Analytical Solutions for Constant-Flux and Constant-Head Tests at a Finite-Diameter Well in a Wedge-Shaped Aquifer

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Abstract: Neglecting the effect of well radius may lead to a significant error in the predicted drawdown distribution near the pumping well area. New analytical solutions describing aquifer responses to a constant pumping or a constant head maintained at a finite-diameter well in a wedge-shaped aquifer are derived based on the image-well method and applicable to an arbitrarily located well in the system. The solutions are useful for quantifying groundwater exploitation from a wedge-shaped aquifer and for determining the hydrogeological parameters of a wedge-shaped aquifer in inverse problems.

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Introduction

The image-well method is useful in evaluating the impact of topographic boundaries existing around a well on the groundwater flow system. It removes aquifer boundaries and places image wells at judicious locations in order to reflect the effect of boundaries. The drawdown in a test well can then be superposed as the sum of the individual drawdown due to the real well and the image wells (Ferris et al. 1962; Streltsova 1988; Kuo et al. 1994). Previous studies of the drawdown solutions are based on the superposition of the Theis solution (1935), which describes the transient hydraulic response of an infinite confined aquifer to a line pumping.

Alternative drawdown solutions accounting for topographic boundaries can be based on integral transform methods (Chan et al. 1976, 1978; Yeh and Chang 2006). These can solve the problem of a wedge-shaped aquifer for the constant-flux test (CFT) with any wedge angle, but solutions are complicated and rather difficult to accurately evaluate (Chan et al. 1976). Further, in those studies, the Dirac delta function was used to represent the pumping at the test well. Previous studies devoted to analytical solutions for the CFT in the vicinity of topographic boundaries all assume infinitesimal-diameter well, but ignoring the finite test-well diameter may lead to errors in the predicted drawdown near the test well.

Several studies have also examined the problem of constant-head test (CHT) in the vicinity of an aquifer boundary. Unlike the treatment in the CFT, one cannot directly apply the image-well method to the CHT, because superposing the drawdown produced by each image well would violate the condition of constant drawdown at the test well. Based on an instantaneous source function and the convolution method, Murdoch and Franco (1994) and Renard (2005) presented approximate expressions for various geometries of flow problem, which are in accordance with the condition of constant well drawdown. Analytical solutions for the CHT in a wedge-shaped aquifer, to the writers' knowledge, have never been addressed.

This technical note develops analytical solutions for the drawdown distribution of CFT as well as well discharge rate and drawdown distribution of CHT in a wedge-shaped aquifer considering a finite well diameter. These new solutions describe the aquifer response to a finite-diameter well arbitrarily located in a wedge-shaped aquifer bounded by various combinations of recharging and impermeable boundaries. They are relatively easy to formulate and evaluate once the test condition and the aquifer parameter values are known. The obtained drawdown solution for a CFT will be compared with two solutions in which the well diameter is neglected.

Mathematical Formulation

Mathematical Models of CFT and CHT in an Infinite Extent Aquifer

The assumptions related to the aquifer and well configurations are: (1) the confined aquifer is homogeneous, isotropic, and uniform in thickness; and (2) the test well with a finite diameter penetrates the entire thickness of the aquifer. Define the dimensionless variables as

$$s_D = \frac{s}{r_w}, \quad r_D = \frac{r}{r_w}, \quad t_D = \frac{Tt}{r_w^2 S} \tag{1}$$

where s_D=dimensionless drawdown; r_D=dimensionless radial distance; t_D=dimensionless time; s=drawdown; r_w=radius of test well; r=radial distance from the centerline of test well; T=transmissivity of aquifer; S=storativity of aquifer; and t=time from the start of test. In radial coordinates, the governing equation for the groundwater flow in a dimensionless form is

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$$\frac{\partial^2 s_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial s_D}{\partial r_D}=\frac{\partial s_D}{\partial t_D} \tag{2}$$

