J. Math. Anal. Appl. 352 (2009) 669–671

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A simple proof for persistence of snap-back repellers

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ARTICLE INFO

Article history:
Received 4 April 2008
Available online 13 November 2008
Submitted by Y. Huang

Keywords:
Snap-back repeller
Implicit function theorem
Li–Yorke chaos
Topological entropy

ABSTRACT

In this article, we show that if f has a snap-back repeller then any small C^1 perturbation of f has a snap-back repeller, and hence has Li–Yorke chaos and positive topological entropy, by simply using the implicit function theorem. We also give some examples.

Recently, Marotto [16] redefined snap-back repellers and stated that his early result in [12]: "a snap-back repeller implies Li–Yorke chaos" is still correct. Both definitions of snap-back repellers in [12,16] depend on the norms of the phase space. Based on Marotto's argument, Blanco García [2] showed that a snap-back repeller implies positive topological entropy.

In this article, we give a slightly different definition so that it is independent of norms and the mentioned results of Marotto and Blanco García still hold obviously. By using the implicit function theorem in Banach spaces (refer to Lang's textbook [8, Theorem 6.2.1]), we give a simple proof that any small C^1 perturbation of a (possibly noninvertible) system with a snap-back repeller has a snap-back repeller and exhibits chaos. Some examples are demonstrated as applications.

In [13–15], Marotto has studied perturbations of snap-back repellers and mainly showed that if the scalar problem $x_{n+1} = f(x_n, 0)$ has a snap-back repeller then the problem $x_{n+1} = f(x_n, \lambda x_{n-1})$ has a transverse homoclinic point, hence has chaotic dynamics, whenever λ is close to 0. His methodology heavily replies on the Birkhoff–Smale theorem on transverse homoclinic points in two gradients: one is smooth perturbations of stable and unstable manifolds (see [7, Theorem 5.1]) and the other one is persistence of transverse intersection of stable and unstable manifolds (see [1, Theorem 18.2]). For the case when the map is noninvertible and has no global stable/unstable manifolds, one needs to further use a generalization of the Birkhoff–Smale theorem (refer to [6, Theorem 5.2] and [19, Theorem 5.1]).

There are recent developments on multidimensional perturbations of lower dimensional systems; refer to [10] for snap-back repellers, [5] for chaotic interval maps, and [11] for chaotic difference schemes.

First, we give the definition of a snap-back repeller.

Definition 1. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a differentiable function. A fixed point w_0 for f is called a *snap-back repeller* if (i) all eigenvalues of $Df(w_0)$ are greater than one in absolute value and (ii) there exists a sequence $\{w_{-n}\}_{n\in\mathbb{N}}$ such that $w_{-1} \neq w_0$, $\lim_{n\to\infty} w_{-n} = w_0$, and for all $n \in \mathbb{N}$, $f(w_{-n}) = w_{-n+1}$ and $\det(Df(w_{-n})) \neq 0$.

Roughly speaking, a snap-back repeller of a map is a repelling fixed point associated with which there is a transverse homoclinic point. Note that if there exists a norm $|\cdot|_*$ on $\mathbb{R}^k$ such that for some constants $\delta > 0$ and $\lambda > 1$, one has that $|f(x) - f(y)|_* > \lambda|x - y|_*$ for all $x, y \in B(w_0, \delta)$, where $B(w_0, \delta) = \{x \in \mathbb{R}^k\colon\ |x - w_0|_* < \delta\}$, then f is one-to-one on $B(w_0, \delta)$ and $f(B(w_0, \delta)) \supset B(w_0, \delta)$; hence item (ii) of the above definition can be satisfied if there is a point $q \in B(w_0, \delta)$ such that

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.11.021

$f^m(q) = w_0$ and $\det(Df^m(q)) \neq 0$ for some positive integer m. In fact, item (i) implies that such a norm must exist (refer to [18, Theorem V.6.1]). Furthermore, if all eigenvalues of $(Df(w_0))^T Df(w_0)$ are greater than one, then such a norm can be chosen to be the Euclidean norm on $\mathbb{R}^k$ (see [9, Lemma 5]).

It was proved by Marotto [12] and Blanco García [2] that a snap-back repeller implies Li–Yorke chaos and positive topological entropy, respectively.

