Wavelet Transform Method for Coupled Map Lattices

Cheng Juang, Chin-Lung Li, Y. H. Liang, and Jonq Juang

***Abstract*—The purpose of this paper is twofold. First, we derive a necessary and sufficient condition for local synchronization in coupled map lattices (CMLs) with symmetric coupling. In particular, we also identify the best choice of coupling strength in the sense that such a coupling strength gives the fastest convergence rate of initial values toward the synchronous manifold. Furthermore, such a coupling strength is independent of the choice of the individual chaotic map. In addition, it is demonstrated that the wavelet transform method, which is proposed by Wei *et al.*, can greatly increase the applicable ranges of coupling strengths, the parameters of the individual chaotic map, and the number of nodes for local synchronization of CMLs.**

***Index Terms*—Connectivity topology, synchronization, wavelet transform.**

I. INTRODUCTION

SIMULATION of natural phenomena is one of the most important research fields, and coupled map lattices (CMLs) are a paradigm for studying fundamental questions in spatially extended dynamical systems. This is because of their wide range of applications such as in turbulence, pattern formation in natural systems, and solitons. They also exhibit a very rich phenomenology, including a wide variety of both spatial and temporal periodic structures, intermittence, chaos, domain walls, kink dynamics, etc. As a matter of fact, one of the most interesting aspects of CMLs is the presence of attracting manifolds. Such attracting manifolds lead to notions such as partial synchronization [1], weak and strong synchronization [2], [3], and (complete) synchronization [4]–[9].

In continuous coupled chaotic systems, the stably synchronous motion was first studied in [10]. Later, Pecora and Carroll [11] proposed a master stability function (MSF) to symmetrically determine the stability of their synchronous manifold. In the case of full-state coupling, the MSF can be further reduced to a single inequality $L_{\max} + \varepsilon\lambda_2 < 0$ [12], [13], where $L_{\max}$ is the largest Lyapunov exponent of the single chaotic oscillator, ε is the coupling strength, and λ_2 is the second largest eigenvalue of the coupling matrix ($\lambda_1 = 0$). This *necessary and sufficient stability condition* then allows one to achieve the synchronization of N identical dynamical systems whenever ε is greater than the critical value $\varepsilon_c = L_{\max}/-\lambda_2$. However, for a typical connectivity topology such as the diffusively coupled matrix, its λ_2 moves closer to the origin as the number of nodes increases. Consequently, synchronization is more difficult to realize as N increases. In [8], Wei et al. proposed a wavelet transform method to alter the connectivity topology. In doing so, $\lambda_2 = \lambda_2(K)$ becomes a quantity depending on wavelet parameter K. It is found there that a critical wavelet parameter K_c can be chosen to move $\lambda_2(K_c)$ away from the origin regardless of the number of nodes. This, in turn, greatly reduces the size of the critical coupling strength ε_c. Such phenomena are analytically verified in [14]–[16]. We also remark that, in the case of partial-state coupling for continuous coupled chaotic systems, even though the MSF can no longer be further reduced to the single inequality, as given earlier, much progresses are still made for both local synchronization theory (see, e.g., [17] and the references cited therein) and global synchronization theory (see, e.g., [18] and the references cited therein). It should also be noted that a theory for multiresolution signal decomposition by using wavelet representation is addressed in [19].

The development of synchronization theory of the discrete dynamical coupled systems is still at the primitive stage as compared to that of their continuous counterparts. The reason for the gap between the theory developed in the lattices of coupled chaotic systems and that of CMLs lies mostly on the fact that it is more natural to have a nonlinear coupling between oscillators in CMLs. This is because a nonlinear coupling with suitable range of coupling strength tends to yield an invariant region for the corresponding CMLs, while linear coupling cannot. It should be noted that there is no such problem for the lattices of coupled chaotic systems. It should also be mentioned that the analytical results of the lattice of the coupled chaotic systems stated earlier are linearly coupled. As a result of such nonlinear coupling, the following different phenomena of CMLs as compared to those of the lattices of coupled chaotic systems are observed (see, e.g., [5]). A stronger coupling strength does not ensure synchronization, but it will cause the trajectory of the oscillator to grow out of bound. Furthermore, synchronization becomes numerically unobservable even for a modest number of oscillators.

