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A systematic procedure to obtain a preferable and robust ranking of units

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Abstract

This paper introduces one systematic procedure for the manager of an organization to assess units under its governance using multiple performance indices. The goal of this systematic procedure is to assist the manager in obtaining a preferable and robust ranking result for units. In this procedure, for all units, one common set of weights attached to the performance indices is determined in order to maximize the group's comprehensive score. Then, using the common set of weights, each unit's comprehensive score is evaluated and compared for ranking. In order to obtain the preferable ranking, the manager's subjective preference is considered and formulated by the virtual weights restrictions while determining the common weights in the procedure. The procedure is applied in order to obtain a robust ranking by modifying the boundary of the feasible region of virtual weights restrictions in each assessment. The final statistical ranking of all assessments provides the manager with one robust ranking, which is invariant in different feasible regions of virtual weights restrictions in the numerical example.

Keywords: Ranking; Comprehensive score; Common set of weights; Virtual weights restrictions

1. Introduction

It is a frequent task for managers to assess units that differ in their values of *to-be-minimized* and *to-be-maximized* performance indices. Intuitively, the larger values in the to-be-maximized indices and the smaller values in the to-be-minimized indices indicate superior units. In order to integrate the values of different indices into one comprehensive score for each unit, there is a need for the manager to assign a weight to each performance index in the computation of the score. Then, the comprehensive score is defined as the ratio of the weighted sum of to-be-maximized indices to the weighted sum of to-be-minimized indices. In this paper, we developed a systematic procedure to determine a common set of weights in computing units' comprehensive scores that can reflect the ranking outcome. Simultaneously, based on the consideration of the manager's preference and the statistical ranking of repeatable assessments used in the procedure, we further obtained a preferable and robust ranking for the manager.

Our problem is similar to the problem of data envelopment analysis (DEA) proposed by Charnes et al. [1], which viewed input indices and output indices as attributes for evaluating decision making units (DMUs), with the minimization of input indices and/or the maximization of output indices as associated objectives. However, the discrimination power of DEA is constrained to only classifying the DMUs, and lacks a complete and clear ranking system. There is much

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follow-up research on increasing the DEA's discrimination power, especially the use of weight restrictions—*absolute* weights restrictions and *virtual* weights restrictions (VWR)—to reduce the number of efficient DMUs.

Absolute weights restrictions were first proposed by Thompson et al. [2], imposing acceptable bounds on ratios of weights in DEA that is known as the assurance region method. Dyson et al. [3] proposed that meaningful bounds are directly imposed upon individual weights. The other famous method, the cone ratio method, proposed and discussed by Charnes et al. [4,5], is more general than the assurance method. The disadvantage of using absolute weights restrictions is that the bounds setting are dependent upon the units of the indices and the orders of magnitude in the indices values.

It is not easy for a human to intuitively express their preference for weights restrictions. In order to make it easier for a human to quantify value judgments in terms of percentage values, VWR was first proposed by Wong et al. [6], setting the lower and/or upper bounds into the ratio of virtual variables. Sarrico et al. [7] further brought the concept of assurance regions into VWR. They showed that the use of the assurance region of virtual weights restrictions is more general and preferable to the use of proportional VWR, Bernroider et al. [8] proposed discussion about the interaction between bound setting in the assurance region method and the validity of ranking outcomes in the assessment of an information system. However, because of the infeasibility problem occurring in the incorporation of lots of weights restrictions, Estellita Lins et al. [9] proposed the existence theorem, which establishes feasibility conditions for DEA with multiple weights restrictions.

As for the other approaches for ranking in DEA, a review of several ranking methods was proposed by Alder et al. [10]. An interesting approach for us to obtain the ranking of units in the DEA structure is to apply the common weights to all DMUs. The use of common weights in DEA was first proposed by Cook et al. [11] and Roll et al. [12] to evaluate highway maintenance units. Cook et al. [13,14] gave a subjective ordinal preference ranking with common weights obtained by closing the gap between the upper and lower limits of the weights while they proceed a series of bounded DEA runs. Ganley et al. [15] ranked each DMU by using the common weights for all the DMUs by maximizing the sum of efficiency ratios of all the DMUs. Sinuany–Stern et al. [16] used linear discriminant analysis to find a score function which ranks DMUs while given the DEA efficient and inefficient sets. Sinuany-Stern et al. [17] ranked all the DMUs on the same scale by developing DR/DEA to provide the best common weights attached to the input indices and output indices to.

Liu et al. [18] introduced common weight analysis (CWA) to determine the single most favorable common set of weights for DMUs on the DEA frontier in view of maximizing the group's efficiency score. The assessment that proceeded based on the original DEA models shows that each DMU determines the efficiency score under its most favorable weights attached to its input indices and output indices. The model used in the current procedure was proposed by Liu et al. [18], different to the original DEA, and shows that the manager chooses the most favorable weights for the group that compromises all DMUs under the manager's governance. In other words, one set of weights that maximizes the group's comprehensive score is used as the common set of weights for all the units to obtain each individual's comprehensive score.

The paper is structured as follows. In Section 2, we review the literature on VWR and CWA. Section 3 introduces our developed systematic procedure to determine the common weights in obtaining preferable and robust ranking by adopting the VWR. In Section 4, we take two numerical examples to illustrate our procedure. The first one shows the procedure is workable in terms of the values of the units' performance indices across large scale ranges. The second one shows the realistic assessment for a retail manager governing eight branches. Finally, Section 5 gives our conclusions, including the notice of application and future research suggestions.

2. Literature review

2.1. Common weights analysis

CWA proposed by Liu et al. [18] aims to assist the manager in determining one set of weights attached to the performance indices, in order to have the best efficiency score for the group of efficient DMUs. Then, the set of weights is regarded as one common set of weights across each efficient DMU, in order to compute its absolute efficiency score for the ranking of DMUs. In order to solve the ranking problem thoroughly, the assessed target is further expanded to all DMUs, including the inefficient DMUs. The CWA model to determine the common set of weights for n units with m input indices and s output indices is formulated as (P1):

