國 立 交 通 大 學 理 學 院 應 用 數 學 研 究 所

碩 士 論 文

Isospectral problem on simplicial tori

單形環面上的等譜問題

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A Thesis Submitted to Department of Applied Mathematics National Chiao Tung University in partial Fulfillment of the Requirements for the Degree of Master in Statistics

中 華 民 國 一 百 零 三 年 三 月

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Abstract

The goal of this paper is to discuss the classical isospectral problem in graph theory for a particular type of graphs. We focus on finite regular simplicial tori arising from the Bruhat-Tits building of $\mathrm{PGL}_3(\mathbb{Q}_1)$ and study which kinds of simplicial tori can be uniquely determined by their spectra via graph zeta functions.

單形環面上的等譜問題

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摘要

這篇論文主要是討論特定圖型的等譜性這個古老的圖論問題。其中，我們將目標放在從 $\mathrm{PGL}(\mathbb{Q}_1)$ 的 Bruhat-Tits 結構得到的單形環面，並從他們的 zeta 函數中所得的譜來討論哪些單形環面可以被他們的譜所決定。

致謝

現在回想起來，讀數學系這件事真的是高中時想都沒想過的事。當時高中指考完後結果非常不理想，自暴自棄的心情下選了高師大數學系逃避，沒想到在那邊讀得不錯而重新找到了信心，研究所也順利考到交大應數系所。雖然很可惜沒能應屆推上交大博班，再來也因為比較現實的理由要開始考老師，學生生涯就此告一段落，不過這 6 年來的體驗真的是學生生涯中最充實的 6 年。

這 6 年中，真的很感謝父母願意讓我走我想嘗試的路，感謝老師們的鼓勵也謝謝同學們的友情支持，也非常謝謝康明軒老師願意那麼仔細挑我論文的錯誤（挑到我自己都覺得不好意思了）。最後我想對我自己說一句:”恭喜！畢業了～～”

Contents

1. Introduction

The graph zeta function is a geodesic counting function on a finite graph which was first introduced by Ihara [YI96]. Ihara's original zeta function is defined on a discrete torsion-free co-compact subgroup Γ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and later Serre [JPS03] reformulates it as a function on the graph obtained from the building of $\mathrm{PGL}_2(\mathbb{Q}_p)$ quotient by Γ. The building $\mathcal{B}_1$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a $(p+1)$-regular tree whose vertices are left $\mathrm{PGL}_2(\mathbb{Z}_p)$-cosets and the adjacency operator is the Hecke operator

$$A = \mathrm{PGL}_2(\mathbb{Z}_p)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\mathrm{PGL}_2(\mathbb{Z}_p).$$

The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on the tree $\mathcal{B}_1$ by left multiplication. For a torsion-free discrete co-compact subgroup Γ of $\mathrm{PGL}_2(\mathbb{Q}_p)$, the quotient $\Gamma\backslash\mathcal{B}_1$ is a $(p+1)$-regular graph.

For $\mathrm{PGL}_n(\mathbb{Q}_p)$, its building $\mathcal{B}_{n-1}$ is a $(n-1)$-dimensional simplicial complex. Similar to $\mathrm{PGL}_2(\mathbb{Q}_p)$, for a torsion-free discrete co-compact subgroup Γ of $\mathrm{PGL}_n(\mathbb{Q}_p)$, $\Gamma\backslash\mathcal{B}_{n-1}$ is a finite complex.

One can also consider that the case that when $p = 1$, so that the residue field is the field with one element. In this case, the building of $\mathrm{PGL}_n(\mathbb{Q}_1)$ becomes a single apartment and the torsion-free discrete co-compact subgroup Γ is a free abelian group of rank $n-1$. Especially, when $n = 3$, the building $\mathcal{B}_2$ of $\mathrm{PGL}_3(\mathbb{Q}_1)$ is a simplicial euclidean plane.

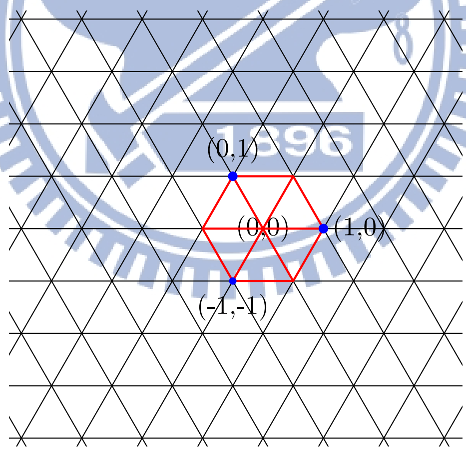

Figure 1: The building of $\mathrm{PGL}_3(\mathbb{Q}_1)$

The 1-skeleton of $\mathcal{B}_2$ can be described as a Cayley graph on $\mathbb{Z}^2$ with the generated set $S = \{(\pm1, 0), (0, \pm1), \pm(1, 1)\}$ shown in Figure 1.

The spectrum of a graph is the set of eigenvalues of the adjacency matrix and two graphs are isospectral if they have the same spectrum. Moreover, two graphs are isospectral if they have the same zeta function (see section 3), so there are many non-isomorphic graphs with the same zeta function.

In this thesis, we would like to study if a finite quotient X of $\mathcal{B}_2$ (which is a torus) can be uniquely determined by its graph zeta function $Z_X(u)$ (up to isomorphism).

Let K_n be the collection of complete representatives of isomorphic classes of finite quotients of $\mathcal{B}_2$ with n vertices. Consider the two numbers $\mathbb{T}_1(n) = |K_n|$ and $\mathbb{T}_2(n) = |\{Z_X(u), X \in K_n\}|$.

By straightforward computation (using Matheamtica), we have the following result

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathbb{T}_1(n)$	1	2	3	2	3	3	5	4	4	3	8	4	5	6	9	4	8	5	10
$\mathbb{T}_2(n)$	1	2	2	2	3	3	4	4	4	3	7	4	5	6	8	4	8	5	9

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
$\mathbb{T}_1(n)$	8	7	5	15	7	8	9	13	6	14	7	15	10	10	10	20	8
$\mathbb{T}_2(n)$	8	7	5	14	7	8	9	12	6	14	7	14	10	10	10	19	8

n	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
$\mathbb{T}_1(n)$	11	12	20	9	18	9	17	16	13	9	28	12	17	14	20	10	22
$\mathbb{T}_2(n)$	11	12	19	9	18	9	16	16	13	9	27	12	17	14	19	10	22

From the above, we have the following conjectures on $\mathbb{T}_1(n)$ and $\mathbb{T}_2(n)$:
(1) $\mathbb{T}_1(p) = [\frac{p+11}{6}]$, where p is an odd prime.
(2) $\mathbb{T}_2(n) = \mathbb{T}_1(n) - \delta$ where δ is equal to 1 if $4|n$ and equal to zero otherwise.

The rest of the thesis is organized as follows. In the Section two and three, we review some basic concepts in graph theory and introduce graph zeta functions. In Section four and five, we describe the group $\mathrm{PGL}_3(\mathbb{Q}_1)$ and it's building explicitly. In the end, we prove (1) and a part of (2) of the conjecture.

2. Ihara zeta function

A graph $X = (V, E)$ is an ordered pair where V is the set whose elements are called the vertices of X; E is a multi-subset of $V \times V$, which elements are called the oriented edges of X. Moreover, if (u, v) is an element of E, so is (v, u) and their multiplicities must be the same.

For a oriented edge $e = (v_1, v_2)$, v_1 is called the starting point of e, denoted by $o(e)$; v_1 is called the end point of e, denoted by $t(e)$. A walk C on X is a sequence of oriented edges

$$C = (e_1, e_2, \ldots, e_n)$$

satisfying $t(e_i) = o(e_{i+1})$ for all $i = 1$ to $n-1$. Here n is called the length of C, denoted by $l(C)$. C has a backtrack if $o(e_i) = t(e_{i+1})$ for some i. C is closed if $o(e_1) = t(e_n)$. When C is closed, we say C has a tail if $t(e_1) = o(e_n)$. Two closed walks are equivalent if one can be obtained from the other by cyclically permuting its sequence of edges. Denote the equivalence class of C by $[C]$. We denote by C^j the multiple of the closed walk C with j times. A closed walk is called prime if it is non-backtracking, tailless and is not a multiple of other shorter closed walk.

We denote $d(v)$ by the number of oriented edges e with $o(e) = v$. Furthermore, we say X is regular if $d(v)$ are the same for all $v \in V$.

The adjacency matrix A of X is a matrix whose rows and columns are indexing by vertices. The (v, v')-entry of A is the number of oriented edges from v to v'. Note that the trace of A^n is the number of closed walks of X of length n.