The purpose of this paper is twofold. First, we derive an optimal synchronization theory for CMLs (1). In particular, we have the following results. A *necessary and sufficient condition* on coupling strength for local synchronization is obtained. Furthermore, the coupling strength ε giving the fastest convergence rate of initial values toward the synchronous state is explicitly obtained. Such ε is shown to be independent of the choice of the individual chaotic map. Moreover, the maximum number of oscillators for which the corresponding network would still yield synchronization can be explicitly obtained. Second, by using such results, we demonstrate that the wavelet transform method can greatly increase the applicable range of coupling strength

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and the number of oscillators for local synchronization of coupled chaotic map lattices. The family of quadratic maps is used as numerical examples.

II. Theoretical Considerations

A. Stability Results Without Wavelet Transform Method

The case of N CMLs can be described in vector form [6], [7]

$$\mathbf{X}(n+1) = (\mathbf{I}+\varepsilon\mathbf{A})\mathbf{F}(\mathbf{X}(n)) \tag{1}$$

where $\mathbf{X}(n) = (x_1(n), \ldots x_N(n))^{\mathrm{T}}$; $\mathbf{I}$ is the unit matrix; ε is the coupling strength; $\mathbf{A}$ is a symmetric coupling matrix having zero row sums, with zero being a simple eigenvalue; and $\mathbf{F}(x_1, \ldots, x_N) = (f(x_1), \ldots, f(x_N))^{\mathrm{T}}$. Here, $f_{(x)}$ describes the chaotic dynamics of an individual oscillator. Let $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of the coupling matrix $\mathbf{A}$. It was shown, e.g., [6], that if

$$L_{\max} + \ln|1+\varepsilon\lambda_i| < 0, \qquad i = 2, \ldots, N \tag{2}$$

for all the nonzero eigenvalues λ_i. Note that the second largest eigenvalue alone is not enough to ensure that all other eigenvalues satisfied (3). To achieve synchronization of CMLs, we need to find ε so that the maximum of $|1+\varepsilon\lambda_i|, i = 2, \ldots, N$, is a minimum, i.e., to solve a min–max problem of the form

$$\min_{\varepsilon\in\Re} \max_{2\leq i\leq N} |1+\varepsilon\lambda_i|. \tag{3}$$

To find such saddle point ε, let $\varepsilon_{i,j}$, $2 \leq i, j \leq N$, be real numbers so that

$$|1+\varepsilon_{i,j}\lambda_i| = |1+\varepsilon_{i,j}\lambda_j|. \tag{4}$$

Solving for (3), we have $\varepsilon_{i,j} := -2/\lambda_i + \lambda_j$ or $\varepsilon_{i,j} = 0$. It then follows that, for fixed

$$\min_{\varepsilon\in\Re} \max\{|1+\varepsilon\lambda_i|, |1+\varepsilon\lambda_j|\} = \left|\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\right| =: t_{i,j} \tag{5}$$

occurred at $\varepsilon_{i,j} = -2/\lambda_i + \lambda_j$. Let k, l be indexes so that $t_{k,l} \geq t_{i,j}$ for all $2 \leq i \leq N$. We have that $\min_{\varepsilon\in\Re} \max_{2\leq i\leq N} |1+\varepsilon\lambda_i| = t_{k,l}$ occurred at $\varepsilon_{k,l}$. Taking the fact that λ_i, $i = 2, \ldots, N$, are all negative, we conclude that $t_{k,l} = t_{2,N}$, which is to be called the synchronization index of the system. The aforementioned results are summarized as follows.