(P1) CWA-FP

$$\begin{aligned} \Delta^* = \min \quad & \sum_{j=1}^{n} (\Delta_j^{\mathrm{O}} + \Delta_j^{\mathrm{I}}), \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^{\mathrm{O}}}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^{\mathrm{I}}} = 1, \quad j = 1, \ldots, n, \\ & \Delta_j^{\mathrm{O}}, \Delta_j^{\mathrm{I}} \geqslant 0, \quad j = 1, \ldots, n, \\ & U_r \geqslant \varepsilon > 0, r = 1, \ldots, s, \\ & V_i \geqslant \varepsilon > 0, i = 1, \ldots, m. \end{aligned}$$

x_{ij} is the value of input index i of DMU_j, y_{rj} the value of output index r of DMU_j, V_i the common weights for all DMUs attached to the input index i, U_r the common weights for all DMUs attached to the output index r, $\sum_{i=1}^{m} x_{ij} V_i$ the virtual input of DMU_j, $\sum_{r=1}^{s} y_{rj} U_r$ the virtual output of DMU_j, Δ_j^{I} and Δ_j^{O} denote the input virtual gap and output virtual gap of DMU_j to the benchmark, which are expressed by the maximal efficiency score 1.0. ε is a positive Archimedean infinitesimal constant, which is used in order to avoid the appearance of zero weights. The criteria of (P1) are to minimize the total virtual gaps of all DMUs to benchmark in consideration of group performance.

The optimal common set of weights U_r^* $(r = 1, 2, \ldots, s)$ and V_i^*, $(i = 1, 2, \ldots, m)$ would be solved in (P1) and then used to obtain the absolute efficiency score for each DMU as the standard for comparison. Then, the ranking of all DMUs would be completed.

2.2. *Virtual weights restrictions*

VWR means that the restrictions are imposed on virtual input/output, comprising the product of input/output level and optimal weight for the input/output, rather than on weights directly. It is noted that VWR are developed with reference to the original absolute weights restrictions in DEA formulation. Different to the difficult ascertainment of meaningful bounds in absolute weights restrictions, VWR make it intuitive and easy for a manager to express their subjective preference in the assessment.

The proportional VWR and virtual assurance regions separately provide a different expression in the preference relationships among performance indices. The former represents the importance of one certain input/output attached to the input/output measure, and the latter further expresses the known relationship between any two indices, even among more indices. In this subsection, we give a brief review of virtual assurance regions and proportional VWR.

2.2.1. *Virtual assurance regions*

Sarrico et al. [7] proposed that all the VWR can be described by the general set of restrictions expressed by

$$\sum_{i=1}^{m} \alpha_{iw} x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw} y_{rj} U_r \geqslant k_w, \quad w = 1, \ldots, W, \quad j = 1, \ldots, n, \tag{1}$$

where x_{ij} is the amount of input i to DMU_j, y_{rj} the amount of output r to DMU_j, V_i the weight given to input i, U_r the weight given to output r, m the number of inputs, s the number of outputs, W the number of VWR, n the number of DMUs, $\sum_{i=1}^{m} x_{ij} V_i$ the virtual input given to DMU_j, $\sum_{r=1}^{s} y_{rj} U_r$ the virtual output given to DMU_j, α_{iw} the preference of virtual input to restriction w, β_{rw} the preference of virtual output to restriction w, and k_w the intercept of line restriction w.

While we set $\alpha_{iw} = 0$ (for all i) or $\beta_{rw} = 0$ (for all r) with $k_w = 0$, Eq. (1) translates an ordering of preference in outputs or inputs, as expressed in the following equations:

$$\sum_{r=1}^{s} \alpha_{rw} y_{rj} U_r \geqslant 0, \quad w = 1, \ldots, W, \quad j = 1, \ldots, n, \tag{2}$$

$$\sum_{i=1}^{m} \beta_{iw} x_{ij} V_i \geqslant 0, \quad w = 1, \ldots, W, \quad j = 1, \ldots, n. \tag{3}$$

These kinds of restrictions mentioned above in Eqs. (2) and (3) are useful while managers concentrate the preferences on indices of the same measure.

Besides, if there is at least one $\alpha_{iw} \neq 0$ (for all i) and one $\beta_{rw} \neq 0$ (for all r) with $k_w = 0$, Eq. (1) can be translated as an ordering of preference in input–output, as expressed in the following equation:

$$\sum_{i=1}^{m} \alpha_{iw} x_{ij} V_i + \sum_{r=1}^{s} \beta_{rw} y_{rj} U_r \geqslant 0, \quad w = 1, \ldots, W, \quad j = 1, \ldots, n, \tag{4}$$

Eq. (4) is used to express a known relationship between a pair of inputs and outputs. For instance, to produce one unit of output, one needs to consume at least a certain level of an input.

2.2.2. The proportional VWR

Wong et al. [6] proposed the use of VWR, and in particular, proportional VWR, which were intended to make it easier for managers to quantify value judgments in terms of contribution percentage in the same measure, that is, input measure or output measure. Conceptually the proportional virtual output r of DMU_j represents the importance attached to the output measure (a similar reasoning can be applied to the virtual input i). Let P_{rj}^{O} and P_{ij}^{I}, respectively denote the proportional virtual output r and input i of DMU_j, as follows in Eqs. (5) and (6). Thus, the manager can intuitively set limits on this proportion to reflect value judgments, as follows in Eqs. (7) and (8).

$$P_{rj}^{\mathrm{O}} = \frac{y_{rj} U_r}{\sum_{r=1}^{s} y_{rj} U_r}, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \tag{5}$$

$$P_{ij}^{\mathrm{I}} = \frac{x_{ij} V_i}{\sum_{i=1}^{m} x_{ij} V_i}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{6}$$

$$a_r \leqslant P_{rj}^{\mathrm{O}} \leqslant b_r, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n, \tag{7}$$

$$c_i \leqslant P_{ij}^{\mathrm{I}} \leqslant d_i, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{8}$$

The constant values a_r, b_r, c_i, d_i are the subjective preference limits provided by the manager for output r and input i. Sarrico et al. [7] discussed the possible infeasibility of multiple proportional VWR resulting from the setting of lower and upper bounds, while there exists a large scale range in the index value across all units. They proposed one formulation to determine the feasible lower (upper) bound according to the given upper (lower) bound. Estellita Lins et al. [9] proposed one model to test the feasibility in DEA models with given weight restrictions, including the absolute and VWR, and further modified the bounds using their hyperplane adjusting model while infeasibility occurs.

