國立交通大學

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碩士論文

三維橢圓介面問題的數值研究

An Immersed Interface Method for 3D Elliptic Interface Problems

研　究　生：郭柏均

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摘　要

在本篇論文中，係使用內嵌介面法把一個二維橢圓介面問題推廣到三維。此方法為調整過的有限差分法，其修正項透過介面上的不連續條件而得。藉由由法向量方向所展開的泰勒展開式，在介面附近的網格點，離散差分方程會被調整。因為在介面上不連續條件型態的關係， 在解橢圓介面問題的過程中，必須要解一個線性系統。首先用重新初始化水平集方法來找網格點在界面上的正交投影，接下來我們運用內嵌介面法，廣義最小殘量方法，和最小平方法來解橢圓介面問題。

An Immersed Interface Method for
3D Elliptic Interface Problems

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Abstract

In this thesis, we extend the immersed interface method for a 2D elliptic interface problem to 3D. The numerical method is a finite difference method modified with some correction terms from the jump conditions. By using Taylor's expansions along the normal direction, the discrete difference equation is modified at the gird point close to interface. Because of the types of the jump condition, we have to solve a linear system in the process of solving the elliptic interface problem. We first use the reinitialization of level set method to find the orthogonal projection of the grid point and perform it to check its accuracy. Then, we solve the elliptic interface problem by the immersed interface method, GMRES and least squares method, and make some numerical tests to check the rate of convergence.

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1 Introduction

Elliptic interface problems have many applications in science and engineering, such as multi-material problems, two-phase problems, fluid problems. We focus on elliptic interface problems with variable coefficients. Along the interface, there exist the jump conditions of solution and flux across the interface. For example, this can applied in electrohydrodynamics [2, 15].

Peskin used the immersed boundary method [1] to simply describe the fluid motion interacting with complicated interface at the Cartesian grid. Many methods can solve elliptic interface problems, such as finite element method [5, 10], boundary integral method [13], ghost fluid method [3, 12], sharp interface method [7].

In this thesis, we use the immersed interface method [8, 15] to solve the problem from 2D to 3D. We have to find the orthogonal projection to the interface. In 2D, we can use polynomials to interpolate the interface by picking some point and use these approximation to find the orthogonal projection, such as cubic spline representation [4]. However, it is difficult to use this method in 3D. Therefore, we use reinitialization of level set method [9, 11, 14].

In Section 2, we introduce the elliptic interface equation to study how to use the immersed interface method to solve this equation. In Section 3, we can see how to find the orthogonal projection to the interface by reinitialization of level set method. In Section 4, since we do not have $[u_n]$, we have to get the solution through Poisson solver and solve a linear system. In Section 5, a method to modify the equation s.t. $[u] = 0$ is introduced. Finally, some conclusions are displayed in Section 6.

2 Elliptic interface equation and the immersed interface method

2.1 Elliptic interface equation

We consider a rectangular domain $\Omega = [x_\ell, x_r] \times [y_\ell, y_r] \times [z_\ell, z_r]$ with a closed interface Γ. Since the interface Γ divides the domain Ω into two regions, we denote the inside region by Ω^- and the outside region by Ω^+, and with piecewise constant coefficient, σ^- on Ω^- and σ^+ on Ω^+. Along Γ, there exist the jump conditions of solution and flux across Γ.

We solve the following elliptic interface equation

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = f & \text{in } \Omega - \Gamma, \\ [u](\mathbf{X}) = \omega(\mathbf{X}) & \text{on } \Gamma, \\ [\sigma u_n](\mathbf{X}) = \nu(\mathbf{X}) & \text{on } \Gamma, \\ u = u_b & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $u_n = \frac{\partial u}{\partial \mathbf{n}}$.

Since σ is a piecewise constant, we can rewrite the elliptic interface equation as

$$\Delta u = \overline{f} \quad \text{in } \Omega - \Gamma$$

where $\overline{f} = f/\sigma$, i.e.

$$\begin{cases} \Delta u = \overline{f} & \text{in } \Omega - \Gamma, \\ [u](\mathbf{X}) = \omega(\mathbf{X}) & \text{on } \Gamma, \\ [\sigma u_n](\mathbf{X}) = \nu(\mathbf{X}) & \text{on } \Gamma, \\ u = u_b & \text{on } \partial\Omega. \end{cases} \tag{2}$$

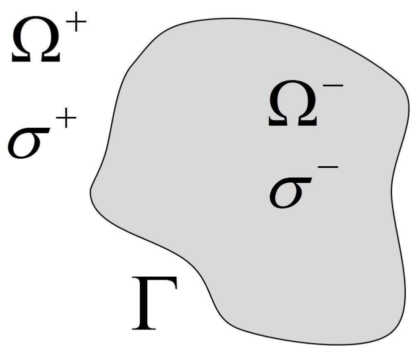

Figure 1: The interface Γ divides the domain into Ω^+ and Ω^-

2.2 The interpolation of the jump condition

To solve this problem, we use the immersed interface method developed in [8, 15]. We use a uniform Cartesian grid in Ω with the mesh width $h = \Delta x = \Delta y = \Delta z$, and use $\mathbf{x}_{i,j,k} = (x_l + i\Delta x, y_l + j\Delta y, z_l + k\Delta z)$ to denote the mesh grid point and denote discretized solution at the mesh grid point by $u_{i,j,k}$. Since we have the jump condition on the interface Γ, we determine whether the grid point is a regular point or an irregular point. If the seven-point Laplacian of a grid point uses some points in Ω^- and some points in Ω^+ simultaneously, the grid point is called the irregular point. Otherwise, when the grid point uses either points in Ω^- or points in Ω^+, it is the regular point. At the irregular point, because of the jump conditions, we have to change the approximations according to the jump condition on the interface. For example, we let $\mathbf{x}_{i,j,k}, \mathbf{x}_{i+1,j,k}, \mathbf{x}_{i,j+1,k}, \mathbf{x}_{i,j,k+1}$ be in the inside region, $\mathbf{x}_{i-1,j,k}, \mathbf{x}_{i,j-1,k}, \mathbf{x}_{i,j,k-1}$ in the outside region, then $\mathbf{x}_{i,j,k}$ is an irregular point, and the modified seven-point Laplacian $\Delta_h u$ is written as follows.

