國 立 交 通 大 學 理 學 院 應 用 數 學 系

博 士 論 文

圖消圈數的研究

Decycling Number on Graphs and Digraphs

研究生：連敏筠
Student: Min-Yun Lien

指導教授：傅恆霖 教授
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中 華 民 國 一 百 零 三 年 六 月
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摘要

所謂的消圈集或反饋點集，是無向圖或有向圖裡的一個點集合，滿足扣掉這個點集合後，圖上沒有圈。一個圖的消圈數是最小的消圈集的數目。

決定一個一般圖的消圈數已經被證明是 NP 完備 (NP-complete)，甚至在平面圖，二部圖以及完美圖中，找它們的消圈數的複雜度也不會降低。

在圖上破壞圈的問題，一開始是應用在組合設計電路。接著也發現可以應用在作業系統中預防死結、約束補償問題、人工智慧上的貝斯推論、完全控制同步分散式系統、在光學網路上布置變波器，以及在超大型積體電路的晶片設計。

在這篇論文中，我們討論了在有向圖及無向圖的消圈數。在無向圖中，我們考慮了外部平面圖和格子圖。對於第一類圖，我們利用圈包裝數來刻劃消圈數，對於格子圖，我們改善了已知結果，使得上下界更靠近，在某些群組中，我們得到了消圈數的確切值。在有向圖中，我們考慮了廣義考茨有向圖以及廣義迪布恩有向圖，我們給了一個有系統的方法來獲得消圈集，進而得到消圈數的上界，這個方法對所有的有向圖都是可行的。

Abstract

A set of vertices of a graph or an digraph whose removal induces an acyclic graph is referred as a *decycling set*, or a *feedback vertex set*, of the graph. The minimum cardinality of a decycling set of a graph G is referred to as the *decycling number* of G.

The problem of determining the decycling number has been proved to be *NP*-complete for general graphs, which also shows that even for planar graphs, bipartite graphs and perfect graphs, the computation complexity of finding their decycling numbers is not reduced.

The problem of destroying all cycles in a graph by deleting a set of vertices originated from applications in combinatorial circuit design. Also, it has found applications in deadlock prevention in operating systems, the constraint satisfaction problem and Bayesian inference in artificial intelligence, monopolies in synchronous distributed systems, the converters' placement problem in optical networks, and VLSI chip design.

In this thesis, we study the decycling number of graphs and also digraphs. The graphs we consider are outerplanar graphs and grid graphs $P_m \Box P_n$. For the first class of graphs, we char-

acterize their decycling number by way of the cycle packing number and for grid graphs, we improve the known results to obtain either tight bounds or exact values. On digraphs, we consider generalized Kautz digraphs and generalized de Bruijn digraphs. Mainly, we use a novel idea in which we find a sequence of subsets of vertex set satisfying certain conditions and then obtain a decycling set. This provides an upper bound of the decycling number of digraphs we consider. Note that this idea can be applied to find the decycling set of general digraphs.

誌謝

在這漫長的研究生生涯中，首先要感謝我的指導教授 – 傅恆霖老師。在研究上，傅老師會提供我很多的想法，當我在研究上遇到瓶頸的時候，也會給予我很多的意見，為了讓我增加視野，老師也幫我申請到福州大學訪問三個月。在運動方面，老師鼓勵我繼續打球，也會指導我該如何讓運動表現更好。除了學術以及運動上的教導，老師更是我的人生導師，教了我很多做人處世的道理以及態度。非常感謝傅老師所教授的一切。

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版面，還有很多未能提及的夥伴們，非常謝謝他們，讓我的研究生生涯多采多姿。

最後，我要感謝我的家人，謝謝你們的支持及體諒，讓我可以堅持到最後，順利拿到博士學位，謹以此論文獻給你們。

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Chapter 1

Introduction and Preliminaries

1.1 Motivation

A set of vertices of a graph or an digraph, whose removal leaves an acyclic graph, is referred as a *decycling set* [3], or a *feedback vertex set* [37], of the graph. The minimum cardinality of a decycling set of a graph G is referred to as the *decycling number* of G.

The problem of destroying all cycles in a graph by deleting a set of vertices originated from applications in combinatorial circuit design [19]. Also, it has found applications in deadlock prevention in operating systems [34, 37], the constraint satisfaction problem and Bayesian inference in artificial intelligence [1], monopolies in synchronous distributed systems [28, 29], the converters' placement problem in optical networks [22], and VLSI chip design [16].