However, there exists another trap to set the constant value of a_r, b_r, c_i, d_i while the manager has no idea about the implicit restrictions $\sum_{r=1}^{s} P_{rj}^{\mathrm{O}} = 1$ and $\sum_{i=1}^{m} P_{ij}^{\mathrm{I}} = 1$. For instance, there exits two output indices y_{1j} and y_{2j} in the output measure for all DMU_j. While the manager sets $a_1 = 0.2$ and $b_1 = 0.4$ with $20\% \leqslant P_{1j}^{\mathrm{O}} \leqslant 40\%$, $a_2 = 0.2$ and $b_2 = 0.4$ with $20\% \leqslant P_{2j}^{\mathrm{O}} \leqslant 40\%$ for intuitive convenience, the setting obviously cannot satisfy the implicit restriction $P_{1j}^{\mathrm{O}} + P_{2j}^{\mathrm{O}} = 1$. In other words, there exists no such feasible P_{1j}^{O} and P_{2j}^{O} to satisfy these proportional VWR in the output measure. The current paper introduces one method to set initial feasible bounds on the virtual weights to avoid the possible infeasibility in the VWR mentioned above.

3. Our procedure

This section introduces one procedure to use the VWR in the CWA model for the purpose of obtaining one preferable and robust ranking in the assessment results. For the proportional VWR, the infeasibility which occurs in the setting of the lower and upper bound is discussed and solved in the current procedure. In order to express the scenario of units under the manager's governance, referred to by Sarrico et al. [7], we call the unit a unit of assessment (UOA) in this paper in order to discriminate it from the DMU in the DEA model. Besides, in order to unify the terms in the context of the following sections, we regard the to-be-minimized and to-be-maximized indices in our initial problem as input and output indices, respectively.

3.1. The new setting of bounds in proportional VWR

In order to solve the potential infeasibility issue that occurs in the proportional VWR, the current paper proposes one systematic setting in the lower and upper bounds of the proportional virtual inputs and outputs to ensure the feasibility of proportional VWR. Besides solving the infeasibility problems, we use the systematic setting to analyze the relationship between ranking and proportional VWR in Section 4.

Exploring the reasoning of infeasibility in proportional VWR, under the same measure, the sum of the upper bound to all outputs ($\sum_{r=1}^{s} b_r$) cannot reach 1.0 or the sum of the lower bound to all outputs ($\sum_{r=1}^{s} a_r$) exceeds 1.0. In order to avoid this problem, we rewrite Eqs. (7) and (8) to Eqs. (9) and (10), respectively, by formulating the lower bound and upper bound of input and output with the function of parameters τ_i^-, τ_i^+, δ_r^- and δ_r^+. For the purposes of ensuring the proportion is between 0 and 1.0, we give the following range $0 \leqq \tau_i^- \leqq 1$, $0 \leqq \tau_i^+ \leqq m-1$, $0 \leqq \delta_r^- \leqq 1$ and $0 \leqq \delta_r^+ \leqq s-1$:

$$\frac{1}{s}(1-\delta_r^-) \leqslant P_{rj}^{\mathrm{O}} \leqslant \frac{1}{s}(1+\delta_r^+), \quad r=1,\ldots,s, \quad j=1,\ldots,n, \tag{9}$$

$$\frac{1}{m}(1-\tau_i^-) \leqslant P_{ij}^{\mathrm{I}} \leqslant \frac{1}{m}(1+\tau_i^+), \quad i=1,\ldots,m, \quad j=1,\ldots,n. \tag{10}$$

Eqs. (9) and (10) can then be rewritten as Eqs. (11) and (12), respectively. $B_r^{\mathrm{OL}}(B_r^{\mathrm{OU}})$ is a function of $\delta_r^-(\delta_r^+)$, that is $B_r^{\mathrm{OL}} = \frac{1}{s}(1-\delta_r^-)$ ($B_r^{\mathrm{OU}} = \frac{1}{s}(1+\delta_r^+)$), the lower (upper) bound of the P_{rj}^{O} of UOA_j. Similarly, $B_i^{\mathrm{IL}} = \frac{1}{m}(1-\tau_i^-)$ and $B_i^{\mathrm{IU}} = \frac{1}{m}(1+\tau_i^+)$ are the lower bound and upper bound of the P_{ij}^{I} of UOA_j. In other words, P_{rj}^{O} and P_{ij}^{I} can only vary within the interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$ and $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$, respectively.

$$B_r^{\mathrm{OL}} \leqslant P_{rj}^{\mathrm{O}} \leqslant B_r^{\mathrm{OU}}, \quad r=1,\ldots,s, \quad j=1,\ldots,n, \tag{11}$$

$$B_i^{\mathrm{IL}} \leqslant P_{ij}^{\mathrm{I}} \leqslant B_i^{\mathrm{IU}}, \quad i=1,\ldots,m, \quad j=1,\ldots,n. \tag{12}$$

By using Eqs. (9) and (10), a manager easily avoids the infeasibility problem discussed in Section 2.2.2 generated by the implicit restrictions $\sum_{r=1}^{s} P_{rj}^{\mathrm{O}} = 1$ and $\sum_{i=1}^{m} P_{ij}^{\mathrm{I}} = 1$. This is because, for instance, Eq. (9) implies the property that the sum of lower bound is not greater than 1.0, and the sum of upper bound is not less than 1.0, with the highest value $1/s$ to B_r^{OL} and lowest value $1/s$ to B_r^{OU}. The above property also ensures that there exists P_{rj}^{O} satisfying the implication restriction $\sum_{r=1}^{s} P_{rj}^{\mathrm{O}} = 1$. The illustration is similar to Eqs. (10) and (12).

In order to match the virtual assurance region, we have rewritten Eqs. (11) and (12) as Eq. (4) with appropriate values α_{iw} and β_{rw}. For instance, Eq. (11) can be divided into two parts, $P_{rj}^{\mathrm{O}} \geqslant B_r^{\mathrm{OL}}$ and $P_{rj}^{\mathrm{O}} \leqslant B_r^{\mathrm{OU}}$. The former and the latter can be rewritten as Eq. (4) with the setting of parameters α_{iw} and β_{rw} shown in Eqs. (13) and (14), respectively:

$$\alpha_{iw} = 0, \quad i=1,\ldots,m,$$

$$\beta_{rw} = \begin{cases} -B_z^{\mathrm{OL}}, & r \neq z \\ 1-B_z^{\mathrm{OL}}, & r = z \end{cases} \quad r=1,\ldots,s, \quad z=1,\ldots,s, \tag{13}$$

$$\alpha_{iw} = 0, \quad i=1,\ldots,m,$$

$$\beta_{rw} = \begin{cases} B_z^{\mathrm{OU}}, & r \neq z \\ B_z^{\mathrm{OU}}-1, & r = z \end{cases} \quad r=1,\ldots,s, \quad z=1,\ldots,s. \tag{14}$$

