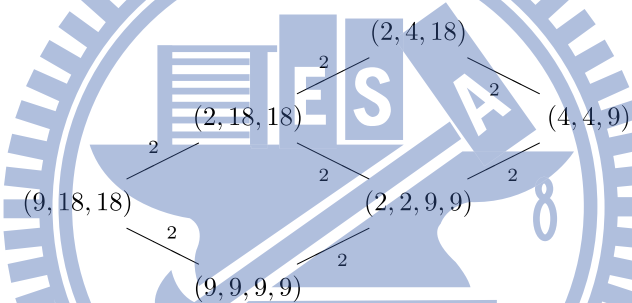
Class X



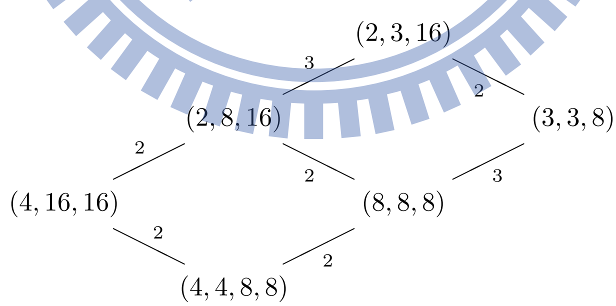
Class XI



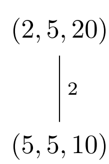
Class XII



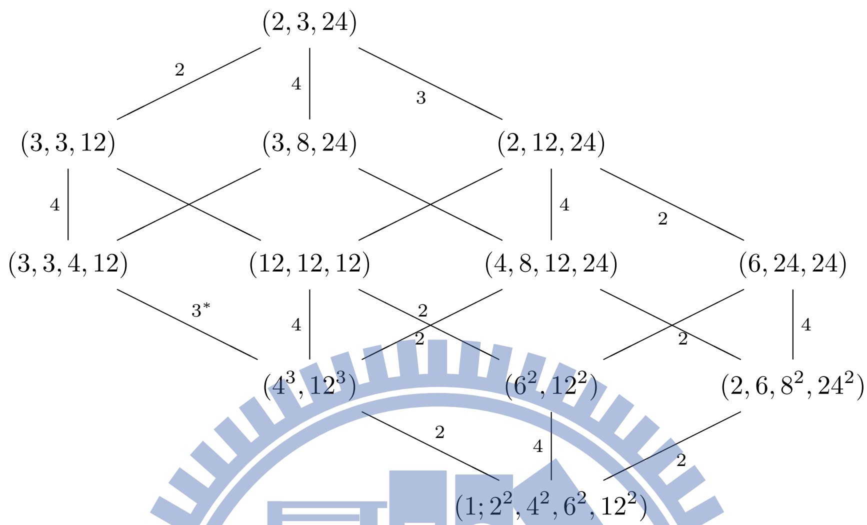
Class XIII



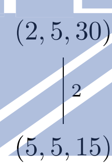
Class XIV



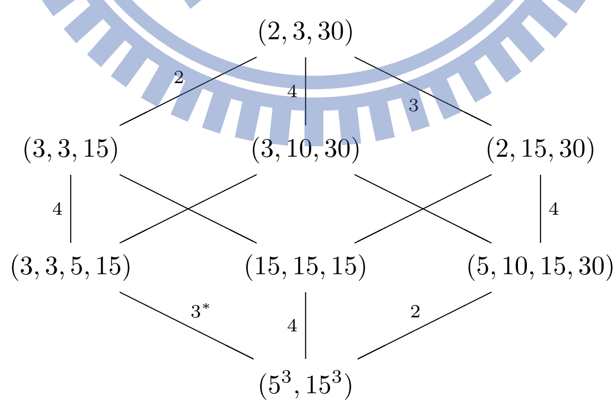
Class XV



Class XVI



Class XVII



Class XVIII

$$(2,5,8)$$

$$\Big|\,{\scriptstyle 2}$$

$$(4,5,5)$$

Class XIX

$$(2,3,11)$$

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