In 1986, Erdös, Saks and Sós [14] considered the problem of finding a maximum subset of G that would induce a tree. Meanwhile, the more general problem of finding the size of maximum subset of G that would induced a forest was also beginning to receive attention. Determining the decycling number of a graph G is equivalent to finding the maximum induced forest of G, since the sum of these two numbers are equal to the number of vertices of G.

The problem of determining the decycling number has been proved to be NP-complete for general graphs [20], which also shows that even for planar graphs, bipartite graphs and perfect graphs, the computation complexity of finding their decycling numbers is not reduced.

Besides searching for the value (or an upper bound) of the decycling number in the order of a graph, another parameter that is closely related to the decycling number is the *cycle packing number*, which is the maximum number of vertex-disjoint cycles. A trivial relation between the decycling number and the cycle packing number is the decycling number is not less than the cycle packing number. Moreover, the investigation of the decycling number and cycle packing number on graphs and digraphs is closed related to learn the structure of the studied graphs. The above facts motivate us to make a careful study.

1.2 Graphs

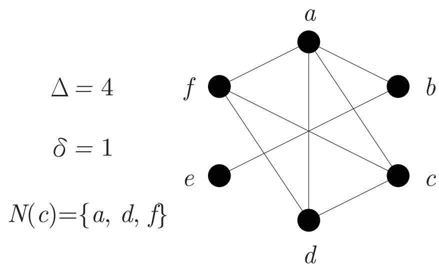
First, we introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory" by D. B. West [39].

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$ and a relation that associate each edge with two vertices called its *endpoints*. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of G, and the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of G. In this section, we focus on the undirected graphs in which all the edges have no directions.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops and multiple edges. We specify a simple graph by its vertex set and edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) as an edge e with endpoints u and v.

If $e = uv$ is an edge of G, then e is said to be *incident* to u and v. We also say that u and v are *adjacent*. For each $v \in V(G)$, $N(v)$ denotes the neighbors of v; that is, all vertices of $N(v)$ are adjacent to v. The *degree* of v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v. For the sake of brevity, a vertex of degree d is denoted by a *d-vertex*. The maximum degree is $\Delta(G)$ and the minimum degree is $\delta(G)$. Moreover,

G is *regular* if $\Delta(G) = \delta(G)$, and it is said to be k-*regular* if the common degree is k.



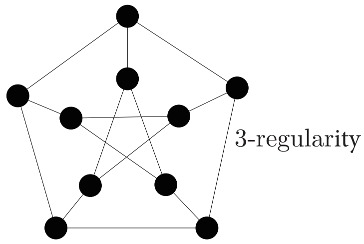


Figure 1.1: Degree, neighborhood and regularity

An *independent set* in a graph is a set of pairwise nonadjacent vertices.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path with n vertices is denoted by P_n. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A *spanning subgraph* of G is a subgraph with vertex set $V(G)$. Given S be a subset of vertex set $V(G)$, the *induced subgraph determined by* S, denoted by $G[S]$, is a subgraph of G such that for any $u, v \in S$, u is adjacent to v in $G[S]$ if u is adjacent to v in G.

The *components* of a graph G are its maximal connected subgraph. We use $c(G)$ to denote the number of components of G. An *isolated vertex* is a vertex of degree 0.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n.

A graph is called *triangle-free* if it contains no C_3 as its subgraph.

In contrast, a graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph.

A *separating set* or *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The connectivity of G, written $\kappa(G)$, is the minimum size of vertex set S such that $G - S$ is disconnected or has only one vertex. A graph is *k-connected* if its connectivity is at least k.

In a graph G, a *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w. A subdivision of H is a graph obtained from a graph H by successive subdivision of edges. Two graphs G_1, G_2 are *homeomorphic* if G_1 can be transformed into G_2 via a finite sequence of subdivisions.

The cartesian product of G and H, written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

The *k-dimensional cube* or *hypercube* Q_k is the simple graph whose vertices are the k-tuples with entries in $\{0, 1\}$ and whose edges are the pair of k-tuples that differ in exactly one position.