Similarly, while Eq. (12) is divided into two parts, $P_{ij}^{\mathrm{I}} \geqslant B_i^{\mathrm{IL}}$ and $P_{ij}^{\mathrm{I}} \leqslant B_i^{\mathrm{IU}}$, they can be rewritten as Eq. (4) with the setting of parameters α_{iw} and β_{rw} shown in Eqs. (15) and (16), respectively.

$$\alpha_{iw} = \begin{cases} -B_z^{\mathrm{IL}}, & i \neq z, \\ 1-B_z^{\mathrm{IL}}, & i = z, \end{cases} \quad i=1,\ldots,m, \quad z=1,\ldots,m,$$

$$\beta_{rw} = 0, \quad r=1,\ldots,s, \tag{15}$$

$$\alpha_{iw} = \begin{cases} B_z^{\mathrm{IU}}, & i \neq z \\ B_z^{\mathrm{IU}} - 1, & i = z \end{cases} \quad i = 1, \ldots, m, \quad z = 1, \ldots, m,$$

$$\beta_{rw} = 0, \quad r = 1, \ldots, s. \tag{16}$$

As for the amount of restrictions, if there exist m inputs and s outputs, the proportional VWR, both Eqs. (11) and (12), can be written as $2s + 2m$ restrictions of Eq. (4) with $W = 2s + 2m$.

The advantage of bound setting in the proportional VWR mentioned above is that the manager can systematically choose the different τ_i^-, τ_i^+, δ_r^- and δ_r^+ to analyze the relationship between ranking and the interval, which is composed of the lower and upper bound. For instance, the manager can start the analysis from the unconstrained case, $\tau_i^- = 1$, $\tau_i^+ = m - 1$, $\delta_r^- = 1$ and $\delta_r^+ = s - 1$, with the interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}] = [0\%, 100\%]$ and $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}] = [0\%, 100\%]$ to P_{rj}^{O} and P_{ij}^{I}, and step by step shorten the interval to the extreme cases, $\tau_i^- = 0$, $\tau_i^+ = 0$, $\delta_r^- = 0$ and $\delta_r^+ = 0$, where each input or output index has equal proportion determined separately.

3.2. CWA with VWR (VWR-CWA)

Because the proportional weights restrictions are one case of the virtual assurance regions, we add the general form of VWR Eq. (1) into the constraints of CWA fractional programming (P1). Then, (P1) can be translated into (P2):

(P2) VWR-CWA-FP

$$\begin{aligned} \Delta^* = \min \quad & \sum_{j=1}^{n} (\Delta_j^{\mathrm{O}} + \Delta_j^{\mathrm{I}}), \\ \text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^{\mathrm{O}}}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^{\mathrm{I}}} = 1, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} \alpha_{iw} V_i x_{ij} + \sum_{r=1}^{s} \beta_{rw} U_r y_{rj} \geqslant k_w, \quad j = 1, \ldots, n, \quad w = 1, \ldots, W, \\ & \Delta_j^{\mathrm{O}}, \Delta_j^{\mathrm{I}} \geqslant 0, \quad j = 1, \ldots, n, \\ & U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\ & V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m. \end{aligned}$$

Following the transformation of the CWA model, the ratio from (P2) can be rewritten in a linear form (P3) and (P4), step by step:

(P3) VWR-CWA-LP1

$$\begin{aligned} \Delta^* = \min \quad & \sum_{j=1}^{n} \Delta_j, \\ \text{s.t.} \quad & \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \Delta_j = 0, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} \alpha_{iw} V_i x_{ij} + \sum_{r=1}^{s} \beta_{rw} U_r y_{rj} \geqslant k_w, \quad j = 1, \ldots, n, \quad w = 1, \ldots, W, \\ & U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\ & V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m, \\ & \Delta_j \geqslant 0, \quad j = 1, \ldots, n. \end{aligned}$$

(P4) VWR-CWA-LP2

$$
\begin{aligned}
-\Delta^* = \max \quad & \sum_{r=1}^{s} Y_r U_r - \sum_{i=1}^{m} X_i V_i, \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \leqslant 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} \alpha_{iw} V_i x_{ij} + \sum_{r=1}^{s} \beta_{rw} U_r y_{rj} \geqslant k_w, \quad j = 1, \ldots, n, \quad w = 1, \ldots, W, \\
& Y_r = \sum_{j=1}^{n} y_{rj}, \quad r = 1, \ldots, s, \\
& X_i = \sum_{j=1}^{n} x_{ij}, \quad i = 1, \ldots, m \\
& U_r \geqslant \varepsilon > 0, \quad r = 1, \ldots, s, \\
& V_i \geqslant \varepsilon > 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

Assume that variable value π_j^* is the *shadow price* of the first set of constraints in linear programming (P4). Then, according to the definition of shadow price, the variations of criterion Eq. (17) will result in the variation of constraint Eq. (18). That is, if the right-hand side of the *j*th constraint increases 1 unit, then the criterion Eq. (18) gets the variation π_j^*:

$$\sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i \leqslant 0 + 1, \tag{17}$$

$$\left(\sum_{r=1}^{s} \left(\sum_{j=1}^{n} y_{rj} \right) U_r - \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) V_i \right) + \pi_j^*(0+1), \tag{18}$$

π_j^* represents the marginal influence on the criteria of linear programming (P4), that is, the marginal influence upon the group's overall performance. It gives another priority reference while UOAs possess equivalent comprehensive score. In the following subsections, we analyze further the ranking rules of those UOAs.

3.3. Ranking rules

In this subsection, we define the ranking rules by comparing the absolute comprehensive score and the shadow price mentioned above with the comprehensive score ζ_j^* of UOA_j, as defined as follows:

$$\zeta_j^* = \frac{\sum_{r=1}^{s} y_{rj} U_r^*}{\sum_{i=1}^{m} x_{ij} V_i^*}, \quad j = 1, \ldots, n, \tag{19}$$

V_i^* and U_r^* denote the optimal common weights obtained in (P3) for all UOAs attached to the input index *i* and output index *r*. One can easily distinguish the UOAs according to the following properties.

