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On the spanning fan-connectivity of graphs☆

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ABSTRACT

Let G be a graph. The connectivity of G, $\kappa(G)$, is the maximum integer k such that there exists a k-container between any two different vertices. A *k-container* of G between u and v, $C_k(u, v)$, is a set of k-internally-disjoint paths between u and v. A spanning container is a container that spans $V(G)$. A graph G is *k^*-connected* if there exists a spanning k-container between any two different vertices. The *spanning connectivity* of G, $\kappa^*(G)$, is the maximum integer k such that G is w^*-connected for $1 \le w \le k$ if G is 1^*-connected.

Let x be a vertex in G and let $U = \{y_1, y_2, \ldots, y_k\}$ be a subset of $V(G)$ where x is not in U. A *spanning $k-(x, U)$-fan*, $F_k(x, U)$, is a set of internally-disjoint paths $\{P_1, P_2, \ldots, P_k\}$ such that P_i is a path connecting x to y_i for $1 \le i \le k$ and $\cup_{i=1}^{k} V(P_i) = V(G)$. A graph G is *k^*-fan-connected* (or *k_f^*-connected*) if there exists a spanning $F_k(x, U)$-fan for every choice of x and U with $|U| = k$ and $x \notin U$. The spanning fan-connectivity of a graph G, $\kappa_f^*(G)$, is defined as the largest integer k such that G is w_f^*-connected for $1 \le w \le k$ if G is 1_f^*-connected.

In this paper, some relationship between $\kappa(G)$, $\kappa^*(G)$, and $\kappa_f^*(G)$ are discussed. Moreover, some sufficient conditions for a graph to be k_f^*-connected are presented. Furthermore, we introduce the concept of a spanning pipeline-connectivity and discuss some sufficient conditions for a graph to be k^*-pipeline-connected.

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1. Introduction

For graph definitions and notations, we follow [1]. A graph $G = (V, E)$ consists of a finite set V $(=(V(G)))$ and a subset E $(=(E(G)))$ of $\{(u, v) \mid u \neq v$ and (u, v) is an unordered pair of elements of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. We use $n(G)$ to denote $|V(G)|$. Two vertices u and v are *adjacent* if $(u, v) \in E$. A graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a subset of $V(G)$. The subgraph of G *induced* by S, denoted $G[S]$, is the graph with the vertex set S and the edge set $\{(u, v) \mid (u, v) \in E(G)$ and $u, v \in S\}$. We use $G - S$ to denote the graph $G[V(G) - S]$. A *vertex cut* is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. A graph is *k-connected* if every vertex cut has at least k vertices. The *connectivity* of G, $\kappa(G)$, is the minimum size of a vertex cut. In other words, $\kappa(G)$ is the maximum k such that G is k-connected. A complete graph has no cut set. We adopt the convention that $\kappa(K_n) = n - 1$ where K_n is the complete graph with n vertices. A *path* is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k \rangle$ with no repeated vertex and (v_i, v_{i+1}) is an edge of G for $0 \le i \le k - 1$. We also write the path $\langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle$ where Q is a subpath from v_i to v_j. A *hamiltonian path* of a graph G is a path that contains all vertices of $V(G)$. A graph G is *hamiltonian connected* if there is a hamiltonian path between every two different vertices. A *cycle* is a path with at least three vertices

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such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverse every vertex of G. A graph is *hamiltonian* if it has a hamiltonian cycle. Let $P = \langle x_1, x_2, \ldots, x_k\rangle$ be a path of the graph G connecting x_1 and x_k. We use P^{-1} to denote the path $\langle x_k, x_{k-1}, \ldots, x_1\rangle$. We use $V(P)$ to denote the set $\{x_1, x_2, \ldots, x_k\}$ and $I(P)$ to denote the set $V(P) - \{x_1, x_k\}$. Let P_1 and P_2 be two paths of a graph G. We say that P_1 and P_2 are *internally-disjoint* if $I(P_1) \cap I(P_2) = \emptyset$.

Let u and v be two vertices of a graph G. A *k-container* of G between u and v, $C_k(u, v)$, is a set of k-internally-disjoint paths between u and v [2]. It follows from the Menger Theorem [3] that there is a k-container between any two distinct vertices of G if and only if G is k-connected. A k-container $C_k(u, v) = \{P_1, P_2, \ldots, P_k\}$ of G is a k^*-container if $\cup_{i=1}^{k} V(P_i) = V(G)$. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices. The *spanning connectivity* of G, $\kappa^*(G)$, is defined as the largest integer k such that G is w^*-connected for $1 \le w \le k$ if G is a 1^*-connected graph. It is obvious that a 1^*-connected graph is actually a hamiltonian connected graph and that a 2^*-connected graph is actually a hamiltonian graph. Moreover, any 1^*-connected graph except K_1 and K_2 is 2^*-connected. Thus, the concept of a k^*-connected graph is a hybrid concept of connectivity and hamiltonicity. Recently, the spanning connectivity of graphs have been studied extensively [4–10].