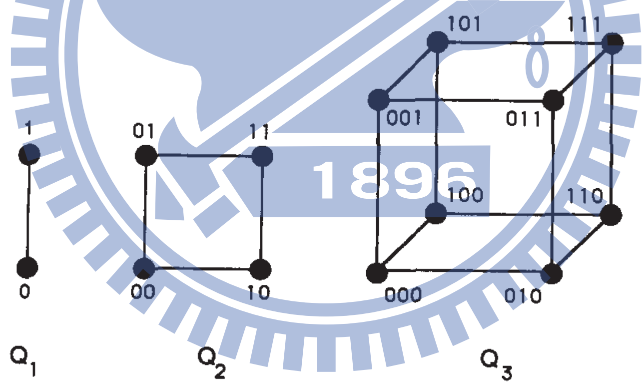


Figure 1.2: Hypercube Q_k for $k = 1, 2, 3$

A graph is *planar* if it has a drawing in the plane without any edge crossing. Such a drawing is a *planar embedding* of G. The *faces* of a planar graph are the maximal regions of the plane that contain no point used in the embedding. A *face* f of a planar graph is a circuit that surrounds a region bounded by edges; let ℓ_f denote the length of f, i.e., the number of surrounding edges. For a planar graph G, let $F(G)$ be the set of faces of the embedding. A finite planar graph G has one unbounded face (also called the *outer face*).

Euler's formula states that for every plane graph G,

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

A graph is *outerplanar* if it has an embedding with every vertex on the boundary of the unbounded face.

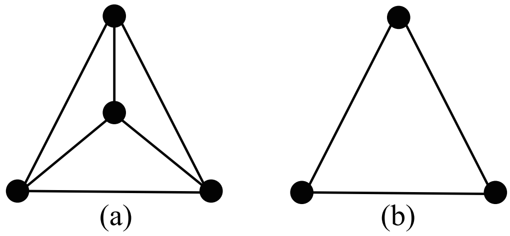


Figure 1.3: (a) planar, not outerplanar (b) outerplanar

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is *isomorphic to* H", written $G \cong H$, if there is an isomorphism from G to H.

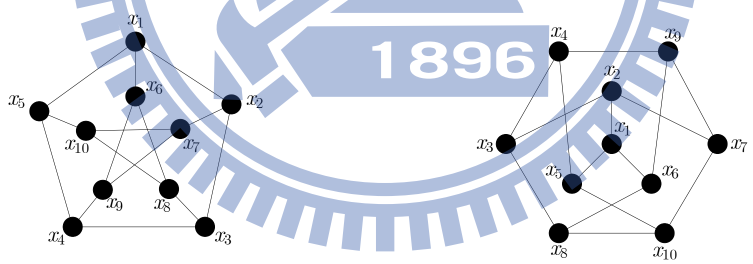


Figure 1.4: Two isomorphic graphs

1.3 Directed Graphs

A *directed graph* or *digraph* D is a triple consisting of a vertex set $V(D)$, an edge set $E(D)$ and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the *tail* of the edge and the second is the *head*; together, they are *endpoints*. The terms "head" and "tail" come from an arrow used to draw directed graphs. As with graphs, we assign each vertex a point in the plane and each edge a curve joining its endpoints. When drawing a directed graph, the direction of a curve is from the tail to the head. Figure 1.5 shows a directed graph D with vertex set $V(D) = \{0, 1, \cdots, 7\}$ and edge set $E(D) = \{(0,0), (0,1), (1,2), (1,3), (2,4), (2,5), (3,6), (3,7), (4,0), (4,1), (5,2), (5,3), (6,4), (6,5), (7,6), (7,7)\}$.

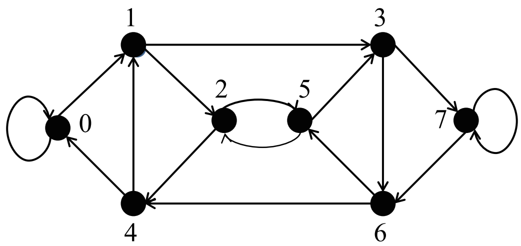


Figure 1.5: A directed graph $G_B(2, 8)$

In a directed graph, a *loop* is an edge whose endpoints are equal, such as $(0,0), (7,7)$ in Figure 1.5. *Multiple edges* are edges having the same ordered pair of endpoints. A directed graph is *simple* if each ordered pair of vertices have at most one edge; one loop may be present at each vertex. Therefore, Figure 1.5 is a simple directed graph.

In a simple directed graph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a *successor* of u and u is a *predecessor* of v. We write $u \to v$ for "there is an edge from u to v".

A directed graph is a *path* if it is a simple directed graph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v is immediately follows u in the vertex ordering. A *cycle* is defined similarly using an ordering of the vertices on a circle.