For both the CFT and CHT, the initial dimensionless drawdown for the whole aquifer equals zero, that is

$$s_D(r_D,0)=0 \tag{3}$$

The dimensionless drawdown at infinity is

$$s_D(\infty,t_D)=0 \tag{4}$$

The boundary condition at the wellbore for the CFT is defined differently from the CHT. For the CFT, the boundary condition follows Darcy's law and is described as

$$Q_{Df}=-\left.\frac{\partial s_{Df}(r_D,t_D)}{\partial r_D}\right|_{r_D=1}, \quad t_D>0 \tag{5}$$

where the subscript f represents a CFT. Eq. (5) expresses the mass conservation at the rim of the wellbore. The dimensionless discharge Q_{Df} is defined as

$$Q_{Df}=\frac{Q}{2\pi r_w T} \tag{6}$$

where Q=constant pumping rate. The Laplace-domain solution $\tilde{s}_{Df}$ to Eq. (2) subject to Eqs. (3)–(5) is (van Everdingen and Hurst 1949; Carslaw and Jaeger 1959)

$$\tilde{s}_{Df}=\frac{Q_{Df}K_0(r_D\sqrt{p})}{p\sqrt{p}K_1(\sqrt{p})} \tag{7}$$

where p=Laplace variable, and K_0 and K_1=modified Bessel functions of the second kind of order zero and one, respectively.

For the CHT, the boundary condition at the wellbore is expressed as

$$s_{Dh}(r_D=1,t_D)=s_{0D}, \quad t_D>0 \tag{8}$$

or expressed in the Laplace domain as

$$\tilde{s}_{Dh}(r_D=1,p)=\frac{s_{0D}}{p} \tag{9}$$

with the subscript h denoting the CHT and s_{0D}=dimensionless initial drawdown defined as the initial drawdown normalized by r_w. Eqs. (8) and (9) state that the piezometric head at the wellbore is constant.

Based on Duhamel's formula (Murdoch and Franco 1994), the drawdown produced by a CHT can be written as

$$s_{Dh}(r_D,t_D)=\int_0^{t_D} Q_{Dh}(\tau)\frac{\partial[s_{Dfu}(r_D,t_D-\tau)]}{\partial t_D}d\tau \tag{10}$$

where Q_{Dh}=dimensionless discharge at the test well during the CHT; s_{Dfu}=dimensionless unit step response function, which is defined as the drawdown under constant unit discharge (s_{Df}/Q_{Df}). Taking the Laplace transform of Eq. (10) yields

$$\tilde{s}_{Dh}(r_D,p)=p\tilde{Q}_{Dh}(p)\tilde{s}_{Dfu}(r_D,p) \tag{11}$$

Eq. (11) relates the solution of the CHT to that of the CFT. This relationship holds in any well model.

Configuration of an Image-Well System

The wedge angle of the wedge-shaped aquifer is assumed to be an aliquot part of 360°. It must be an aliquot part of 90° for aquifers

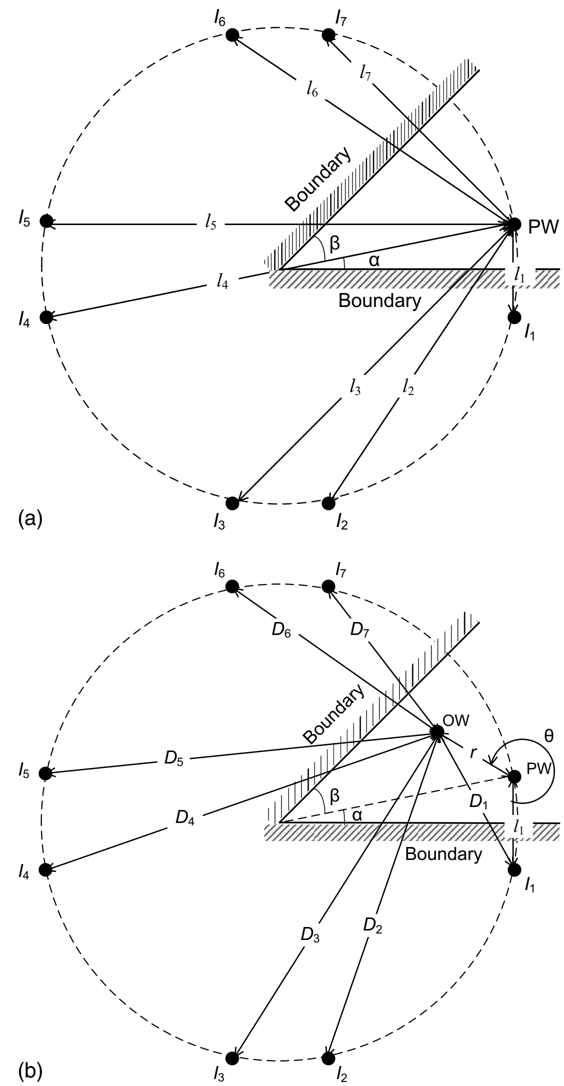

Fig. 1. Configuration of an image-well system: (a) without observation wells; (b) with an observation well. Note that PW and OW represent pumping well and observation well, respectively.

with boundaries that are either like or unlike; otherwise, it must be an aliquot part of 180° for aquifers with like boundaries (Ferris et al. 1962). This rule has its exception in the case where the wedge angle is an odd aliquot part of 360°, the test well is on the bisector of wedge angle, and the boundaries are both impermeable (Ferris et al. 1962). The number of the image wells, n, required in analyzing the flow toward the test well is given by the relation (Ferris et al. 1962)

$$n=\frac{360°}{\phi}-1 \tag{12}$$

where ϕ refers to the wedge angle.