There is a Menger type theorem similar to the spanning connectivity of a graph. Let x be a vertex in a graph G and let $U = \{y_1, y_2, \ldots, y_t\}$ be a subset of $V(G)$ where x is not in U. A *t-(x, U)-fan*, $F_t(x, U)$, is a set of internally-disjoint paths $\{P_1, P_2, \ldots, P_t\}$ such that P_i is a path connecting x and y_i for $1 \le i \le t$. It is proved by Dirac [11] that a graph G is k-connected if and only if it has at least $k + 1$ vertices and there exists a t-(x, U)-fan for every choice of x and U with $|U| \le k$ and $x \notin U$. Similarly, we can introduce the concept of a spanning fan. A *spanning k-(x, U)-fan* is a k-(x, U)-fan $\{P_1, P_2, \ldots, P_k\}$ such that $\cup_{i=1}^{k} V(P_i) = V(G)$. A graph G is *k^*-fan-connected* (also written as *k_f^*-connected*) if there exists a spanning k-(x, U)-fan for every choice of x and U with $|U| = k$ and $x \notin U$. The *spanning fan-connectivity* of a graph G, $\kappa_f^*(G)$, is defined as the largest integer k such that G is w_f^*-connected for $1 \le w \le k$ if G is a 1_f^*-connected graph. In this paper, some relationship among $\kappa(G)$, $\kappa^*(G)$, and $\kappa_f^*(G)$ are discussed. Moreover, some sufficient conditions for a graph to be k_f^*-connected are presented.

There is another Menger type theorem similar to the spanning connectivity and spanning fan-connectivity of a graph. Let $U = \{x_1, x_2, \ldots x_t\}$ and $W = \{y_1, y_2, \ldots, y_t\}$ be two t-subsets of $V(G)$. A *(U, W)-pipeline* is a set of internally-disjoint paths $\{P_1, P_2, \ldots, P_t\}$ such that P_i is a path connecting x_i to $y_{\pi(i)}$ where π is a permutation of $\{1, 2, \ldots, t\}$. It is known that a graph G is k-connected if and only if it has at least $k + 1$ vertices and there exists a (U, W)-pipeline for every choice of U and W with $|U| = |W| \le k$ and $U \ne W$. Similarly, we can introduce the concept of spanning pipeline. A *spanning (U, W)-pipeline* is a (U, W)-pipeline $\{P_1, P_2, \ldots, P_k\}$ such that $\cup_{i=1}^{k} V(P_i) = V(G)$. A graph G is *k^*-pipeline-connected* (or *k_p^*-connected*) if there exists a spanning (U, W)-pipeline for every choice of U and W with $|U| = |W| \le k$ and $U \ne W$. The *spanning pipeline-connectivity* of a graph G, $\kappa_p^*(G)$, is defined as the largest integer k such that G is w_p^*-connected for $1 \le w \le k$ if G is a 1_p^*-connected graph.

In Section 2, we establish some relationships among $\kappa(G)$, $\kappa^*(G)$, and $\kappa_f^*(G)$. Section 3 gives sufficient conditions for a graph to be k_f^*-connected. In Section 4, spanning pipeline-connectivity is included. Section 5 gives an example to illustrate the differences between $\kappa(G)$, $\kappa^*(G)$, $\kappa_f^*(G)$, and $\kappa_p^*(G)$.

2. Relationship among $\kappa(G)$, $\kappa^*(G)$, and $\kappa_f^*(G)$

Let u be a vertex of G and let H be a subgraph of G. The *neighborhood* of u with respect to H, denoted by $N_H(u)$, is $\{v \in V(H) \mid (u, v) \in E(G)\}$. We use $d_H(u)$ to denote $|N_H(u)|$. For any vertex u, the *degree* of u in G is $d_G(u)$. The *minimum degree* of G, written $\delta(G)$, is $\min\{d_G(x) \mid x \in V\}$. Let u and v be any two non-adjacent vertices of G, we use $G + (u, v)$ to denote the graph obtained from G by adding the edge (u, v).

Lemma 1. *Every 1^*-connected graph is 1_f^*-connected. Moreover, every 1_f^*-connected graph that is not K_2 is 2_f^*-connected. Thus, $\kappa_f^*(G) \ge 2$ if G is a hamiltonian connected graph with at least three vertices.*

Proof. Let G be a 1^*-connected graph with at least three vertices and let x be any vertex of G. Assume that $U = \{y\}$ with $x \ne y$. Obviously, there exists a hamiltonian path P_1 joining x and y. Apparently, $\{P_1\}$ forms a spanning 1-(x, U)-fan. Thus, G is 1_f^*-connected. Assume that $U = \{y_1, y_2\}$ with $x \notin U$. Let Q be a hamiltonian path of G connecting y_1 and y_2. We write Q as $\langle y_1, Q_1, x, Q_2, y_2\rangle$. We set P_1 as $\langle x, Q_1^{-1}, y_1\rangle$ and P_2 as $\langle x, Q_2, y_2\rangle$. Then $\{P_1, P_2\}$ forms a spanning 2-(x, U)-fan. Thus, G is 2_f^*-connected and $\kappa_f^*(G) \ge 2$. □

Theorem 1. *$\kappa_f^*(G) \le \kappa^*(G) \le \kappa(G)$ for any 1_f^*-connected graph. Moreover, $\kappa_f^*(G) = \kappa^*(G) = \kappa(G) = n(G) - 1$ if and only if G is a complete graph.*

Proof. Obviously, $\kappa^*(G) \le \kappa(G)$. Now, we prove that $\kappa_f^*(G) \le \kappa^*(G)$. Assume that $\kappa_f^*(G) = k$. Let x and y be any two vertices of G. We need to show that there is a k^*-container of G between x and y.