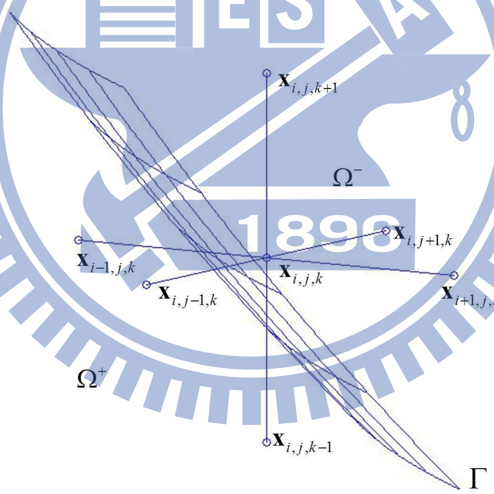

Figure 2: The seven-point Laplacian of the irregular point $\mathbf{x}_{i,j,k}$

$$
\begin{aligned}
\Delta_h u(\mathbf{x}_{i,j,k}) := & \frac{u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}}{h^2} \\
& + \frac{u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}}{h^2} \\
& + \frac{u_{i,j,k-1} - 2u_{i,j,k} + u_{i,j,k+1}}{h^2} \\
= & \frac{u^+_{i-1,j,k} - 2u^-_{i,j,k} + u^-_{i+1,j,k}}{h^2} \\
& + \frac{u^+_{i,j-1,k} - 2u^-_{i,j,k} + u^-_{i,j+1,k}}{h^2} \\
& + \frac{u^+_{i,j,k-1} - 2u^-_{i,j,k} + u^-_{i,j,k+1}}{h^2} \\
= & \frac{u^-_{i-1,j,k} - 2u^-_{i,j,k} + u^-_{i+1,j,k}}{h^2} + \frac{u^+_{i-1,j,k} - u^-_{i-1,j,k}}{h^2} \\
& + \frac{u^-_{i,j-1,k} - 2u^-_{i,j,k} + u^-_{i,j+1,k}}{h^2} + \frac{u^+_{i,j-1,k} - u^-_{i,j-1,k}}{h^2} \\
& + \frac{u^-_{i,j,k-1} - 2u^-_{i,j,k} + u^-_{i,j,k+1}}{h^2} + \frac{u^+_{i,j,k-1} - u^-_{i,j,k-1}}{h^2} \\
= & u_{xx}(\mathbf{x}_{i,j,k}) + u_{yy}(\mathbf{x}_{i,j,k}) + u_{zz}(\mathbf{x}_{i,j,k}) + O(h^2) \\
& + \frac{u^c_{i-1,j,k}}{h^2} + \frac{u^c_{i,j-1,k}}{h^2} + \frac{u^c_{i,j,k-1}}{h^2} \\
= & \overline{f}_{i,j,k} + \frac{1}{h^2}\left(u^c_{i-1,j,k} + u^c_{i,j-1,k} + u^c_{i,j,k-1}\right) + O(h^2)
\end{aligned} \tag{3}
$$

where $u^c_{i,j,k} = u^+_{i,j,k} - u^-_{i,j,k}$.

To compute the correction term $u^c_{i,j,k}$, we define the foot $\mathbf{X}^*_{i,j,k}$ of $\mathbf{x}_{i,j,k}$. It means that $\mathbf{X}^*_{i,j,k}$ is the orthogonal projection to the interface of $\mathbf{x}_{i,j,k}$. By Taylor's expression, we can write

$$
\begin{aligned}
u^c_{i,j,k} = & u^+_{i,j,k} - u^-_{i,j,k} \\
= & \left(u^+ + \alpha\frac{\partial u^+}{\partial \mathbf{n}} + \frac{\alpha^2}{2}\frac{\partial^2 u^+}{\partial \mathbf{n}^2}\right)\Bigg|_{\mathbf{X}^*_{i,j,k}} + O(h^3) \\
& - \left(u^- + \alpha\frac{\partial u^-}{\partial \mathbf{n}} + \frac{\alpha^2}{2}\frac{\partial^2 u^-}{\partial \mathbf{n}^2}\right)\Bigg|_{\mathbf{X}^*_{i,j,k}} + O(h^3) \\
= & [u]_{\mathbf{X}^*_{i,j,k}} + \alpha\left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\mathbf{X}^*_{i,j,k}} + \frac{\alpha^2}{2}\left[\frac{\partial^2 u}{\partial \mathbf{n}^2}\right]_{\mathbf{X}^*_{i,j,k}} + O(h^3).
\end{aligned} \tag{4}
$$

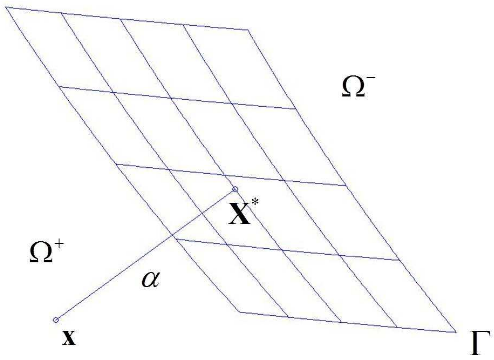

Figure 3: The foot $\mathbf{X}^*$ of the irregular point $\mathbf{x}$

The $|\alpha|$ is the distance between $\mathbf{x}_{i,j,k}$ and $\mathbf{X}^*_{i,j,k}$. If $\mathbf{x}_{i,j,k}$ is in the inside region, then α is positive. Otherwise, if $\mathbf{x}_{i,j,k}$ is in the outside region, then α is negative. (i.e., α is the signed distance between $\mathbf{x}_{i,j,k}$ and $\mathbf{X}^*_{i,j,k}$)

Instead of finding $\left[\frac{\partial^2 u}{\partial \mathbf{n}^2}\right]_{\mathbf{X}^*_{i,j,k}}$, we use this equation

$$\Delta u = \frac{\partial^2 u}{\partial \mathbf{n}^2} + \kappa \frac{\partial u}{\partial \mathbf{n}} + \Delta_{ss} u \tag{5}$$

where $\kappa = 2H = \nabla \cdot \mathbf{n}$, H is the mean curvature. $\Delta_{ss} u$ is the surface laplacian operator of u. Then, we can write

$$\left[\frac{\partial^2 u}{\partial \mathbf{n}^2}\right]_{\mathbf{X}^*_{i,j,k}} = [\,\overline{f}\,]_{\mathbf{X}^*_{i,j,k}} - \kappa_{\mathbf{X}^*_{i,j,k}} \left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\mathbf{X}^*_{i,j,k}} - \Delta_{ss}[u]_{\mathbf{X}^*_{i,j,k}}. \tag{6}$$