Theorem 1: The min–max problem (3) can be achieved when $\varepsilon = \varepsilon_2$, $N = -2/\lambda_2 + \lambda_N$. Consequently, system (1) is (locally) synchronized if and only if

$$L_{\max} + \ln|t_{2,N}| =: \delta_{N,f} < 0. \tag{6}$$

If (6) holds, then there exists an optimal neighborhood $N_{N,f}$ of $\varepsilon_{2,N}$ so that (1) is (locally) synchronized whenever $\varepsilon \in N_{N,f}$. Here

$$N_{N,f} = \left(\frac{1-e^{-L_{\max}}}{-\lambda_2}, \frac{1+e^{-L_{\max}}}{-\lambda_N}\right). \tag{7}$$

The interval $N_{N,f}$ (if it exists) is optimal in the sense that if ε is not in $N_{N,f}$, then system (1) will not acquire (local) synchronization. Moreover, $\varepsilon_{2,N}$, which is independent of the choice of the individual chaotic map, is the best choice of coupling strength for local synchronization of (1) in the sense that such a coupling strength gives the fastest convergence rate of initial values toward the synchronous manifold.

Remark 1: The interval $N_{N,f}$ is first derived in [5]. Our approach here gives the optimal coupling strength $\varepsilon_{2,N}$, which is independent of the choice of the individual chaotic map. Since $t_{2,N} = |\lambda_2 - \lambda_N/\lambda_2 + \lambda_N|$, it is then clear that as the number of oscillators increases, $t_{2,N}$ approaches to one. Consequently, (6) cannot be fulfilled, provided that N is large. Hence, to achieve synchronization of (1), one has to place a limit on the number of oscillators considered. Using (6), one should be able to find the maximum number N of oscillators satisfying (6). From here on, $N_{N,f}$ and $\delta_{N,f}$ are to be called the synchronization interval and the Lyapunov index of the system, respectively. The length of $N_{N,f}$ is denoted by $l_{N,f}$. Apparently, a larger $l_{N,f}$ gives a better applicable range of coupling strength.

B. Stability Results With Wavelet Transform Method

How the wavelet transform method [12] affects the stability of synchronous manifold of (1) is discussed in this part. Let the number N of nodes be equal to $n2^i$. Write $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \ldots & \mathbf{A}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \ldots & \mathbf{A}_{nn} \end{pmatrix}_{n\times n}.$$

Here, the dimension of each block matrix $\mathbf{A}_{kl}$ is $2^i \times 2^i$. By an i-scale wavelet operator W[12], [20], the matrix $\mathbf{A}$ is transformed into $W(\mathbf{A})$ of the form

$$W(\mathbf{A}) = \begin{pmatrix} \tilde{\mathbf{A}}_{11} & \ldots & \tilde{\mathbf{A}}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{A}}_{n1} & \ldots & \tilde{\mathbf{A}}_{nn} \end{pmatrix}_{n\times n}$$

where each entry of $\tilde{\mathbf{A}}_{kl}$ is the average of entries of $\mathbf{A}_{kl}$, $1 \leq k$, $l \leq n$. After reconstruction [12], the coupling matrix $\mathbf{A}$ becomes $\mathbf{A}+KW\mathbf{A}$. Here, K is a wavelet parameter. In summary, the effects of the wavelet transform method can be viewed as the changes of the eigenvalues of the coupling matrix [13] and vary dramatically for different N's. The eigenvalues of $\mathbf{A}+KW\mathbf{A}$ are denoted by $\lambda_i(K)$, with $0 = \lambda_1(K) > \lambda_2(K) \geq \cdots \geq \lambda_N(K)$. Clearly, Theorem 1 is still valid for such new coupling matrix. Note that the corresponding $\delta_{N,f}$, $\varepsilon_{2,N}$, $N_{N,f}$, and $t_{2,N}$ now depend on the wavelet parameter K as well. To emphasize such dependence, we shall write $\delta_{N,f}$, $\varepsilon_{2,N}$, $N_{N,f}$, and $t_{2,N}$ as $\delta_{N,f}(K)$, $\varepsilon_{2,N}(K)$, $N_{N,f}$, and $t_{2,N}(K)$, respectively.