Property 1. *The performance of UOA_j is better than UOA_i if $\zeta_j^* > \zeta_i^*$.*

Property 2. *If $\zeta_j^* = \zeta_i^* < 1.0$, then the performance of UOA_j is better than UOA_i if $\Delta_j^* < \Delta_i^*$.*

Property 3. *If $\zeta_j^* = \zeta_i^* = 1.0$, then the performance of UOA_j is better than UOA_i if $\pi_j^* > \pi_i^*$.*

4. Numerical example

In this section, there are two numerical examples to be discussed. We first give a test example to demonstrate the discrimination power of the proposed approach in the current paper. The example with the characteristic of a large scale range in the values of performance indices across UOAs could appeal to the intuitive ranking of UOAs by merely observing the value in the performance indices. Then, it showed that VWR-CWA obtained the consistent ranking with the intuitive ranking. Secondly, one illustrative example shows how the manager of a retailer could obtain preferable and robust ranking results for all branches.

4.1. Test example

Table 1 gives the simulated data set proposed by Liu et al. [18], with two inputs and one output for seven UOAs. The test example possesses the characteristic of a large scale range in the value across UOAs, such that UOA_1, UOA_2, UOA_3, UOA_5, UOA_6 and UOA_7 are shown to be many times larger than UOA_4. These UOAs are ranked intuitively as UOA_1, UOA_2, UOA_3, UOA_4, UOA_5, UOA_6, UOA_7 by comparing the value of input index x_{1j}. The rankings assessed in CWA, as shown in Table 2, are consistent with intuitive ranking in Table 1. By observing the proportion in input measure, x_{2j} plays a more important role than x_{1j} for all UOAs ($P^{\mathrm{I}}_{1j} < P^{\mathrm{I}}_{2j}$) according to the assessment results of CWA. We try to add the preference of the performance indices to understand whether VWR-CWA works to obtain the consistent ranking with intuitive ranking.

The general form of virtual assurance region Eq. (4) can be rewritten as Eq. (20) for the test example with two input indices and one output index:

$$\alpha_1 x_{1j} V_1 + \alpha_2 x_{2j} V_2 + \beta_1 y_{1j} U_1 \geqslant 0, \quad j = 1, \ldots, 7. \tag{20}$$

If we have the preference that the proportion of x_{1j} is larger than twice of x_{2j}, then the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are substituted by $(1, -2, 0, 0)$. Eq. (20) is further rewritten as Eq. (21) for all UOA_j:

$$x_{1j} V_1 - 2x_{2j} V_2 \geqslant 0, \quad j = 1, \ldots, 7. \tag{21}$$

Table 1
Test example with large scale ranges across UOAs

UOA_j	Input index		Output index	Intuitive ranking
	x_{1j}	x_{2j}	y_{1j}	
UOA_1	470 000	700 000	200 000	1
UOA_2	4800	7000	2000	2
UOA_3	49	70	20	3
UOA_4	5	7	2	4
UOA_5	510	700	200	5
UOA_6	52 000	70 000	20 000	6
UOA_7	530 000	700 000	200 000	7

Table 2
The assessment results of CWA and VWR-CWA in test example

UOA_j	(1) CWA $(V_1^*, V_2^*, U_1^*) = (1.00, 1.00, 5.85)$					(2) VWR-CWA with Eq. (21) $(V_1^*, V_2^*, U_1^*) = (2.98, 1.00, 10.50)$				
	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j}	P^{O}_{1j} (%)	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j} (%)	P^{O}_{1j} (%)
UOA_1	1.000	1	40	60	100	1.000	1	67	33	100
UOA_2	0.998	2	41	59	100	0.986	2	67	33	100
UOA_3	0.996	3	41	59	100	0.972	3	68	32	100
UOA_4	0.994	4	42	58	100	0.959	4	68	32	100
UOA_5	0.992	5	42	58	100	0.946	5	68	32	100
UOA_6	0.990	6	43	57	100	0.933	6	69	31	100
UOA_7	0.988	7	43	57	100	0.921	7	69	31	100

Table 3
The indices data in illustrative example

Branch j	Input index		Output index	
	Employee x_{1j}	Cost x_{2j}	Turnover y_{1j}	Profit y_{2j}
A	20	6583	7929	419
B	21	7713	8414	406
C	18	6980	8020	359
D	24	8273	9947	373
E	28	8566	9741	412
F	23	8397	9408	500
G	29	7011	7890	621
H	26	8680	9701	705

Similar to CWA, as Table 2 depicted, VWR-CWA obtains a consistent ranking in the large scale range in the value of performance indices across UOAs with our preference in input measure. It implies that VWR-CWA provides the available discrimination power in assessing the UOAs.

4.2. Illustrative example

A manager of a retail company governs eight branches and periodically assesses them by observing four performance indices: number of Employees, Cost, Turnover and Profit, as depicted in Table 3. Employees and Cost are treated as input indices, while Turnover and Profit are the output indices. Lower inputs and higher outputs are preferred to generate a higher comprehensive score. Different to the first example, the characteristic of a large scale range in the value is across indices, not UOAs (branches). In the following subsections, we illustrate how to obtain the preferable ranking and robust ranking for the manager.

4.2.1. Preferable ranking

In order to discuss the proportion of each index in different models, we assess these branches by using DEA (CCR input-oriented model), VWR-DEA (CCR input-oriented model with VWR), CWA and VWR-CWA models. The general form of VWR Eq. (4) can be rewritten as Eq. (22) for the current numerical example, with two input and two output indices:

$$\alpha_1 x_{1j} V_1 + \alpha_2 x_{2j} V_2 + \beta_1 y_{1j} U_1 + \beta_2 y_{2j} U_2 \geqslant 0, \quad j = A, \ldots, H. \tag{22}$$

If the manager has the preference that the proportion of Profit is no less than half of Turnover, then the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are substituted by $(0, 0, -1, 2)$. Eq. (22) is further rewritten as Eq. (23) for all branches j:

$$-y_{1j} U_1 + 2 y_{2j} U_2 \geqslant 0, \quad j = A, \ldots, H. \tag{23}$$

The proportion allocation of each index obtained from the original DEA model, as depicted in column (1) of Table 4, is extremely disproportional in most branches, even though we add the VWR Eq. (23) in the DEA model (VWR-DEA), as depicted in column (2) of Table 4.For instance, despite the preferable VWR Eq. (23), branches *G* and *H* still choose their favorable weight to create a feasible disproportion in Turnover (0%) and Profit (100%). Besides, comparing DEA with CWA, as depicted in column (1) of Table 5, the proportion allocation in the DEA model is more unstable than the CWA model, without large variation in all branches. The comparison between VWR-DEA and VWR-CWA, as depicted in column (2) of Tables 4 and 5, would have similar results. These results imply that the proportion allocation obtained, whether in the DEA or VWR-DEA models of all branches, cannot reflect the manager's preference altogether.