Definition 2.1. *The Ihara zeta function for X is defined to be the following function of complex number u, with $|u|$ sufficiently small:*

$$Z_X(u) = \prod_{[P]} (1 - u^{\ell(P)})^{-1}$$

where the product is over all primes $[P]$ in X.

Based on Ihara's work [YI96], Bass [BH92] prove the following theorem .

Theorem 2.1. *Let X be a finite $(q+1)$-regular graph. The following identities hold*

$$Z_X(u) = \exp(\sum_{n=1}^{\infty} \frac{N_n}{n} u^n) = \frac{1}{(1-u^2)^{|E|/2-|V|} \det(I - Au + qu^2)},$$

where N_n is the number of non-backtracking and tailless cycles of length n; $|V|$ is the number of vertices of X; $|E|/2$ is the number of non-oriented edges of X.

Corollary 2.1. *The following features of the graph X are uniquely determined by its zeta function:*

(1) *The number of vertices $|V|$.*
(2) *The number of closed walks of length n in X for all n.*

Proof. (1) is followed by that the degree of $Z_X^{-1}(u) = |E| = (q+1)|V|$. For (2), the number of the closed walks of X of length n equals to the trace of A^n, which is the sum of n-th power of eigenvalues of A. On the other hand, the spectrum of A can be determined by the zeta function from the above theorem.

□

3. Field with one element and 1-adic field

In the coming two sections, we call the definition of the field with one element F_1 and the building of PGL_n over 1-padic field from [DK13]. The field with one element, denoted by $F_1 = \{\bar{\pi}\}$, is a suggestive name for an object that should behave similarly to a finite field with a single element, if such a field could exist. The only operator on F_1 is multiplication so that $\bar{\pi} \cdot \bar{\pi} = \bar{\pi}$.

Now we consider the 1-adic field $\mathbb{Q}_1 = \{\pi^i, i \in \mathbb{Z}\}$ so that F_1 is the residue field of $\mathbb{Q}_1$ and the ring of integer for $\mathbb{Q}_1$ is $\mathbb{Z}_1 = \{\pi^i, i \geq 0\}$. Since there is no addition, the n-dimension space over $\mathbb{Q}_1$ is defined as

$$\mathbb{Q}_1^n = \mathbb{Q}_1 \coprod \mathbb{Q}_1 \coprod \dots \coprod \mathbb{Q}_1 = \{\pi_j^i, j = 1, 2, \dots, n, i \in \mathbb{Z}\}.$$

The group of automorphisms of $\mathbb{Q}_1^n$, denoted by $\mathrm{GL}_n(\mathbb{Q}_1)$, consists of all bijections f on $\mathbb{Q}_1^n$ so that $f(\pi x) = \pi f(x), \forall x \in \mathbb{Q}_1^n$. Moreover, the group $\mathrm{PGL}_n(\mathbb{Q}_1)$ is the quotient of $\mathrm{GL}_n(\mathbb{Q}_1)$ by its center.

Theorem 3.1. $\mathrm{GL}_n(\mathbb{Q}_1) \cong \mathbb{Z}^n \rtimes S_n$, *where the semi-direct product is given by the natural permutation of S_n on n-coordinates of $\mathbb{Z}^n$.*

Proof. For $f \in \mathrm{GL}(\mathbb{Q}_1^n)$, write $f(\pi_j^0) = \pi_{\sigma(j)}^{a_{\sigma(j)}}$ for some $\sigma \in S_n, a_{\sigma(j)} \in \mathbb{Z}$ and $j = 1, 2, \dots, n$. Note that f is uniquely determined by the permutation σ and n-tuple of integers $a_{\sigma(j)}$.

Let $\phi : \mathrm{GL}_n(\mathbb{Q}_1) \to \mathbb{Z}^n \rtimes S_n$ given by $f \mapsto ((a_1, a_2, \dots, a_n), \sigma)$ which is bijective. Now for $f_1, f_2 \in \mathrm{GL}_n(\mathbb{Q}_1)$,

$$f_1 \circ f_2(\pi_j^0) = f_1(\pi_{\sigma_2(j)}^{b_{\sigma_2(j)}}) = \pi^{b_{\sigma_2(j)}} f_1(\pi_{\sigma_2(j)}^0) = \pi_{\sigma_1\sigma_2(j)}^{a_{\sigma_1\sigma_2(j)}+b_{\sigma_2(j)}},$$

so

$$\phi(f_1 \circ f_2) = \left((a_1 + b_{\sigma_1^{-1}(1)}, a_2 + b_{\sigma_1^{-1}(2)}, \ldots, a_n + b_{\sigma_1^{-1}(n)}, \sigma_1\sigma_2\right).$$

On the other hand,

$$\begin{aligned}\phi(f_1) \cdot \phi(f_2) &= ((a_1, a_2, \ldots, a_n), \sigma_1) \cdot ((b_1, b_2, \ldots, b_n), \sigma_2) \\ &= ((a_1 + b_{\sigma_1^{-1}(1)}, a_2 + b_{\sigma_1^{-1}(2)}, \ldots, a_n + b_{\sigma_1^{-1}(n)}), \sigma_1\sigma_2).\end{aligned}$$

Therefore, ϕ is a group isomorphism. □

Immediately, we have

Corollary 3.1. $\mathrm{PGL}_n(\mathbb{Q}_1) \cong \mathbb{Z}^{n-1} \rtimes S_n$.

Remark: In the theory of the field with one element, $\mathrm{PGL}_n(F_1)$ is the group S_n, which is the Weyl group of $\mathrm{GL}(F_p)$. Here a similar phenomenon is occurred, so that $\mathrm{PGL}_n(\mathbb{Q}_1)$ is isomorphic to the affine Weyl group of $\mathrm{PGL}_n(\mathbb{Q}_p)$. [KLW10]

4. The building of $\mathrm{PGL}_3(\mathbb{Q}_1)$

Let $\mathbb{Z}_1$ be the ring of integer of $\mathbb{Q}_1$. A lattice L of rank 3 in $\mathbb{Q}_1^3$ is $\mathbb{Z}_1$-invariant subset so that $\mathbb{Q}_1 L = \mathbb{Q}_1^3$. This implies that the lattice $L = \{\pi_j^i, j = 1, 2, 3, i \geq a_j\}$ for some unique $a_1, a_2, a_3 \in \mathbb{Z}$. Denote L by (a_1, a_2, a_3), then we identify lattices in $\mathbb{Q}_1^3$ with $\mathbb{Z}^3$.

The equivalence classes of L is

$$[L] = \{\alpha L, \alpha \in \mathbb{Q}_1\} = \{(a_1 + k, a_2 + k, a_3 + k), k \in \mathbb{Z}\}$$

and denoted by $[a_1, a_2, a_3]$, which is an element in $\mathbb{Z}^3/\mathbb{Z}(1, 1, 1)$.

The building $\mathcal{B}_2$ of $\mathrm{PGL}_3(\mathbb{Q}_1)$ is a 2-dimensional simplicial complex as follows. As an abstract complex, its vertices are equivalence classes of lattices $[L]$; three vertices $[L_0], [L_1], [L_2]$ form a 2-simplex if there is a representative

$$L_0 \supseteq L_1 \supseteq L_2 \supseteq \pi L_0.$$

(The detail can be seen in [BH92], [KLW10] and [JPS03].)

Moreover, for $L = (a_1, a_2, a_3)$, the adjacency vertices of $[L]$ are $\{[L']$ with $L' = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ where $b_i \in \{0, 1\}$ and $\sum b_i = 1$ or 2. We define the i-th adjacency operator (which index is labelled by vertices of the building).

$$A_{i[L][L']} = \begin{cases} 1 & \text{if } [L] = [a_1, a_2, a_3] \text{ and } [L'] = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \\ & \text{ for some } b_i \in \{0, 1\}, \sum b_i = i \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A_1 + A_2$ is the adjacency operator of the underlying graph (1-skeleton) of $\mathcal{B}_2$. Observe that the set $\mathbb{Z}^3/\mathbb{Z}(1, 1, 1)$ has a canonical additive structure, which is a free abelian group generated by $e_1 = [1, 0, 0]$ and $e_2 = [0, 1, 0]$. In this case, the vertices of $\mathcal{B}_2$ are elements in $\mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and the underlying graph of $\mathcal{B}_2$ is the Cayley graph on $\mathbb{Z}^2$ with the generating set $S = \{(\pm 1, 0), (0, \pm 1), \pm(1, 1)\}$.