Hence, the correction term can be written as

$$\begin{aligned} u^c_{i,j,k} = \; & [u]_{\mathbf{X}^*_{i,j,k}} + \alpha \left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\mathbf{X}^*_{i,j,k}} + \frac{\alpha^2}{2}\left[\frac{\partial^2 u}{\partial \mathbf{n}^2}\right]_{\mathbf{X}^*_{i,j,k}} + O(h^3) \\ = \; & [u]_{\mathbf{X}^*_{i,j,k}} + \alpha \left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\mathbf{X}^*_{i,j,k}} \\ & + \frac{\alpha^2}{2}\left([\,\overline{f}\,]_{\mathbf{X}^*_{i,j,k}} - \kappa_{\mathbf{X}^*_{i,j,k}} \left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\mathbf{X}^*_{i,j,k}} - \Delta_{ss}[u]_{\mathbf{X}^*_{i,j,k}} \right) + O(h^3) \\ = \; & [u]_{\mathbf{X}^*_{i,j,k}} + \alpha [u_n]_{\mathbf{X}^*_{i,j,k}} \\ & + \frac{\alpha^2}{2}\left([\,\overline{f}\,]_{\mathbf{X}^*_{i,j,k}} - \kappa_{\mathbf{X}^*_{i,j,k}} [u_n]_{\mathbf{X}^*_{i,j,k}} - \Delta_{ss}[u]_{\mathbf{X}^*_{i,j,k}} \right) + O(h^3). \end{aligned} \tag{7}$$

3 Find the foot of an irregular point using reinitialization of level set method

3.1 The reinitialization of level set method

For finding irregular points, we use reinitialization of level set method developed in [9, 11, 14]. First, we have to find signed-distance function $\phi : \Omega \to \mathbb{R}$ where

$$\phi(\mathbf{x}) = \begin{cases} d(\mathbf{x}, \Gamma) & \text{in } \Omega^+, \\ 0 & \text{on } \Gamma, \\ -d(\mathbf{x}, \Gamma) & \text{on } \Omega^-. \end{cases} \tag{8}$$

Now, we consider a level set $\phi_0 : \Omega \to \mathbb{R}$ where

$$\phi_0(\mathbf{x}) \begin{cases} > 0 & \text{in } \Omega^+, \\ = 0 & \text{on } \Gamma, \\ < 0 & \text{on } \Omega^-. \end{cases} \tag{9}$$

using reinitialization equation

$$\phi_t + S(\phi_0)(| \bigtriangledown \phi| - 1) = 0 \tag{10}$$

where $S(\phi_0)$ is signed function. (i.e. $S(\phi_0(\mathbf{x})) = 1$ when $\mathbf{x} \in \Omega^+$, $S(\phi_0(\mathbf{x})) = -1$ when $\mathbf{x} \in \Omega^-$ and $S(\phi_0(\mathbf{x})) = 0$ when $\mathbf{x} \in \Gamma$.)
It will converge to signed-distance function. Since the equation converges steady state quickly very much at the points near the Γ, we only use a few time steps to get the results. For getting better results, we modify the $S(\phi_0)$ as

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + (\Delta x)^2}}. \tag{11}$$

as a numerical approximation. In this thesis, we use TVD-RK and WENO to compute Eq. (10). Now, we find the foot $\mathbf{X}^*$ of $\mathbf{x}$. Since $\mathbf{X}^*$ is the foot of $\mathbf{x}$, we have the equation

$$\mathbf{X}^* = \mathbf{x} - \phi(\mathbf{x})\nabla\phi(\mathbf{X}^*). \tag{12}$$

Since $\mathbf{x}$ is close to $\mathbf{X}^*$, we use this following approximation for simplicity

$$\mathbf{X}^* \approx \mathbf{x} - \phi(\mathbf{x})\nabla\phi(\mathbf{x}). \tag{13}$$

3.2 Numerical results

In these numerical experiments, we consider the rectangular domain $\Omega = [-1,1] \times [-1,1] \times [-1,1]$. In Example 1 and 2, we perform the numerical tests for finding the foots by the reinitialization of level set method.

Example 1

In this example, we consider the interface Γ as a sphere. The surface equation is

$$\Sigma \ : \ x^2 + y^2 + z^2 = r_0^2 \tag{14}$$

$$\text{where } r_0 = \sqrt{\frac{2}{13}}.$$

The initial level set function is chosen to be

$$\phi_0(x,y,z) = x^2 + y^2 + z^2 - r_0^2. \tag{15}$$

The following numerical results are the errors about finding the foot by reinitialization of level set method. The length of the domain is 2 so we choose $h = \frac{2}{N}$ with $N = 32, 64, 128, 256$. Since the interface is a sphere, the exact distance and the exact foot of the irregular points are easy to obtain. Besides, we can choose a level set function ϕ to describe the error of the foot to the interface. Since Γ is a sphere, the signed-distance function ϕ is known. ϕ is suited to be the level set function describing the error of the foot. Therefore, we use ϕ to be

$$\phi(x,y,z) = \sqrt{x^2 + y^2 + z^2} - r_0. \tag{16}$$

The following table shows the L^∞ errors at the irregular points and the steps of the iteration. The errors are the distance, the foot, and ϕ for different N. In this case, we can see that the method has the first order accuracy.

N	distance	Order	foot	Order	ϕ	Order	step
32	9.5416e-03	-	2.0864e-02	-	2.0861e-02	-	10
64	4.1199e-03	1.2116	8.6686e-03	1.2671	8.6671e-03	1.2672	10
128	2.2089e-03	0.8993	4.5503e-03	0.9298	4.5502e-03	0.9296	10
256	1.1053e-03	0.9989	2.2309e-03	1.0283	2.2309e-03	1.0283	10

Table 1: The L^∞ errors and the order of accuracy of distance, foot and ϕ on irregular points for different N

Example 2
In this example, we consider the interface Γ as an ellipsoid. The surface equation is

$$\Sigma \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{17}$$

$$\text{where } a = \sqrt{\frac{2}{13}}, \ b = \sqrt{\frac{11}{43}}, \ c = \sqrt{\frac{3}{31}}.$$

A problem occurs when the initial level set function is chosen to be

$$\phi_0(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1. \tag{18}$$

Using this ϕ_0 to compute ϕ, a state happens in the iteration process. For some iteration step n, there exists some $\mathbf{x}_{i,j,k}$, s.t.