CWA provided the assessment results in column (1) of Table 5. They show that branch *A* and *B* are the best and worst, respectively. Following these common weights $(V_1^*, V_2^*, U_1^*, U_2^*) = (1.00, 1.27, 1.00, 1.00)$ used in CWA, as depicted in column (1) of Table 5, the manager would observe a large difference in relative proportion, whether between the virtual inputs $(P_{1j}^{\mathrm{I}}, P_{2j}^{\mathrm{I}})$ or outputs $(P_{1j}^{\mathrm{O}}, P_{2j}^{\mathrm{O}})$; for instance, in the row of branch *A*, Employee (0.02%) vs. Cost (99.8%) and Turnover (95.0%) vs. Profit (5.0%).

Table 4
The proportion results of DEA and VWR-DEA in illustrative example

Branch j	(1) DEA				(2) VWR-DEA with Eq. (23)			
	P^{I}_{1j}, v^*_{1j}	P^{I}_{2j}, v^*_{2j}	P^{O}_{1j}, u^*_{1j}	P^{O}_{2j}, u^*_{2j}	P^{I}_{1j}, v^*_{1j}	P^{I}_{2j}, v^*_{2j}	P^{O}_{1j}, u^*_{1j}	P^{O}_{2j}, u^*_{2j}
A	35.2%, 17.58	64.8%, 0.10	95.1%, 0.12	4.9%, 0.12	13.0%, 06.49	87.0%, 0.13	58.7%, 0.07	41.3%, 0.92
B	45.8%, 21.82	54.2%, 0.07	82.6%, 0.09	17.4%, 0.40	100.0%, 47.62	0.0%, 0.00	60.8%, 0.06	39.2%, 0.86
C	65.4%, 36.33	34.6%, 0.05	78.1%, 0.10	21.9%, 0.61	100.0%, 55.56	0.0%, 0.00	62.6%, 0.08	37.4%, 1.01
D	34.1%, 14.21	65.9%, 0.08	96.5%, 0.10	3.5%, 0.09	100.0%, 41.67	0.0%, 0.00	66.6%, 0.06	33.4%, 0.76
E	0.0%, 00.00	100.0%, 0.12	100.0%, 0.10	0.0%, 0.00	0.0%, 00.00	100.0%, 0.15	63.9%, 0.07	36.1%, 0.88
F	46.0%, 19.98	54.0%,0.06	81.2%, 0.08	18.8%, 0.37	11.7%, 06.44	88.3%, 0.13	58.4%, 0.07	41.6%, 0.93
G	10.1%, 03.49	89.9%, 0.13	72.4%, 0.09	27.6%, 0.44	30.1%, 10.38	69.9%, 0.10	0.0%, 0.00	100.0%, 1.61
H	7.5%, 02.90	92.5%, 0.11	73.9%, 0.08	26.1%, 0.37	100.0%, 38.46	0.0%, 0.00	0.0%, 0.00	100.0%, 1.42

Table 5
The assessment results of CWA and VWR-CWA in illustrative example

Branch j	(1) CWA $(V_1^*, V_2^*, U_1^*, U_2^*) =$ (1.00, 1.27, 1.00, 1.00)						(2) VWR-CWA with Eq. (23) $(V_1^*, V_2^*, U_1^*, U_2^*) =$ (94.25, 1.92, 1.00, 13.30)					
	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j} (%)	P^{O}_{1j} (%)	P^{O}_{2j} (%)	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j} (%)	P^{O}_{1j} (%)	P^{O}_{2j} (%)
A	1.000	1	0.02	99.8	95.0	5.0	1.000	1	13.0	87.0	58.7	41.3
B	0.902	8	0.02	99.8	95.4	4.6	0.935	6	11.8	88.2	60.9	39.1
C	0.947	4	0.02	99.8	95.7	4.3	0.996	3	11.2	88.8	62.6	37.4
D	0.984	2	0.02	99.8	96.4	3.6	0.964	5	12.5	87.5	66.6	33.4
E	0.934	6	0.03	99.7	95.9	4.1	0.878	7	13.8	86.2	63.9	36.1
F	0.931	7	0.02	99.8	95.0	5.0	0.975	4	11.9	88.1	58.5	41.5
G	0.956	3	0.03	99.7	92.7	7.3	0.875	8	16.9	83.1	48.8	51.2
H	0.945	5	0.02	99.8	93.2	7.8	0.998	2	12.8	87.2	50.8	49.2

From a managerial perspective, it reveals that the input index Cost and output index Turnover take a considerably large proportion of branch *A*'s rating. The other branches appear to be in a similar situation. This kind of extreme disproportion may not be accepted under specific practical exercises, even though the manager expects quick business development. In fact, in any case, Profit still plays an important role in rating. The virtual assurance region can assist the manager in easily adding his preference in Profit.

The manager reassesses these branches using the VWR-CWA model. The assessment results of VWR-CWA are arranged in column (2) of Table 5 by using the other common weights $(V_1^*, V_2^*, U_1^*, U_2^*) = (94.25, 1.92, 1.00, 13.3)$. Focusing on the row of branch *A* in Table 5, the proportion of Turnover (P^{O}_{1j}) vs. Profit (P^{O}_{2j}) changes from the CWA disproportion 95.0% vs. 5.0% to the 58.7% vs. 41.3% in VWR-CWA. Similar changes also can be seen in other branches. The rankings of the eight branches under CWA and VWR-CWA are completely different. However, the ranking obtained from VWR-CWA is more preferable and reliable to the manager because its preference is considered.

Obviously, the virtual restriction Eq. (23) has an influence upon the final ranking of the branches. In the above case, Eq. (23) is one of general form of Eq. (4) with the parameter $W = 1$. In order to strengthen the preference for the manager, they can add more restrictions to obtain its most preferable ranking for all branches in VWR-CWA.

