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A note on fault-free mutually independent Hamiltonian cycles in hypercubes with faulty edges

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Abstract In the paper "Fault-free Mutually Independent Hamiltonian Cycles in Hypercubes with Faulty Edges" (J. Comb. Optim. 13:153–162, 2007), the authors claimed that an n-dimensional hypercube can be embedded with $(n-1-f)$-mutually independent Hamiltonian cycles when $f \leq n-2$ faulty edges may occur accidentally. However, there are two mistakes in their proof. In this paper, we give examples to explain why the proof is deficient. Then we present a correct proof.

Keywords Interconnection network · Hypercube · Fault tolerance · Hamiltonian cycle

1 Introduction

In many parallel computer systems, processors are connected on the basis of *interconnection networks* such as hypercubes, star graphs, meshes, bubble-sort networks, etc. For the sake of simplicity, a network topology is usually represented by a *graph*, in which vertices correspond to processors and edges correspond to connections or communication links. Hence, we use the terms, graph and network, interchangeably. Throughout this paper, we concentrate on loopless undirected graphs. For the graph definitions and notations we follow the ones defined in (Bondy and Murty 1980). A graph G consists of a set $V(G)$ and a subset $E(G)$ of $\{(u, v) \mid$

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(u, v) is an unordered pair of $V(G)\}$. The set $V(G)$ is called the *vertex set* and $E(G)$ is called the *edge set*. Two vertices u and v of G are adjacent if $(u, v) \in E(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty subset of $V(G)$. The subgraph *induced* by S is the subgraph of G with its vertex set S and with its edge set which consists of those edges joining any two vertices in S. We use $G - S$ to denote the subgraph of G induced by $V(G) - S$. Analogously, let F be a nonempty subset of $E(G)$. We use $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$. The *degree* of a vertex u in G is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint partite sets $V_0(G)$ and $V_1(G)$ such that every edge will join a vertex of $V_0(G)$ and a vertex of $V_1(G)$.

A *path* P of length k from a vertex x to a vertex y in a graph G is a sequence of distinct vertices $\langle v_1, v_2, \dots, v_{k+1}\rangle$ such that $x = v_1$, $y = v_{k+1}$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \le i \le k$. For convenience, we write P as $\langle v_1, \dots, v_i, Q, v_j, \dots, v_{k+1}\rangle$ where $Q = \langle v_i, v_{i+1}, \dots, v_j\rangle$. Note that we allow Q to be a path of length zero. The i-th vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. To emphasize the beginning and ending vertices of P, we also write P as $P[x, y]$. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \dots, v_k, v_1\rangle$. A path of a graph G is a *Hamiltonian path* if it spans G. Similarly, a cycle of a graph G is a *Hamiltonian cycle* if it spans G. A bipartite graph is Hamiltonian laceable (Simmons 1978) if there is a Hamiltonian path between any two vertices which are in different partite sets. Moreover, a Hamiltonian laceable graph G is *hyper Hamiltonian laceable* (Lewinter and Widulski 1997) if for any vertex $v \in V_i(G)$ with $i \in \{0, 1\}$, there is a Hamiltonian path of $G - \{v\}$ between any two vertices of $V_{1-i}(G)$.

The *n-dimensional hypercube* (or *n-cube* for short) is one of the most popular topologies yet discovered for parallel computation (Leighton 1992). Thus, many attractive properties of hypercubes have been studied in the literature (Akers and Krishnameurthy 1989; Chang et al. 2004; Johnsson and Ho 1989; Leighton 1992; Leu and Kuo 1999; Tsai et al. 2002; Yang et al. 1994). The formal definition of an n-cube is given as follows. Let $u = b_n \dots b_i \dots b_1$ be an n-bit binary string. For $1 \le i \le n$, we use $(u)^i$ to denote the binary string $b_n \dots \bar{b}_i \dots b_1$. Moreover, we use $(u)_i$ to denote the i-th bit b_i of u. The *Hamming weight* of u, denoted by $w(u)$, is $|\{i \mid (u)_i = 1, 1 \le i \le n\}|$. The n-cube Q_n consists of all n-bit binary strings representing its vertices. Two vertices u and v are adjacent if and only if $v = (u)^i$ with some i and we call the edge $(u, (u)^i)$ an i-dimensional edge. Note that Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{u \in V(Q_n) \mid w(u) \text{ is even}\}$ and $V_1(Q_n) = \{u \in V(Q_n) \mid w(u) \text{ is odd}\}$.

