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Particle trajectories of nonlinear gravity waves in deep water

Hsien-Kuo Chang [a], Yang-Yi Chen [b], Jin-Cheng Liou [a,*]

[a] Department of Civil Engineering, National Chiao Tung University, Hsinchu 300, Taiwan
[b] Department of Marine Environment and Engineering, National Sun Yat-Sen University, Kaohsiung 804, Taiwan

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ABSTRACT

The paper presents a numerical method for calculating the particle trajectories of nonlinear gravity waves in deep water. Particle trajectories, mass-transport velocity and Lagrangian wave period can be accurately determined by the proposed method. The high success rate of the proposed method is examined by comparing the present results with those of (a) Longuet-Higgins, M.S., 1986, 1987. Eulerian and Lagrangian aspects of surface waves. Journal of Fluid Mechanics 173, 683–707 and (b) Lagrangian moments and mass transport in Stokes waves. Journal of Fluid Mechanics 179, 547–555. It is shown that the dimensionless mass-transport velocity can exceed 10% for large waves, and the Lagrangian wave period is much larger than the Eulerian wave period for large waves.

1. Introduction

Stokes (1847) initiated the study of a Stokes wave with a two-dimensional symmetric profile travels at a constant speed. The problem of Stokes waves is one of classic hydrodynamics. Stokes (1847) initiated the Stokes wave problem can be formulated mathematically for nonlinear free boundary conditions. When Stokes waves were first regarded as small disturbances of a flat water surface, the leading analysis of the problem of finite-amplitude Stokes waves was pursued in the literature (Lighthill, 1978; Crapper, 1984; Dean and Dalrymple, 1993; Johnson, 1997; Okamoto and Shoji, 2001). There are two approaches to describe a flow field. In the Eulerian approach, the flow quantities are defined as a function of space and time. In the Lagrangian description, particles are identified by the positions they occupy at a given moment in time. Most analytical investigations have been performed using either the Eulerian description of motion or conformal mapping of two-dimensional irrotational flows. However, the number of papers using the Lagrangian description has increased in recent years (Clamond, 2007; Chang et al., 2007). The Lagrangian approach to describing particle trajectories remains convenient (Chang et al., 2007; Kapinski, 2006).

The orbits of the particles in progressive gravity waves with very small amplitude are known to be either elliptical or circular, depending on their linear solution. Stokes (1847) derived a second-order Lagrangian approximation for irrotational waves and found that the particle trajectories are not closed and become quite distorted due to nonlinearity. In a phenomenon called Stokes drift, the particle trajectories lead to a net mass-transport in the direction of wave propagation for irrotational waves (Stokes, 1847; Longuet-Higgins, 1953). However, the well-known Gerstner's wave theory, discovered by Gerstner (1802) and re-discovered by Rankine (1863), Constantin (2001), and Craik (2004), showed that all particles move on circles and Gerstner's wave in water of finite depth with vorticity is rotational.

Buldakov et al. (2006) followed Stokes' perturbation scheme to derive a third-order solution that includes unexpected and unphysical secular terms indicating that the wave amplitude grows indefinitely in time. Thus, Buldakov et al. (2006) proposed the fixed-point method to overcome this problem. Chen (1994) succeeded in presenting a third-order Lagrangian solution considering how the Lagrangian wave frequency varies with water elevation. Chang et al. (2007) used an alternative perturbation parameter different from that of Chen and proposed a fifth-order Lagrangian approximation for Stokes waves in finite water depth. Clamond (2007) derived a mathematically correct formulation in the Lagrangian description for both Stokes waves and Gerstner-like waves, which are irrotational waves and rotational waves, respectively. Constantin (2006) proved that the trajectories of particles demonstrate no closed paths in an irrotational inviscid Stokes wave traveling at the surface of water over a flat bed. Constantin and Villari (2008) showed that even for linear periodic gravity water waves the particles in fluid do not have closed orbits and each particle has a backward–forward motion per period that yields overall a forward drift.