4.2.2. Robust ranking

Column (2) of Table 5 shows the single preference that the manager assigned. It is common that there exists a situation in which the manager has no preference about the relationship among indices. What they are concerned about is one acceptable and feasible proportion of virtual inputs and virtual outputs in the same measure. The manager can determine a set of values (δ_r^-, δ_r^+) and (τ_i^-, τ_i^+) in Eqs. (9) and (10) to obtain the acceptable interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$ and $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$ for P^{O}_{rj} and P^{I}_{ij}, respectively. For the current numerical example with two inputs ($m=2$) and two outputs ($s=2$), all the values of δ_r^-, δ_r^+, τ_i^- and τ_i^+ are set within 0 and 1 to ensure that P^{O}_{rj} and P^{I}_{ij} are between 0 and 1. For the

Table 6
The 25 combinations of interval limitation for P^{O}_{rj} and P^{I}_{ij}

Combination symbol		$[B^{\mathrm{IL}}_i, B^{\mathrm{IU}}_i]$				
		[0%, 100%]	[10%, 90%]	[20%, 80%]	[30%, 70%]	[40%, 60%]
$[B^{\mathrm{OL}}_r, B^{\mathrm{OU}}_r]$	[0%, 100%]	C1	C2	C3	C4	C5
	[10%, 90%]	C6	C7	C8	C9	C10
	[20%, 80%]	C11	C12	C13	C14	C15
	[30%, 70%]	C16	C17	C18	C19	C20
	[40%, 60%]	C21	C22	C23	C24	C25

Table 7
The assessment results in VWR-CWA of C12 and C22

	(1) VWR-CWA of C12 (V_1, V_2, U_1, U_2) = (191.30, 1.09, 1.00, 6.67)						(2) VWR-CWA of C22 (V_1, V_2, U_1, U_2) = (121.38, 2.20, 1.00, 17.78)					
Branch j	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j} (%)	P^{O}_{1j}(%)	P^{O}_{2j}(%)	Score	Ranking	P^{I}_{1j} (%)	P^{I}_{2j} (%)	P^{O}_{1j} (%)	P^{O}_{2j} (%)
A	0.975	2	35	65	74	26	0.909	3	14	86	52	48
B	0.895	7	32	68	76	24	0.801	6	13	87	54	46
C	0.942	3	31	69	77	23	0.821	5	12	88	56	44
D	0.914	5	34	66	80	20	0.785	7	14	86	60	40
E	0.850	8	36	64	78	22	0.767	8	15	85	57	43
F	0.940	4	32	68	74	26	0.860	4	13	87	51	49
G	0.912	6	42	58	66	34	1.000	1	19	81	42	58
H	1.000	1	34	66	67	33	0.998	2	14	86	44	56

purposes of clearly illustrating our approach, we set $\delta^-_r = \delta^+_r = 0.2$ for $r = 1, 2$ to obtain the lower bound $B^{\mathrm{OL}}_r = 0.4$ and upper bound $B^{\mathrm{OU}}_r = 0.6$, respectively. In other words, P^{O}_{rj} would be limited within the interval [40%, 60%]. If a larger interval is allowed, one may set $\delta^-_r = \delta^+_r = 0.6$ to have the interval [20%, 80%].

From a managerial perspective, while managers desire to understand the ranking of branches under variant kinds of limitations for P^{O}_{rj} and P^{I}_{ij}, Eqs. (9) and (10) provide one systematic setting of lower bound and upper bound. For the cases where δ^-_r and δ^+_r are set at five levels 0.2, 0.4, 0.6, 0.8 and 1.0, P^{O}_{rj} would be limited in the gradually wider intervals [40%, 60%], [30%, 70%], [20%, 80%], [10%, 90%] and [0%, 100%], respectively. With the same setting for τ^-_i and τ^+_i, P^{I}_{ij} would have the same limitations as above.

As depicted in Table 6, there are 25 combinations of interval limitation for P^{O}_{rj} and P^{I}_{ij}. Obviously, different interval limitations for P^{O}_{rj} and P^{I}_{ij} may have different assessment results for the ranking. In this numerical example, we can employ the VWR-CWA model in carrying out an assessment for each combination with corresponding intervals $[B^{\mathrm{IL}}_i, B^{\mathrm{IU}}_i]$ and $[B^{\mathrm{OL}}_r, B^{\mathrm{OU}}_r]$. For instance, the results for C12 and C22 are depicted in Table 7. For the combination C12, the general VWR Eq. (22) can be rewritten as Eqs. (24)–(27) by removing four of the same and repeatable restrictions for all branches from the setting of parameters in Eqs. (13)–1(6):

$$0.8y_{1j}U_1 - 0.2y_{2j}U_2 \geqslant 0, \quad j = A, \ldots, H, \tag{24}$$

$$-0.2y_{1j}U_1 + 0.8y_{2j}U_2 \geqslant 0, \quad j = A, \ldots, H, \tag{25}$$

$$0.9x_{1j}V_1 - 0.1x_{2j}V_2 \geqslant 0, \quad j = A, \ldots, H, \tag{26}$$

$$-0.1x_{1j}V_1 + 0.9x_{2j}V_2 \geqslant 0, \quad j = A, \ldots, H, \tag{27}$$

As Table 7 depicted, the ranking is inconsistent between the two combinations C12 and C22. For managers, it is expected that more ranking outcomes form all kinds of combinations that can help them make more accurate and robust judgments in the ranking of branches.

Table 8
The summary of the 25 ranking results with C1–C25 restrictions

Branch j	Ranking							
	1st	2nd	3rd	4th	5th	6th	7th	8th
A	10	8	7	0	0	0	0	0
B	0	0	0	0	0	18	7	0
C	0	0	15	1	9	0	0	0
D	0	0	0	0	15	3	7	0
E	0	0	0	0	0	0	10	15
F	0	0	1	24	0	0	0	0
G	2	5	2	0	1	4	1	10
H	13	12	0	0	0	0	0	0
Total	25	25	25	25	25	25	25	25
Robust Ranking	H	A	C	F	D	B	E	G

Table 9
The ranking of G in the 25 ranking results with the C1–C25 restrictions

Ranking of branch G		$[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$				
		[0%, 100%]	[10%, 90%]	[20%, 80%]	[30%, 70%]	[40%, 60%]
$[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$	[0%, 100%]	8th	8th	8th	8th	8th
	[10%, 90%]	8th	8th	8th	8th	8th
	[20%, 80%]	6th	6th	6th	6th	7th
	[30%, 70%]	2nd	2nd	2nd	3rd	5th
	[40%, 60%]	1st	1st	2nd	2nd	3rd

While compiling statistics from 25 combinations, we obtained the percentage of occurrence frequency in each ranking, as depicted in Table 8. It is not hard to observe that except for branch G, the high occurrence frequency centralizes in a few rankings for each branch. For instance, branch H is only ranked 1st and 2nd. For branch E, the ranking of 7th and 8th occurs in all combinations. Undoubtedly, branch H is always better than branch E. If managers choose the highest occurrence frequency as the representative branch of each ranking level, the ranking list for 1st–8th is H, A, C, F, D, B, E and G.