Because the components of a network may fail accidentally, it is demanded to consider the fault-tolerance on a network. The faults in a network may take various forms such as hardware failures, software errors, or even missing of transmitted packets. In this paper, faulty edges, one kind of hardware failures, are addressed. More precisely, a set F of faulty edges in a graph G contains those edges which will be removed from G. When all faulty edges are removed, we investigate the properties of the fault-free graph $G - F$. In particular, we concern the *mutually independent Hamiltonian cycles*, initially proposed by Sun et al. (2006), on a faulty n-cube. The mutually independent Hamiltonian cycles are defined as follows. Let G be a graph with N vertices. A Hamiltonian cycle C of G is described by $\langle u_1, u_2, \ldots, u_N, u_1\rangle$ to emphasize the order of vertices on C. Accordingly, u_1 is referred to as the beginning vertex. Two Hamiltonian cycles of G beginning from a given vertex s, namely $C_1 = \langle u_1, u_2, \ldots, u_N, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_N, v_1\rangle$, are *independent* if $u_1 = v_1 = s$ and $u_i \neq v_i$ for $2 \leq i \leq N$. Two Hamiltonian paths of G, $P_1 = \langle u_1, u_2, \ldots, u_N\rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_N\rangle$, are *independent* if $u_1 = v_1$, $u_N = v_N$, and $u_i \neq v_i$ for every $1 < i < N$; P_1 and P_2 are *fully independent* if $u_i \neq v_i$ for every $1 \leq i \leq N$. We say a set of m Hamiltonian cycles $\{C_1, \ldots, C_m\}$ of G, beginning from the same vertex, is m*-mutually independent* if C_i and C_j are independent whenever $i \neq j$. A set of m Hamiltonian paths $\{P_1, \ldots, P_m\}$ of G are m*-mutually independent* (resp. m*-mutually fully independent*) if any two different Hamiltonian paths in the set are independent (resp. fully independent). Moreover, the *mutually independent hamiltonicity* of G, denoted by $\mathcal{IHC}(G)$, is defined as the maximum integer m such that for any vertex u there exist m-mutually independent Hamiltonian cycles of G beginning from u. The concept of mutually independent Hamiltonian cycles can be applied in many different areas like those introduced in (Sun et al. 2006; Hsieh and Yu 2007).

Suppose that Q_n denotes an n-cube. Sun et al. (2006) proved that $\mathcal{IHC}(Q_n) = n - 1$ if $n \in \{1, 2, 3\}$ and $\mathcal{IHC}(Q_n) = n$ if $n \geq 4$. Later, Hsieh and Yu (2007) further addressed this issue and claimed that Q_n contains $(n - 1 - f)$-mutually independent Hamiltonian cycles when $f \leq n - 2$ faulty edges may occur accidentally. However, there are two mistakes in (Hsieh and Yu 2007); one is related to the proof of “Lemma 2” and the other is related to the proof of “Theorem 2”. In this paper, we give counterexamples to indicate why their argument fails and then we present a correct proof.

The rest of this paper is organized as follows. The basic properties of hypercubes are given in Sect. 2. In Sect. 3, we explain why the proof in (Hsieh and Yu 2007) is deficient. The correct proof is given in Sect. 4. Finally, the future work is discussed in Sect. 5.

2 Preliminaries

By definition, an n-cube Q_n is n-regular. It is well known that Q_n has a recursive construction; that is, it can be decomposed into two $(n-1)$-dimensional subcubes. Let Q_n^j be the subgraph of Q_n induced by $\{u \in V(Q_n) \mid (u)_n = j\}$ for $j \in \{0, 1\}$. Obviously, Q_n^j is isomorphic to Q_{n-1}. Then an n*-partition* of Q_n partitions the Q_n along dimension n into $\{Q_n^0, Q_n^1\}$. The set of crossing edges between Q_n^0 and Q_n^1, denoted by $E_c = \{(u, v) \in E(Q_n) \mid u \in V(Q_n^0), v \in V(Q_n^1)\}$, consists of all n-dimensional edges of Q_n. It is also known that Q_n is vertex-transitive and edge-transitive. For convenience, we use $\mathbf{e}$ to denote the identity vertex 0^n of Q_n.

The following results are fault-tolerant properties on hypercubes.

Theorem 1 (Tsai et al. 2002) *Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ is a set of at most $n - 2$ faulty edges. Then $Q_n - F$ is Hamiltonian laceable.*

Theorem 2 (Tsai et al. 2002) *Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ is a set of at most $n-3$ faulty edges. Then $Q_n - F$ is hyper Hamiltonian laceable.*

Lemma 1 (Sun et al. 2006) *Let $n \geq 4$. Suppose that x and y are any two vertices from different partite sets of Q_n. Then $Q_n - \{x, y\}$ is Hamiltonian laceable.*

3 Mistakes in the previous work

As Hsieh and Yu (2007) claimed, Q_n can be embedded with $(n-1-f)$-mutually independent Hamiltonian cycles when $f \leq n-2$ faulty edges may occur accidentally. Their proof is by induction on n and mainly relies on "Lemma 2" of (Hsieh and Yu 2007). However, there is a major mistake in their proof of "Lemma 2". To be precise, this mistake corresponds to the statements within lines 3–12 of p. 159 in (Hsieh and Yu 2007):

> We prove this Lemma by induction on n. The base case where $n = 3$ clearly holds. We now consider an n-cube for $n \geq 4$. Let $d_l = |\{(b_i, w_i) : (b_i, w_i)$ be an edge of dimension $l, 1 \leq i \leq \delta\}|$. Without loss of generality, we assume that $d_1 \geq d_2 \geq \ldots \geq d_n$. Obviously, $d_n = 0$. We then execute an n-partition of Q_n to obtain Q^0_{n-1} and Q^1_{n-1}. Note that each (b_i, w_i) is in either Q^0_{n-1} or Q^1_{n-1}. Let $r_0 = |\{(b_i, w_i) \in E(Q^0_{n-1}) : 1 \leq i \leq \delta\}|$ and $r_1 = |\{(b_i, w_i) \in E(Q^1_{n-1}) : 1 \leq i \leq \delta\}|$. Clearly, $r_0 + r_1 = \delta$. Without loss of generality, we assume that $\{(b_1, w_1), (b_2, w_2), \ldots, (b_{r_0}, w_{r_0})\} \subset E(Q^0_{n-1})$ and $\{(b_{r_0+1}, w_{r_0+1}), (b_{r_0+2}, w_{r_0+2}), \ldots, (b_\delta, w_\delta)\} \subset E(Q^1_{n-1})$. Since $|F_0| \leq |F| - 1$ and $r_0 \leq n - |F| - 1 \leq (n-1) - |F_0| - 1$, by the inductive hypothesis, there exist r_0-mutually fully independent Hamiltonian paths $P_1[b_1, w_1], P_2[b_2, w_2], \ldots, P_{r_0}[b_{r_0}, w_{r_0}]$ in $Q^0_{n-1} - F_0$.