Some early investigators studied particle trajectories of Stokes waves (Gerstner, 1802; Miche, 1944; Pierson, 1962; Moe et al., 1998). The superiority of the Lagrangian approximations to the corresponding Eulerian approximations in surface wave profile or physical properties was demonstrated by Buldakov et al. (2006),

* Corresponding author. Tel./fax: +886 3 5131487.
E-mail addresses: victorliou@gmail.com, jcliou.cv87g@nctu.edu.tw (J.-C. Liou).

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Chang et al. (2007), and Clamond (2007). Although these analytical approximations could provide insight for physical interpretation and were accurate for practical applications, some numerical calculations are required to obtain more accurate projections for the problem of high nonlinear interaction in steep Stokes waves.

A physical plane of $z = x+iy$ defined by a function of complex potential is commonly used to express wave motion in a steady state. The wave motion becomes steady by superimposing a velocity equal and opposite to the wave celerity on the flow. The complex Fourier expansion and Stokes coefficients were used to solve a closed set of nonlinear equations by an iterative computing algorithm (Schwartz, 1974; Cokelet, 1977; Williams, 1981; Longuet-Higgins, 1979, 1984, 1987). Longuet-Higgins and Fox (1978) and Longuet-Higgins (1979) discussed particle trajectories at the free surface for steep Stokes waves through these computing algorithms. The basic definition of a Lagrangian description for the path of a particle provides an alternative way for computing particle trajectories of Stokes waves. This method can solve a set of two ordinary differential equations when the flow of a Stokes wave in a fixed Eulerian system is known (Constantin, 2006; Constantin and Villari, 2008).

First introduced by Rienecker and Fenton (1981), the Fourier approximation method accurately calculates the physical quantities of Stoke waves in a moving Eulerian system. In the present paper, the Fourier approximation method effectively calculated the velocity of a particle in a deep water Stokes wave at any position in an Eulerian system. A Runge–Kutta–Verner numerical algorithm (Jackson et al., 1978; William, 1971) was used to solve a set of two ordinary differential equations for the trajectory of any Stokes wave particle. Additionally, an algorithm for computing mass-transport velocity and Lagrangian wave period was employed.

2. The governing equations for a Stokes wave

A Stokes wave propagates at a constant speed, c, on the surface of the sea. The Cartesian coordinates (X, Y) shown in Fig. 1 are used to describe the wave motion; the Y-axis is vertical, and the X-axis represents the direction of wave propagation. The origin of Cartesian coordinates lies on the mean water level. Let $U(t, X, Y)$ and $V(t, X, Y)$ be two components of the velocity field of the flow below the free surface, $\eta(t, X)$. Furthermore, physical quantities U, V, and η are periodic in the X-variable and time and act in the form of (X–ct). The motion of a Stokes wave is almost identical in any direction parallel to the crest line and its wave profile keeps a constant form. For this type of Stokes wave, it is convenient to eliminate time from the problem by using a moving frame with the same speed of wave celerity in the positive X-axis. The moving frame denoted by (x, y) can be related to fixed coordinates as

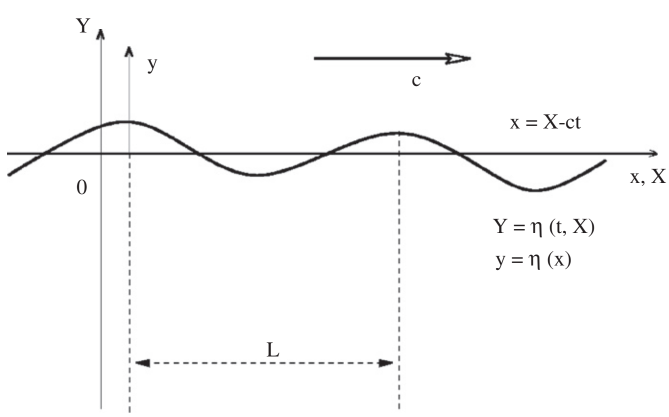

Fig. 1. Coordinates definition and symbolic notations for a Stokes wave.