Theorem 2. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ possess a snap-back repeller. Then f exhibits* Li–Yorke chaos, *that is, there exist*

1. *a positive integer N such that if $m \geqslant N$ is an integer, the map f has a point of period m;*
2. *an uncountable set S containing no periodic points of f such that*
   (a) *if $x, y \in S$ with $x \neq y$, then $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$;*
   (b) *if $x \in S$ and y is a periodic point for f, then $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$;*
   (c) *$f(S) \subset S$; and*
3. *an uncountable subset S_0 of S such that if $x, y \in S_0$, then $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$.*

Moreover, f has positive topological entropy; here the topological entropy of f is defined to be the supremum of topological entropies of f restricted to compact invariant sets; refer to [18] *for the latter.*

We show the persistence of snap-back repellers for small C^1 perturbations. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^k$ and $\|\cdot\|$ be the operator-norm on the space of linear maps on $\mathbb{R}^k$ induced by $|\cdot|$.

Theorem 3. *Let f be a C^1 map on $\mathbb{R}^k$ with a snap-back repeller. If g is a C^1 map on $\mathbb{R}^k$ such that $|f - g| + \|Df - Dg\|$ is small enough, then g has a snap-back repeller, exhibits Li–Yorke chaos, and has positive topological entropy.*

Proof. Let x_0 be a snap-back repeller of f and $\{x_{-n}\}_{n\in\mathbb{N}}$ be its corresponding homoclinic orbit with $x_{-1} \neq x_0$, $\lim_{n\to\infty} x_{-n} = x_0$, and for all $n \in \mathbb{N}$, $f(x_{-n}) = x_{-n+1}$ and $\det(Df(x_{-n})) \neq 0$. Since x_0 is a fixed point of f and all eigenvalues of $Df(x_0)$ are greater than one in absolute value, there exists a norm $|\cdot|_*$ on $\mathbb{R}^k$ such that for some constants $\delta_0 > 0$ and $\lambda_0 > 1$, one has that $|f(x) - f(y)|_* > \lambda_0|x - y|_*$ for all $x, y \in B(x_0, \delta_0)$, where $B(x_0, \delta_0) = \{x \in \mathbb{R}^k\colon\ |x - x_0|_* < \delta_0\}$. Thus f is one-to-one on $B(x_0, \delta_0)$ and $f(B(x_0, \delta_0)) \supset B(x_0, \delta_0)$. Let $\|\cdot\|_*$ denote the operator-norm in the space of linear maps on $\mathbb{R}^k$ induced by $|\cdot|_*$. Let λ_1 be a constant with $1 < \lambda_1 < \lambda_0$ and let $U(f, \lambda_0 - \lambda_1)$ denote the set of all C^1 maps g on $\mathbb{R}^k$ with $|f - g|_* + \|Df - Dg\|_* < \lambda_0 - \lambda_1$. Then for any $g \in U(f, \lambda_0 - \lambda_1)$ and $x, y \in B(x_0, \delta_0)$, we have that

$$\big|g(x) - g(y)\big|_* \geqslant \big|f(x) - f(y)\big|_* - \big|(g - f)(x) - (g - f)(y)\big|_* > \big[\lambda_0 - (\lambda_0 - \lambda_1)\big]|x - y|_* = \lambda_1|x - y|_*; \tag{1}$$

hence, g is one-to-one on $B(x_0, \delta_0)$.

Let $\delta > \delta_0$ be a constant so that $\{x_{-n}\}_{n\in\mathbb{N}} \subset B(x_0, \delta)$. Denote by W the closure of $B(x_0, \delta)$. Then W is a compact subset of $\mathbb{R}^k$. Let S be the space of C^1 functions from W to $\mathbb{R}^k$ endowed with the usual C^1 topology d_{C^1} which is induced from the norm $|\cdot|_*$ on $\mathbb{R}^k$. Then S is a Banach space and the restriction of any C^1 map g on $\mathbb{R}^k$ to W, denoted by $g|W$, is in S. Since x_0 is a snap-back repeller of f and all eigenvalues of $Df(x_0)$ are greater than one in absolute value, there exist positive constants λ_2, δ_1 and a positive integer M such that $\lambda_1 < \lambda_2 < \lambda_0$, $\delta_1 < \delta_0$, $x_{-M} \in B(x_0, \delta_1)\backslash\{x_0\}$, $\det(Df^M(x_{-M})) \neq 0$, $x_0 \in int(f^M(B(x_0, \delta_1)\backslash\{x_0\}))$ and for all $g \in U(f, \lambda_0 - \lambda_2)$ and $x \in B(x_0, \delta_1)$, all eigenvalues of $Dg(x)$ are greater than one in absolute value. Let λ_3 be a constant such that