III. Numerical Results

A. $N = 4$ (No Improvement)

The effects of the wavelet transform method can be viewed as the changes of the eigenvalues of the coupling matrix and vary dramatically for different N''s. Consider the quadratic map $f_\mu(x) = \mu\mathrm{x}(1-x)$ diffusively coupled with periodic boundary conditions for $N = 4$. Fig. 1(a) shows the calculated eigenvalues $\lambda_i(K)$ of the coupling matrix as a function of wavelet parameter K. The coupling matrices before and after reconstruc-

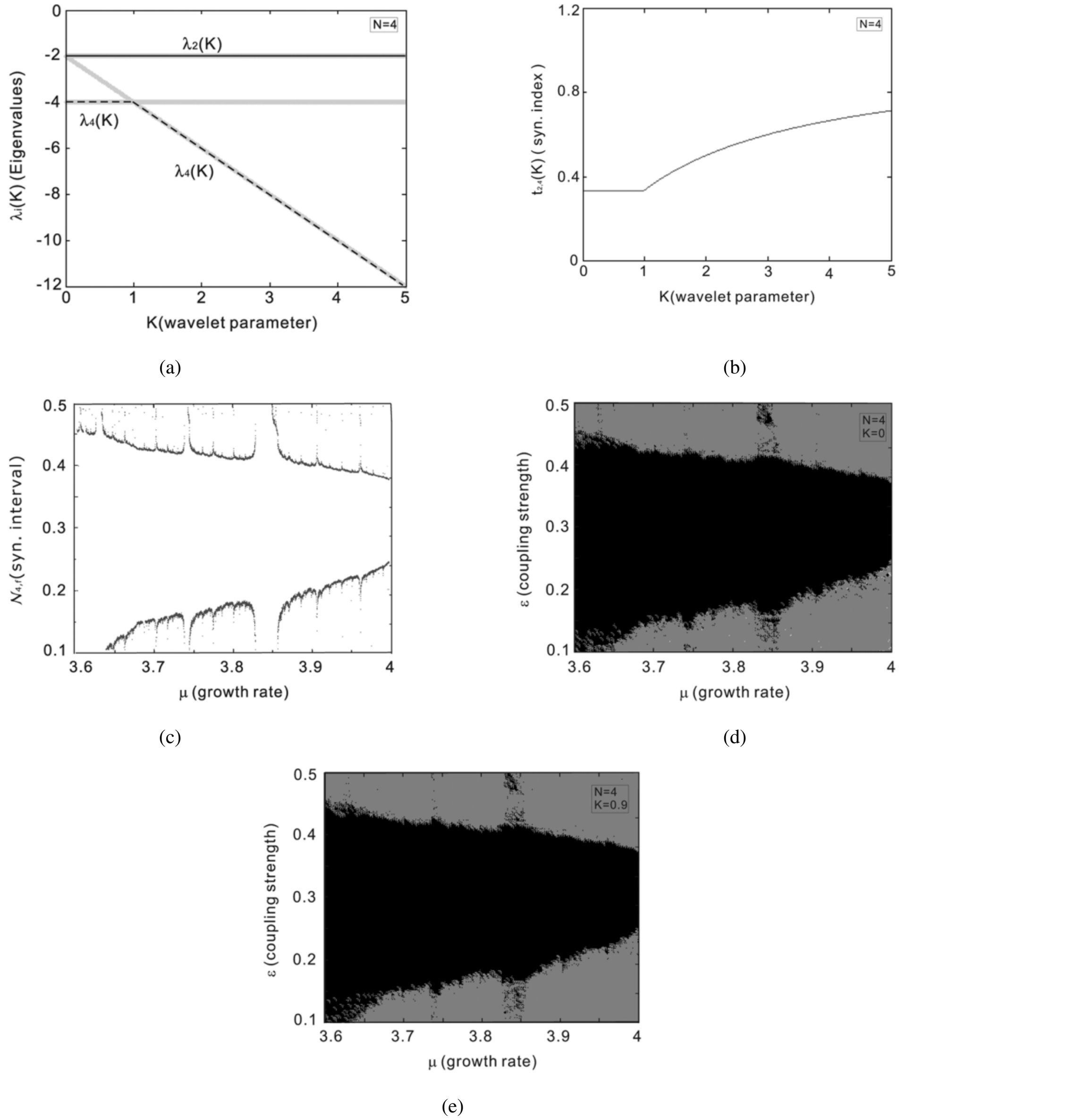

Fig. 1. (a) Eigenvalues $\lambda_i(K)$ of the coupling matrix as a function of wavelet parameter K for $N = 4$. The solid line is $\lambda_2(K)$, while the dotted line is $\lambda_4(K)$. (b) Synchronization index of the coupling matrix as a function of wavelet parameter K. (c) Theoretical predicted synchronization intervals which are in agreement with Fig. 1(d) and (e). (d) Numerically produced intervals of synchronization without the wavelet transform method. (e) Numerically produced intervals of synchronization with $(k = 0.9)$ the wavelet transform method. The theoretically predicted synchronization interval is shown in Fig. 1(c). Fig. 1(e) shows synchronization intervals $(K = 0)$ with and without $(K = 0.9)$ the wavelet transform method via computer simulation. The dark, gray, and white regions represent in complete synchronization, partial synchronization, and out of synchronization, respectively. The dark areas around $\mu = 3.64$ and $\mu = 3.85$ are caused by periodic windows. They are consistent with the results of Fig. 1(a)–(c), where $K_{N,\min} \in [0, 1]$.