$$\phi_0(\mathbf{x}_{i,j,k})\phi^n(\mathbf{x}_{i,j,k}) < 0, \tag{19}$$

i.e. for some grid points in the outside region, they are translated to the inside region in the iteration process, or the opposite situation. To deal with this problem, we discuss $|\nabla\phi| - 1$ near the interface. On Γ,

$$\begin{aligned} |\nabla\phi(x, y, z)| &= \left|\left(2\frac{x}{a^2}, 2\frac{y}{b^2}, 2\frac{z}{c^2}\right)\right| \\ &= 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} \\ &\geq 2\sqrt{\frac{1}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)} \\ &= \frac{2}{b}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} \\ &= 2\sqrt{\frac{43}{11}} \\ &\simeq 4. \end{aligned} \tag{20}$$

Hence, the reason of this occasion may be that $|\nabla\phi| - 1$ near the interface is too large.
Here, we scale the initial level set function as

$$\phi_0(x, y, z) = \left(\frac{a^2 + b^2 + c^2}{3}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right). \tag{21}$$

We take $h = \frac{2}{N}$ with $N = 32, 64, 128, 256$. Since the exact distance and the exact foot of the irregular points can not be obtained directly, we describe the error of the foot to the interface by level set function Φ. The level set function is chosen to be

$$\Phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1. \tag{22}$$

Table 2 shows the L^∞ errors of Φ at the irregular points and the steps of the iteration for different N. In this table, we can see that the rate of convergence is first order.

Size	Φ	Order	step
32	1.1443e-01	-	10
64	5.4037e-02	1.0824	10
128	2.5760e-02	1.0688	10
256	1.2700e-02	1.0203	10

Table 2: The L^∞ errors and the order of accuracy of Φ on irregular points for different N

4 Solve the elliptic interface problem

4.1 Solve the elliptic interface equation

We follow these steps developed in [15] to solve the equation (2). Since we only know $[\sigma u_n]$ instead of $[u_n]$, we use these equations

$$[u_n] = \frac{1}{\sigma^-}\left([\sigma u_n] - [\sigma]u^+\right) \qquad \text{when } \sigma^- > \sigma^+, \tag{23}$$

$$[u_n] = \frac{1}{\sigma^+}\left([\sigma u_n] - [\sigma]u^-\right) \qquad \text{when } \sigma^+ > \sigma^-. \tag{24}$$

We will explain why we choose the equation in Eq. (44).
We let $C_{i,j,k}$ be the correction term, then we can write Eq. (3) as

$$\Delta_h u_{i,j,k} = \overline{f}_{i,j,k} + C_{i,j,k}. \tag{25}$$

And we let $\overline{C}_{i,j,k}$ be the correction term without $[u_n]$ term, i.e., $\overline{C}_{i,j,k}$ does not have $[u_n]$ term, and according to the interpolation of the jump condition, there exists a linear operator A s.t.

$$\overline{C} + A\,[u_n] = C. \tag{26}$$

Therefore,

$$\Delta_h u = \overline{f} + \overline{C} + A\,[u_n]\,. \tag{27}$$

From Eq. (23) and (24), since we do not know the u^+, u^- on the interface Γ, we must approximate u^+ from the grid points in Ω^+ , and

u^- from the grid points in Ω^-. The approximation method is written in the following.

Suppose we want to approximate $u^+(\mathbf{X}^*_{i_0,j_0,k_0})$. First, we set a square number SN. Then, we collect a set S written as

$$\begin{aligned} S = \{\mathbf{x}_{i,j,k} \in \Omega^+ | i &= i_0 - sn, \ldots, i_0 + (SN - sn - 1), \\ j &= j_0 - sn, \ldots, j_0 + (SN - sn - 1), \\ k &= k_0 - sn, \ldots, k_0 + (SN - sn - 1)\} \end{aligned} \tag{28}$$

where sn is the largest integer not greater than $\frac{SN-1}{2}$.

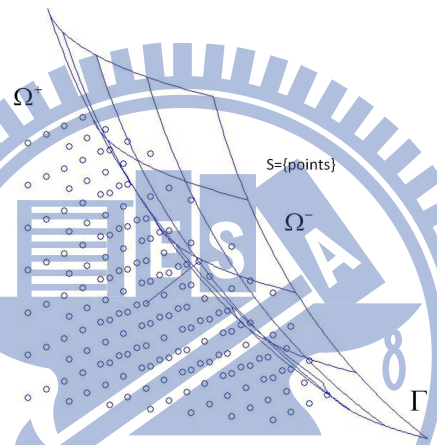

Figure 4: The set S of $\mathbf{X}^*$ with $SN = 7$

Given a polynomial $P_3(x, y, z)$ with degree 3, we determine the coefficients of $P_3(x, y, z)$ by least squares method approximating $u^+(x, y, z)$ on S. So we approximate $u^+(\mathbf{X}^*_{i_0,j_0,k_0})$ by

$$u^+_n(\mathbf{X}^*_{i_0,j_0,k_0}) \approx \nabla P_3(\mathbf{X}^*_{i_0,j_0,k_0}) \cdot \mathbf{n}(\mathbf{X}^*_{i_0,j_0,k_0}). \tag{29}$$

Similarly, we can approximate $u^-(\mathbf{X}^*_{i_0,j_0,k_0})$ by the above method. Since the approximation is linear for $u_{i,j,k}$, we can obtain two linear operators B^+, B^- s.t.

$$u^+_n \approx B^+ u, \tag{30}$$

$$u^-_n \approx B^- u. \tag{31}$$

And with the approximation, we rewrite Eq. (23) and (24) as

$$[u_n] = \frac{1}{\sigma^-}\left([\sigma u_n] - [\sigma] B^+ u\right) \qquad \text{when } \sigma^- > \sigma^+, \tag{32}$$

$$[u_n] = \frac{1}{\sigma^+}\left([\sigma u_n] - [\sigma] B^- u\right) \qquad \text{when } \sigma^+ > \sigma^-. \tag{33}$$

$$B^+u + \frac{\sigma^-}{[\sigma]}[u_n] = \frac{[\sigma u_n]}{[\sigma]}, \tag{34}$$

$$B^-u + \frac{\sigma^+}{[\sigma]}[u_n] = \frac{[\sigma u_n]}{[\sigma]}. \tag{35}$$

From Eq. (27), (34) and (35), we get a linear system as

$$\begin{bmatrix} \Delta_h & -A \\ B^{\pm} & \frac{\sigma^{\mp}}{[\sigma]} \end{bmatrix} \begin{bmatrix} u \\ [u_n] \end{bmatrix} = \begin{bmatrix} \overline{f} + \overline{C} \\ \frac{[\sigma u_n]}{[\sigma]} \end{bmatrix}. \tag{36}$$