The underlying space of the whole building $\mathcal{B}_2$ is $\mathbb{Z}^2 \otimes \mathbb{R} = \mathbb{R}^2$ endowed an inner product characterized by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & if\ i = j \\ -\frac{1}{2} & otherwise. \end{cases}$$

Moreover, $\mathrm{PGL}_3(\mathbb{Q}_1) \cong \mathbb{Z}^2 \rtimes S_3$ acts on $\mathcal{B}_2$ as isometries as follows. With respect to the basis e_1 and e_2, the subgroup S_3 is generated by the reflection $\tau = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and the rotation $\sigma = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. The group $\mathbb{Z}^2$ consists of translations so that for $(a, b) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$, (a, b) maps (x, y) to $(x + a, y + b)$. Especially, the geometric of $\mathcal{B}_2$ is relatively to Figure 1.

5. Simplicial tori and their isomorphism classes

Let Γ be a finite index subgroup of $\mathbb{Z}^2$, so that the quotient X_Γ of $\mathcal{B}_2$ by Γ is a simplicial torus which is locally isometric to $\mathcal{B}_2$.

Now, the vertices of X_Γ are $\Gamma-$cosets in $\mathbb{Z}^2$ and its 1-skeleton is the Cayley group on $\mathbb{Z}^2/\Gamma$ with the same generating set S modulo Γ. By abuse of notation, we still denote S modulo Γ by S.

Two simplicial tori X_Γ and $X_{\Gamma'}$ are called isomorphic, if there is an isometric simplicial isomorphism ρ between them. It is clear that the group of automorphisms of X_Γ acts transitively on vertices, so we may assume that ρ maps the vertex Γ to the vertex Γ'. On the other hand, since $\mathcal{B}_2$ is simply connected, there is a unique lifting $\tilde{\rho}$ so that $\tilde{\rho}(0, 0) = (0, 0)$ and the following diagram commute.

$$\begin{tikzcd}
\mathcal{B}_2 \arrow[r, dashed, "\tilde{\rho}"] \arrow[d, "\mathrm{mod}\ \Gamma"'] & \mathcal{B}_2 \arrow[d, "\mathrm{mod}\ \Gamma'"] \\
X_\Gamma \arrow[r, "\rho"'] & X_{\Gamma'}
\end{tikzcd}$$

Since the two quotient maps and ρ are locally isometric simplicial maps so is $\tilde{\rho}$. We conclude that $\tilde{\rho}$ is a linear isometry on $\mathcal{B}_2$ which maps Γ to Γ'. All linear isometries on $\mathcal{B}_2$ forms the dihedral group D_6 with the center $Z(D_6) = \{I_2, -I_2\}$. We can factor D_6 as a product of S_3 and $Z(D_6)$ where S_3 is the symmetric group on 3 letters consisting the following elements

$$\{I_2, \left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\}.$$

Since $Z(D_6)$ fixes any translation subgroup Γ, we conclude that

Theorem 5.1. *Two simplicial tori X_Γ and $X_{\Gamma'}$ are isomorphic if and only if $g(\Gamma) = \Gamma'$ for some $g \in S_3$.*

Now we are able to compute the number of isomorphic classes of simplicial tori of size n. Let Λ_n be set of all index n subgroup of $\mathbb{Z}^2$, then the group S_3 defined above acts on Λ_n canonically. From the above theorem, we have

$$\begin{aligned} \mathbb{T}_1(n) &= \#\text{ of isomorphic classes of finite quotients of } \mathcal{B}_2 \text{ with } n \text{ vertices.} \\ &= \#\ S_3\text{-orbit of } \Lambda_n. \end{aligned}$$

Let $\tau = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ be a reflection and $\sigma = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ be a rotation in S_3. Note that there are three reflections in τ conjugate to S_3 and two nontrivial rotations conjugate to σ. By Burnside's lemma

$$\#\ S_3\text{-orbit of } \Lambda_n = \frac{1}{6}(|\Lambda_n| + 2|\Lambda_n^\sigma| + 3|\Lambda_n^\tau|). \tag{1}$$

Here Λ_n^g is a subset of Λ_n fixed by g.

6. Main Theorem

Recall the conjecture in Section one: let K_n be the collection of complete representatives of isomorphic classes of finite quotients of Cayley$(\mathbb{Z}^2, S)$ with n vertices. Let $\mathbb{T}_1(n) = |K_n|$ and $\mathbb{T}_2(n) = |Z_X(u), X \in K_n|$, then
(1) $\mathbb{T}_1(p) = [\frac{n+11}{6}]$ where p is an odd prime.
(2) $\mathbb{T}_2(n) = \mathbb{T}_1(n) - \delta$, where δ is equal to 1 if $4|n$ and equal to zero otherwise.

Note that every group N in Λ_n contains $(n\mathbb{Z})^2$. Denote by Λ_n^{cyc} the set of all index n subgroups N of $\mathbb{Z}^2$ with $N/(n\mathbb{Z})^2$ being cyclic. Furthermore, S_3 maps Λ_n^{cyc} onto Λ_n^{cyc}. For $g \in S_3$, denote by $\Lambda_n^{cyc,g}$ the subset of Λ_n^{cyc} fixed by g. Then, we have the following lemma.

Lemma 6.1. *For a prime p, $|\Lambda_{p^k}| = \displaystyle\sum_{t=0}^{[\frac{k}{2}]} |\Lambda^{cyc}_{p^{k-2t}}|$.*

Proof. For $m|n$, let
$$\Lambda_{n,m} = \{N \in \Lambda_n, N/(n\mathbb{Z})^2 \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/(n/m)\mathbb{Z}\}.$$
It is clear that every element in Λ_n contained in some $\Lambda_{n,m}$ and such m is unique provided $m < n/m$. Therefore,
$$\Lambda_n = \coprod_{m|n, m^2 \leq n} \Lambda_{n,m}.$$

Especially, when $n = p^k$, $\Lambda_{p^k} = \displaystyle\coprod_{t=0}^{[\frac{k}{2}]} \Lambda_{p^k,p^t}$. On the other hand, there is a bijective map ϕ from Λ_{p^k,p^t} to $\Lambda_{p^{k-2t},0} = \Lambda^{cyc}_{p^k-2t}$ by $\phi(N) = p^{-t}N$, hence
$$|\Lambda_{p^k}| = \sum_{t=0}^{[\frac{k}{2}]} |\Lambda^{cyc}_{p^{k-2t}}|.$$
□

Corollary 6.1. *For a prime p, $|\Lambda^g_{p^k}| = \displaystyle\sum_{t=0}^{[\frac{k}{2}]} |\Lambda^{cyc,g}_{p^{k-2t}}|$ for any $g \in S_3$.*

To compute $|\Lambda^{cyc,g}_{p^{k-2t}}|$, we recall the well-known Hesnel's lemma.

Theorem 6.1 (Hensel's Lemma)**.** *Let $f(x)$ be a polynomial over $\mathbb{Z}$.*

(1) *If x_0 is a solution of $f(x) \equiv 0$ mod p^k such that $f'(x_0) \not\equiv 0$ mod p, then there is a unique $b \in \{0, 1, 2, \ldots, p-1\}$ such that $x_0 + p^k b$ is a solution of $f(x) \equiv 0$ mod p^{k+1}.*
(2) *If x_1 is a solution of $f(x) \equiv 0$ mod p^k and $f'(x_1) \equiv 0$ mod p, then $f(x_1 + ap^k) \equiv 0$ mod p^{k+1} for all $a \in \{0, 1, \ldots, p-1\}$ if and only if $f(x_1) \equiv 0$ mod p^{k+1}.*

The solution in (1) of the above theorem is called a non-singular solution; the solution in (2) is called a singular solution.

Recall that $\tau = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\sigma = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ are generators of S_3.

Lemma 6.2. *For all positive integer n,*

(1) $|\Lambda_n^{cyc,\tau}| = |\{s \in \mathbb{Z}/n\mathbb{Z}, s^2 \equiv 1 \mod n\}|$.

(2) $|\Lambda_n^{cyc,\sigma}| = |\{s \in \mathbb{Z}/n\mathbb{Z}, s^2+s+1 \equiv 0 \mod n\}|$.

Proof. Let $X_n^{cyc} = \{$ all cyclic subgroups of order n in $(\mathbb{Z}/n\mathbb{Z})^2\}$, then there is a bijective map

$$\eta : \Lambda_n^{cyc} \to X_n^{cyc} \quad \text{by} \quad \eta(\mathrm{N}) = \mathrm{N}/(\mathrm{n}\mathbb{Z})^2.$$

Moreover, the action of S_3 on Λ_n^{cyc} induces an action of S_3 on X_n^{cyc} given by $g(\eta(N)) = \eta(gN)$ for all $g \in S_3$. Let $X_n^{cyc,g}$ be the subset of X_n^{cyc} fixed by g, then $|\Lambda_n^{cyc,g}| = |X_n^{cyc,g}|$.