$$x = X - ct, \quad y = Y. \tag{1}$$

When the fluid is assumed to be incompressible and the wave motion is irrotational, the flow field holds equations for both mass conservation and energy conservation. Hence a stream function satisfying the Laplace equation for this wave field can be given as (Rienecker and Fenton, 1981)

$$\psi_{xx} + \psi_{yy} = 0, \quad -\infty < y < \eta, \quad -\infty < x < \infty, \tag{2}$$

where $\psi(x,y)$ is the stream function. The stream function indicates both the x- and y-components of a particle's the velocity by an expression of $(u(x,y), v(x,y)) = (\psi_y, -\psi_x)$ where the subscripts indicate the partial derivatives. The solution for Eq. (2) subjects to the bottom boundary condition

$$(u, v) = (-c, 0), \quad y \to -\infty, \tag{3}$$

and kinematic and dynamic boundary conditions at the free surface

$$\psi(x, \eta(x)) = -Q, \quad y = \eta(x), \tag{4}$$

and

$$g\eta + \frac{1}{2}(u^2 + v^2) = R, \quad y = \eta(x), \tag{5}$$

respectively. Here $-Q$ is the value of the stream function on the surface, R is the Bernoulli constant and g is the gravitational acceleration. Assuming that the wave is symmetrical about the crest, the governing equations admit a general solution of the form

$$\psi(x, y) = -cy + \sum_{n=1}^{\infty} B_n e^{nky} \cos nkx, \tag{6}$$

where B_n ($n = 1, 2, \ldots$) are the Fourier coefficients for a specified wave, and $k = 2\pi/L$ is the wave number with L as the wavelength. Since the free surface $\eta(x)$ is unknown and the equation of dynamic boundary condition on the free surface is nonlinear, the exact solution of this problem is difficult to obtain.

3. Computing algorithm for trajectories of particles

The Fourier method can give a direct solution in which the values of stream function and surface elevations are obtained as a function of position. Moreover, it does not depend on the wave being small and is valid for all depths. Rienecker and Fenton (1981) showed that the Fourier method gives highly accurate results up to a point equal to about 99% of the maximum wave height, but did not converge to a solution for the highest waves.

To facilitate numerical computation in Eq. (6), the number of terms was limited to N. A set of N+1 equally spaced points on the free surface, from wave crest to trough, was chosen to satisfy Eqs. (4) and (5). Hence, a system of 2N+2 nonlinear equations with 2N+4 unknowns, namely, B_n ($n = 1, 2, \ldots, N$), η_n ($n = 1, 2, \ldots, N+1$), c, Q and R, was obtained. Thus, two additional equations were required to solve the two extra unknowns. That the mean water level is equal to zero as required by mass conservation gives an equation as

$$\int_0^L \eta(x)\, dx = 0. \tag{7}$$

Eq. (7) can be obtained using Simpson's one-third rule for numerical integration of a periodic function. An additional equation for nonbreaking waves of a specified height was introduced as

$$H = \eta(0) - \eta(\pi). \tag{8}$$

Finally, a closed system of $2N+4$ nonlinear equations for the solution of $2N+4$ unknown variables was formed. All quantities were nondimensionalized by virtue of g and k. This set of equations can be solved and programmed using Newton's iteration method, which has the advantage of quadratic convergence (Gerald and Wheatley, 1994). To accomplish this, a Fortran-95 computer program was written in double precision. The computation started with a linear sinusoidal wave as its first approximation. The convergence criterion of all the variables at each iteration and the residuals of the governing equations were set at a value of 10^{-7}. The convergence of the iteration is extremely rapid, and convergence criterion usually is satisfied after about 5 iterations. Rienecker and Fenton (1981) showed that accurate results can be obtained for high and long waves when more than 8 Fourier terms are retained. For $N>40$, the results diverged. In this paper, 32 terms were used to attain even higher accuracy.