Fig. 1 shows the image-well system in a wedge-shaped aquifer where the image wells are named in a clockwise sequence (I_i, $i=1,2,\ldots,n$). The type of image well, whether it extracts water from, or injects water into an aquifer depends on the nature of the aquifer boundaries. Once the dimensionless distances l_{D1} and l_{Dn} are known, the dimensionless distances l_{Dm} from the center of test well to the center of image wells can be obtained as

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$$l_{Dm}=\begin{cases}\sqrt{l_{Dm-1}^2+l_{Dn}^2-2l_{Dm-1}l_{Dn}\cos\left[\pi-\frac{m}{2}\phi\right]}, \\ \text{when } m \text{ is even} \\ \sqrt{l_{D1}^2+l_{Dm-1}^2-2l_{D1}l_{Dm-1}\cos\left[\pi-\frac{m+1}{2}\alpha-\frac{m-1}{2}\beta\right]}, \\ \text{when } m \text{ is odd}\end{cases} \tag{13}$$

where α=angle between the lower topographic boundary and the image line connecting the intersection point of the wedge-shaped aquifer to the test well, and β is defined as $\phi-\alpha$.

The angle θ, shown in Fig. 1(b), is defined as the angle measured from a line connecting the centers of the test well and the first image well (denoted by I_1) to the line connecting the centers of the test well and observation well. The dimensionless distances, D_{Dm}, from the center of observation well to the center of image wells are

$$D_{Dm}=\begin{cases}\sqrt{r_D^2+l_{Dm}^2-2r_Dl_{Dm}\cos\left[\theta+\frac{m-1}{2}\phi\right]}, \\ \text{when } m \text{ is odd} \\ \sqrt{r_D^2+l_{Dm}^2-2r_Dl_{Dm}\cos\left[\theta+\frac{m-2}{2}\alpha+\frac{m}{2}\beta\right]}, \\ \text{when } m \text{ is even}\end{cases} \tag{14}$$

Analytical Solutions for a CFT in a Wedge-Shaped Aquifer

For a CFT performed near aquifer boundaries, the use of the image-well method requires first to determine the location and operation types of the image wells, and then superpose the individual dimensionless drawdown $\tilde{s}_{Df}$ due to the real and each image well. Based on Eq. (7), the resultant $\tilde{s}_{Df}$ can be written as

$$\tilde{s}_{Df}(r_D,\theta,p)=\frac{Q_{Df}}{p\sqrt{p}K_1(\sqrt{p})}\left[K_0(r_D\sqrt{p})+\sum_{m=1}^{n}(-1)^jK_0(D_{Dm}\sqrt{p})\right] \tag{15}$$

where n represents the number of image wells. The variable j equals two for the case where boundaries are both impermeable and equals m for the case where boundaries are both recharging. In addition, j equals $(m+1)/2$ when m is odd and $m/2$ when m is even for the case where the lower boundary is recharging and the upper boundary is impermeable; j equals $(m+3)/2$ when m is odd and $(m+4)/2$ when m is even for the case where the lower boundary is impermeable and the upper boundary is recharging. The lower boundary refers to the boundary next to the first image well and the upper boundary refers to that adjacent to the last image well.

The time-domain solution, s_{Df}, can be obtained by applying the Bromwich integral method (Yeh et al. 2003) to invert Eq. (15) as

$$s_{Df}(r_D,\theta,t_D)=\frac{2Q_{Df}}{\pi}\int_0^\infty e^{-t_Du^2}\frac{A_1(r_D,\theta,u)Y_1(u)-A_2(r_D,\theta,u)J_1(u)}{J_1^2(u)+Y_1^2(u)}\frac{du}{u^2} \tag{16}$$

where

$$A_1(r_D,\theta,u)=J_0(r_Du)+\sum_{m=1}^{n}(-1)^jJ_0(D_{Dm}u) \tag{17}$$

and

$$A_2(r_D,\theta,u)=Y_0(r_Du)+\sum_{m=1}^{n}(-1)^jY_0(D_{Dm}u) \tag{18}$$

where J_0 and Y_0=zero-order Bessel functions of the first and second kinds, respectively, and J_1 and Y_1=first-order Bessel functions of the first and second kinds, respectively.