Under the above ranking rule, the ranking of branch G is debatable due to its average occurrence in multiple ranking levels. In other words, branch G's ranking varies largely under different combinations. We further trace the ranking status of branch G in all combinations, as depicted in Table 9. While fixing the interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$ with [0%, 100%] or [10%, 90%] for P_{rG}^{O}, branch G is ranked the last of all branches, whatever the interval for P_{iG}^{I}. On the contrary, while we shorten the interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$ step by step from [0%, 100%] to [40%, 60%] for P_{rG}^{O}, fixing the interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$ at [0%, 100%] for P_{iG}^{I}, branch G can reach the best form of all branches.

Following the above observation, we observe that the ranking of branch G is deeply affected by the variation of interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$. If the manager is asked to only select some combinations as the reference of assessment, they should concentrate more attention in determining the appropriate interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$. Branch G will obtain a different ranking while the manager determines a different interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$. As for the determination of interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$, in this case it is not necessary for the manager to cost more effort because these combinations show the same ranking while the interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$ varies.

In order to explore the cause of the above phenomenon, we observe the relationship between the ranking variations and proportion variations of branch G while varying interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$ or $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$, as Tables 10 and 11 depicted. It is obvious that the values of P_{1G}^{I} (54%) and P_{2G}^{I} (46%) obtained in C1 are simultaneously satisfied with a narrower interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$ in C2, C3, C4 and C5. Therefore, as depicted in Table 10, while fixing the interval $[B_r^{\mathrm{OL}}, B_r^{\mathrm{OU}}]$ at [0%, 100%] and shortening the interval $[B_i^{\mathrm{IL}}, B_i^{\mathrm{IU}}]$, we still obtain the invariant values of proportion and ranking

Table 10
The proportion variations of indices of branch G while varying $[B_i^{IL}, B_i^{IU}]$

Combination	Ranking	$[B_i^{IL}, B_i^{IU}]$	P_{1G}^{I} (%)	P_{2G}^{I} (%)	$[B_r^{OL}, B_r^{OU}]$	P_{1G}^{O} (%)	P_{2G}^{O} (%)
C1	8th	[0%, 100%]	54	46	[0%, 100%]	74	26
C2	8th	[10%, 90%]	54	46	[0%, 100%]	74	26
C3	8th	[20%, 80%]	54	46	[0%, 100%]	74	26
C4	8th	[30%, 70%]	54	46	[0%, 100%]	74	26
C5	8th	[40%, 60%]	54	46	[0%, 100%]	74	26

Table 11
The proportion variations of indices of branch G while varying $[B_r^{OL}, B_r^{OU}]$

Combination	Ranking	$[B_i^{IL}, B_i^{IU}]$	P_{1G}^{I} (%)	P_{2G}^{I} (%)	$[B_r^{OL}, B_r^{OU}]$	P_{1G}^{O} (%)	P_{2G}^{O} (%)
C1	8th	[0%, 100%]	54	46	[0%, 100%]	74	26
C6	8th	[0%, 100%]	54	46	[10%, 90%]	74	26
C11	6th	[0%, 100%]	42	58	[20%, 80%]	66	34
C16	2nd	[0%, 100%]	23	77	[30%, 70%]	53	47
C21	1st	[0%, 100%]	19	81	[40%, 60%]	42	58

for branch G. However, as depicted in Table 11, P_{1G}^{O} (74%) and P_{2G}^{O} (26%) obtained in C1 are not satisfied with the narrower interval $[B_r^{OL}, B_r^{OU}]$ in C6, C11, C16 and C21. In order to satisfy narrower intervals $[B_r^{OL}, B_r^{OU}]$, the smaller P_{1G}^{O} and P_{2G}^{O} are necessary. Therefore, the above variation in interval $[B_r^{OL}, B_r^{OU}]$ easily results in the variations of P_{1G}^{I}, P_{2G}^{I} and ranking.

Following the above discussion, we conclude that given the fixed interval $[B_r^{OL}, B_r^{OU}]$, if the value of P_{1G}^{I} and P_{2G}^{I} obtained in C1 is feasible in the narrowest interval $[B_i^{IL}, B_i^{IU}]$ of C5, then the value of P_{1G}^{I} and P_{2G}^{I} is also feasible in C2, C3 and C4. Most importantly, the ranking is invariant with the same proportion in these combinations. If the manager needs to complete all combinations, it is helpful for them to deduce the times of assessment by omitting C2, C3 and C4 while fixing the interval $[B_r^{OL}, B_r^{OU}]$ at [0%, 100%].

5. Conclusion

The systematic procedure proposed in this paper provides both preferable and robust ranking for UOAs. The original CWA model provides one common set of weights for the complete and clear ranking of UOAs. The preferable ranking for the manager is obtained by considering the manager's preference among performance indices with a virtual assurance region. Besides, the systematic setting of upper and lower bounds in proportional weights restriction provides the UOAs with different ranking levels. The ranking statistics finally give the manager one robust ranking which is invariant with the different values of lower and upper bounds.

There are some suggestions to be noted in the application of the current procedure. In carrying out the assessment, if the manager categorizes the performance indices as input indices and output indices according to its subject or intuitive judgment, including the casual relationship among the performance indices, it is easy to confuse input indices and to-be-minimized indices, or output indices and to-be-maximized indices. For instance, the lower defective rate of a product is superior and desirable for a manager of factories. The manager may think the defective rate of the product is the output indices in assessing the production efficiency of factories with input resources. However, based on the current procedure, the defective rate of a product is a to-be-minimized index. This is a possible conflict in the category of performance index in two systems. Therefore, the category of performance indices is suggested first by the idea of to-be-maximized/to-be-minimized indices, and then they are transferred to the output/input indices used in the model. Besides, for the assessment without a to-be-minimized index (input index), an artificial input index is added for all units with value 1.0. It is similar to the assessment without a to-be-maximized index (output index).

As for the future development of the current procedure, we offer the following directions. The application of the procedure proposed in this paper is simplified in order to set the same interval in input measure and output measure, for the convenience of observing the interaction between the ranking and boundary of the feasible region. In reality, the application can be expanded to the case that sets different intervals for variant indices in the same measure, even in different measures. Besides, one appropriate size of combinations is a point of further analysis in the approach. The influence on the size of combinations plays an important role in the robustness of the ranking.

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