When the coefficients B_n ($n = 1, 2,\ldots, N$) are evaluated, the velocity at any position (x,y) can be directly obtained by the definition of $(u(x,y),v(x,y)) = (\psi_y, -\psi_x)$ associated with Eq. (6). The corresponding system in the fixed frame is the Hamiltonian system and has expressions for the velocity at any position (X, Y) for any time

$$U(X,Y,t) = u(x,y) + c, \quad (x \mapsto X - ct, y \mapsto Y) = \sum_{n=1}^{N} nkB_n e^{nkY} \cos nk(X - ct), \tag{9}$$

and

$$V(X,Y,t) = v(x,y), \quad (x \mapsto X - ct, y \mapsto Y) = \sum_{n=1}^{N} nkB_n e^{nkY} \sin nk(X - ct). \tag{10}$$

The path $(X(t), Y(t))$ of a particle with an initial position $(X(0), Y(0))$ can be determined by solving the following system of two differential equations in an implicit form as

$$\frac{dX}{dt} = \sum_{n=1}^{N} nkB_n e^{nkY} \cos n(kX - \sigma_E t), \tag{11}$$

and

$$\frac{dY}{dt} = \sum_{n=1}^{N} nkB_n e^{nkY} \sin n(kX - \sigma_E t), \tag{12}$$

where $\sigma_E = 2\pi/T_E$ is the Eulerian wave frequency that is physically related to the wave number and wave celerity. Thus, it holds that $\sigma_E = kc$. Eqs. (11) and (12) form an initial problem of two ordinary differential equations. Subroutine DIVPRK in the IMSL software programmed based on the Runge–Kutta–Verner fifth and sixth order method was chosen due to its high efficiency for nonstiff systems where the derivative evaluations are not expensive and where the solution is not required at a large number of finitely spaced points (IMSL, 1986). The routine attempts to keep the global error proportional to a user-specified tolerance. The tolerance of maximum absolute error is specified to be 10^{-9}. For intensively plotting the particle trajectories, a fine time step is set to be T_E/N.

4. The fifth-order Lagrangian approximation

An fifth-order Lagrangian approximation (Chang et al., 2007) for Stokes waves in water of finite depth provided explicit expressions for describing particle trajectories. The fifth-order Lagrangian approximation can be limited to a case of deep water when $h \to \infty$ was set in the approximation. The theories and results were briefly introduced in this section.

Lagrangian variables (a,b) designate a label for individual particles in the Lagrangian approach to a physical problem. For a regular train of irrotational gravity waves in a uniform water depth, any particle at a specified mean level is expected to equal b after it advances for a wavelength. Thus, the free surface can be specified as $b = 0$. Based on the results (Ursell, 1953; Longuet-Higgins, 1986), the Lagrangian period is reasonably assumed to be a function of the designated position of each individual particle. The fifth-order Lagrangian approximation for Stokes waves in deep water can be expressed by

$$X(a,b,t) = a + \sum_{n=1}^{3}\left[\sum_{m=1}^{5} X_{mn}(b)\right] \sin n(ka - \sigma_L t) + \sum_{m=2,4} X_{m0}(b,t), \tag{13}$$

$$Y(a,b,t) = b + \sum_{n=1}^{3}\left[\sum_{m=1}^{5} Y_{mn}(b)\right] \cos n(ka - \sigma_L t) + \sum_{m=2,4} Y_{m0}(b), \tag{14}$$

and

$$\sigma_L(b) = \sigma_0(1 + \sigma_{E2} + \sigma_{E4}) + \sigma_0(\sigma_{L2}(b) + \sigma_{L4}(b)), \tag{15}$$

where $\sigma_0 = gk$ is the wave frequency of first-order and the coefficients on the right hand of Eqs. (13)–(15) listed in Table 1 are obtained by setting $h \to \infty$ to the corresponding coefficients (Chang et al., 2007; Liou, 2005).