tion are denoted by $\mathbf{A} + KW(\mathbf{A}) = \mathbf{A} + K[\tilde{A}_{ij}]$, with $K = 0$ and $K > 0$, respectively, where

$$\tilde{\mathbf{A}}_{11} = \tilde{\mathbf{A}}_{22} = \begin{bmatrix} \frac{-(4+k)}{2} & \frac{2-k}{2} \\ \frac{2-k}{2} & \frac{-(4+k)}{2} \end{bmatrix}$$
$$\tilde{\mathbf{A}}_{12} = \tilde{\mathbf{A}}_{21} = \begin{bmatrix} \frac{k}{2} & \frac{2+k}{2} \\ \frac{2+k}{2} & \frac{k}{2} \end{bmatrix}.$$

The solid line is $\lambda_2(K)$, while the dotted line is $\lambda_4(K)$. Note that as K is increased, a crossing appears at $K = 1$. This crossing makes the analytical identification of $\lambda_i(K)$ a difficult task. Thus, the optimal K is numerically determined from Fig. 1(b), where $t_{2,N}$, the synchronization index, is obtained from Fig. 1(a). According to (6), $K = K_{N,\min}$ is the number for which $t_{2,N}(K_{N,\min})$ is a minimum. Fig. 1(b) shows that $K_{N,\min} \in [0, 1]$. Thus, it is clear that no enhancement of synchronization is expected.

B. $N = 8$ *(Significantly Improved)*

For $N = 8$, the enhancement of synchronization is shown. Fig. 2(a) shows the eigenvalues $\lambda_i(K)$ of the coupling matrix as a function of wavelet parameter K, where

$$\tilde{\mathbf{A}}_{ij} = \begin{bmatrix} \frac{-(4+K)}{2} & \frac{2-K}{2} \\ \frac{2-K}{2} & \frac{-(4+K)}{2} \end{bmatrix}, \qquad \text{if } i = j$$
$$\tilde{\mathbf{A}}_{ij} = \begin{bmatrix} \frac{K}{4} & \frac{K}{2} \\ \frac{K+4}{2} & \frac{K}{4} \end{bmatrix} = \tilde{\mathbf{A}}_{ji}^T = \tilde{\mathbf{A}}_{41} = \tilde{\mathbf{A}}_{14}^T, \qquad \text{if } j - i = 1$$
$$\tilde{\mathbf{A}}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{otherwise.}$$