Suppose that $k = 1$. Since G is 1_f^*-connected, there is a spanning 1-$(x, \{y\})$-fan, $\{P_1\}$, of G. Then $\{P_1\}$ forms a spanning container of G between x and y.

Suppose that $k \ge 2$. Let $U' = \{y_1, y_2, \ldots, y_{k-1}\}$ be a set of $(k-1)$ neighbors of y not containing x. We set $U = U' \cup \{y\}$. By assumption, there exists a spanning k-(x, U)-fan. Obviously, we can extend the spanning k-(x, U)-fan by adding the edges

$\{(y_i, y) \mid y_i \in U'\}$ to obtain a k^*-container between x to y. Hence, G is k^*-connected. Therefore, $\kappa_f^*(G) \le \kappa^*(G)$ for every 1_f^*-connected graph.

Suppose that G is not a complete graph. There exists a vertex cut S of size $\kappa(G)$. Let x and y be any two vertices in different connected components of $G - S$. Obviously, y is not in any (x, S)-fan of G. Thus, $\kappa_f^*(G) < \kappa(G)$. □

3. Some sufficient conditions for a graph to be k_f^*-connected

Since the concept of spanning fan-connectivity is a generalization of hamiltonicity, we review some previews results concerning hamiltonian graphs and hamiltonian connected graphs.

Lemma 2 ([12]). *Every graph G with at least three vertices and $\delta(G) \ge \frac{n(G)}{2}$ is 2^*-connected. Moreover, every graph G with at least four vertices and $\delta(G) \ge \frac{n(G)}{2} + 1$ is 1^*-connected.*

Lemma 3 ([13,14]). *Let u and v be two non-adjacent vertices of G with $d_G(u) + d_G(v) \ge n(G)$. Then G is 2^*-connected if and only if $G + (u, v)$ is 2^*-connected. Moreover, suppose that $d_G(u) + d_G(v) \ge n(G) + 1$, then G is 1^*-connected if and only if $G + (u, v)$ is 1^*-connected.*

Lemma 4 ([15]). *A graph G is 2^*-connected if $d_G(u) + d_G(v) \ge n(G)$ for all non-adjacent vertices u and v. Moreover, a graph G is 1^*-connected if $d_G(u) + d_G(v) \ge n(G) + 1$ for all non-adjacent vertices u and v.*

For comparison, we list the preview results concerning spanning connectivity.

Lemma 5 ([8]). *$\kappa^*(G) \ge 2\delta(G) - n(G) + 2$ if $\frac{n(G)}{2} + 1 \le \delta(G) \le n(G) - 2$.*

Lemma 6 ([9]). *Let k be a positive integer. Suppose that u and v are two non-adjacent vertices of G with $d_G(u)+d_G(v) \ge n(G)+k$. Then $\kappa^*(G) \ge k + 2$ if and only if $\kappa^*(G + (u, v)) \ge k + 2$.*

Lemma 7 ([9]). *Let k be a positive integer. Then $\kappa^*(G) \ge k + 2$ if $d_G(u) + d_G(v) \ge n(G) + k$ for all non-adjacent vertices u and v.*

Note that Lemma 5 (Lemmas 6 and 7, respectively) generalizes the result of Lemma 2, (Lemmas 3 and 4, respectively) in spanning connectivity. The following theorem on spanning fan-connectivity is analogous to that on spanning connectivity in Lemma 6 [9].

Lemma 8. *Let u and v be two non-adjacent vertices of G with $d_G(u) + d_G(v) \ge n(G) + 1$, and let x and y be any two distinct vertices of G. Then G has a hamiltonian path joining x to y if and only if $G + (u, v)$ has a hamiltonian path joining x to y.*

Proof. Since every path in G is a path in $G+(u, v)$, there is a hamiltonian path of $G+(u, v)$ joining x to y if G has a hamiltonian path joining x to y.