The first summation on the right-hand side of both Eqs. (13) and (14) includes some terms that indicate trajectories of a particle moving in a periodic function. The second summation on the right-hand side of Eq. (13) shows an aperiodic function increasing linearly in time, implying that a particle marches forward continuously and horizontally in time. The second summation on the right-hand side of Eq. (14) indicates a function of b only, independent of time. This summation indicates a high order vertical correction, decreasing with depth, on the vertical displacement.

Eq. (15) shows that the Lagrangian wave frequency includes the Eulerian wave frequency, $\sigma_E = \sigma_0(1+\sigma_{E2}+\sigma_{E4})$, that is constant for all particles and involves the second-order and fourth-order corrections that are negative and decay with depth. The obtained Eulerian wave frequency is equivalent to that of Fenton's fifth-order Stokes wave theory (Chang et al., 2007; Liou, 2005). The Lagrangian–Eulerian wave frequency relation, Eq. (15), is applicable to all particles at different elevation. Thus, the relation indicates a more general expression than that of Longuet-Higgins (1986), which is valid only at the free surface. Negative values of the corrections, $\sigma_0(\sigma_{L2}(b)+\sigma_{L4}(b))$, in Eq. (15) imply that the Lagrangian wave frequency is smaller than the Eulerian wave frequency. An alternative interpretation is that the Lagrangian wave period (T_L) is longer than the Eulerian wave period (T_E)

Table 1
Formulas for coefficients in Eqs. (13)–(15).

$X_{11} = -(H/2)e^{kb}$	$Y_{11} = (H/2)e^{kb}$
$X_{20} = (kH^2/4)e^{2kb}\sigma_0 t$	$Y_{20} = kH^2/8$
$X_{31} = k^2H^3((-e^{kb}/8)-(e^{3kb}/4))$	$Y_{31} = k^2H^3((-e^{kb}/8)+(-e^{3kb}/4))$
$X_{40} = k^3H^4((-3e^{2kb}/32)+(e^{4kb}/8)\sigma_0 t)$	$Y_{40} = k^3H^4((-e^{2kb}/16)+(-3e^{4kb}/32))$
$X_{42} = k^3H^4((-e^{2kb}/32)+(e^{4kb}/96))$	$Y_{42} = k^3H^4((e^{2kb}/32)-(e^{4kb}/48))$
$X_{51} = k^4H^5((7e^{kb}/192)+(7e^{kb}/64)-(53e^{3kb}/256))$	$X_{51} = k^4H^5((-7e^{kb}/192)-(3e^{3kb}/64)+(21e^{3kb}/256))$
$X_{53} = k^4H^5((-e^{3kb}/384)+(e^{3kb}/2304))$	$Y_{53} = k^4H^5((e^{3kb}/384)-(e^{3kb}/768))$
$\sigma_{E2} = k^2H^2/8$	$\sigma_{L2} = k^2H^2e^{2kb}/4$
$\sigma_{E4} = k^4H^4/128$	$\sigma_{L4} = k^4H^4((2e^{2kb}/32)-(e^{4kb}/8))$

(Longuet-Higgins, 1986). The fifth-order Lagrangian approximation can provide comprehensible physical interpretations of the particle trajectories of Stokes waves.

5. The results and discussion

5.1. Trajectories of particles

Using both the numerical algorithm in Section 3 and the fifth-order Lagrangian approximation, the trajectory of a particle initially located at the crest during a period of $2T_E$ is plotted in Fig. 2. The case of $kH = 0.84$ indicates a very large wave in deep water and is also a case used to compute the mass-transport by Longuet-Higgins (1987). The solid and dashed lines in Fig. 2 denote the path by two methods. Circles on the lines show the positions of the particle's movement for every half Eulerian period.