Let v be a vertex in a digraph. The *outdegree* $d^+(v)$ is the number of edges with tail v. The *indegree* $d^-(v)$ is the number of edges with head v. The *out-neighborhood* or *successor set* $N^+(v)$ is $\{x \in V(D) : v \to x\}$. The *in-neighborhood* or *predecessor set* $N^-(v)$ is $\{x \in V(D) : x \to v\}$.

1.4 Notations and Definitions

A set of vertices of a graph or an digraph whose removal leaves an acyclic graph is referred to as a decycling set of the graph. The minimum cardinality of a decycling set of G denoted by $\nabla(G)$, is referred to as the decycling number of G.

An *acyclic coloring* of a graph G is a coloring of its vertices, satisfying the following two rules:

(1) No two neighboring vertices are assigned the same color (this is also denoted as *proper coloring*).

(2) Let $V_a \subseteq V(G)$ be the set of vertices of G that are assigned color a. Then for any $a \neq b$, the induced subgraph $G[V_a \cup V_b]$ must be acyclic.

The minimum number of colors necessary to color G is called the *acyclic chromatic number* of G, and is denoted $a(G)$.

The cycle packing number $\nu(G)$ is the maximum number of vertex disjoint cycles.

The grid $P_m \square P_n$ has vertex set $V(P_m \square P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(P_m \square P_n) = \{(v_{i,j}, v_{i+1,j}) : 1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{(v_{i,j}, v_{i,j+1}) : 1 \leq i \leq m, 1 \leq j \leq n-1\}$.

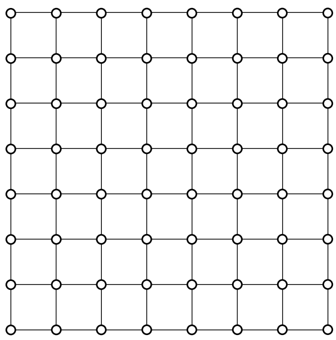


Figure 1.6: $P_8 \square P_8$

A *k-dimension butterfly* is a graph $B_k = (V, E)$ composed of $(k+1)2^k$ vertices organized in $k+1$ levels of 2^k vertices each, where $v_{i,j}$ denotes the jth vertex at level i, with $0 \leq i \leq k$

and $0 \le j \le 2^k - 1$. For $i > 0$, $v_{i,j}$ is connected with the two vertices $v_{i-1,j}$ and v_{i,j_i}, where j_i denotes the integer whose binary representation differs from that of j in only the ith position from right.

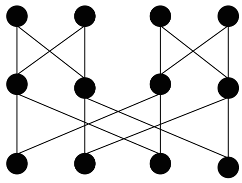


Figure 1.7: Butterfly B_2

For convenience, we use $[a, b] = [a, a+1, \cdots, b]$ for $a \le b$ and Z_d for the representation of $\{0, 1, \cdots, d-1\}$.

For $X, Y \subseteq V(G)$, an X, Y-path is a path having one endpoint in X, the other one in Y, and no other vertices in $X \cup Y$. A $\{v\}, Y$-path is simply written as a v, Y-path. Similarly, we use u, v-path to represent a path from u to v in G.

For $x \in \mathbb{R}$, the *floor* $\lfloor x \rfloor$ is the greatest integer less or equal to x. The *ceiling* $\lceil x \rceil$ is the smallest integer greater than or equal to x.

A *q-nary code* C of length n is a set of q-nary n-tuples and the *Hamming distance* between two strings of equal length (codewords) is the number of positions at which the corresponding symbols are different.

Let n and l be two positive integers with $n \ge 2l$. For any two numbers i, j where $1 \le i, j \le n$, we define a function (difference) χ by

$$\chi(i-j) = \begin{cases} |i-j| & \text{if } |i-j| \le \frac{n}{2}, \\ n - |i-j| & \text{otherwise.} \end{cases}$$

A *circular graph* $G = C(n, l)$ of order n is one spanned by n-cycle $C_n = (1, 2, \cdots, n)$ together with the chords $(i, j) \in E(G)$ if and only if $\chi(i-j) = l (l > 1)$.

The *Euler totient function* $\varphi(n)$ is the number of integers less than n that are relatively prime to n.

1.5 de Bruijn Digraphs, de Bruijn Undirected Graphs and Generalized de Bruijn Digraphs

Graphs are widely used in the design and analysis of parallel computer network systems. A vertex in the graph denotes a node (or processor) in the corresponding network, and an edge represents a communication link between two nodes. We will not discuss the difference between network and graph in this thesis.