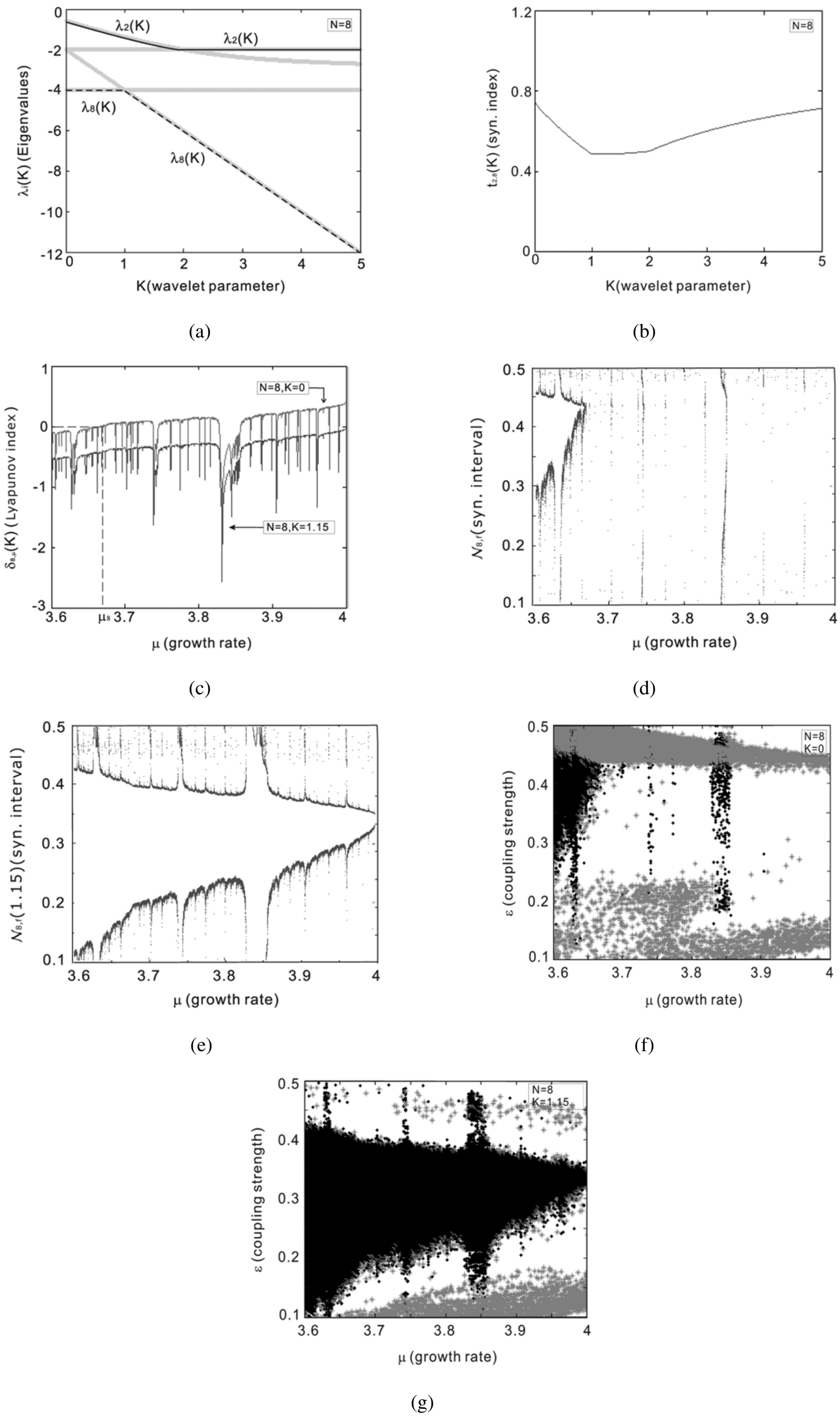

Fig. 2. (a) Eigenvalues $\lambda_i(K)$ of the coupling matrix as a function of wavelet parameter K for $N = 8$. (b) Synchronization index of the coupling matrix as a function of wavelet parameter K. (c) Lyapunov index versus growth rate. (d) White region gives the synchronization intervals, as stated in (8), for $K = 0$. (e) White region gives the synchronization intervals, as stated in (8), with $k = 1.15$. (f) Numerically produced synchronization intervals without the wavelet transform method. (g) Numerically produced synchronization intervals with $(K = 1.15)$ the wavelet transform method for $N = 8$.

Note that two crossing points appear. Similarly, the synchronization index $t_{2,8}(K)$ is shown in Fig. 2(b). It is observed that $K_{8,\min} \in (1.13, 1.204)$. Thus, using an optimal K (in the min region), it is expected to have a significant improvement over $K = 0$ according to Theorem 1.

Given the optimal K, the effects of the wavelet transform method on synchronization with different growth rates and coupling strengths are further investigated. The quantities obtained in Theorem 1 produced those shown Fig. 2(b) and (c). Fig. 2(c) shows the Lyapunov index $\delta_{8,\mu}(K)$ of system (1) with $K = 0$

and $K = 1.15$. These two graphs are identical with a vertical shift. It is seen, via Fig. 2(c), that if $\mu > \mu_8 \approx 3.67$, so as $\delta_{8,\mu}(0) > 0$, then the local synchronization is lost without the wavelet transform method. However, with the wavelet transform method $(K = 1.15)$, $\delta_{8,\mu}(1.15) < 0$ for all μ's, and so, the local synchronization of (1) is preserved. Figs. 2(d) and (e) shows the optimal length of the coupling strength as a function of growth rate without the transform $(K = 0)$ and with the optimal transform $(K = 1.15)$, respectively.