Analytical Solutions for a CHT in a Wedge-Shaped Aquifer

For a CHT performed adjacent to aquifer boundaries, it is required to determine the location and operation types of the image wells before applying the convolution and the image-well method to solve for the discharge at the test well or the drawdown at the observation well. Mathematically, the well discharge is assigned at the center of test well, but the singularity of the unit step response function $\tilde{s}_{Dfu}$ at $r_D=0$ precludes the evaluation. Instead, $\tilde{s}_{Dfu}$ is evaluated at the rim of the test well ($r_D=1$), and D_{Dm} is approximated as $l_{Dm}-1$ for all the image wells. The following steps are therefore taken in the solutions.

- Step 1. Determine $\tilde{s}_{Dfu}(r_D,\theta,p)$ from Eq. (15) corresponding to the same aquifer condition as the CHT.
- Step 2. Determine $\tilde{Q}_{Dh}(p)$ at the test well by substituting Eq. (9) and $\tilde{s}_{Dfu}(r_D,\theta,p)$ with $r_D=1$ and $D_{Dm}=l_{Dm}-1$ into Eq. (11).
- Step 3. Determine $\tilde{s}_{Dh}(r_D,\theta,p)$ at the observation well by inserting $\tilde{Q}_{Dh}(p)$ and its corresponding $\tilde{s}_{Dfu}(r_D,\theta,p)$ into Eq. (11).

The dimensionless discharge at the test well obtained from Step 2 is expressed as

$$\tilde{Q}_{Dh}(p)=\frac{s_{0D}K_1(\sqrt{p})}{\sqrt{p}\{K_0(\sqrt{p})+\Sigma_{m=1}^{n}(-1)^jK_0[(l_{Dm}-1)\sqrt{p}]\}} \tag{19}$$

which is inverted as

$$Q_{Dh}(t_D)=\frac{2s_{0D}}{\pi}\int_0^\infty e^{-t_Du^2}\frac{J_1(u)B_2(u)-Y_1(u)B_1(u)}{B_1^2(u)+B_2^2(u)}du \tag{20}$$

where

$$B_1(u)=J_0(u)+\sum_{m=1}^{n}(-1)^jJ_0[(l_{Dm}-1)u] \tag{21}$$

and

$$B_2(u)=Y_0(u)+\sum_{m=1}^{n}(-1)^jY_0[(l_{Dm}-1)u] \tag{22}$$

Following Step 3, the dimensionless drawdown at the observation well is

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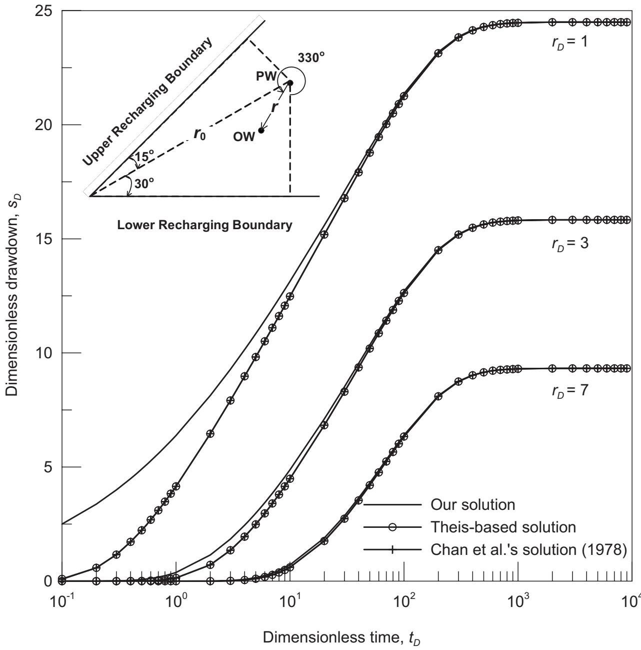

Fig. 2. Comparison of present solution with the Theis-based solution, and Chan et al.'s solution (1978). The above-left sketch shows the configuration of well and aquifer system. Note that PW and OW represent pumping well and observation well, respectively.