Once an n-partition is executed on Q_n, the proof provided by Hsieh and Yu (2007) is merely fitted to the special case when both Q^0_n and Q^1_n contain $n-3$ or less faulty edges; i.e., the requirements $|F_0| \leq |F| - 1$ and $|F_1| \leq |F| - 1$ need to be satisfied. In this case, $Q^0_n - F_0$ and $Q^1_n - F_1$ still contain $(n-1-|F|)$-mutually fully independent Hamiltonian paths. For example, let $F = \{(00000, 10000)\}$ and $A = \{(00011, 00010), (00110, 00100), (00101, 00001)\}$. Obviously, F contains a 5-dimensional edge. Moreover, the edges of A are 1-dimensional, 2-dimensional, and 3-dimensional, respectively. According to the proof of "Lemma 2" in (Hsieh and Yu 2007), we may partition Q_5 into $\{Q^0_5, Q^1_5\}$ along dimension 5. Since the faulty edge is 5-dimensional, both Q^0_5 and Q^1_5 are fault-free. Therefore, we have $r_0 = 3$, $r_1 = 0$, and $\delta = n - 1 - |F| = n - 2 - |F_0| = 3$. Since Q^0_5 is isomorphic to Q_4, the inductive hypothesis guarantees that Q^0_5 has 3-mutually fully independent Hamiltonian paths, namely $P_1[00011, 00010]$, $P_2[00110, 00100]$, $P_3[00101, 00001]$.

However, one should notice that the faulty edges may occur accidentally. For example, we suppose the faulty edge is 1-dimensional rather than a 5-dimensional edge. Then either Q^0_5 or Q^1_5 is no longer fault-free, so the final result cannot be directly derived from the inductive hypothesis. For clarity, we give a counterexample against the argument given in (Hsieh and Yu 2007).

Example 1 Let $F = \{(00000, 00001)\}$ consist of a 1-dimensional faulty edge in Q_5 and let $A = \{(00011, 00010), (00110, 00100), (00101, 00001)\}$. Obviously, the edges of A are 1-dimensional, 2-dimensional, and 3-dimensional, respectively. According to the proof of "Lemma 2" in (Hsieh and Yu 2007), we may partition Q_5 into $\{Q_5^0, Q_5^1\}$ along dimension 5. Then we have $F \cup A \subset E(Q_5^0)$; that is, we have $F_0 = F$, $\delta = r_0 = 3$, and $r_1 = 0$. Since Q_5^0 is isomorphic to Q_4, $Q_5^0 - F_0$ has at most 2, not 3, mutually fully independent Hamiltonian paths by the inductive hypothesis. Indeed, no matter which dimension is used to partition Q_5, the inductive argument proposed in (Hsieh and Yu 2007) always fails.

In summary, Hsieh and Yu (2007) did not ever consider the case when all faulty edges are unfortunately located in the same $(n-1)$-dimensional subcube of Q_n. In this case, the final result cannot be derived directly from the inductive hypothesis since $Q_n^0 - F_0$ has at most $r_0 - 1$, instead of r_0, mutually fully independent Hamiltonian paths. More generally, we give the next example to show this deficiency.

Example 2 Let n be a multiple of 5. Suppose that $F = \{(0^n, 0^{n-i}10^{i-1}) \mid 1 \le i \le \frac{n}{5}\}$ consists of $\frac{n}{5}$ faulty edges in Q_n. Moreover, suppose that $w_i = 0^{n-1-i}110^{i-1}$ and $b_i = 0^{n-1-i}10^i$ for $1 \le i \le \frac{4n}{5} - 2$. Besides, let $w_{\frac{4n}{5}-1} = 0^{\frac{n}{5}+1}10^{\frac{4n}{5}-3}1$ and $b_{\frac{4n}{5}-1} = 0^{n-1}1$. Then $A = \{(w_i, b_i) \mid 1 \le i \le \frac{4n}{5} - 1\}$ is a set of $\frac{4n}{5} - 1$ edges with no shared endpoints. Obviously, the edges of A are over dimensions $1, 2, \ldots, (\frac{4n}{5} - 1)$, respectively. According to the proof of "Lemma 2" in (Hsieh and Yu 2007), we may partition Q_n into $\{Q_n^0, Q_n^1\}$ along dimension n. Then we have $F \cup A \subset E(Q_n^0)$; that is, we have $F_0 = F$, $\delta = r_0 = \frac{4n}{5} - 1$, and $r_1 = 0$. By the inductive hypothesis, $Q_n^0 - F_0$ has at most $(\frac{4n}{5} - 2)$, instead of $(\frac{4n}{5} - 1)$, mutually fully independent Hamiltonian paths.