For $G \in X_n^{cyc}$, G is cyclic of order n and it is generated by some $(x,y) \in (\mathbb{Z}/n\mathbb{Z})^2$.

If G is fixed by τ, then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix} \mod n$$

for some $s \in \mathbb{Z}$. It implies that $y = s^2 y$, so $s^2 \equiv 1 \bmod n$.

Since $(x,y) = (x, sx)$ is of order n, x has to be coprime to n. Therefore,

$$G = \langle (x, sx) \rangle = \langle (1, s) \rangle.$$

Conversely, for s satisfy $s^2 \equiv 1 \bmod n$, τ fix $\langle (1,s) \rangle$. We conclude that for each solution of $s^2 \equiv 1 \bmod n$, there is a unique group in X_n^{cyc} generated by $(1,s)$ fixed by τ and all groups in X_n^{cyc} fixed by τ come from this manner.

If G is fixed by σ, then

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+y \\ -x \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix} \mod n$$

for some $s \in \mathbb{Z}$. It implies that $s^2 y = -(s+1)y$ and $s^2 + s + 1 \equiv 0 \bmod n$.

Since $(x,y) = (-sy, y)$ is of order n, y has to be coprime to n. Therefore,

$$G = \langle (-sy, y) \rangle = \langle (-s, 1) \rangle.$$

Conversely, for s satisfy $s^2+s+1 \equiv 0 \bmod n$, σ fix $\langle (-s,1) \rangle$. We conclude that for each solution of $s^2+s+1 \equiv 0 \bmod n$, there is a unique group in X_n^{cyc} generated by $(-s,1)$ fixed by σ and all groups in X_n^{cyc} fixed by σ come from this manner. $\square$

Lemma 6.3. *For a prime p,*

(1) $|\Lambda_{p^k}^{cyc}| = p^k + p^{k-1}$.

(2) $|\Lambda_{2^k}^{cyc,\tau}| = \begin{cases} 1 & \textit{if } k=1 \\ 2 & \textit{if } k=2 \\ 4 & \textit{if } k \geq 3 \end{cases}$ *and* $|\Lambda_{p^k}^{cyc,\tau}| = 2$ *if* $p \neq 2$.

(3) $|\Lambda_{3^k}^{cyc,\sigma}| = \begin{cases} 1 & \textit{if } k=1 \\ 0 & \textit{if } k \geq 2 \end{cases}$ *and* $|\Lambda_{p^k}^{cyc,\sigma}| = \begin{cases} 2 & \textit{if } p \equiv 1 \textit{ mod } 3 \\ 0 & \textit{if } p \equiv 2 \textit{ mod } 3 \end{cases}$.

Proof. First, for $G \in \Lambda_{p^k}^{cyc}$, since G is cyclic, $G = \langle (x,y) \rangle$. Suppose x is coprime to p^k, then $G = \langle (1,z) \rangle$ with $z = x^{-1}y$. In this case, G is uniquely determined by z and z runs through all elements in $\mathbb{Z}/p^k\mathbb{Z}$. Therefore, we have p^k groups of this type in $\Lambda_{p^k}^{cyc}$. If x is not coprime to p^k, since G is cyclic, y has to be coprime to n and $G = \langle (z,1) \rangle$ with $z = y^{-1}x$. In this case, G is uniquely determined by z and z runs through all elements in $\mathbb{Z}/p^k\mathbb{Z}$ not coprime to p^k. Therefore, we have p^{k-1} groups of this type in $\Lambda_{p^k}^{cyc}$. Hence $|\Lambda_{p^k}^{cyc}| = p^k + p^{k-1}$.

Second, let us compute the cardinality of $A_{p^k} = \{s \in \mathbb{Z}_{p^k}, s^2 \equiv 1 \mod p^k\}$.

By direct computation, we have

$$A_2 = \{1 \mod 2\}, A_{2^2} = \{\pm 1 \mod 2^2\} \quad \text{and } A_{2^3} = \{\pm 1, \pm 1 + 2^2 \mod 2^3\}.$$

Claim $A_{2^k} = \{\pm 1, \pm 1 + 2^{k-1} \mod 2^k\}$ for $k \geq 3$ which means $|A_{2^k}| = 4$ for $k \geq 3$.

Assume $A_{2^{k-1}} = \{\pm 1, \pm 1 + 2^{k-2} \mod 2^{k-1}\}$. Note that all solutions in $A_{2^{k-1}}$ are singular solution of $s^2 - 1 \equiv 0 \mod 2^{k-1}$ and

$$\begin{cases} s^2 - 1 \equiv 0 \mod 2^k & \text{, if } s = \pm 1 \\ s^2 - 1 \equiv 2^{k-1} \not\equiv 0 \mod 2^k & \text{, if} s = \pm 1 + 2^{k-2}. \end{cases}$$

By Theorem 6.1, we have $A_{2^k} = \{\pm 1, \pm 1 + 2^{k-1} \mod 2^k\}$ and the claim is followed by induction.

For $p \neq 2$, it is clear that $A_p = \{\pm 1 \mod p\}$. Since ± 1 are non-singular solutions, by Theorem 6.1, $|A_{p^k}| = |A_p| = 2$, $\forall k \in \mathbb{N}$.

Next, let us compute the cardinality of $B_{p^k} = \{s \in \mathbb{Z}_{p^k}, s^2 + s + 1 \equiv 0 \mod p^k\}$.

For $p = 2$, B_2 is empty and so is B_{2^k} for all k.

For $p = 3$, $B_3 = \{1 \mod 3\}$, B_{3^2} is empty and so is B_{3^k} for all $k \geq 2$.

For $p \neq 2$ or 3,

$$\begin{aligned} B_p &= \{s \in \mathbb{Z}_p, 4s^2 + 4s + 4 \equiv 0 \mod p\} \\ &= \{s \in \mathbb{Z}_p, (2s+1)^2 \equiv -3 \mod p\} \end{aligned}$$

which cardinality is given by the Legendre symbol $(\frac{-3}{p}) + 1$. Recall that $(\frac{-3}{p}) = (\frac{-1}{p})(\frac{3}{p})$ and

$$(\frac{-1}{p}) = \begin{cases} 1 & \text{if p} \equiv 1 \text{ mod } 4 \\ -1 & \text{if p} \equiv 3 \text{ mod } 4 \end{cases} \quad \text{and} \quad (\frac{3}{\text{p}}) = \begin{cases} 1 & \text{if p} \equiv \pm 1 \text{ mod } 12 \\ -1 & \text{if p} \equiv \pm 5 \text{ mod } 12 \end{cases},$$

which implies

$$|B_p| = 1 + (\frac{-3}{p}) = \begin{cases} 2 & \text{if } p \equiv 1 \text{ mod } 3 \\ 0 & \text{if } p \equiv 2 \text{ mod } 3 \end{cases}.$$

Note that when $s^2 + s + 1 \equiv 0$ has two solutions, these two solutions are non-singular. By Theorem 6.1, we have $|B_{p^k}| = |B_p|$ for all $p \neq 2, 3$ and all k.

Together with Lemma 6.2, we complete the proof.

□

Combining Corollary 6.1 and Lemma 6.3, we have

Theorem 6.2. *For a prime p,*

(1) $|\Lambda_{p^k}| = \displaystyle\sum_{t=0}^{k} p^t$

(2) $|\Lambda^{\tau}_{p^k}| = \begin{cases} 2k - 1 & \textit{if } p = 2 \\ k + 1 & \textit{if } p \neq 2 \end{cases}$

(3) $|\Lambda^{\sigma}_{p^k}| = \begin{cases} 1 & \textit{if } \ p = 3 \\ k + 1 & \textit{if } \ p \equiv 1 \textit{ mod } 3 \\ 0 & \textit{if } \ p \equiv 2 \textit{ mod } 3 \end{cases}.$

From Theorem 6.2, we can compute $\mathbb{T}_1(p)$, where p is an odd prime. Since

$$\begin{aligned}\mathbb{T}_1(p) &= \frac{1}{6}(|\Lambda_p| + 2|\Lambda_p^\sigma| + 3|\Lambda_p^\tau|) \\ &= \begin{cases} \frac{1}{6}(3+0+3) & \text{if } p=2 \\ \frac{1}{6}(4+2+6) & \text{if } p=3 \\ \frac{1}{6}((p+1)+4+6) & \text{if } p\equiv 1 \mod 3 \\ \frac{1}{6}((p+1)+0+6) & \text{if } p\equiv 2 \mod 3 \text{ and } p\neq 2 \end{cases}.\end{aligned}$$

and

$$\begin{cases} p\equiv 1 \mod 6 & \text{if } p\equiv 1 \mod 3 \\ p\equiv 5 \mod 6 & \text{if } p\equiv 2 \mod 3 \text{ and } p\neq 2. \end{cases}$$