The numerical solution shows that the particle at the crest marches forward in a non-closed loop and stays around the trough at time $2T_E$, implying that more time than $2T_E$ is needed for the particle to reach the crest again. If the time for a particle at the crest takes to move forward and reach the next crest is defined as the Lagrangian period, it is obvious from the present result that the Lagrangian wave period is longer than the Eulerian period. The result agrees with the conclusions of Chang et al. (2007) and Longuet-Higgins (1986). Fig. 2 indicates that a Lagrangian wave period may be between T_E and $1.5T_E$. Longuet-Higgins (1986) derived a relationship between the Lagrangian wave period and the Eulerian wave period for the limiting wave in deep water, indicating that $T_L = 1.38T_E$. The trajectories of Stokes wave particles obtained by the fifth-order Lagrangian approximation form a similar path to that by the proposed method, but move slightly slower. The proposed Fourier approximation method, which includes 32 harmonic components, is of a much higher order than the fifth-order Lagrangian approximation. Therefore proposed method accounts for the highly nonlinear interaction between each component as bringing about the fast movement of particles. The fifth-order approximation of Chang et al. (2007) obtained by the perturbation method is valid for a gravity wave of finite-amplitude so that a large difference between the numerical solution and the approximation occurs for the case of the very large wave. The trajectories of Stokes wave particles from small amplitude almost to the highest one are calculated to have non-closed paths, indicating that any particle of a Stokes wave moves a horizontal distance over one wave period. The distance divided by the Lagrangian period is called the mass-transport velocity or Stokes drift. The result corresponds to the conclusion of Constantin (2006) and Constantin and Villari (2008), proving that there are no closed particle orbits for Stokes waves of small or large amplitude. Constantin and Escher (2007) proved that in a solitary water wave there is no backward motion, meaning all particles in a solitary wave move in the direction of wave propagation at a positive speed.

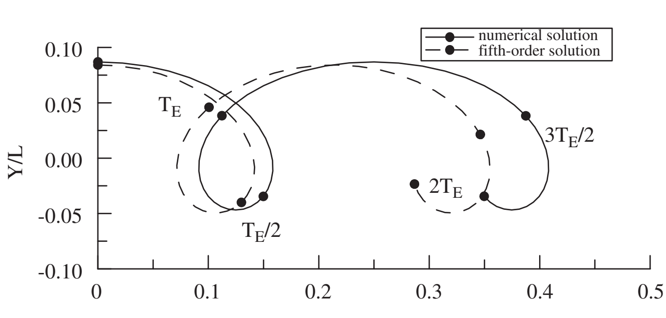

Fig. 2. Trajectory of a particle initially located at the crest during a period of $2T_E$ for the wave case of $kH = 0.84$.

5.2. Mass-transport velocity

Stokes (1847) showed that in a fixed frame the fluid motion in a wave train is oscillatory and that there is a slow drift in the direction of wave propagation that is positive near the surface. Conversely, it decreases toward the bottom in deep water and becomes negative in the case of shallow water. Ursell (1953) and Longuet-Higgins (1984) presented strong proof that the mass-transport velocity has a zero net transport of water. The mass-transport velocity is important in determining the Lagrangian residual velocity from the current measurement at a fixed point in the sea (Zimmerman, 1979).

In this paper, the trajectory of a particle is implicitly expressed by (X,Y) in terms of time t, and solved by using the Runge–Kutta–Verner method for two first-order differential equations. The position of a particle at a given time step can be obtained by the Runge–Kutta–Verner method when the earlier position at the proceeding time is given. Three positions whose vertical elevations are denoted by Y_1, Y_2, and Y_3 are calculated at time t_1, t_2 and t_3, respectively. A set of data with three pairs of Y- and $t = f(Y)$-values is depicted as three solid circles in Fig. 3. The time of a particle moving to Y_c can be estimated by interpolation scheme using a specified function with undetermined coefficients for the collected data. A suggested function in terms of polynomials for these data is drawn in Fig. 3. A Lagrangian form in which uniform spacing is not required is an easy way to establish interpolating polynomials. Through these three data pairs a quadratic Lagrangian polynomial can be given as (Gerald and Wheatley, 1994)