In addition to this mistake, another minor one corresponds to the proof of "Theorem 2" in (Hsieh and Yu 2007). From line 10 to line 13 of page 160, the authors claimed that there exist δ edges $(C_i(t), C_i(t + 1 (\mathrm{mod}\ 2^{n-1})))$ for all $1 \le i \le \delta$ such that $(C_i(t), (C_i(t))^d)$ and $(C_i(t + 1(\mathrm{mod}\ 2^{n-1})), (C_i(t + 1(\mathrm{mod}\ 2^{n-1})))^d)$ are fault-free. They further mentioned that if these edges do not exist, then $|F| \ge |F_c| \ge 2^{n-2} > n - 2$ for $n > 3$. However, this argument is wrong because every faulty edge of F_c repeats δ times. In contrast, it should be argued that if such edges do not exist, we will have $\delta|F_c| \ge 2^{n-2}$; that is, $|F_c| \ge 2^{n-2}/\delta > |F|$ will lead to an immediate contradiction for $n \ge 3$.

4 Fault-free mutually independent Hamiltonian cycles of faulty Q_n

To derive the main theorem of this paper, we need the following results.

Lemma 2 (Sun et al. 2006) *Let Q_n be an n-cube for $n \ge 2$. Suppose that $\{(w_i, b_i) \in E(Q_n) \mid w_i \in V_0(Q_n), b_i \in V_1(Q_n), 1 \le i \le n-1\}$ consists of $n-1$ distinct edges with no shared endpoints. Then Q_n contains $(n-1)$-mutually fully independent Hamiltonian paths $P_1[w_1, b_1], \ldots, P_{n-1}[w_{n-1}, b_{n-1}]$.*

Theorem 3 (Sun et al. 2006) *$\mathcal{IHC}(Q_n) = n-1$ if $n \in \{1,2,3\}$ and $\mathcal{IHC}(Q_n) = n$ if $n \geq 4$.*

Let F be a set of faulty edges of Q_n. Suppose that Q_n is partitioned along dimension n into $\{Q_n^0, Q_n^1\}$ and E_c is the set of crossing edges between Q_n^0 and Q_n^1. Then we define $F_0 = F \cap E(Q_n^0)$, $F_1 = F \cap E(Q_n^1)$ and $F_c = F \cap E_c$. Moreover, we set $\delta = n - 1 - |F|$ in the remainder of this paper. To tolerate faulty edges in hypercubes, we have the next lemma.

Lemma 3 *Let $F \subseteq E(Q_n)$ be a set of at most $n-2$ faulty edges for $n \geq 3$. Suppose that $A = \{(w_i, b_i) \in E(Q_n) \mid w_i \in V_0(Q_n), b_i \in V_1(Q_n), 1 \leq i \leq \delta\}$ consists of δ distinct edges with no shared endpoints. Then $Q_n - F$ contains δ-mutually fully independent Hamiltonian paths $P_1[w_1, b_1], \ldots, P_\delta[w_\delta, b_\delta]$.*

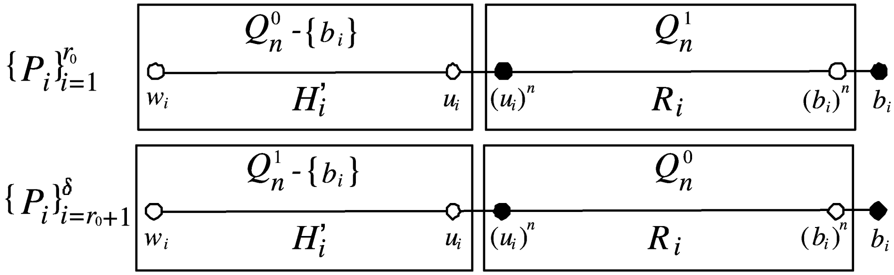
Proof This proof proceeds by induction on n. First suppose $|F| = 0$. Then this case follows from Lemma 2. Suppose $|F| = n-2$. Then we have $\delta = n-1-(n-2) = 1$. By Theorem 1, $Q_n - F$ has a Hamiltonian path between any two vertices from different partite sets. Obviously, the statement holds for Q_3, as the induction basis. In what follows, we only consider $1 \leq |F| \leq n-3$ and $n \geq 4$. As the inductive hypothesis, suppose that the statement is true for Q_{n-1}.

Since $\delta + |F| = n - 1 < n$, there must exist a dimension d of $\{1, 2, \ldots, n\}$ such that $A \cup F$ contains no d-dimensional edges. Since Q_n is edge-transitive, we can assume $d = n$. Then we partition Q_n into $\{Q_n^0, Q_n^1\}$ along dimension n. Thus, each edge of $A \cup F$ is in either Q_n^0 or Q_n^1. Let $r_0 = |\{(w_i, b_i) \in E(Q_n^0) \mid 1 \leq i \leq \delta\}|$ and $r_1 = |\{(w_i, b_i) \in E(Q_n^1) \mid 1 \leq i \leq \delta\}|$. Clearly, $r_0 + r_1 = \delta$. Without loss of generality, we assume $\{(w_1, b_1), \ldots, (w_{r_0}, b_{r_0})\} \subset E(Q_n^0)$ and $\{(w_{r_0+1}, b_{r_0+1}), \ldots, (w_\delta, b_\delta)\} \subset E(Q_n^1)$. Since $n - 1 = \delta + |F| = r_0 + r_1 + |F_0| + |F_1|$, we have $r_i + |F_j| \leq n-1$ for any $i, j \in \{0, 1\}$. Then we have to take the following cases into account.