To solve Eq. (36), we use Schur-complement method, i.e., we solve u and $[u_n]$ separately. In this method, we have to use Poisson solver. In this thesis, we use the public software package MUDPACK to solve Poisson equation. The details are written as follows.
First, we solve a Poisson equation u^* for

$$\Delta_h u^* = \overline{f} + \overline{C}, \tag{37}$$

$$u^* = u_b \qquad \text{on } \partial\Omega. \tag{38}$$

Second, we compare u and u^*

$$\Delta_h u - A\,[u_n] = \overline{f} + \overline{C}, \tag{39}$$

$$\Delta_h u^* = \overline{f} + \overline{C}, \tag{40}$$

$$u = u^* = u_b \qquad \text{on } \partial\Omega. \tag{41}$$

$$\Delta_h(u - u^*) - A\,[u_n] = 0, \tag{42}$$

$$u - u^* = 0 \qquad \text{on } \partial\Omega. \tag{43}$$

$$\begin{aligned}
(u - u^*) - \Delta_h^{-1} A\,[u_n] &= 0, \\
u - \Delta_h^{-1} A\,[u_n] &= u^*, \\
B^{\pm}u - B^{\pm}\Delta_h^{-1} A\,[u_n] &= B^{\pm}u^*, \\
-\frac{\sigma^{\mp}}{[\sigma]}[u_n] + \frac{[\sigma u_n]}{[\sigma]} - B^{\pm}\Delta_h^{-1} A\,[u_n] &= B^{\pm}u^*,
\end{aligned}$$

$$\left(-B^{\pm}\Delta_h^{-1} A - \frac{\sigma^{\mp}}{[\sigma]} I\right)[u_n] = B^{\pm}u^* - \frac{[\sigma u_n]}{[\sigma]}. \tag{44}$$

When $\sigma^- > \sigma^+$, $\left|\frac{\sigma^-}{[\sigma]}\right|$ is larger than $\left|\frac{\sigma^+}{[\sigma]}\right|$. Since $\frac{\sigma^{\mp}}{[\sigma]}$ affects the coefficient of diagonal term, Eq. (34) is better than Eq. (35). Similarly, when $\sigma^+ > \sigma^-$, for solving the linear system Eq. (44) ,choosing Eq. (35) is better than Eq. (34). To solve the linear system Eq. (44), we use the iterative method GMRES. Using GMRES has an advantage, this is, we do not have to construct the matrix $-B^{\pm}\Delta_h^{-1}A$. We can use the poission solver to replace Δ_h^{-1}, and do not need to write the explicit form of A and $B^{\pm}$. For the iteration, we let the stopping criteria be h^2. We can do this setting for the following reasons. The error of $[u_n]$ is $O(h^2)$ and d is $O(h)$, so the error of $[u_n]$ makes an error $O(h)$ in the correction term. The correction term originally has an error $O(h)$. Hence, The error of $[u_n]$ does not affect the overall accuracy. Once we know $[u_n]$, then we can solve u by

$$\Delta_h u = \overline{f} + \overline{C} + A\,[u_n]\,, \tag{45}$$

$$u = u_b \qquad \text{on } \partial\Omega. \tag{46}$$

In summary, we write the following steps of the method solving Eq. (36).

Step 1.

Solving the Poisson equation u^* for

$$\Delta_h u^* = \overline{f} + \overline{C}, \tag{47}$$

$$u^* = u_b \qquad \text{on } \partial\Omega. \tag{48}$$

Step 2.

Using GMRES with the stopping criteria h^2 to solve $[u_n]$ in the below linear system

$$\left(-B^{\pm}\Delta_h^{-1}A - \frac{\sigma^{\mp}}{[\sigma]}I\right)[u_n] = B^{\pm}u^* - \frac{[\sigma u_n]}{[\sigma]} \tag{49}$$

where Δ_h^{-1} is poisson solver with zero Dirichlet boundary condition.

Step 3.

Solving the Poisson equation u

$$\Delta_h u = \overline{f} + \overline{C} + A\,[u_n]\,, \tag{50}$$

$$u = u_b \qquad \text{on } \partial\Omega. \tag{51}$$

4.2 Numerical results

Considering the rectangular domain $\Omega = [-1,1]\times[-1,1]\times[-1,1]$, we display the numerical results of the immersed interface method for the elliptic interface equation (2).

Example 3

We consider an interface and an exact solution as

$$\Sigma \; : \; x^2+y^2+z^2 = r_0^2, \tag{52}$$

$$u^+ = x^2+y^2+z^2, \tag{53}$$

$$u^- = \sin(x^2+y^2+z^2) \tag{54}$$

$$\text{where } r_0 = \sqrt{\frac{2}{13}}.$$

The force $\overline{f}$, curvature κ and the boundary conditions can be calculated as

$$\begin{cases} u_n^+ &= 2\sqrt{x^2+y^2+z^2}, \\ u_n^- &= 2\sqrt{x^2+y^2+z^2}\cos(x^2+y^2+z^2), \\ \overline{f}^+ &= 6, \\ \overline{f}^- &= 6\cos(x^2+y^2+z^2) - 4(x^2+y^2+z^2)\sin(x^2+y^2+z^2), \\ \kappa &= \frac{2}{\sqrt{x^2+y^2+z^2}}. \end{cases} \tag{55}$$

Therefore, the equation (2) is written as

$$\begin{cases} \Delta u = 6 & \text{in } \Omega^+, \\ \Delta u = 6\cos(x^2+y^2+z^2) - 4(x^2+y^2+z^2)\sin(x^2+y^2+z^2) & \text{in } \Omega^-, \\ [u] = r_0^2 - \sin(r_0^2) & \text{on } \Gamma, \\ [\sigma u_n] = 2r_0\left(\sigma^+ - \sigma^-\cos(r_0^2)\right) & \text{on } \Gamma, \\ u = x^2+y^2+z^2 & \text{on } \partial\Omega. \end{cases} \tag{56}$$

Here, $[u]$ is constant on Γ, so $\Delta_{ss}[u] = 0$. We take $h = \frac{2}{N}$ with different N. Since the interface is a sphere, the exact foot of the irregular point is known. Two methods are used to solve Eq. (56)

Method 1:	solve u by the exact foots of the irregular points
Method 2:	solve u by the foots obtained by reinitialization of level set method

About σ^+ and σ^-, we choose two cases. They are $\sigma^+ = 2$, $\sigma^- = 23$ and $\sigma^+ = 23$, $\sigma^- = 2$. The exact u is known, so we can compute the error between the numerical u_N and the exact u. The L^∞, L^1, L^2

errors, the order of accuracy and the iteration steps for different N, σ^+, σ^- are displayed in the following tables.