Suppose that there is a hamiltonian path P of $G + (u, v)$ joining x to y. We need to show that there is a hamiltonian path of G between x and y. If $(u, v) \notin E(P)$, then P is a hamiltonian path of G between x and y. Thus, we consider that $(u, v) \in E(P)$. Without loss of generality, we write P as $\langle z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{n(G)}\rangle$ where $z_1 = x, z_i = u, z_{i+1} = v$, and $z_{n(G)} = y$. Since $d_G(u) + d_G(v) \ge n(G) + 1$, there is an index k in $\{1, 2, \ldots, n(G)\} - \{i - 1, i, i + 1\}$ such that $(z_i, z_k) \in E(G)$ and $(z_{i+1}, z_{k+1}) \in E(G)$. We set $R = \langle z_1, z_2, \ldots, z_k, z_i, z_{i-1}, \ldots, z_{k+1}, z_{i+1}, z_{i+2}, \ldots, z_{n(G)}\rangle$ if $1 \le k \le i-2$ and $R = \langle z_1, z_2, \ldots, z_i, z_k, z_{k-1}, \ldots, z_{i+1}, z_{k+1}, z_{k+2}, \ldots, z_{n(G)}\rangle$ if $i + 2 \le k \le n(G)$. Then R is a hamiltonian path of G between x and y. □

Theorem 2. *Assume that k is a positive integer. Let u and v be two non-adjacent vertices of G with $d_G(u) + d_G(v) \ge n(G) + k$. Then $\kappa_f^*(G) \ge k + 1$ if and only if $\kappa_f^*(G + (u, v)) \ge k + 1$.*

Proof. Obviously, $\kappa_f^*(G + (u, v)) \ge k + 1$ if $\kappa_f^*(G) \ge k + 1$. Suppose that $\kappa_f^*(G + (u, v)) \ge k + 1$. Let x be any vertex of G and $U = \{y_1, y_2, \ldots, y_t\}$ be any subset of $V(G)$ such that $x \notin U$ and $t \le k + 1$. We need to find a spanning t-(x, U)-fan of G.

Since $G + (u, v)$ is $(k + 1)_f^*$-connected, there exists a spanning t-(x, U)-fan $\{P_1, P_2, \ldots, P_t\}$ of $G + (u, v)$ with P_i joining x to y_i for $1 \le i \le t$. Obviously, $\{P_1, P_2, \ldots, P_t\}$ is a spanning t-(x, U)-fan of G if (u, v) is not in $\cup_{i=1}^t E(P_i)$. Thus, we consider $(u, v) \in \cup_{i=1}^t E(P_i)$. By Lemma 3, we can find a spanning (x, U)-fan of G if $t = 1, 2$. Thus, we consider the case $t \ge 3$. Without loss of generality, we may assume that $(u, v) \in P_1$. Therefore, we can write P_1 as $\langle x, H_1, u, v, H_2, y_1\rangle$. Let $P_i' = \langle w_i, P_i', y_i\rangle$ be the path obtained from P_i by deleting x. Thus, we can write P_i as $\langle x, w_i, P_i', y_i\rangle$ for $1 \le i \le t$. Note that $x \ne w_i$ and $P_i = \langle y_i\rangle$ if $w_i = y_i$ for every $2 \le i \le t$.

Case 1: $d_{P_i'}(u) + d_{P_i'}(v) \ge n(P_i') + 2$ for some $2 \le i \le t$. Without loss of generality, we may assume that $d_{P_2'}(u) + d_{P_2'}(v) \ge n(P_2') + 2$. Obviously, $n(P_2') \ge 2$. We write $P_2' = \langle w_2 = z_1, z_2, \ldots, z_r = y_2\rangle$. We claim that there exists an index j in $\{1, 2, \ldots, r - 1\}$ such that $(z_j, v) \in E(G)$ and $(z_{j+1}, u) \in E(G)$. Suppose that this is not the case. Then $d_{P_2'}(u) + d_{P_2'}(v) \le r + r - (r - 1) = r + 1 = n(P_2') + 1$. We get a contradiction.

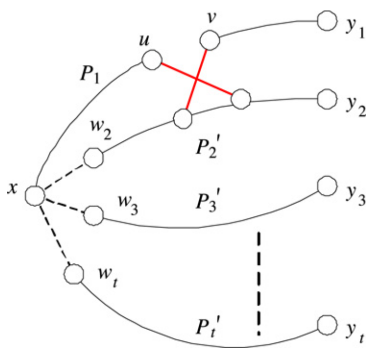

Fig. 1. Illustration of case 1.

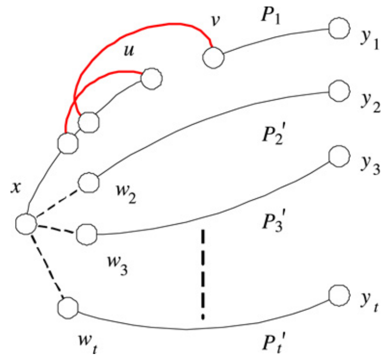

Fig. 2. Illustration of case 2.1.

We set $Q_1 = \langle x, w_2 = z_1, z_2, \ldots, z_j, v, H_2, y_1\rangle$, $Q_2 = \langle x, H_1, u, z_{j+1}, z_{j+2}, \ldots, z_r = y_2\rangle$, and $Q_i = P_i$ for $3 \le i \le t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning t-(x, U)-fan of G. See Fig. 1 for illustration.

Case 2: $d_{P_i'}(u) + d_{P_i'}(v) \le n(P_i') + 1$ for every $2 \le i \le t$.