$$\tilde{s}_{Dh}(r_D,\theta,p) = \frac{s_{0D}}{p} \cdot \frac{K_0(r_D\sqrt{p}) + \Sigma_{m=1}^{n}(-1)^j K_0(D_{Dm}\sqrt{p})}{K_0(\sqrt{p}) + \Sigma_{m=1}^{n}(-1)^j K_0[(l_{Dm}-1)\sqrt{p}]} \tag{23}$$

Similarly, the dimensionless time-domain drawdown at the observation well is obtained as

$$s_{Dh}(r_D,\theta,t_D) = s_{0D} + \frac{2s_{0D}}{\pi}\int_0^\infty e^{-t_D u^2} \frac{A_1(r_D,\theta,u)B_2(u) - A_2(r_D,\theta,u)B_1(u)}{B_1^2(u)+B_2^2(u)} \frac{du}{u} \tag{24}$$

Comparison with Other Solutions

The present solution for a CFT is compared with two other solutions obtained in similar aquifer systems but assuming infinitesimal-diameter wells. The first solution is based on the superposition principle and the Theis solution (1935), whereas the second solution is given by Chan et al. (1978) for a wedge-shaped aquifer bounded by two recharging boundaries. Consider a CFT performed in the vicinity of two recharging boundaries intersecting at an angle of 45° as shown in Fig. 2. The aquifer data of $T = 10^{-3}$ m^2/s and $S=2.5\times 10^{-4}$ are used in the simulation. Define r_0 as the length from the test well to the intersection point of the wedge-shaped aquifer. The test well is constructed at $r_0=10$ m and $\alpha=30°$ with $Q=10^{-2}$ m^3/s and $r_w=0.2$ m. The dimensionless drawdowns, s_{Df}, at $r_D=1$, 3, and 7 with $\theta=330°$ are calculated. The values of s_{Df}, predicted by these three solutions plotted in Fig. 2, increase with t_D and then reach to their steady-state values as t_D approaches 10^3. The curve of s_{Df} produced by the Theis-based solution is identical to that produced by Chan et al.'s solution (1978), but differ significantly from the present solution when $t_D \leqslant 50$ at $r_D=1$. The differences in s_{Df} assuming an infinitesimal diameter in the Theis-based and Chan et al.'s model are most important for a small r_D and at early time.

Concluding Remarks

New solutions describing the drawdown distribution of CFT as well as discharge rate and drawdown distribution of CHT in a wedge-shaped confined aquifer are developed by considering the influence of well diameter. The present drawdown solution for a CFT is compared with the solutions obtained by neglecting the well diameter. It appears that ignoring the influence of the presence of well diameter leads to a significant error in the predicted early time drawdown near the pumping well area.

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Notation

The following symbols are used in this technical note:

D_{Di} = dimensionless distance from observation well to image well i;
I_i = image well i;
J_0, J_1 = Bessel function of the first kind of order zero and order one, respectively;
K_0, K_1 = modified Bessel function of the second kind of order zero and order one, respectively;
l_{Di} = dimensionless distance from test well to image well i;
p = Laplace variable;
Q, Q_{Df} = constant discharge and dimensionless discharge produced by a CFT, respectively;
$Q_{Dh}, \tilde{Q}_{Dh}$ = dimensionless discharge produced by a CHT in the time domain and Laplace domain, respectively;
r, r_D = radial distance and dimensionless radial distance starting from the centerline of test well, respectively;
r_w = effective radius of test well;
r_0 = distance from the centerline of test well to the intersection point of the wedge-shaped aquifer;
S = storativity of aquifer;
s, s_D = drawdown and dimensionless drawdown, respectively;
$s_{Df}, \tilde{s}_{Df}$ = dimensionless drawdown produced by a CFT in the time domain and Laplace domain, respectively;
$s_{Dfu}, \tilde{s}_{Dfu}$ = dimensionless unit step response drawdown in the time domain and Laplace domain, respectively;
$s_{Dh}, \tilde{s}_{Dh}$ = dimensionless drawdown produced by a CHT in the time domain and Laplace domain, respectively;

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s_{0D} = dimensionless initial drawdown;
T = transmissivity of aquifer;
t, t_D = real time and dimensionless time from the start of test, respectively;
u = dummy variable;
Y_0, Y_1 = Bessel function of the second kind of order zero and order one, respectively;
α = angle between the lower topographic boundary and an image line connecting the intersected point of the wedge-shaped aquifer to test well;
β = $\phi - \alpha$;
θ = angle estimated from the line connecting test well and I_1; and
ϕ = wedge angle of wedged aquifer.

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