Method 1

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	7.6125e-05	-	8.0125e-06	-	1.4937e-05	-	2
64	3.1055e-05	1.2936	3.4919e-06	1.1982	6.2259e-06	1.2625	3
128	1.0722e-05	1.5343	6.8645e-07	2.3468	1.4433e-06	2.1089	5

Table 3: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$ by method 1

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	1.0930e-03	-	1.7164e-04	-	2.9306e-04	-	2
64	2.6165e-04	2.0625	4.0319e-05	2.0898	6.7259e-05	2.1234	2
128	2.1928e-05	3.5768	2.9699e-06	3.7630	4.8823e-06	3.7841	2
256	1.1043e-06	4.3116	6.8618e-08	5.4357	1.2182e-07	5.3247	2

Table 4: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$ by method 1

Method 2

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	1.1487e-04	-	1.4585e-05	-	2.6055e-05	-	2
64	3.7705e-05	1.6071	5.1938e-06	1.4896	8.9480e-06	1.5419	2
128	1.1889e-05	1.6652	1.5597e-06	1.7356	2.6672e-06	1.7463	3
256	3.3990e-06	1.8064	4.9160e-07	1.6657	8.2544e-07	1.6921	4

Table 5: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$ by method 2

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	7.4238e-04	-	1.1580e-04	-	1.9781e-04	-	1
64	1.8352e-04	2.0162	2.8322e-05	2.0316	4.7207e-05	2.0670	2
128	1.4103e-05	3.7018	1.8714e-06	3.9197	3.0544e-06	3.9500	2
256	1.1318e-06	3.6393	8.3163e-08	4.4920	1.6270e-07	4.2306	3

Table 6: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$ by method 2

Example 4

In this example, an interface and an exact solution are chosen to be

$$\Sigma \ : \ x^2+y^2+z^2 = r_0^2, \tag{57}$$
$$u^+ = e^{x+y+z}, \tag{58}$$
$$u^- = \sin(x+y+z) \tag{59}$$
$$\text{where } r_0 = \sqrt{\frac{2}{13}}.$$

The force $\overline{f}$, curvature κ and the boundary conditions are calculated as

$$\begin{cases} u_n^+ &= e^{x+y+z}\frac{x+y+z}{\sqrt{x^2+y^2+z^2}}, \\ u_n^- &= \sin(x+y+z)\frac{x+y+z}{\sqrt{x^2+y^2+z^2}}, \\ \overline{f}^+ &= 0, \\ \overline{f}^- &= -3\sin(x+y+z), \\ \kappa &= \frac{2}{\sqrt{x^2+y^2+z^2}}. \end{cases} \tag{60}$$

The equation (2) is written as

$$\begin{cases} \Delta u = e^{x+y+z} & \text{in } \Omega^+, \\ \Delta u = -3\sin(x+y+z) & \text{in } \Omega^-, \\ [u] = e^{x+y+z} - \sin(x+y+z) & \text{on } \Gamma, \\ [\sigma u_n] = \frac{x+y+z}{r_0}\left(\sigma^+ e^{x+y+z} - \sigma^- \cos(x+y+z)\right) & \text{on } \Gamma, \\ u = e^{x+y+z} & \text{on } \partial\Omega. \end{cases} \tag{61}$$

Since the interface Γ is a sphere, $\Delta_{ss}[u]$ is available.

$$\begin{aligned} \Delta_{ss}[u] = 2\cos(x+y+z)\frac{x+y+z}{r_0^2} + 2\sin(x+y+z)\left(1-\frac{xy+yz+zx}{r_0^2}\right) \\ + 2e^{x+y+z}\left(1-\frac{xy+yz+zx+x+y+z}{r_0^2}\right). \end{aligned} \tag{62}$$

We choose $h = \frac{2}{N}$ with different $N = 32, 64, 128, 256$. Since the interface is a sphere, two methods are used to solve Eq. (61)

Method 1: solve u by the exact foots of the irregular points
Method 2: solve u by the foots obtained by reinitialization of level set method

About σ^+ and σ^-, we choose $\sigma^+ = 2$, $\sigma^- = 23$ and $\sigma^+ = 23$, $\sigma^- = 2$ to be the cases in this example. Since the exact u is known, the error between the numerical u_N and the exact u is available. The following tables show the L^∞, L^1, L^2 errors, the order of accuracy and the iteration steps.

Method 1

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	4.7880e-03	-	5.0982e-04	-	9.5825e-04	-	4
64	3.7523e-03	0.3516	4.8873e-04	0.0610	8.4660e-04	0.1787	6
128	1.9643e-03	0.9338	2.5085e-04	0.9622	4.2371e-04	0.9986	8
256	5.1709e-04	1.9255	6.4666e-05	1.9558	1.0849e-04	1.9655	10

Table 7: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$ by method 1

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	6.3817e-04	-	9.7196e-05	-	1.3414e-04	-	4
64	1.9205e-04	1.7325	2.4062e-05	2.0141	3.1806e-05	2.0764	4
128	3.2468e-05	2.5644	6.4305e-06	1.9037	9.0349e-06	1.8157	4
256	1.0633e-05	1.6104	1.6876e-06	1.9299	2.4299e-06	1.8946	5

Table 8: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$ by method 1

Method 2

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	2.5791e-03	-	2.0105e-04	-	4.2491e-04	-	3
64	1.3330e-03	0.9522	1.2981e-04	0.6311	2.4516e-04	0.7935	5
128	5.2778e-04	1.3367	4.1144e-05	1.6577	7.8304e-05	1.6465	7
256	1.6009e-04	1.7211	1.0226e-05	2.0085	2.0025e-05	1.9673	8

Table 9: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$ by method 2

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	7.0015e-04	-	1.0703e-04	-	1.4639e-04	-	3
64	1.8356e-04	1.9314	2.3551e-05	2.1842	3.2243e-05	2.1828	4
128	8.5021e-05	1.1103	5.0232e-06	2.2291	8.0645e-06	1.9993	5
256	7.4797e-05	0.1848	2.0646e-06	1.2827	4.6048e-06	0.8085	6

Table 10: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$ by method 2

Example 5

Considering an interface and an exact solution as

$$\Sigma \quad : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{63}$$

$$u^+ = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \tag{64}$$

$$u^- = \sin\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right), \tag{65}$$

$$\text{where } a = \sqrt{\frac{2}{13}},\ b = \sqrt{\frac{11}{43}},\ c = \sqrt{\frac{3}{31}}.$$