Case 2.1: $d_{P_i'}(u) + d_{P_i'}(v) < n(P_i') + 1$ for some $2 \le i \le t$. Without loss of generality, we may assume that $d_{P_2'}(u) + d_{P_2'}(v) \le n(P_2')$. Thus,

$$\begin{aligned} d_{P_1}(u) + d_{P_1}(v) &= d_G(u) + d_G(v) - \sum_{i=2}^{t}(d_{P_i'}(u) + d_{P_i'}(v)) \\ &= d_G(u) + d_G(v) - (d_{P_2'}(u) + d_{P_2'}(v)) - \sum_{i=3}^{t}(d_{P_i'}(u) + d_{P_i'}(v)) \\ &\ge n(G) + k - n(P_2') - \sum_{i=3}^{t}(n(P_i') + 1) \\ &= n(P_1) + k - (t-2) \\ &\ge n(P_1) + 1. \end{aligned}$$

By Lemma 8, there is a hamiltonian path Q_1 of $G[P_1]$ joining x to y_1. We set $Q_i = P_i$ for $2 \le i \le t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning t-(x, U)-fan of G. See Fig. 2 for illustration.

Case 2.2: $d_{P_i'}(u) + d_{P_i'}(v) = n(P_i') + 1$ for every $2 \le i \le t$. We have

$$\begin{aligned} d_{P_1}(u) + d_{P_1}(v) &= d_G(u) + d_G(v) - \sum_{i=2}^{t}(d_{P_i'}(u) + d_{P_i'}(v)) \\ &= n(G) + k - \sum_{i=2}^{t}(n(P_i') + 1) \\ &= n(P_1) + k - (t-1) \\ &\ge n(P_1). \end{aligned}$$

Let $R = \langle y_1, H_2^{-1}, v, u, H_1^{-1}, x, w_2, P_2', y_2\rangle$. Then

$$\begin{aligned} d_R(u) + d_R(v) &= d_{P_1}(u) + d_{P_1}(v) + d_{P_2'}(u) + d_{P_2'}(v) \\ &\geq n(P_1) + n(P_2') + 1 \\ &= n(R) + 1. \end{aligned}$$

By Lemma 8, there is a hamiltonian path W of $G[R]$ joining y_1 to y_2. Thus, W can be written as $\langle y_1, W_1, x, W_2, y_2\rangle$. We set $Q_1 = \langle x, W_1^{-1}, y_1\rangle$, $Q_2 = \langle x, W_2, y_2\rangle$, and $Q_i = P_i$ for $3 \leq i \leq t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning (x, U)-fan of G. □

We note that Theorem 2 analogizes the result of Lemma 3 in spanning fan-connectivity. By Theorem 2, we can obtain the following theorem.

Theorem 3. *Let k be a positive integer. Then $\kappa_f^*(G) \geq k + 1$ if G is not the complete graph and $d_G(u) + d_G(v) \geq n(G) + k$ for all non-adjacent vertices u and v.*

Proof. Let $E^c(G)$ be the set $\{e \mid e \notin E(G)\}$. Without loss of generality, we write $E^c = \{e_1, e_2, \ldots, e_m\}$. We set $H_0 = G$ and H_i being the graph with $V(H_i) = V(H_{i-1})$ and $E(H_i) = E(H_{i-1}) \cup \{e_i\}$ for every $1 \leq i \leq m$. Since H_m is isomorphic to the complete graph with $n(G)$ vertices, $\kappa_f^*(H_m) \geq k + 1$. By Theorem 2, $\kappa_f^*(G) = \kappa_f^*(H_0) \geq k + 1$. □

Note that Theorem 3 is an analogous result of Lemma 4 in spanning fan-connectivity.

Theorem 4. *$\kappa_f^*(G) \geq 2\delta(G) - n(G) + 1$ if $\frac{n(G)}{2} + 1 \leq \delta(G)$.*

Proof. Suppose that $\frac{n(G)}{2} + 1 \leq \delta(G)$. Obviously, $\delta(G) \leq n(G) - 1$ and $n(G) \geq 4$. Suppose that $n(G) = 2m$ for some integer $m \geq 2$. Then $\delta(G) = m + k$ for some integer k with $1 \leq k \leq m - 1$. Obviously, $d_G(u) + d_G(v) \geq 2\delta(G) = 2m + 2k$ for all two distinct vertices u and v in G. By Theorem 3, $\kappa_f^*(G) \geq 2k + 1 = 2\delta(G) - n(G) + 1$. Suppose that $n(G) = 2m + 1$ for some integer $m \geq 2$. Then $\delta(G) = m + k + 1$ for some integer k with $1 \leq k \leq m - 1$, and $d_G(u) + d_G(v) \geq 2\delta(G) = 2m + 2k + 2$ for all two distinct vertices u and v in G. By Theorem 3, $\kappa_f^*(G) \geq 2k + 2 = 2\delta(G) - n(G) + 1$ and the theorem follows. □

Theorem 4 analogizes the result of Lemma 2 in spanning fan-connectivity. Moreover, when $\delta(G) = n(G) - 2$, we have the following corollary.