$$t(Y) = \frac{(Y-Y_2)(Y-Y_3)}{(Y_1-Y_2)(Y_1-Y_3)}t_1 + \frac{(Y-Y_1)(Y-Y_3)}{(Y_2-Y_1)(Y_2-Y_3)}t_2 + \frac{(Y-Y_1)(Y-Y_2)}{(Y_3-Y_1)(Y_3-Y_2)}t_3. \tag{16}$$

When $Y = Y_c$ is specified for the elevation of the wave crest, the estimated Lagrangian period, t_L, can be obtained by Eq. (16). Substitution of end time t_L into the Runge–Kutta–Verner algorithm associated with a proceeding point, $(X_2(t_2), Y_2(t_2))$ yields a new end point $(X_3(t_L), Y_3(t_L))$ that would be closer to the expected point (X_c, Y_c) than the original point $(X_3(t_3), Y_3(t_3))$. The updated data set of three points, $(X_1(t_1), Y_1(t_1))$, $(X_2(t_2), Y_2(t_2))$, and $(X_3(t_L), Y_3(t_L))$ is inserted into Eq. (16) again to compute a newly estimated Lagrangian wave period, t_L. Using the new Lagrangian wave period in the Runge–Kutta–Verner algorithm a new $(X_3(t_L), Y_3(t_L))$ more

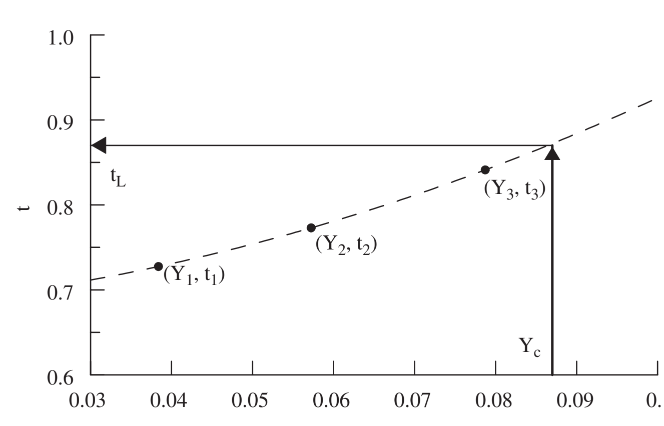

Fig. 3. Graphical interpretation of the Lagrangian form of quadratic polynomials for estimating the Lagrangian wave period.

closely approximates to (X_c, Y_c) than the previous one. After some such iterations, an accurate ($X_3(t_L)$, $Y_3(t_L)$) is decided under a requested tolerance of the absolute value of the difference between the estimated $Y_3(t_L)$ and Y_c. The tolerance is specified by 10^{-7} in the computation. A computational case of $ak = 0.42$ is used for demonstrating the procedure. The first data set at time $t = T_E$, (1+2/32) T_E, and (1+5/32) T_E is given for the particle at the crest of which $Y_c = 0.08692$. When the computation is terminated, the time t_L is set as the Lagrangian period and the estimated horizontal distance, X_c, divided by t_L is then denoted the mass-transport velocity, U_m.

A dimensionless mass-transport velocity, U_m/c, is used to compare the present solution with that of Longuet-Higgins (1987), denoted by U_{mN}, and U_{mL}, respectively, for a particle at the free surface in deep water. The comparison is listed in Table 2. The fourth column lists the absolute errors between U_{mN}/c to U_{mL}/c. If the result of Longuet-Higgins (1987) is taken as a standard, small errors occurring at the fourth decimal digit shown in the fourth column of Table 2 indicates that the obtained U_{mN}/c slightly deviates from U_{mL}/c. Thus, the proposed method is an accurate way for calculating the mass-transport velocity.