It is equivalent to the formula $\mathbb{T}_1(p) = \left[\frac{p+11}{6}\right]$, where p is an odd prime. Moreover, to compute $\mathbb{T}_1(n)$ for general n, we need the following lemma

Lemma 6.4. *For $g \in S_3, |\Lambda_n^g|$ is a multiplicative function in n.*

Proof. Suppose two positive integers n and m are coprime. Claim

$$|\Lambda_{mn}| = |\Lambda_m| \times |\Lambda_n|$$

Let $X_n = \{$ the subgroups of order n in $(\mathbb{Z}/n\mathbb{Z})^2\}$, then there is a bijective mapping

$$\psi_n : \Lambda_n \to X_n \text{ by } N \mapsto N/(n\mathbb{Z})^2.$$

Since the map $\phi : X_{mn} \to X_m \times X_n$ by $H/(mn\mathbb{Z})^2 \mapsto H/(m\mathbb{Z})^2 \times H/(n\mathbb{Z})^2$ is bijective (by Chinese Remainder Theorem) and the fact $\psi_m, \psi_n, \psi_{mn}$ are bijective, we have $|\Lambda_{mn}| = |\Lambda_m| \times |\Lambda_n|$. Moreover, we can restrict ψ_n to Λ_n^g and hence $|\Lambda_n^g|$ is a multiplicative.

□

From Lemma 6.4, we can compute $\mathbb{T}_1(n)$ for general n. For the second conjecture, we have some partial results given in next section.

7. Geometric information encoded in the graph zeta function

Given a subgroup Γ of $\mathbb{Z}^2$, let X_Γ be the quotient of $\mathcal{B}_2$ by Γ which number of vertices is $V_0 = V_0(\Gamma) = [\mathbb{Z}^2 : \Gamma]$. Let Y_Γ be the 1-skeleton of X_Γ, which is the Cayley graph on $\mathbb{Z}_2/\Gamma$ with the generator set $S = \{(\pm 1, 0), (0, \pm 1), \pm(1,1)\}$.

From Corollary 2.1, knowing the graph zeta function of Y_Γ is equivalent to knowing the spectrum of the adjacent matrix A of Y_Γ.

On the other hand, knowing the spectrum of A is equivalent to knowing the trace of A^n for all n, which is the number of closed walks of length n in Y_Γ.

Therefore, we would like to study if one can determine the structure of X_Γ through these numbers $\mathrm{tr}(A^n)$.

Let P_n be the collection of closed walks in Y_Γ of length n starting from the vertices Γ. Observe that $\mathbb{Z}^2$ acts transitively on vertices of Y_Γ. Therefore, $\mathrm{tr}(A^n) = V_0|P_n|$. Now for each closed walks c in P_n, it can be uniquely lifted to a walks in Y_Γ starting from $(0,0)$ to some element in Γ, denoted by γ_c. Then we can decompose P_n as

$$P_n = \coprod_{\gamma\in\Gamma} P_n(\gamma), \qquad \text{where } P_n(\gamma) = \{c \in P_n, \gamma_c = \gamma\}.$$

On the other hand, one can compute $N_n(\gamma) = |P_n(\gamma)|$ by

$$N_n(\gamma) = |(s_1, \cdots, s_n) \in S^n, s_1 + \cdots + s_n = \gamma\}|.$$

Recall that

$$S_3 = \{I_2, \left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\}$$

which acts on the set of vertices $\mathbb{Z}^2$ of $\mathcal{B}_2$. We can decompose $\mathbb{Z}^2$ as S_3-orbits as

$$\mathbb{Z}^2 = \coprod_{a,b \in \mathbb{Z}_{\geq 0}} S_3(a+b, b)$$

and we say γ in $\mathbb{Z}^2$ is of type (a, b) if $\gamma \in S_3(a+b, b)$.

Theorem 7.1. *If $\gamma \neq (0,0)$ is of type (a, b), then*

(1) $N_n(\gamma) = 0$ *if* $n < a + b$.
(2) $N_{a+b}(\gamma) = \binom{a+b}{a}$.
(3) $N_{a+b+1}(\gamma) = \begin{cases} (a+b+1)\left(\binom{a+b}{b-1} + \binom{a+b}{b+1}\right) & \textit{if } ab \neq 0 \\ (a+b+1)(a+b) \textit{ if one of } a, b = 0 \end{cases}$

Proof. Note that if γ and γ' lie in the same S_3-orbit, then $N_n(\gamma) = N_n(\gamma')$. Therefore, we may assume $\gamma = (a+b, b)$. Let c be a closed walk from $(0,0)$ to γ of length n which uses $r_1, \cdots, r_6$ times of generators $(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)$ respectively. Then we have the equations

$$\begin{cases} r_1 + r_2 - r_4 - r_5 = a + b \\ r_2 + r_3 - r_5 - r_6 = b \\ r_1 + r_2 + \ldots + r_6 = n \end{cases}$$

and the number of such c is given by $\frac{n!}{r_1! r_2! \ldots r_6!}$.

Observing that its length n must be greater than or equal to $a + b$, and if $n = a + b$, then $r_1 = a$ and $r_2 = b$. Therefore, we have $N_n(\gamma) = 0$ if $n < a + b$ and $N_n(\gamma) = \binom{a+b}{a}$ if $n = a + b$.

Next we consider the case of $n = a + b + 1$, then we have

$$\begin{cases} r_1 + r_2 - r_4 - r_5 = a + b \\ r_2 + r_3 - r_5 - r_6 = b \\ r_1 + r_2 + \ldots + r_6 = a + b + 1 \end{cases}.$$

Subtracting the third equation by the first equation above, we obtain $r_3 + 2r_4 + 2r_5 + r_6 = 1$. Thus

$$\begin{cases} r_4 = r_5 = 0 \\ r_1 + r_2 = a + b \\ r_3 + r_6 = 1 \end{cases}.$$

If $a, b \neq 0$, then we have $(r_1, r_2, \ldots, r_6) = (a+1, b-1, 1, 0, 0, 0)$ or $(a-1, b+1, 0, 0, 0, 1)$. Thus $N_{a+b+1}(\gamma) = (a+b+1)\left(\binom{a+b}{b-1} + \binom{a+b}{b+1}\right)$.

If $a = 0$ and $b \neq 0$, then $r_2 + r_3 - r_6 = r_1 + r_2 = b$. It implies that $r_3 = r_1 + r_6$ and $r_1 + 2r_6 = 1$. Therefore, we have $(r_1, r_2, \ldots, r_6) = (1, b-1, 1, 0, 0, 0)$ and $N_{b+1}(\gamma) = b(b+1)$.

If $b = 0$ and $a \neq 0$, then $r_2 + r_3 = r_6$ and $r_2 + 2r_3 = 1$. It implies $(r_1, r_2, \ldots, r_6) = (a-1, 1, 0, 0, 0, 1)$ and $N_{a+1}(\gamma) = a(a+1)$. □

For an element γ of type (a, b), the graph distance of γ and the origin is $a + b$ and we say γ is of length $a + b$. Observe that for a positive integer n, the set of elements of length n in $\mathbb{Z}^2$ is

$$R(n) = \{\pm(n, m), \pm(n - m, n), \pm(-m, n - m), \text{ where } 0 \le m \le n\}$$

which forms a regular hexagon with perimeter $6n$. Let n be the shortest length among all nonzero elements of Γ and N_0 be the number of elements in Γ of length n. N_0 is an even integer since if γ is of length n so is $-\gamma$. Since Γ is a subgroup of $\mathbb{Z}^2$, the graph distance of any two distinct vertices in Γ is greater than or equal to n. Therefore, on the regular hexagon $R(n)$ with perimeter $6n$, there are at most six vertices contained in Γ. We conclude that N_0 is equal to 2, 4 or 6 and when $N_0 = 6$, the six vertices in $\Gamma \cap R(n)$ are

$$\pm(n, m), \pm(n - m, n) \text{ and } \pm(m, n - m)$$

for some $0 \le m < n$.

We define the number $N_0/2$ to be the type of Γ so that Γ is of type 1, 2 or 3. In the rest of the section, we study that whether the lattices of type 3 could be uniquely determined by its graph zeta function.

First, we list all Γ satisfying $V_0 \le 4$ upto the action of S_3 and their corresponding zeta functions.