Corollary 1. *$\kappa_f^*(G) = n(G) - 3$ if $\delta(G) = n(G) - 2$ and $n(G) \geq 5$.*

Proof. By Lemma 5, $\kappa^*(G) \geq n(G) - 2$. Since $n(G) - 2 \leq \kappa^*(G) \leq \kappa(G) \leq \delta(G) = n(G) - 2$, $\kappa(G) = n(G) - 2$. By Theorem 4, $\kappa_f^*(G) \geq n(G) - 3$. By Theorem 1, $\kappa_f^*(G) < \kappa(G)$. Thus, $\kappa_f^*(G) = n(G) - 3$. □

4. Spanning pipeline-connectivity

Similar to some recent works on the spanning connectivity [4–10] and the spanning fan-connectivity, studied in Section 2, we study spanning pipeline-connectivity in this section. Lemma 9 and Theorem 5 are analogous to Lemma 1 and Theorem 1 respectively.

Lemma 9. *Every 1^*-connected graph is 1_p^*-connected.*

Theorem 5. *$\kappa_p^*(G) \leq \kappa_f^*(G) \leq \kappa^*(G) \leq \kappa(G)$ for any 1_p^*-connected graph. Moreover, $\kappa_p^*(G) = \kappa_f^*(G) = \kappa^*(G) = \kappa(G)$ if and only if G is a complete graph.*

Theorem 6. *Assume that k is a positive integer. Let u and v be two non-adjacent vertices of G. Suppose that $d_G(u) + d_G(v) \geq n(G) + k$. Then $\kappa_p^*(G) \geq k$ if and only if $\kappa_p^*(G + (u, v)) \geq k$.*

Proof. Obviously, $\kappa_p^*(G + (u, v)) \geq k$ if $\kappa_p^*(G) \geq k$. Suppose that $\kappa_p^*(G + (u, v)) \geq k$. Let $U = \{x_1, x_2, \ldots, x_t\}$ and $W = \{y_1, y_2, \ldots, y_t\}$ be any two subsets of G such that $U \neq W$ and $t \leq k$. We need to find a spanning (U, W)-pipeline of G.

Since $G + (u, v)$ is k_p^*-connected, there exists a spanning (U, W)-pipeline of $G + (u, v)$. Let $\{P_1, P_2, \ldots, P_t\}$ be a spanning (U, W)-pipeline with P_i joining x_i to $y_{\pi(i)}$ for $1 \leq i \leq t$. Without loss of generality, we assume that $\pi(i) = i$. Obviously, $\{P_1, P_2, \ldots, P_t\}$ is a spanning (U, W)-pipeline of G if (u, v) is not in P. Thus, we consider the case that (u, v) is in P. By Lemma 3, we can find a spanning (U, W)-pipeline of G if $t = 1$. Thus, we consider the case $t \geq 2$.

Case 1: $U \cap W = \emptyset$. Without loss of generality, we may assume that $(u, v) \in P_1$. Thus, we can write P_1 as $\langle x_1, H_1, u, v, H_2, y_1\rangle$. (Note that $H_1 = \langle x\rangle$ if $x = u$, and $H_2 = \langle y\rangle$ if $y = v$.) Let P_i' be the path obtained from P_i by deleting x and y_i. Thus, we can write P_i as $\langle x_i, P_i', y_i\rangle$ for $1 \leq i \leq t$.

Case 1.1: $d_{P_1}(u) + d_{P_1}(v) \geq n(P_1) + 1$. With Lemma 8, there is a hamiltonian path Q_1 of $G[P_1]$ joining x_1 to y_1. We set $Q_i = P_i$ for $2 \leq i \leq t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning (U, W)-pipeline of G. See Fig. 3 for illustration.

Case 1.2: $d_{P_1}(u) + d_{P_1}(v) \leq n(P_1)$. We claim that $d_{P_i}(u) + d_{P_i}(v) \geq n(P_i) + 2$ for some $2 \leq i \leq t$.

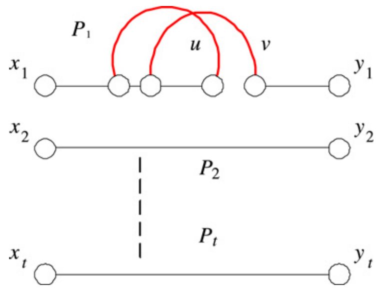

Fig. 3. Illustration of Case 1.1.

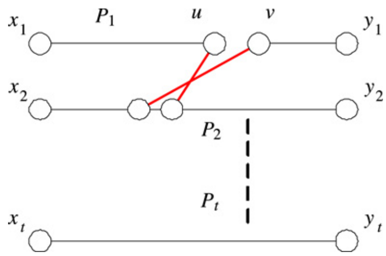

Fig. 4. Illustration of Case 1.2.