A relationship between the relative mass-transport velocity at the free surface and a ratio of two wave frequencies was also derived by Longuet-Higgins (1986, 1987) as

$$\frac{U_m}{c} + \frac{T_E}{T_L} = 1. \tag{17}$$

The formula can be used to examine the accuracy of the present numerical solution. The estimated values of (U_{mL}/c)+(T_E/T_L) are shown in the last column of Table 2 and are very close to one, showing the high accuracy of the proposed solution. The estimated U_m/c shows that the nondimensionized mass-transport velocity may exceed 10% for $ak > 0.32$.

An efficient method of general Lagrangian mean theory proposed by Ardhuin et al. (2008) provides for explicit wave-averaged primitive equations for general wave–turbulence–mean flow interactions.

5.3. Lagrangian period

Computed values of T_L/T_E of all Stokes waves are plotted in Fig. 4. Fig. 4 shows that the values of T_L/T_E obtained by both methods are greater, and these values increase with wave steepness for very large waves much faster than those for small waves. The former conclusion once again identified that the Lagrangian period is larger than the Eulerian period. Very small discrepancy between both results for small waves is shown in Fig. 4 but a gradually increasing difference occurs for large waves. The increasing variation with wave steepness results from more highly nonlinear interaction between components considered in the proposed numerical method than that in the fifth-order Lagrangian approximation.

Table 2
A comparison of the mass-transport velocity at the free surface of Stokes waves in deep water.

$kH/2$	U_{mL}/c	U_{mN}/c	$\lvert U_{mL}-U_{mN}\rvert/c$	(U_{mN}/c)+(T_L/T_E)
0.10	0.01000	0.01018	0.00018	0.99843
0.20	0.04009	0.04034	0.00025	0.99910
0.30	0.09137	0.09114	0.00023	1.00059
0.35	0.12691	0.12723	0.00032	0.99956
0.40	0.17369	0.17331	0.00038	1.00031
0.42	0.19988	0.19996	0.00008	0.99995

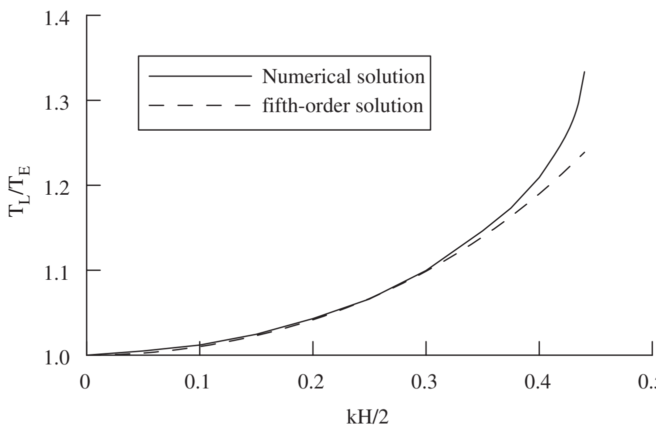

Fig. 4. Ratio of the Lagrangian wave period to the Eulerian wave period of Stokes waves varying with wave steepness.

6. Conclusions

Accurate evaluation of the mass-transport of Stokes waves is significant in determining the Lagrangian residual velocity in practical use. The proposed method combines the Fourier approximation method for computing the flow of Stokes waves in a moving frame and the Runge–Kutta–Verner algorithm for solving the particle position at any time from a set of two differential equations in an implicit form based on the definition of Lagrangian approach to the particle trajectories of a flow. The proposed algorithm can compute the particle trajectories, mass-transport velocity and Lagrangian wave period of any particle of a Stokes wave in deep water. Good agreement between the computed results and those of Longuet-Higgins (1986, 1987) shows the high accuracy of the proposed method. It is found that the dimensionless mass-transport velocity may be exceeded 10% for $ak > 0.32$ and the ratio T_L/T_E increases with the wave steepness.

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