$$\max\left\{\lambda_2, \frac{\lambda_0 + \delta_1}{1 + \delta_1}\right\} < \lambda_3 < \lambda_0. \tag{2}$$

Then for any $g \in U_W(f, \lambda_0 - \lambda_3)$, we have that g is one-to-one on $B(x_0, \delta_1)$. In addition, if $x \in \mathbb{R}^k$ with $|x - x_0|_* = \delta_1$, by Eq. (1) with λ_1 replaced by λ_3 and Eq. (2), we get that

$$\big|g(x) - x_0\big|_* \geqslant \big|f(x) - x_0\big|_* - \big|g(x) - f(x)\big|_* > \lambda_3\delta_1 - (\lambda_0 - \lambda_3) > \delta_1.$$

Moreover, the continuity of g implies that $g(B(x_0, \delta_1)) \supset B(x_0, \delta_1)$. Let $V = B(x_0, \delta_1)\backslash\{x_0\}$ and $U_W(f, \lambda_0 - \lambda_3) = \{g|W\colon\ g \in U(f, \lambda_0 - \lambda_3)\}$.

For the first desired result, we need to show the existence of a snap-back repeller for any $g \in U_W(f, \lambda_0 - \lambda_3)$ near f. Define $H : U_W(f, \lambda_0 - \lambda_3) \times W \times V \to \mathbb{R}^k \oplus \mathbb{R}^k$ by $H(g, x, y) = (g(x) - x, g^M(y) - x)$. Then $H(f, x_0, x_{-M}) = 0$ and H is C^1 on its domain; refer to [4, Appendix B]. Since all eigenvalues of $Df(x_0)$ are greater than one in absolute value, we have $\det(Df(x_0) - I_k) \neq 0$, where I_k denotes the identity matrix of size k; refer to [18, Lemma V.5.7.2]. By the chain rule, $\det(Df^M(x_{-M})) = \prod_{i=1}^M \det(Df(x_{-i})) \neq 0$. Hence, by writing $z = (x, y) \in W \times V$, we have

$$\det\left(\frac{\partial H}{\partial z}(g, z)\bigg|_{g=f,\, z=(x_0,x_{-M})}\right) = \det\begin{bmatrix} Df(x_0) - I_k & 0 \\ -I_k & Df^M(x_{-M}) \end{bmatrix} \neq 0;$$

refer to [17, Proposition 0.0]. By the implicit function theorem applied to the function H, there exist positive constants $\lambda_4, \delta_2, \eta$ and a C^1 map $h : U_W(f, \lambda_0 - \lambda_4) \to B(x_0, \delta_2) \times B(x_{-M}, \eta)$ such that $\lambda_3 < \lambda_4 < \lambda_0$, $\delta_2 < \delta_1, B(x_{-M}, \eta) \subset V$,

$B(x_0, \delta_2) \cap B(x_{-M}, \eta) = \emptyset$, and for every $g \in U_W(f, \lambda_0 - \lambda_4)$, one has that $h(g) \equiv (h_1(g), h_2(g))$ is the unique solution for the system of equations $g(x) = x$ and $g^M(y) = x$ in $B(x_0, \delta_2) \times B(x_{-M}, \eta)$, and $\det(Dg^M(h_2(g))) \neq 0$. In particular, $h(f) = (x_0, x_{-M})$.