Suppose that $d_{P_i}(u) + d_{P_i}(v) \le n(P_i) + 1$ for every $2 \le i \le t$. Then

$$\begin{aligned} d_G(u) + d_G(v) &= d_{P_1}(u) + d_{P_1}(v) + \sum_{i=2}^{t}(d_{P_i}(u) + d_{P_i}(v)) \\ &\le n(P_1) + \sum_{i=2}^{t}(n(P_i) + 1) \\ &= n(G) + t - 1 \\ &\le n(G) + k - 1. \end{aligned}$$

We obtain a contradiction. Thus, $d_{P_i}(u) + d_{P_i}(v) \ge n(P_i) + 2$ for some $2 \le i \le t$. Without loss of generality, we assume that $d_{P_2}(u) + d_{P_2}(v) \ge n(P_2) + 2$. Obviously, $n(P_2') \ge 2$. We write $P_2' = \langle x_2 = z_1, z_2, \ldots, z_r = y_2\rangle$. We claim that there exists an index j in $\{1, 2, \ldots, r-1\}$ such that $(z_j, v) \in E(G)$ and $(z_{j+1}, u) \in E(G)$. Suppose this is not the case. Then $d_{P_2'}(u) + d_{P_2'}(v) \le r + r - (r-1) = r + 1 = n(P_2') + 1$. We get a contradiction.

We set $Q_1 = \langle x_2 = z_1, z_2, \ldots, z_j, v, H_2, y_1\rangle$, $Q_2 = \langle x_1, H_1, u, z_{j+1}, z_{j+2}, \ldots, z_r = y_2\rangle$, and $Q_i = P_i$ for $3 \le i \le t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning (U, W)-pipeline of G. See Fig. 4 for illustration.

Case 2: $U \cap W \ne \emptyset$. Let $|U \cap W| = r$. Without loss of generality, we assume that $x_i = y_i$ for $t - r + 1 \le i \le t$. Let $G' = G[V(G) - (U \cap W)]$, $U' = U - W$, and $W' = W - U$. Obviously, $d_{G'}(u) + d_{G'}(v) \ge d_G(u) + d_G(v) - 2r \ge n(G) + k - 2r = n(G') + k - r$, $|U'| = |W'| = t - r \le k - r$, and $U' \cap W' = \emptyset$. By Case 1, there exists a spanning (U', W')-pipeline $\{Q_1, Q_2, \ldots, Q_{t-r}\}$ of G'. We set $Q_i = \langle x_i\rangle$ for $t - r + 1 \le i \le t$. Then $\{Q_1, Q_2, \ldots, Q_t\}$ forms a spanning (U, W)-pipeline of G. □

We note that Theorems 6–8 are analogous to Lemmas 3, 4 and 2 in spanning pipeline-connectivity respectively. By Theorem 6, we can obtain the following theorem.

Theorem 7. *Let k be a positive integer. Then $\kappa_p^*(G) \ge k$ if G is not the complete graph and $d_G(u) + d_G(v) \ge n(G) + k$ for all non-adjacent vertices u and v.*

Proof. Let $E^c(G)$ be the set $\{e \mid e \notin E(G)\}$. Without loss of generality, we write $E^c = \{e_1, e_2, \ldots, e_m\}$. We set $H_0 = G$ and H_i being the graph with $V(H_i) = V(H_{i-1})$ and $E(H_i) = E(H_{i-1}) \cup \{e_i\}$ for every $1 \le i \le m$. Since H_m is isomorphic to the complete graph with $n(G)$ vertices, $\kappa_p^*(H_m) \ge k$. By Theorem 6, $\kappa_p^*(G) = \kappa_p^*(H_0) \ge k$. □

Theorem 8. *$\kappa_p^*(G) \ge 2\delta(G) - n(G)$ if $\frac{n(G)}{2} + 1 \le \delta(G)$.*

Proof. Since $\frac{n(G)}{2} + 1 \le \delta(G)$ and $\delta(G) \le n(G) - 1$, $n(G) \ge 4$. Suppose that $n(G) = 2m$ for some integer $m \ge 2$. Then $\delta(G) = m + k$ for some integer k with $1 \le k \le m - 1$. Thus, $d_G(u) + d_G(v) \ge 2\delta(G) = 2m + 2k$ for all two distinct vertices u and v in G. By Theorem 7, $\kappa_p^*(G) \ge 2k = 2\delta(G) - n(G)$. Suppose that $n(G) = 2m + 1$ for some integer $m \ge 2$. Then

$\delta(G) = m + k + 1$ for some integer k with $1 \le k \le m - 1$, and $d_G(u) + d_G(v) \ge 2\delta(G) = 2m + 2k + 2$ for all two distinct vertices u and v in G. By Theorem 3, $\kappa_f^*(G) \ge 2k + 1 = 2\delta(G) - n(G)$ and the theorem follows. □

Corollary 2. *$\kappa_p^*(G) = n(G) - 4$ if $\delta(G) = n(G) - 2$ and $n(G) \ge 5$.*

Proof. By Theorem 8, $\kappa_p^*(G) \ge n(G) - 4$. Let $V(G) = \{x_1, x_2, \dots, x_{n(G)}\}$. Without loss of generality, we assume that $(x_1, x_2) \notin E(G)$. We set $U = \{x_3, x_5, x_6, \dots, x_{n(G)}\}$ and $W = \{x_4, x_5, x_6, \dots, x_{n(G)}\}$. Obviously, $U \ne W$ and $|U| = |W| = n(G) - 3$.