The numerical simulation for obtaining the interval of synchronization recorded in Fig. 2(f) and (g) again confirms our theoretical prediction earlier. Without the transform $(K = 0)$, there is a narrow region for complete synchronization. In dark areas around $\mu = 3.64$ and 3.85, each cell shows a periodic window type of behavior. In gray areas, different initial conditions give rise to different scenarios of partial synchronization (for example, even/odd cells are synchronized). With the wavelet transform method, there is a very significant increase in dark areas as compared to those in Fig. 2(f). The applicable ranges of coupling strengths and growth rates are significantly improved. The numerically produced Fig. 2(f) and (g) is in agreement with our theoretically predicted Fig. 2(d) and (e).

C. $N = 40$ *(Effects on Large N)*

The wavelet transform method is most dramatic for a large number of oscillators. Fig. 3(a) shows the maximum number of oscillators for which the local synchronization of the system with or without the wavelet transform method can still be sustained. The numbers are obtained by solving (6). It is seen, via Fig. 3(a), that the good improvement on the maximum number of oscillators allowed is there even without choosing the optimal K. The graphs in Fig. 2(a) are decreasing with respect to the growth rate μ of the map, except at those μ's yielding the window behavior.

As N increases, the dominant eigenvalue approaches zero. Hence, local synchronization becomes unobservable. Furthermore, the change of the dominant eigenvalue due to the wavelet transform method is very significant. Fig. 3(b) shows that if $\mu\infty = 3.5699456 < \mu < \mu_1 = 3.5708$, then system (1) acquires synchronization with $N = 40$ and $K = 1$. However, it is easily verified from (6) that if $\mu = \mu_1$, then the maximum number of oscillators allowed for synchronization without the wavelet transform method is $N = 24$. From Fig. 3(b), it is also seen that if $\mu < \mu < \mu_1 = 3.5704$, then system (1) achieves synchronization with $K = 0$. It should be noted that in producing Fig. 3(b). only the end points of synchronization intervals are recorded. For those μ's, where $\mu > 3.571$, exhibiting the window behavior, the end points of synchronization intervals lie outside the interval (0.1, 0.5).