To conclude that the point $h_1(g)$ is a snap-back repeller of g, it remains to show that $h_2(g)$ has a backward orbit converging to $h_1(g)$. Let $g \in U_W(f, \lambda_0 - \lambda_4)$ and denote $y_{-M+i} = g^i(h_2(g))$ for all $0 \leqslant i \leqslant M - 1$. Then $y_{-M} \neq h_1(g)$ and $g^M(y_{-M}) = h_1(g)$. Since g is one-to-one on $B(x_0, \delta_1)$, $g(B(x_0, \delta_1)) \supset B(x_0, \delta_1)$ and $h_2(g) \in B(x_0, \delta_1)$, we can define $y_{-M-i} = \hat{g}^{-i}(h_2(g))$ inductively for $i \geqslant 1$, where $\hat{g}^{-1} = (g|B(x_0, \delta_1))^{-1}$ denotes the inverse of the restriction of g to $B(x_0, \delta_1)$ and $\hat{g}^{-i}$ denotes the ith iterate of $\hat{g}^{-1}$. Then the sequence $\{y_{-i}\}_{i \in \mathbb{N}}$ forms a backward orbit of $h_1(g)$ such that $y_{-n} \in B(x_0, \delta_1)$ for all $n \geqslant M$. From Eq. (1), we obtain that for any $x, y \in B(x_0, \delta_1)$,

$$\left|\hat{g}^{-1}(x) - \hat{g}^{-1}(y)\right|_* < \lambda_1^{-1}|x - y|_*. \tag{3}$$

By considering inequality (3) inductively, we have that for any $i \geqslant 1$,

$$\left|y_{-M-i} - h_1(g)\right|_* = \left|\hat{g}^{-i}(y_{-M}) - \hat{g}^{-i}\big(h_1(g)\big)\right|_* < \lambda_1^{-i}\left|y_{-M} - h_1(g)\right|_*.$$

This shows that $\lim_{n \to \infty} y_{-n} = h_1(g)$.

Since the norms $|\cdot|$ and $|\cdot|_*$ on $\mathbb{R}^k$ are equivalent, the proof of the first desired result is now complete. The second and third assertions immediately follow from Theorem 2. □

Notice that from the above proof of Theorem 3, it is sufficient to require a smallness of $|f - g| + \|Df - Dg\|$ locally in a neighborhood of the homoclinic orbit associated to the snap-back repeller, instead of globally in $\mathbb{R}^k$.

As an immediate consequence of the above theorem, we have the following result for a parametrized family.

Corollary 4. *Let $f_\mu(x)$ be a one-parameter family of C^1 maps with variable $x \in \mathbb{R}^k$ and parameter $\mu \in \mathbb{R}^\ell$. Assume that $f_\mu(x)$ is C^1 as a function jointly of x and μ and that f_{μ_0} has a snap-back repeller. Then for all μ sufficiently close to μ_0, the map f_μ has a snap-back repeller, exhibits Li–Yorke chaos, and has positive topological entropy.*

Next is another application to perturbations of a decoupled system.

Corollary 5. *Let $f_\epsilon : \mathbb{R}^k \to \mathbb{R}^k$ be a one-parameter family of C^1 maps with components $(f_\epsilon)_i(x) = h_i(x_i) + \epsilon_i g_i(x)$ for each $1 \leqslant i \leqslant k$; here we denote the variable $x = (x_1, \ldots, x_k)$ and the parameter $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ in $\mathbb{R}^k$. If the number of snap-back repellers for each map h_i is $m_i \geqslant 1$, then for all sufficiently small $|\epsilon|$, the number of snap-back repellers for the map f_ϵ is at least $\prod_{i=1}^k m_i$.*

Gardini et al. [3] studied the double logistic map $T_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_\lambda(x, y) = \big((1 - \lambda)x + 4\lambda y(1 - y), (1 - \lambda)y + 4\lambda x(1 - x)\big), \quad \lambda \in [0, 1]; \tag{4}$$

therein the basins of attraction of the absorbing areas are determined together with their bifurcations. Moreover, it was mentioned that $T_1^2(x, y) = (h^2(x), h^2(y))$, where $h(x) = 4x(1 - x)$, has a snap-back repeller at the origin. Therefore, applying Corollary 5, we have the following result.

Corollary 6. *For all λ near one, the second iterate of system* (4) *has a snap-back repeller, exhibits Li–Yorke chaos, and has positive topological entropy.*

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