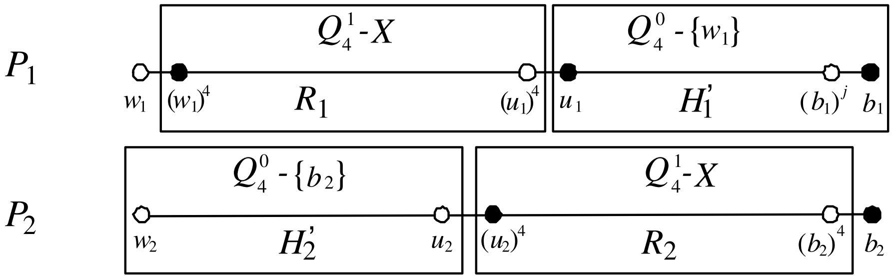
Case 1: Suppose $r_i + |F_j| \leq n-2$ for any $i, j \in \{0,1\}$. Since $r_0 + |F_0| \leq n-2$, $r_0 \leq n-2-|F_0| = (n-1)-1-|F_0|$. By the inductive hypothesis, $Q_n^0 - F_0$ has r_0-mutually fully independent Hamiltonian paths $H_i[w_i, b_i]$, $1 \leq i \leq r_0$. Obviously, $H_i[w_i, b_i]$ can be represented as $\langle w_i, H_i', u_i, b_i \rangle$, in which u_i is some vertex adjacent to b_i. Similarly, $Q_n^1 - F_1$ has r_1-mutually fully independent Hamiltonian paths $H_i[w_i, b_i] = \langle w_i, H_i', u_i, b_i \rangle$, $r_0 + 1 \leq i \leq \delta$.

Next, we construct r_0 paths in $Q_n^1 - F_1$ to incorporate the previously established r_0 paths of $Q_n^0 - F_0$. Since $r_0 + |F_1| \leq n-2$, we have $r_0 \leq n-2-|F_1|$. By the inductive hypothesis, $Q_n^1 - F_1$ also contains r_0-mutually fully independent Hamiltonian paths $R_1[(u_1)^n, (b_1)^n], \ldots, R_{r_0}[(u_{r_0})^n, (b_{r_0})^n]$. Similarly, $Q_n^0 - F_0$ also contains r_1-mutually fully independent Hamiltonian paths $R_{r_0+1}[(u_{r_0+1})^n, (b_{r_0+1})^n], \ldots, R_\delta[(u_\delta)^n, (b_\delta)^n]$. Accordingly, we set $P_i[w_i, b_i] = \langle w_i, H_i', u_i, (u_i)^n, R_i, (b_i)^n, b_i \rangle$ for every $1 \leq i \leq \delta$. Thus, $\{P_1, \ldots, P_\delta\}$ forms a set of δ-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(a) for illustration.

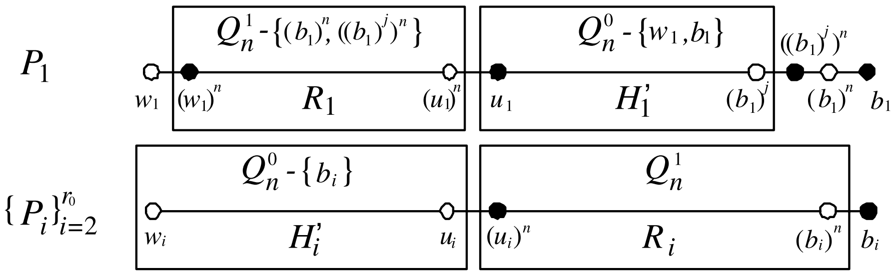
Case 2: Suppose $r_i + |F_i| = n-1$ for some $i \in \{0, 1\}$. Without loss of generality, we assume $r_0 + |F_0| = n-1$. Since $r_0 = n-1-|F_0| \geq n-1-|F| = \delta$, we must have $r_0 = \delta$ and $|F_0| = |F| \leq n-3$. Note that $r_0 - 1 = \delta - 1 = n-2-|F| = (n-1)-1-|F_0|$. By the inductive hypothesis, $Q_n^0 - F_0$ has $(r_0 - 1)$-mutually



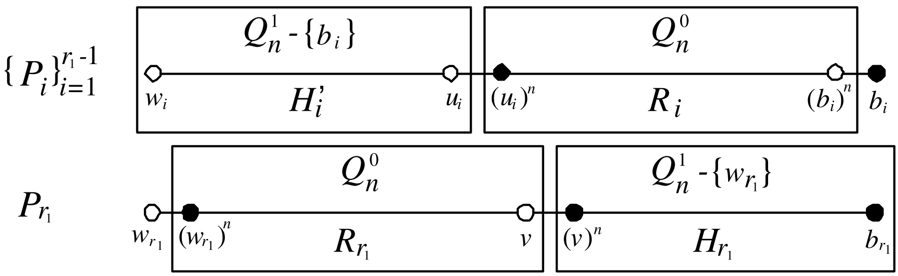
(a) Case 1



(b) Subcase 2.1 of Case 2



(c) Subcase 2.2 of Case 2



(d) Case 3

Fig. 1 Illustration for the proof of Lemma 3 fully independent Hamiltonian paths $H_i[w_i, b_i]$, $2 \le i \le r_0$. Again, $H_i[w_i, b_i]$ can be represented as $\langle w_i, H_i', u_i, b_i\rangle$, in which u_i is some vertex adjacent to b_i.