The *de Bruijn interconnection network* is modeled by the *de Bruijn digraph*, which is named after N. G. de Bruijn for his work in counting d-ary sequences of maximal period [5]. The de Bruijn digraph was widely studied as a communication network model, and was proposed as a suitable processor interconnection network for VLSI implementation [33].

The de Bruijn digraphs have good properties such as it is regular, eulerian, hamiltonian, and has small diameter, nearly optimal connectivity, simple recursive structure, simple routing algorithm, contains some other useful topologies as its subgraphs (see [40]) and, thus, been thought of as a good candidate for the next generation of parallel system architectures, after the hypercube network [6].

The de Bruijn digraphs $B(d, n)$ $(d \geq 2, n \geq 1)$ is defined as follows. The de Bruijn digraph has vertex set

$V(B(d, n)) = \{x_1 x_2 \cdots , x_n : x_i \in \{0, 1, \cdots , d - 1\}, 1 \leq i \leq n\}$

and a directed edge set $E(B(d, n))$, where $x = x_1 x_2 \cdots x_n, y = y_1 y_2 \cdots y_n \in V(B(d, n))$, $xy \in E(B(d, n))$ if and only if $y_i = x_{i+1}$ for $i = 1, 2, \cdots n - 1$. Figure 1.8 is the de Bruijn digraphs $B(2, 3)$.

The de Bruijn undirected graph, denoted by $UB(d, n)$, is an undirected graph obtained from $B(d, n)$ by deleting the orientation of all edges and omitting multiple edges.

However, one of the disadvantage of $B(d, n)$ is the restriction on the number of vertices [12]. From $B(d, n)$ to $B(d + 1, n)$, the number of vertices will increase from d^n to d^{n+1}. As d or n increased, the gap between d^n and d^{n+1} becomes larger and larger, which also

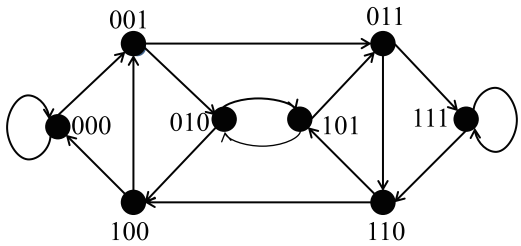


Figure 1.8: A de Bruijn digraph $B(2,3)$

poses the problem of smooth expansion. Therefore, this increases the difficulty for its applications.

In 1981, Imase and Itoh [17] propsed a generalization of de Bruijn digraphs to include any number of vertices. Reddy, Pradhadn and Kuhl [32] also proposed the same graph independently in 1980. We use $G_B(d,n)$ to denote the generalized de Bruijn graphs.

For $n \geq d \geq 2$, the generalized de Bruijn digraph $G_B(d,n)$ is defined by congruence equations as follows: $V(G_B(d,n)) = \{0,1,2,\cdots,n-1\}$ and $A(G_B(d,n)) = \{(x,y)|y \equiv dx + i \pmod{n}, 0 \leq i < d\}$. Figure 1.9 shows the generalized de Bruijn digraph $G_B(2,7)$. Clearly, if $n = d^D$, $G_B(d,n)$ is the de Bruijn digraph $B(d,D)$. Figure 1.10 represent the generalized de Bruijn digraph $G_B(2,8)$. Figure 1.8 and Figure 1.10 show that $B(2,3) \cong G_B(2,8)$.

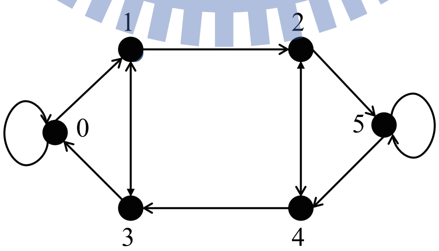


Figure 1.9: A generalized de Bruijn digraph $G_B(2,7)$

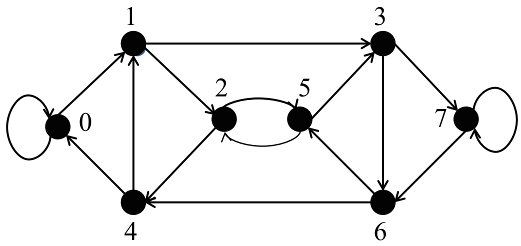


Figure 1.10: A generalized de Bruijn digraph $G_B(2, 8)$

1.6 Kautz Digraphs, Kautz Undirected Graphs and Generalized Kautz Digraphs

In this section, we will introduce the Kautz digraphs and generalized Kautz digraphs. The Kautz digraphs are also an important class of intersection networks first proposed by Kautz in 1969 [21].