Since there is no hamiltonian path of $G[V(G) - \{x_5, x_6, \dots, x_{n(G)}\}]$ joining x_3 to x_4, there is no spanning (U, W)-pipeline of G. Thus, $\kappa_p^*(G) = n(G) - 4$. □

5. An example

We use the following example to illustrate that $\kappa(G)$, $\kappa^*(G)$, $\kappa_f^*(G)$, and $\kappa_p^*(G)$ are really different concepts and, in general, have different values.

Example 1. Suppose that n is a positive integer with $n \ge 2$. Let $H(n)$ be the complete 3-partite graph $K_{2n,2n,n-1}$ with vertex partite sets $V_1 = \{x_1, x_2, \dots, x_{2n}\}$, $V_2 = \{y_1, y_2, \dots, y_{2n}\}$, and $V_3 = \{z_1, z_2, \dots, z_{n-1}\}$. Let $G(n)$ be the graph obtained from $H(n)$ by adding the edge set $\{(z_i, z_j) \mid 1 \le i \ne j < n\}$. Thus, $G[V_3]$ is the complete graph K_{n-1}. Obviously, $n(G(n)) = 5n - 1$, $\delta(G(n)) = 2n + (n-1) = 3n - 1$, and $\kappa(G(n)) = \delta(G(n))$. In the following, we will show that $\kappa^*(G(n)) = n + 1$, $\kappa_f^*(G(n)) = n$, and $\kappa_p^*(G(n)) = n - 1$.

By Lemma 5, $\kappa^*(G(n)) \ge 2\delta(G(n)) - n(G(n)) + 2 = n + 1$. To show $\kappa^*(G(n)) = n + 1$, we claim that there is no $(n+2)^*$-container of $G(n)$ between x_1 and x_2. Suppose this is not the case. Let $\{P_1, P_2, \dots, P_{n+2}\}$ be an $(n+2)^*$-container of $G(n)$ between x_1 and x_2. Obviously, $|V(P_i) \cap (V_1 - \{x_1, x_2\})| \le |V(P_i) \cap (V_2 \cup V_3)| - 1$ for $1 \le i \le n + 2$. Thus, $\sum_{i=1}^{n+2} |V(P_i) \cap (V_1 - \{x_1, x_2\})| = (\sum_{i=1}^{n+2} |(V_2 \cup V_3) \cap V(P_i)|) - (n + 2)$. Therefore, $|V_1 - \{x, y\}| \le |V_2 \cup V_3| - (n + 2)$. However, $|V_1 - \{x, y\}| = 2n - 2$ but $|V_2 \cup V_3| - (n + 2) = 2n - 3$. This leads to a contradiction.

By Theorem 4, $\kappa_f^*(G(n)) \ge 2\delta(G(n)) - n(G(n)) + 1 = n$. To show $\kappa_f^*(G(n)) = n$, we claim that there is no spanning (x_1, U)-fan of $G(n)$ where $U = \{y_1, y_2, \dots, y_{n+1}\}$. Suppose this is not the case. Let $\{P_1, P_2, \dots, P_{n+1}\}$ be a spanning (x_1, U)-fan of $G(n)$. Without loss of generality, we assume that P_i is a path joining x_1 to y_i for $1 \le i \le n + 1$. Obviously, $|(V_1 - \{x_1\}) \cap V(P_i)| \le |((V_2 \cup V_3) - \{y_i\}) \cap V(P_i)|$. Thus, $\sum_{i=1}^{n+1} |(V_1 - \{x_1\}) \cap V(P_i)| \le \sum_{i=1}^{n+1} |((V_2 \cup V_3) - \{y_i\}) \cap V(P_i)|$. Therefore, $|V_1 - \{x_1\}| \le |(V_2 \cup V_3) - U|$. However, $|V_1 - \{x_1\}| = 2n - 1$ but $|(V_2 \cup V_3) - U| = 2n - 2$. We get a contradiction.

By Theorem 8, $\kappa_p^*(G(n)) \ge 2\delta(G(n)) - n(G(n)) = n - 1$. To prove $\kappa_p^*(G(n)) = n - 1$, we claim that there is no spanning (U, W)-pipeline where $U = \{x_1, x_2, \dots, x_n\}$ and $W = \{x_{n+1}, x_{n+2}, \dots, x_{2n}\}$. Suppose that there exists a spanning (U, W)-pipeline $\{P_1, P_2, \dots, P_n\}$. Obviously, $|V_2 \cap V(P_i)| - 1 \le |V_3 \cap V(P_i)|$ for $1 \le i \le n$. Then $\sum_{i=1}^{n}(|V_2 \cap V(P_i)| - 1) \le \sum_{i=1}^{n} |V_3 \cap V(P_i)|$. Therefore, $(\sum_{i=1}^{n} |V_2 \cap V(P_i)|) - n \le \sum_{i=1}^{n} |V_3 \cap V(P_i)|$. However, $(\sum_{i=1}^{n} |V_2 \cap V(P_i)|) - n = |V_2| - n = n$ but $\sum_{i=1}^{n} |V_3 \cap V(P_i)| = |V_3| = n - 1$. We get a contradiction.

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