We calculate the force $\overline{f}$, curvature κ and the boundary conditions

$$\begin{cases} u_n^+ &= 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}, \\ u_n^- &= 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} \cos\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right), \\ \overline{f}^+ &= 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right), \\ \overline{f}^- &= 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\cos\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \\ & \quad -4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\sin\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right), \\ \kappa &= \frac{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) - \left(\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}\right)}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}. \end{cases} \tag{66}$$

Therefore, the equation (2) is written as

$$\begin{cases} \Delta u = 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) & \text{in } \Omega^+, \\ \Delta u = 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\cos\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) & \text{in } \Omega^-, \\ [u] = 1 - \sin(1) & \text{on } \Gamma, \\ [\sigma u_n] = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\,(\sigma^+ - \sigma^- \cos(1)) & \text{on } \Gamma, \\ u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} & \text{on } \partial\Omega. \end{cases} \tag{67}$$

Here, $[u]$ is constant on Γ, so $\Delta_{ss}[u] = 0$. We take $h = \frac{2}{N}$ with different $N = 32, 64, 128, 256$. We choose $\sigma^+ = 2$, $\sigma^- = 23$ and $\sigma^+ = 23$, $\sigma^- = 2$ in this test. The exact u is known, so we can compute the error between the numerical u_N and the exact u. The L^∞, L^1, L^2 errors, the order of accuracy and the iteration steps for different N, σ^+, σ^- are displayed in the following tables.

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	2.5842e-02	-	2.8891e-03	-	5.2446e-03	-	3
64	8.8586e-03	1.5445	9.8525e-04	1.5521	1.7558e-03	1.5787	6
128	3.2733e-03	1.4363	3.2026e-04	1.6213	5.8850e-04	1.5771	9
256	9.5019e-04	1.7844	1.0107e-04	1.6639	1.7713e-04	1.7323	11

Table 11: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	3.6022e-01	-	4.2028e-02	-	7.2799e-02	-	4
64	9.3078e-02	1.9524	9.8860e-03	2.0879	1.6791e-02	2.1162	6
128	7.7633e-03	3.5837	6.2775e-04	3.9771	1.0806e-03	3.9578	7
256	3.6076e-04	4.4275	3.2838e-05	4.2567	6.1427e-05	4.1368	8

Table 12: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$

Example 6

In this example, we consider an interface and an exact solution as

$$\Sigma \; : \; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{68}$$

$$u^+ = e^{x+y+z}, \tag{69}$$

$$u^- = \sin(x+y+z) \tag{70}$$

$$\text{where } a = \sqrt{\frac{2}{13}},\; b = \sqrt{\frac{11}{43}},\; c = \sqrt{\frac{3}{31}}.$$

We calculate the force $\overline{f}$, curvature κ and the boundary conditions

$$\begin{cases} u_n^+ &= \dfrac{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} e^{x+y+z}, \\ u_n^- &= \dfrac{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \cos\left(x+y+z\right), \\ \overline{f}^+ &= 3e^{x+y+z}, \\ \overline{f}^- &= -3\sin(x+y+z), \\ \kappa &= \dfrac{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)-\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}\right)}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}. \end{cases} \tag{71}$$

Therefore, the equation (2) is written as

$$\begin{cases} \Delta u = 3e^{x+y+z} & \text{in } \Omega^+, \\ \Delta u = -3\sin(x+y+z) & \text{in } \Omega^-, \\ [u] = e^{x+y+z} - \sin(x+y+z) & \text{on } \Gamma, \\ [\sigma u_n] = \dfrac{\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\left(\sigma^+ e^{x+y+z} - \sigma^- \cos(x+y+z)\right) & \text{on } \Gamma, \\ u = e^{x+y+z} & \text{on } \partial\Omega. \end{cases} \tag{72}$$

Since the interface Γ is an ellipsoid, $\Delta_{ss}[u]$ is available.

$$\begin{aligned} \Delta_{ss}[u] = e^{x+y+z} & \left(2 - \frac{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\alpha_3 + 2\alpha_1}{\alpha_2} + \frac{\alpha_3\alpha_4}{\alpha_2^2}\right) \\ & - 2\left(1 - \frac{\alpha_1}{\alpha_2}\right)\sin(x+y+z) \\ & - \frac{\alpha_3}{\alpha_2^2}\left[\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\alpha_2 - \alpha_4\right]\cos(x+y+z) \end{aligned} \tag{73}$$

where

$$\alpha_1 = \frac{xy}{a^2b^2} + \frac{yz}{b^2c^2} + \frac{zx}{c^2a^2}, \tag{74}$$

$$\alpha_2 = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}, \tag{75}$$

$$\alpha_3 = \frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}, \tag{76}$$

$$\alpha_4 = \frac{x^2}{a^6} + \frac{y^2}{b^6} + \frac{z^2}{c^6}. \tag{77}$$

We choose $\sigma^+ = 2$, $\sigma^- = 23$ and $\sigma^+ = 23$, $\sigma^- = 2$ to be the cases in this example. Since we know the exact u , the error between the numerical u_N and the exact u is available. The following tables show the L^∞, L^1, L^2 errors, the order of accuracy and the iteration steps for different N, σ^+, σ^-.

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	1.0985e-02	-	5.1651e-04	-	1.0907e-03	-	4
64	5.2105e-03	1.0761	3.2198e-04	0.6818	6.2963e-04	0.7927	7
128	2.1972e-03	1.2458	1.2664e-04	1.3462	2.2841e-04	1.4629	10
256	6.5012e-04	1.7569	2.2964e-05	2.4633	4.6245e-05	2.3043	12

Table 13: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 2$, $\sigma^- = 23$

Size	L^∞	Order	L^1	Order	L^2	Order	step
32	1.1158e-02	-	3.5443e-04	-	8.5945e-04	-	5
64	4.0653e-03	1.4566	1.7419e-04	1.0248	3.8375e-04	1.1632	5
128	1.6554e-03	1.2962	6.3940e-05	1.4459	1.3586e-04	1.4981	7
256	8.2041e-04	1.0127	2.9861e-05	1.0984	6.2598e-05	1.1179	8

Table 14: The L^∞, L^1 and L^2 errors and the order of accuracy for different N with $\sigma^+ = 23$, $\sigma^- = 2$

5 Transfer the elliptic interface equation with zero jump condition $[u]$

Now, we consider an elliptic interface equation

$$\begin{cases} \Delta u = f & \text{in } \Omega - \Gamma, \\ [u](\mathbf{X}) = \omega(\mathbf{X}) & \text{on } \Gamma, \\ [\sigma u_n](\mathbf{X}) = \nu(\mathbf{X}) & \text{on } \Gamma, \\ u = u_b & \text{on } \partial\Omega \end{cases} \tag{78}$$

where $\omega \in C^2(\Gamma)$.
In previous sections, if the equation is solved, we must calculate $\Delta_{ss}[u]$. When the interface is known, calculating $\Delta_{ss}[u]$ is feasible. However, for some problems, the interface will move with time. When the interface moves in the problem, the interface is possible to become unknown. Then, calculating $\Delta_{ss}[u]$ become a difficult problem.