Note that for any finite index subgroup Γ of $\mathbb{Z}^2$, there exist a unique basis of the form $\{(a, 0), (c, d)\}$ with $a > c \ge 0$. Therefore, we obtain the following table of S_3-orbit of Γ (denoted by $[\Gamma]_{S_3}$):

V_0	2
$[\Gamma]_{S_3}$	$\{\langle(2,0),(0,1)\rangle, \langle(2,0),(1,1)\rangle\}$
Type of Γ	type 1
$(Z_{X_\Gamma}(u))^{-1}$	$(1-u)(1-5u)(1+2u+5u^2)$

V_0	3	
$[\Gamma]_{S_3}$	$\{\langle(3,0),(0,1)\rangle, \langle(3,0),(1,1)\rangle\}$	$\langle(3,0),(2,1)\rangle$
Type of Γ	type 1	type 3
$(Z_{X_\Gamma}(u))^{-1}$	$(1-u)(1-5u)(1+5u^2)^2$	$(1-u)(1-5u)(1+3u+5u^2)^2$

V_0	4	
$[\Gamma]_{S_3}$	$\langle(2,0),(0,2)\rangle$	$\{\langle(2,0),(1,2)\rangle, \langle(4,0),(2,1)\rangle, \langle(4,0),(3,1)\rangle\}$
Type of Γ	type 3	type 2
$(Z_{X_\Gamma}(u))^{-1}$	$(-1+u)(-1+5u)(1+2u+5u^2)^3$	

V_0	4
$[\Gamma]_{S_3}$	$\{\langle(4,0),(0,1)\rangle, \langle(4,0),(1,1)\rangle\}$
Type of Γ	type 1
$(Z_{X_\Gamma}(u))^{-1}$	$(-1+u)(-1+5u)(1-2u+5u^2)^2(1+2u+5u^2)$

Note that when $V_0 = 2$ or 3, the zeta functions can uniquely determine Γ. But when $V_0 = 4$, there are three type 2 lattice whose zeta function and the zeta function of the type 3 lattice $\langle (2,0),(0,2)\rangle$ are the same. In this rest of the section, we study the case that $V_0 \geq 5$.

Recall that two elements v_1 and v_2 in Γ form a reduced basis if v_1 is a shortest nonzero element in Γ and v_2 is a shortest element in $\Gamma - \{\mathbb{Z}v_1\}$. When $N_0 > 2$, $\Gamma \cap R(n)$ contains a reduced basis $\{v_1, v_2\}$. Furthermore, we can replace Γ by $g\Gamma$ for some $g \in S_3$ if necessary, so that we can always assume $v_1 = (n,m)$ with $0 \leq m \leq \frac{n}{2}$.

Suppose Γ is of type 3 with the shortest nonzero elements $\pm(n,m), \pm(n-m,n)$ and $\pm\,(m,n-m)$, where $0 \leq m \leq \frac{n}{2}$. In this case, the number of vertices of Y_Γ is

$$V_0 = n^2 - nm + m^2.$$

Let

$$T_k(\Gamma) = \frac{1}{2}\left(\frac{\mathrm{tr}(A^n)}{V_0} - N_k(e)\right) = \frac{1}{2}\left(|P_k| - N_k(e)\right).$$

which can be determined by the zeta function of Y_Γ. We shall prove that $\{T_k\}$ can be used to distinguish Γ from other subgroups.

By the above theorem, we have $T_k = 0$ for all $k < n$ and

$$T_n(\Gamma) = 3\binom{n}{m};$$

if $m > 0$,

$$T_{n+1} = 3(n+1)\left[\binom{n}{m-1} + \binom{n}{m+1}\right];$$

if $m = 0$,

$$T_{n+1} = 3n(n+1).$$

Let Γ' be another lattice which has the same zeta function as Γ. Therefore, we have $V_0(\Gamma) = V_0(\Gamma')$ and $T_k(\Gamma) = T_k(\Gamma')$ for all k. Especially, the shortest nonzero elements of Γ' are of length n. Suppose the reduced basis of Γ' is $v_1' = (n, m_1)$ with $0 \leq m_1 \leq \frac{n}{2}$ and v_2' with the first coordinate of v_2' is non-negative.

Case I: Γ' is of type 3.

As we mentioned earlier, we may assume that the shortest nonzero elements of Γ' are $\pm(n,m'), \pm(n-m',n)$ and $\pm\,(m',n-m')$, where $0 \leq m' \leq \frac{n}{2}$. Then we have

$$3\binom{n}{m} = T_n(\Gamma) = T_n(\Gamma') = 3\binom{n}{m'}.$$

Since the binomial coefficient $f(m) = \binom{n}{m}$ is increasing when $0 \leq m \leq \frac{n}{2}$, we have $m = m'$. On the other hand, Γ and Γ' are both generated by their shortest nonzero elements. We conclude that $\Gamma = \Gamma'$.

Case II: Γ' is of type 2.

In this case, $v_2' = (n, m_2)$, (m_2, n) or $(n - m_2, -m_2)$ for some $0 \leq m_2 \leq n$. Observe that if $v_2' = (n, m_2)$ then $v_1' - v_2' = (0, m_1 - m_2)$ which is a non-zero vector of length less than or equal to n. On the other hand, $(0, m_1 - m_2) \neq \pm v_1'$

or $\pm v_2'$ which is a contraction. Therefore, it is sufficient to consider two sub-cases $v_2' = (m_2, n)$ or $v_2' = (n - m_2, -m_2)$.

Case II-A: $v_2' = (m_2, n)$. In this case, we have

$$n^2 - nm + m^2 = V_0(\Gamma) = V_0(\Gamma') = n^2 - m_1 m_2 \quad \Rightarrow \quad m(n-m) = m_1 m_2 \tag{2}$$

and

$$3\binom{n}{m} = T_n(\Gamma) = T_n(\Gamma') = \binom{n}{m_1} + \binom{n}{m_2}. \tag{3}$$

If $m = 0$, then $m_1 = 0$ or $m_2 = 0$. Suppose $m_1 = 0$, then Eq. (3) becomes

$$\binom{n}{m_2} = 2.$$

We conclude that $n = 2$ and $m_2 = 1$ so that $V_0(\Gamma) = V_0(\Gamma') = 4$.

Assume $m_2 = 0$, then similar to the case $m_1 = 0$, we have $n = 2$ and $m_1 = 1$ so that $V_0(\Gamma) = V_0(\Gamma') = 4$. The two cases contradict to our assumption.

Now suppose $m > 0$. If one of m_1 and m_2 is less than or equal to m, then by Eq.(2), the other one is greater than or equal to $n - m$. Since $m < \frac{n}{2}$, in this case, we have

$$\binom{n}{m_1} + \binom{n}{m_2} \leq \binom{n}{m} + \binom{n}{n-m} < 3\binom{n}{m}$$

which contradicts to Eq.(3). Therefore, we have $m < m_1, m_2 < n - m$. Next we consider the following two sub-cases.

Case II-A-(1): Suppose there is no element of length $n+1$ in Γ'. Then

$$3(n+1)\left[\binom{n}{m-1} + \binom{n}{m+1}\right] = T_{n+1}(\Gamma) = T_{n+1}(\Gamma') = (n+1)\left[A + B\right]. \tag{4}$$

where $A = \binom{n}{m_1 - 1} + \binom{n}{m_1 + 1}$, $B = \binom{n}{m_2 - 1} + \binom{n}{m_2 + 1}$.

Applying Pascard identity to 2 Eq.(3) $+ \frac{1}{n+1}$ Eq.(4), we obtain

$$3\binom{n+2}{m+1} = \binom{n+2}{m_1+1} + \binom{n+2}{m_2+1}. \tag{5}$$

On the other hand,

$$\binom{n+2}{m+1} = \frac{(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n}{m}.$$

Therefore, together with Eq.(3) and Eq.(5), we have

$$\begin{aligned} &\frac{(n+2)(n+1)}{(m+1)(n-m+1)}\left[\binom{n}{m_1} + \binom{n}{m_2}\right] \\ =&\frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1} + \frac{(n+2)(n+1)}{(m_2+1)(n-m_2+1)}\binom{n}{m_2}. \end{aligned} \tag{6}$$

Observe that the denominator in the above is of the form

$$h(x) = (x+1)(n-x+1) = -\left(x - \frac{n}{2}\right)^2 + \frac{n^2}{2} + n + 1.$$

Thus,

$$h(x) < f(y) \text{ if } \left|x - \frac{n}{2}\right| > \left|y - \frac{n}{2}\right|. \tag{7}$$

Therefore, $h(m) < h(m_1), h(m_2)$ and the left hand side of the above equation is greater than the right hand side, which is a contradiction.

Case II-A-(2): Suppose there is some element of Γ' of length $n+1$. We need the following lemma.

Lemma 7.1. *The shortest non-zero elements in* $\Gamma' - \{\pm(n, m_1), \pm(m_2, n)\}$ *are* $\pm(n - m_2, m_1 - n)$.