With the wavelet transform method, global synchronization can be achieved for $N = 40$. In the case of $N = 8$, the transform enhances the synchronization effect. In this case, there is a very significant region, as shown in the dark areas in Fig. 3(c). Without the transform, synchronization for such a large number of oscillators would not be possible. This demonstrates the dramatic effects of the transform with relatively large N.

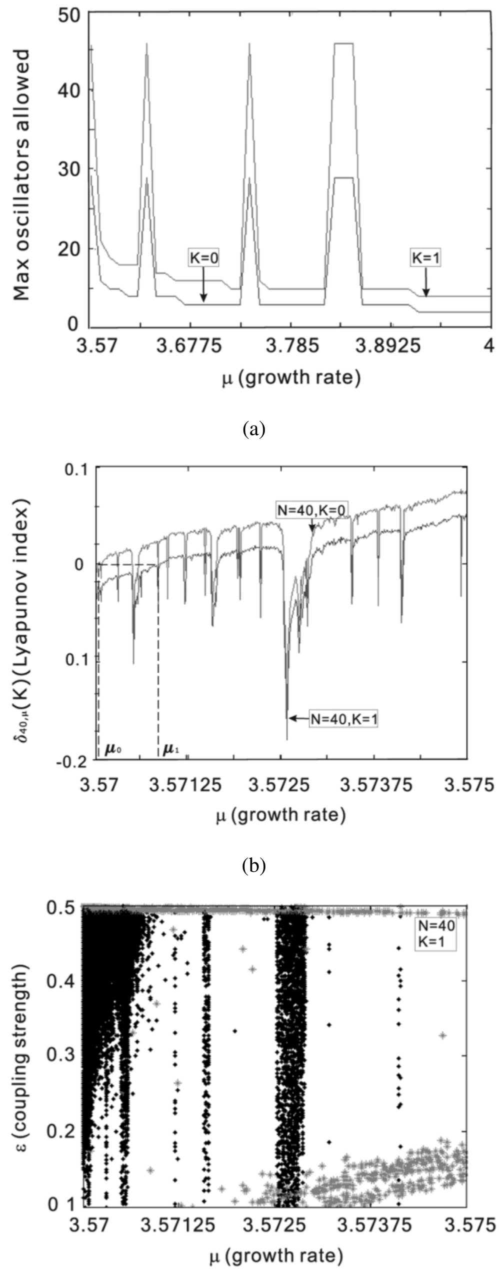

Fig. 3. (a) Maximum number of oscillators allowed for which (1) acquires synchronization. (b) Lyapunov index versus growth rate. Here, $\mu_0 = 3.5704$ and $\mu_1 = 3.57085$. (c) Numerically produced synchronization interval with $K = 1$ and $N = 40$.

IV. Conclusion

The optimal coupling strength of CMLs with symmetric coupling can be analytically obtained. In particular, we also identify the best choice of coupling strength in the sense that such a coupling strength gives the fastest convergence rate of initial values toward the synchronous manifold. Furthermore, such a coupling strength is independent of the choice of the individual chaotic map. It should be noted that due to the nonlinear coupling for CMLs, both the second largest and the smallest values play a role in determining the synchronization interval.

This, in turn, places a limit on the number of oscillators for acquiring synchronization, no matter how we choose the coupling strengths. Based on those results, the theory of the wavelet transform method on system (1) can be predicted numerically. The family of quadratic maps is then used to demonstrate that the wavelet transform method can greatly increase the applicable ranges of coupling strengths, the parameters of the individual chaotic map, and the number of nodes for local synchronization of CMLs.

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