If we do not want to calculate $\Delta_{ss}[u]$, then we must transfer the equation s.t. $[u] = 0$. To do that, we can use this method in [16]. First, we get the signed-distance function $\phi(\mathbf{x})$ as follows

$$\phi(\mathbf{x}) = \begin{cases} d(\mathbf{x}, \Gamma) & \text{in } \Omega^+, \\ 0 & \text{on } \Gamma, \\ -d(\mathbf{x}, \Gamma) & \text{on } \Omega^-. \end{cases} \tag{79}$$

Let us suppose $\phi \in C^3(\Omega)$ and the normal lines of the interface Γ do not intersect in the outside region Ω^+, then every $\mathbf{x}$ in the outside region has only one foot $\mathbf{X}^*$, i.e.

$$\forall \mathbf{x} \in \Omega^+,\ \exists!\ \mathbf{X}^* \in \Gamma, \beta \geq 0 \text{ s.t. } \mathbf{x} = \mathbf{X}^* + \beta \nabla\phi(\mathbf{X}^*). \tag{80}$$

We can define an extension $\omega_e(\mathbf{x})$ of $\omega(\mathbf{x})$ in $\Omega^+ \cup \Gamma$

$$\begin{aligned} \omega_e(\mathbf{x}) &= \omega_e(\mathbf{X}^* + \phi(\mathbf{x})\nabla\phi(\mathbf{X}^*)) \\ &= \omega(\mathbf{X}^*) \end{aligned} \tag{81}$$

and $\omega_e(\mathbf{x}) = 0$ in Ω^-.
Since $\phi \in C^3(\Omega)$ and $\omega \in C^2(\Gamma)$, $\omega_e \in C^2(\Omega \setminus \Gamma)$. Now, we define $\hat{u}(\mathbf{x})$

$$\hat{u}(\mathbf{x}) = H(\phi(\mathbf{x}))\omega_e(\mathbf{x}) = \begin{cases} 0 & \phi(\mathbf{x}) < 0, \\ \frac{1}{2}\omega_e(\mathbf{x}) & \phi(\mathbf{x}) = 0, \\ \omega_e(\mathbf{x}) & \phi(\mathbf{x}) > 0. \end{cases} \tag{82}$$

where $H(\cdot)$ is the Heaviside function.
Let $q(\mathbf{x}) = u(\mathbf{x}) - \hat{u}(\mathbf{x})$, then we can get this equation

$$\begin{cases} \Delta q = f - H(\phi)\Delta\hat{u} & \text{in } \Omega - \Gamma, \\ [q] = 0 & \text{on } \Gamma, \\ [\sigma q_n](\mathbf{X}) = \nu(\mathbf{X}) & \text{on } \Gamma, \\ q = u_b - \hat{u} & \text{on } \partial\Omega. \end{cases} \tag{83}$$

Proof :
If $\mathbf{x} \in \Omega^-$, then $\hat{u}(\mathbf{x}) = 0$. Therefore, $\Delta\hat{u}(\mathbf{x}) = 0$, and $H(\phi(\mathbf{x})) = 0$. Hence,

$$\begin{aligned} \Delta q(\mathbf{x}) &= \Delta u(\mathbf{x}) - \Delta\hat{u}(\mathbf{x}) \\ &= f(\mathbf{x}) - 0 \\ &= f(\mathbf{x}) - H(\phi(\mathbf{x}))\Delta\hat{u}(\mathbf{x}). \end{aligned} \tag{84}$$

If $\mathbf{x} \in \Omega^+$, then $H(\phi(\mathbf{x})) = 1$. Hence,

$$\begin{aligned} \Delta q(\mathbf{x}) &= \Delta u(\mathbf{x}) - \Delta\hat{u}(\mathbf{x}) \\ &= f(\mathbf{x}) - H(\phi(\mathbf{x}))\Delta\hat{u}(\mathbf{x}). \end{aligned} \tag{85}$$

Then, when $\mathbf{X} \in \Gamma$, $[q](\mathbf{X})$ is

$$\begin{aligned} [q](\mathbf{X}) &= [u](\mathbf{X}) - [\hat{u}](\mathbf{X}) \\ &= \omega(\mathbf{X}) - \omega_e(\mathbf{X}^+) \\ &= \omega(\mathbf{X}) - \omega_e(\mathbf{X}) \\ &= 0. \end{aligned} \tag{86}$$

Finally, check $[\sigma q_n]$. ω_e is constant along the normal line. Therefore,

$$\begin{aligned} \left.\frac{\partial\omega_e}{\partial\mathbf{n}}\right|_{\mathbf{X}^+} &= \frac{\partial\omega(\mathbf{X})}{\partial\mathbf{n}} \\ &= 0. \end{aligned} \tag{87}$$

Hence,

$$
\begin{aligned}
[\sigma \hat{u}_n](\mathbf{X}) &= \sigma^+ \frac{\partial \hat{u}}{\partial \mathbf{n}}\bigg|_{\mathbf{X}^+} - \sigma^- \frac{\partial \hat{u}}{\partial \mathbf{n}}\bigg|_{\mathbf{X}^-} \\
&= \sigma^+ \frac{\partial \omega_e}{\partial \mathbf{n}}\bigg|_{\mathbf{X}^+} - 0 \\
&= 0,
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
[\sigma q_n](\mathbf{X}) &= [\sigma u_n](\mathbf{X}) - [\sigma \hat{u}_n](\mathbf{X}) \\
&= \nu(\mathbf{X}) - 0 \\
&= \nu(\mathbf{X}).
\end{aligned} \tag{89}
$$

6 Conclusion

In this thesis, we use the immersed interface method to solve the 3D elliptic interface problem, and reinitialization of level set method to find the foot of the irregular point. To deal with the problem, we only use a simple interpolation, Poisson solver, least squares method and GMRES. It is simple to solve the elliptic interface problem by the present method. There only needs a few GMRES iteration steps. About approximation of $u^{\pm}$, using polynomial with degree 3 by least squares method makes the rates of convergence of the numerical results in this thesis are better than first order. Even some results have high order accuracy.

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