Proof. Observe that the length of $(n - m_2, m_1 - n)$ is $2n - m_1 - m_2$. Suppose $x = a(n, m_1) + b(n, m_2)$ is the desired element in $\Gamma' - \{\pm(n, m_1), \pm(m_2, n)\}$, where $a, b \in \mathbb{Z}$. If one of a and b is equal to zero or both a and b are positive, then the length of $x \geq 2n > 2n - m_1 - m_2$.

Now suppose $ab < 0$. Without loss of generality, we may assume that $a > 0 > b$ and $|a| \geq |b|$, then $x = (a + b)(n, m_1) - b(n - m_2, m_1 - n)$ which length is bounded by $(a + b)n - b(n - m_2)$.

If $a + b > 0$, then $(a + b)n - b(n - m_2) \geq n + (n - m_2) \geq 2n - m_1 - m_2$.

Assume $a+b = 0$, then $x = a(n-m_2, m_1-n)$ is length $a(2n-m_1-m_2)$. Hence the shortest non-zero elements in $\Gamma' - \{\pm(n, m_1), \pm(m_2, n)\}$ are $\pm(n-m_2, m_1-n)$. $\square$

In this case, the above lemma implies $2n - m_1 - m_2 = n + 1$ or equivalently $m_2 = n-m_1-1$. Especially, we have $\binom{n}{m_1}+\binom{n}{m_2} = \binom{n+1}{m_1+1}$. The same computation in Eq.(4) as the previous case provides

$$\begin{aligned}\frac{(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n+1}{m_1+1} &= \frac{(n+2)(n+1)}{(n-m_1)(m_1+2)}\binom{n}{m_1+1}\\ &+ \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1}\\ &+ \frac{1}{n+1}\binom{n+1}{m_1+1}.\end{aligned}$$

Here the extra term comes from the contribution of length $n+1$ elements. Dividing the above equation by $\binom{n+1}{m_1+1}$, we obtain

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)} = \frac{(n+2)}{(n-m_1+1)} + \frac{(n+2)}{(m_1+2)} + \frac{1}{(n+1)}.$$

On the other hand, we have $m(n - m) = m_1 m_2 = m_1(n - m_1 - 1)$ and

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)} = \frac{(n+2)(n+1)}{m(n-m)+n+1} = \frac{(n+2)(n+1)}{m_1(n-m_1-1)+n+1}.$$

Consider that

$$\begin{aligned}&\frac{(n+2)(n+1)}{m_1(n-m_1-1)+n+1} - \frac{(n+2)}{(n-m_1+1)} - \frac{(n+2)}{(m_1+2)} - \frac{1}{(n+1)}\\ &= \frac{E_1 + E_2 n + E_3 n^2 + E_4 n^3 + n^4}{(2+m_1)(1+n)(1-m_1+n)(1-m_1-m_1^2+n+m_1 n)},\end{aligned}$$

where

$$E_1 = -4 + 7m_1 + 6m_1^2 - 2m_1^3 - m_1^4; \quad E_2 = -7 + 2m_1 + 11m_1^2 + 2m_1^3;$$
$$E_3 = -1 - 7m_1 + m_1^2; \quad E_4 = 3 - 2m_1.$$

Let

(8) $$f(x) = E_1 + E_2 x + E_3 n^2 + E_4 x^3 + x^4.$$

then

(9) $f'(x) = -7 + 2m_1 + 11m_1^2 + 2m_1^3 - 2x - 14m_1x + 2m_1^2x + 9x^2 - 6m_1x^2 + 4x^3.$

$$f''(x) = -2 - 14m_1 + 2m_1^2 + 18x - 12m_1x + 12x^2 \tag{10}$$

$$f'''(x) = 18 - 12m_1 + 24x \tag{11}$$

Recall that $1 \leq m_1 \leq \frac{n}{2}$. Since $f'''(x) > 0, \forall x \geq 2m_1$, $f''(x)$ is increasing when $x \geq 2m_1$. Together with

$$f''(2m_1) = -2 + 22m_1 + 26m_1^2 > 0,$$

we have $f'(x)$ is increasing when $x \geq 2m_1$. Moreover,

$$f'(2m_1) = -7 - 2m_1 + 19m_1^2 + 14m_1^3 > 0.$$

Therefore, $f(x)$ is increasing when $x \geq 2m_1$. On the other hand,

$$f(2m_1) = -4 - 7m_1 + 6m_1^2 + 16m_1^3 + 7m_1^4 > 0,$$

we have $f(x) > 0, \forall x \geq 2m_1$ and it contradicts to

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)} = \frac{(n+2)}{(n-m_1+1)} + \frac{(n+2)}{(m_1+2)} + \frac{1}{(n+1)}.$$

Case II-B: $v_2' = (n - m_2, -m_2)$. Similar to Case II-A, we have

$$3\binom{n}{m} = T_n(\Gamma) = T_n(\Gamma') = \binom{n}{m_1} + \binom{n}{m_2}. \tag{12}$$

In this case, we have the same conclusion as the case II-A-(1) if we assume that there is no element of length $n+1$. Therefore, it remains to consider the case that there are some elements of Γ' of length $(n+1)$.

Lemma 7.2. *The shortest non-zero elements in* $\Gamma' - \{\pm(n, m_1), \pm(n - m_2, -m_2)\}$ *are* $\pm(m_2, m_1 + m_2)$.

Proof. When there is some element (x, y) in Γ' with length $n+1$, $(x, y) = a(n, m_1) + b(n - m_2, -m_2)$ for some non-zero integers a and b. It implies that $0 \leq (a+b)n - bm_2 \leq n+1$ and $0 \leq am_1 - bm_2 \leq n+1$. Therefore, (x, y) may be $(m_2, m_1 + m_2)$ or $(m_2 - n, m_1 + 2m_2)$.

Since the length of (m_2, m_1+m_2) is m_1+m_2 and the length of (m_2-n, m_1+2m_2) is $n + m_1 + m_2$, we hence the shortest non-zero elements in $\Gamma' - \{\pm(n, m_1), \pm(n - m_2, -m_2)\}$ are $\pm(m_2, m_1 + m_2)$. □

Now we return to show that there does not exist m_1 and m_2 such that Eq.(12) can be hold.

From above lemma and Eq(3), the equation $T_{n+1}(\Gamma) = T_{n+1}(\Gamma')$ can be written as

$$\frac{3(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n}{m} = \frac{(n+2)(n+1)}{(m+1)(n-m+1)}\left(\binom{n}{m_1} + \binom{n}{m_2}\right)$$
$$= \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1} + \frac{(n+2)(n+1)}{(n-m_2+1)(m_2+1)}\binom{n}{m_2} + \frac{1}{n+1}\binom{m_1+m_2}{m_2},$$

where $m_1 + m_2 = n + 1$. Therefore, we have

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n+1}{m_1} = \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1} + \frac{(n+2)(n+1)}{m_1(n-m_2+2)}\binom{n}{m_1-1} + \frac{1}{n+1}\binom{n+1}{m_1}$$

and it implies that

$$\frac{(n+2)(n+1)}{m_1(n-m_1+1)+n+1} = \frac{(n+2)}{(m_1+1)} + \frac{(n+2)}{(n-m_2+2)} + \frac{1}{n+1}.$$

It is equivalent to

$$\frac{F_1 + F_2 n + F_3 n^2 + F_4 n^3}{(1+m_1)(1+n)(2-m_1+n)(m_1(n-m_1+1)+n+1)} = 0, \tag{13}$$

where

$$F_1 = 4 + 7m_1 - 6m_1^2 - 2m_1^3 + m_1^4; \quad F_2 = 8 + 15m_1 - 6m_1^2 - 2m_1^3;$$
$$F_3 = 5 + 10m_1 - m_1^2; \quad F_4 = 1 + 2m_1.$$

Let

$$g(x) = F_1 + F_2 x + F_3 x^2 + F_4 x^3. \tag{14}$$

Therefore,

$$g'(x) = 8 + 15m_1 - 6m_1^2 - 2m_1^3 + 10n + 20m_1 x - 2m_1^2 x + 3x^2 + 6m_1 x^2. \tag{15}$$

$$g''(x) = 10 + 20m_1 - 2m_1^2 + (6 + 12m_1)x. \tag{16}$$

Recall that $1 \le m_1 \le \frac{n}{2}$. Since $g''(x)$ is increasing and

$$g''(2m_1) = 10 + 32m_1 + 22m_1^2 > 0,$$

we have that $g'(x)$ is increasing when $x \ge 2m_1$. Moreover, since

$$g'(2m_1) = 8 + 35m_1 + 46m_1^2 + 18m_1^3 > 0,$$

it implies that $g(x)$ is increasing when $x \ge 2m_1$. Together with

$$g(2m_1) = 4 + 23m_1 + 44m_1^2 + 34m_1^3 + 9m_1^4 > 0,$$

it contradicts to Eq.(13). Hence the underlying graphs corresponding to the lattice of type 3 and type 2 can be distinguished if $V_0 > 4$.