Subcase 2.1: Suppose $n = 4$. Thus, we have $r_0 = 2$. By Theorem 2, $Q_4^0 - F_0$ has a Hamiltonian path $H_1[w_1, b_1] = \langle w_1, u_1, H_1', (b_1)^j, b_1\rangle$, in which u_1 is a vertex adjacent to w_1 and j is some integer of $\{1, 2, 3, 4\}$. Let $X = \{((u_1)^4, (u_2)^4)\}$. Similarly, there are two Hamiltonian paths $R_1[(w_1)^4, (u_1)^4]$ and $R_2[(u_2)^4, (b_2)^4]$ in $Q_4^1 - X$. Obviously, one can see that $R_1(7) \ne R_2(1)$ and $R_1(8) \ne R_2(2)$. Then we set $P_1[w_1, b_1] = \langle w_1, (w_1)^4, R_1, (u_1)^4, u_1, H_1', (b_1)^j, b_1\rangle$ and $P_2[w_2, b_2] = \langle w_2, H_2', u_2, (u_2)^4, R_2, (b_2)^4, b_2\rangle$. Consequently, $\{P_1, P_2\}$ forms a set of 2-mutually fully independent Hamiltonian paths in $Q_4 - F$. See Fig. 1(b) for illustration.

Subcase 2.2: Suppose $n \ge 5$. We first consider $|F_0| \le n - 4$. By the inductive hypothesis, Q_n^1 has $(r_0 - 1)$-mutually fully independent Hamiltonian paths $R_i[(u_i)^n, (b_i)^n]$, $2 \le i \le r_0$. Then we can choose an integer j of $\{1, \dots, n-1\}$ such that both $(b_1)^j \ne w_1$ and $((b_1)^j)^n \notin \{R_i(2^{n-1} - 1) \mid 2 \le i \le r_0\}$ are satisfied. Since $r_0 = n - 1 - |F| \le n - 2$, such an integer exists. By Theorem 2, $Q_n^0 - (F_0 \cup \{b_1\})$ has a Hamiltonian path $H_1[w_1, (b_1)^j] = \langle w_1, u_1, H_1', (b_1)^j\rangle$, in which u_1 is some vertex adjacent to w_1. By Lemma 1, there exists a Hamiltonian path $R_1[(w_1)^n, (u_1)^n]$ in $Q_n^1 - \{(b_1)^n, ((b_1)^j)^n\}$. Then we set $P_1[w_1, b_1] = \langle w_1, (w_1)^n, R_1, (u_1)^n, u_1, H_1', (b_1)^j, ((b_1)^j)^n, (b_1)^n, b_1\rangle$ and $P_i[w_i, b_i] = \langle w_i, H_i', u_i, (u_i)^n, R_i, (b_i)^n, b_i\rangle$ for $2 \le i \le r_0$. As a result, $\{P_1, \dots, P_{r_0}\}$ forms a set of r_0-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(c) for illustration.

Next, we consider $|F_0| = n - 3$. Thus, we have $r_0 = 2$. By Theorem 1, $Q_n^0 - F_0$ has a Hamiltonian path $H_1[w_1, b_1] = \langle w_1, u_1, H_1', (b_1)^j, b_1\rangle$, in which u_1 is a vertex adjacent to w_1 and j is some integer of $\{1, 2, \dots, n-1\}$. By Lemma 1, there exists a Hamiltonian path $R_1[(w_1)^n, (u_1)^n]$ in $Q_n^1 - \{(b_1)^n, ((b_1)^j)^n\}$. By the inductive hypothesis, $Q_n^1 - \{((b_2)^n, ((b_1)^j)^n)\}$ has a Hamiltonian path $R_2[(u_2)^n, (b_2)^n]$. Obviously, we have $R_2(2^{n-1} - 1) \ne ((b_1)^j)^n$. Again, we set $P_1[w_1, b_1] = \langle w_1, (w_1)^n, R_1, (u_1)^n, u_1, H_1', (b_1)^j, ((b_1)^j)^n, (b_1)^n, b_1\rangle$ and $P_2[w_2, b_2] = \langle w_2, H_2', u_2, (u_2)^n, R_2, (b_2)^n, b_2\rangle$. Hence, $\{P_1, P_2\}$ forms a set of 2-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(c).

Case 3: Suppose that $r_i + |F_{1-i}| = n - 1$ for some $i \in \{0, 1\}$. Without loss of generality, we assume $r_1 + |F_0| = n - 1$. Since $r_1 = n - 1 - |F_0| \ge n - 1 - |F| = \delta$, we have $r_1 = \delta$ and $F_0 = F$. By the inductive hypothesis, Q_n^1 has $(r_1 - 1)$-mutually fully independent Hamiltonian paths $H_i[w_i, b_i] = \langle w_i, H_i', u_i, b_i\rangle$, in which u_i is some vertex adjacent to b_i with $1 \le i \le r_1 - 1$. Since $r_1 - 1 = \delta - 1 = n - 2 - |F| = (n - 1) - 1 - |F_0|$, $Q_n^0 - F_0$ has $(r_1 - 1)$-mutually fully independent Hamiltonian paths $R_i[(u_i)^n, (b_i)^n]$, $1 \le i \le r_1 - 1$. Then we set $P_i[w_i, b_i] = \langle w_i, H_i', u_i, (u_i)^n, R_i, (b_i)^n, b_i\rangle$ with $1 \le i \le r_1 - 1$. Next, we have to choose a vertex v of $V_0(Q_n^0)$ and construct a Hamiltonian path $R_{r_1}[(w_{r_1})^n, v]$ in $Q_n^0 - F_0$ such that $v \ne R_i(2)$ and $R_{r_1}(2^{n-1} - 1) \ne (u_i)^n$ for every $1 \le i \le r_1 - 1$. We distinguish the following subcases.