Case III: Γ' is of type 1.

In this case, the length of v_2' is greater than n. From Theorem 7.1, we have

$$3\binom{n}{m} = T_n(\Gamma) = T_n(\Gamma') = \binom{n}{m_1}, \tag{17}$$

which implies $m_1 \ge 1$ and $m_1 > m$.

Case III-A: Suppose there is no element of length $n+1$ in Γ'.

If $m = 0$, from Theorem 7.1, we have

$$\binom{n}{m_1} = 3 \qquad \text{and} \qquad 3n(n+1) = (n+1)\left[\binom{n}{m_1-1} + \binom{n}{m_1+1}\right].$$

Therefore, $n = 3, m_1 = 1$ and it contradicts to $9 = \binom{3}{0} + \binom{3}{2}$.

If $m \neq 0$, from Theorem 7.1, we have

(18)
$$3(n+1)\left[\binom{n}{m-1}+\binom{n}{m+1}\right]=T_{n+1}(\Gamma)=T_{n+1}(\Gamma')=(n+1)\left[\binom{n}{m_1-1}+\binom{n}{m_1+1}\right].$$

Applying Pascard identity to 2 Eq.(17) + $\frac{1}{n+1}$ Eq.(18), we have

$$3\binom{n+2}{m+1}=\binom{n+2}{m_1+1}. \tag{19}$$

Together with Eq.(17) and Eq.(19), we have

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n}{m_1}=\frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1}$$

and we get a contradiction from Eq.(7) and Eq.(17).

Case III-B: Suppose there is an element of length $n+1$ in Γ'.

In this case, v_2' may be equal to $(n+1, m_2)$, $(m_2, n+1)$ or $(n+1-m_2, -m_2)$ for some $0 \leq m_2 \leq n+1$. Observe that if $v_2' = (n+1, m_2)$, then $v_2' - v_1' = (1, m_2 - m_1)$ which is a nonzero vector of length less than $n+1$ and not equal to $\pm v_1'$. It contradicts to that Γ' is of type 1 and $\pm v_1'$ are the two shortest elements in Γ'. Therefore, we just need to consider the case $v_2' = (m_2, n+1)$ or $(n+1-m_2, -m_2)$.

Suppose $m_2 = 0$. If $v_2' = (0, n+1)$, then $n^2+n = V_0(\Gamma') = V_0(\Gamma) = n^2 - nm + m^2$ and it implies $n = m(m-n)$, which contradicts to the assumption that $0 \leq m \leq \frac{n}{2}$.

If $v_2' = (n+1, 0)$, then similar to the case $v_2' = (n+1, m_2)$ above, we have a contradiction.

Assume $m_2 = n+1$. If $v_2' = (n+1, n+1)$, then $v_2' - v_1' = (1, n+1-m_1)$, whose length is less than or equal to n since $m_1 \geq 1$. It contradicts to that $\pm v_1'$ are the only two shortest elements in Γ' whose lengths are n. If $v_2' = (0, -n-1)$, then $n^2 + n = V_0(\Gamma') = V_0(\Gamma) = n^2 - nm + m^2$ and it contradicts to our assumption that $0 \leq m \leq \frac{n}{2}$. Therefore, we conclude $1 \leq m_2 \leq n$.

Lemma 7.3. *There is no element of length $n+1$ in $\Gamma' - \{\pm v_2'\}$.*

Proof. Suppose there is an element v_3' of length $n+1$ in Γ'. We may assume that the first coordinate of v_3' is non-negative. Then v_3' is equal to $(m_3, n+1)$ or $(n+1-m_3, -m_3)$ for some $1 \leq m_3 \leq n$. Note that both v_2' and v_3' are of length $n+1$ so we can we assume $m_2 \leq m_3$ and switch v_2' and v_3' if necessary.

Recall that $v_2' = (m_2, n+1)$ or $(n+1-m_2, -m_2)$ and we shall prove the lemma holds for both cases. First, we assume $v_2' = (m_2, n+1)$, then $v_3' - v_2' = (m_3 - m_2, 0)$ or $(n+1-m_2-m_3, m_3-m_2)$. For the case $v_3' - v_2' = (m_3 - m_2, 0)$, its length is less or equal to n, which contradicts to that $\pm v_1'$ are the only two shortest elements in Γ' of length n. Now we consider $v_3' - v_2' = (n+1-m_2-m_3, m_3-m_2)$. Observe that if $m_2 + m_3 \leq n+1$, then the length of $v_3' - v_2'$ is $\max\{n+1-m_2-m_3, m_3-m_2\}$. If we assume $m_2 + m_3 > n+1$, then the length of $v_2' - v_3'$ is $2m_3 - n - 1$. The length of $v_2' - v_3'$ in the two cases are of less than n. It contradicts to that $\pm v_1'$ are the shortest elements in Γ' of length n. Hence there is no element of length $n+1$ in $\Gamma' - \{\pm v_2'\}$. $\square$

Case III-B-(1): $v_2' = (m_2, n+1)$. In this case, we have

$$n^2 - nm + m^2 = V_0(\Gamma) = V_0(\Gamma') = n^2 + n - m_1 m_2 \tag{20}$$

and

$$\text{length of } v_1' - v_2' = 2n - m_1 - m_2 + 1 > n + 1$$

i.e.

$$n > m_1 + m_2. \tag{21}$$

From Eq.(20) and Eq.(21), we have

$$m^2 - nm = n - m_1 m_2 > n - m_1(n - m_1) = n - m_1 n + m_1^2$$

and it implies

$$m^2 - m_1^2 = (m + m_1)(m - m_1) > n(1 - m_1 + m). \tag{22}$$

It contradicts to that $m, m_1 \leq \frac{n}{2}$.

Case III-B-(2): $v_2' = (n + 1 - m_2, -m_2)$. In this case, we have

$$3\binom{n+2}{m+1} = T_{n+1}(\Gamma) = T_{n+1}(\Gamma') = \binom{n+2}{m_1+1} + \frac{1}{n+1}\binom{n+1}{m_2}.$$

It is equivalent to

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)}\binom{n}{m_1} = \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)}\binom{n}{m_1} + \frac{1}{n-m_2+1}\binom{n}{m_2}. \tag{23}$$

Since there is no element of length less than or equal to n and

$$\text{length of } v_1' - v_2' = m_1 + m_2 > n + 1, \tag{24}$$

we have

$$\frac{n+1}{n-m_2+1}\binom{n}{m_2} = \binom{n+1}{m_2} = \binom{n+1}{n-m_2+1} < \binom{n+1}{m_1} = \frac{n+1}{n-m_1+1}\binom{n}{m_1}$$

and implies

$$\frac{1}{n-m_2+1}\binom{n}{m_2} < \frac{1}{n-m_1+1}\binom{n}{m_1} \tag{25}$$

On the other hand, together with the fact $0 \leq m < m_1 \leq \frac{n}{2}$, we have

$$\begin{aligned}
&\frac{(n+2)(n+1)}{(m+1)(n-m+1)} - \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)} \\
&= \frac{(n+2)(n+1)[m_1(n-m_1) - m(n-m)]}{(m+1)(n-m+1)(m_1+1)(n-m_1+1)} \\
&> \frac{(n+2)(n+1)[m_1(n-m_1) - m(n-m_1)]}{(m+1)(n-m+1)(m_1+1)(n-m_1+1)} \\
&= \frac{(n+2)(n+1)(m_1-m)(n-m_1)}{(m+1)(n-m+1)(m_1+1)(n-m_1+1)} \\
&> \frac{1}{(n-m_1+1)}.
\end{aligned}$$

Here, the last inequality is followed by the fact that $(n+2)(n+1) > (m_1+1)(n-m+1)$, $n - m_1 \geq m + 1$ and $m_1 - m \geq 1$.

It contradicts to the inequality

$$\frac{(n+2)(n+1)}{(m+1)(n-m+1)} < \frac{(n+2)(n+1)}{(m_1+1)(n-m_1+1)} + \frac{1}{n-m_1+1}.$$

We summarize the above discussions as

Theorem 7.2. *If Γ is of type 3 and $V_0(\Gamma) > 4$, then Y_Γ can be uniquely determined by it's zeta function.*

To prove the conjecture (2), we have to compare the zeta functions of Γ of type 1 and type 2, which is more complicated. We will finish it in the future works.

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