Subcase 3.1: Suppose $n \ne 5$ or $|F| > 1$. One can see that $(u_1)^n, \dots, (u_{r_1-1})^n$ have at most $(r_1 - 1)(n - 1)$ neighbors in Q_n^0. Since $|V_0(Q_n^0)| = 2^{n-2} > (r_1 - 1)(n - 1) = (n - 2 - |F|)(n - 1)$ in this subcase, we can choose v other than all neighbors of $(u_1)^n, \dots, (u_{r_1-1})^n$. Obviously, we have $v \ne R_i(2)$ for $1 \le i \le r_1 - 1$. By Theorem 1, there is a Hamiltonian path $R_{r_1}[(w_{r_1})^n, v]$ in $Q_n^0 - F_0$. Since v is not adjacent to any node of $\{(u_1)^n, \ldots, (u_{r_1-1})^n\}$, we have $R_{r_1}(2^{n-1}-1) \neq (u_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Theorem 2, there is a Hamiltonian path $H_{r_1}[(v)^n, b_{r_1}]$ in $Q_n^1 - \{w_{r_1}\}$. Then we set $P_{r_1} = \langle w_{r_1}, (w_{r_1})^n, R_{r_1}, v, (v)^n, H_{r_1}, b_{r_1}\rangle$. Consequently, $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(d) for illustration.

In the following, we consider $n = 5$ and $|F| = 1$; that is, $r_1 = 3$.

Subcase 3.2: For $n = 5$ and $|F| = 1$, suppose that $(u_1)^n$ and $(u_2)^n$ have at least one common neighbor. Since $|V_0(Q_n^0)| = 2^{n-2} = 8 > 7 = (r_1 - 1)(n - 1) - 1$, we still can choose a vertex v from $V_0(Q_n^0)$ other than all neighbors of $(u_1)^n$ and $(u_2)^n$. Obviously, we have $v \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Theorem 1, there is a Hamiltonian path $R_{r_1}[(w_{r_1})^n, v]$ of $Q_n^0 - F_0$ such that $R_{r_1}(2^{n-1}-1) \neq (u_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Theorem 2, there is a Hamiltonian path $H_{r_1}[(v)^n, b_{r_1}]$ in $Q_n^1 - \{w_{r_1}\}$. Similarly, we set $P_{r_1} = \langle w_{r_1}, (w_{r_1})^n, R_{r_1}, v, (v)^n, H_{r_1}, b_{r_1}\rangle$. Then $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(d).

Subcase 3.3: For $n = 5$ and $|F| = 1$, suppose that $(u_1)^n$ and $(u_2)^n$ have no common neighbors. Then we assign the vertex v as the one that is adjacent to $(u_1)^n$ but is not identical to $R_1(2)$. Obviously, we have $v \neq R_i(2)$ for $1 \leq i \leq r_1 - 1$. By Theorem 1, $Q_n^0 - (F_0 \cup \{(v, (u_1)^n)\})$ remains Hamiltonian laceable. Thus, there is a Hamiltonian path $R_{r_1}[(w_{r_1})^n, v]$ of $Q_n^0 - (F_0 \cup \{(v, (u_1)^n)\})$ such that $R_{r_1}(2^{n-1}-1) \neq (u_i)^n$ for every $1 \leq i \leq r_1 - 1$. By Theorem 2, there is a Hamiltonian path $H_{r_1}[(v)^n, b_{r_1}]$ in $Q_n^1 - \{w_{r_1}\}$. Similarly, we set $P_{r_1} = \langle w_{r_1}, (w_{r_1})^n, R_{r_1}, v, (v)^n, H_{r_1}, b_{r_1}\rangle$. Then $\{P_1, \ldots, P_{r_1}\}$ forms a set of r_1-mutually fully independent Hamiltonian paths in $Q_n - F$. See Fig. 1(d). □

With Lemma 3, the next theorem can be easily derived.

Theorem 4 *Let $n \geq 3$. Suppose that $F \subseteq E(Q_n)$ consists of at most $n - 2$ faulty edges. Then $Q_n - F$ contains $(n - 1 - |F|)$-mutually independent Hamiltonian cycles beginning from any vertex.*

Proof Since Q_n is vertex-transitive, we only need to construct δ-mutually independent Hamiltonian cycles beginning from $\mathbf{e} = 0^n$. Suppose $|F| = 0$. Then the statement follows from Theorem 3. Thus, we only consider the situation that F is nonempty. Furthermore, since Q_n is edge-transitive, we assume that at least one faulty edge is an n-dimensional edge.

The proof idea is based on the partition of Q_n. As discussed previously, Q_n can be partitioned into $\{Q_n^0, Q_n^1\}$. Obviously, $\mathbf{e}$ is located in Q_n^0. Recall that F_0 and F_1 denote the sets of faulty edges in Q_n^0 and Q_n^1, respectively. Then the proof idea is outlined as follows:

(1) We first build δ-mutually independent Hamiltonian cycles $C_1, C_2, \ldots, C_\delta$ beginning from $\mathbf{e}$ in $Q_n^0 - F_0$.
(2) Next, we have to claim that there must exist an integer t, $1 \leq t \leq 2^{n-2}$, so that the crossing edges $(C_i(2t-1), (C_i(2t-1))^n)$ and $(C_i(2t), (C_i(2t))^n)$ are fault-free for all $1 \leq i \leq \delta$. For convenience, let $x_i = C_i(2t-1)$ and $y_i = C_i(2t)$.

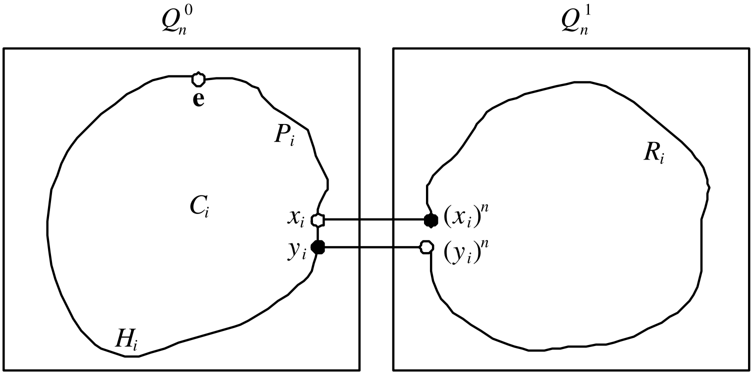


Fig. 2 Illustration for the proof of Theorem 4. Without loss of generality, we assume $x_i \in V_0(Q_n)$ for $1 \le i \le \delta$

(3) By Lemma 3, $Q_n^1 - F_1$ contains δ-mutually fully independent Hamiltonian paths $R_1[(x_1)^n, (y_1)^n], \ldots, R_\delta[(x_\delta)^n, (y_\delta)^n]$.
(4) Finally, we obtain the desired Hamiltonian cycles from combining C_i and R_i, $1 \le i \le \delta$. See Fig. 2 for illustration.

More precisely, the proof is by induction on n. It is trivial that the statement holds for Q_3, as the induction basis. When $n \ge 4$, we assume that the statement holds for Q_{n-1}. Now we consider how to build δ-mutually independent Hamiltonian cycles in $Q_n - F$. Since we assume there is at least one n-dimensional faulty edge, we partition Q_n into $\{Q_n^0, Q_n^1\}$ along dimension n. Accordingly, we have $|F_0| \le |F| - 1 \le n - 3$, $|F_1| \le |F| - 1 \le n - 3$, and $(n-1) - 1 - |F_0| \ge (n-1) - 1 - (|F| - 1) = n - 1 - |F| = \delta$. Thus, by the inductive hypothesis, $Q_n^0 - F_0$ contains δ-mutually independent Hamiltonian cycles $C_1, C_2, \ldots, C_\delta$ beginning from **e**. For convenience, we assume that the vertices on each cycle are indexed sequentially from 1 to 2^{n-1}; that is, the beginning vertex **e** has index 1. Next, we claim that there must exist an integer t, $1 \le t \le 2^{n-2}$, so that the crossing edges $(C_i(2t-1), (C_i(2t-1))^n)$ and $(C_i(2t), (C_i(2t))^n)$ are fault-free for all $1 \le i \le \delta$. If such edges do not exist, then we have $|F| \ge |F_c| \ge 2^{n-2}/\delta > |F|$ for $n \ge 3$, leading to an immediate contradiction. Let $x_i = C_i(2t-1)$ and $y_i = C_i(2t)$. Accordingly, C_i can be represented as $\langle \mathbf{e}, P_i, x_i, y_i, H_i, \mathbf{e}\rangle$, $1 \le i \le \delta$. By the definition of hypercubes, $(x_i)^n$ and $(y_i)^n$ are adjacent in Q_n^1. By Lemma 3, $Q_n^1 - F_1$ contains δ-mutually fully independent Hamiltonian paths $R_1[(x_1)^n, (y_1)^n], \ldots, R_\delta[(x_\delta)^n, (y_\delta)^n]$. Therefore, $\{\langle \mathbf{e}, P_i, x_i, (x_i)^n, R_i, (y_i)^n, y_i, H_i, \mathbf{e}\rangle \mid 1 \le i \le \delta\}$ forms a set of δ-mutually independent Hamiltonian cycles beginning from **e**. □

5 Conclusion

In this paper, we concentrate on the problem of embedding mutually independent Hamiltonian cycles in a faulty n-cube, as previously addressed by Hsieh and Yu

(Hsieh and Yu 2007). However, there are two mistakes in (Hsieh and Yu 2007). Therefore, we first point out why their proof is deficient. Then we prove that Q_n contains $(n-1-f)$-mutually independent Hamiltonian cycles when $f \le n-2$ faulty edges may occur accidentally. Indeed, we believe this result can be further refined; that is, we would like to show Q_n can be embedded with $(n-f)$-mutually independent Hamiltonian cycles beginning from any vertex when $f \le n-2$ faulty edges occur.

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