Structurally, Kautz networks are very similar to de Bruijn networks, and thus contain as many desirable properties as those of de Bruijn networks (see [40]). Moreover, Kautz networks are an improvement over de Bruijn networks, and have also been thought of as good candidates for the next generation of parallel system architectures, after the hypercube networks [6].

For two given integers $d \geq 2$ and $n \geq 1$, the Kautz digraph $K(d, n)$ is defined as follows. The vertex set of $K(d, n)$ is

$V(K(d, n)) = \{x_1x_2 \cdots x_n : x_i \in \{0, 1, \cdots, d\}, x_i \neq x_{i+1}, 1 \leq i \leq n-1\}$

and the edge set $E(K(d, n))$ consists of all edges from $x_1x_2 \cdots x_n$ to d other vertices $x_2x_3 \cdots x_n\alpha$ where $\alpha \in \{0, 1, \cdots, d\}$ and $\alpha \neq x_n$. Figure 1.11 is a Kautz digraph $K(2, 2)$.

The Kautz undirected graph, denoted by $UK(d, n)$, is an undirected graph obtained from $K(d, n)$ by deleting the orientation of all edges and omitting multiple edges.

Similarly the Kautz digraphs have the same restriction on the number of vertices as de Bruijn digraphs. Imase and Itoh [17, 18] generalized the Kautz digraphs in 1981. The generalization removes the restrict on the cardinality of vertex and retains all of the properties of graphs. Thus, these graphs are also good networks for the next generation

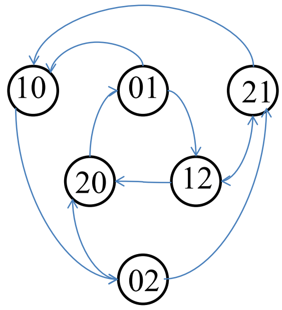


Figure 1.11: A Kautz digraph $K(2, 2)$

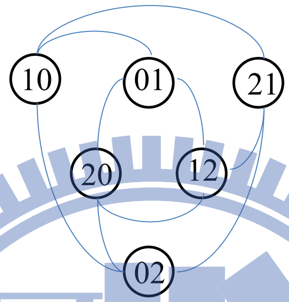


Figure 1.12: A Kautz undirected graph $UK(2, 2)$

of parallel system architecture.

For $n > d \geq 2$, the generalized Kautz digraph $G_K(d, n)$ is defined by congruence equations as follows:

$$\begin{cases} V(G_K(d,n)) = \{0, 1, 2, \cdots, n-1\}; \\ A(G_K(d,n)) = \{(x,y) | y \equiv -dx - i \pmod n, 1 \leq i \leq d\}. \end{cases}$$

In particular, $K(d, D) \cong G_K(d, d^D + d^{D-1})$.

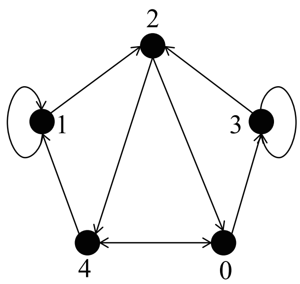


Figure 1.13: A generalized Kautz digraph $G_K(2, 5)$

In this thesis, we study the decycling number on graphs and digraphs, and the thesis is organized as follows. In Chapter 2, we make a survey of all the known results which

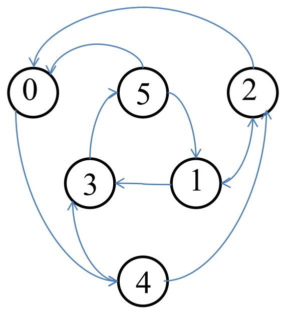


Figure 1.14: A generalized Kautz digraph $G_K(2,6)$

are related to the classes of graphs we focus on. Then, in Chapter 3, we consider the decycling number of outerplanar graphs and grid graph $P_m \Box P_n$. The main results on digraphs will be discussed in Chapter 4. The digraphs we consider are generalized Kautz digraphs and generalized de Bruijn digraphs. Finally, we have a conclusion and a novel idea about total decycling number will be proposed.

Chapter 2

Known Results

From the literatures, there are many studies which focus on determining the decycling number of graphs. In this chapter, we will give an overview of these results.

2.1 In Graphs

In the beginning of this section, we present the general lower bound of graph G.

Lemma 2.1.1. [3] *Let G be a connected graph with p vertices and q edges, and degrees $d_1, d_2, \cdots, d_p$ in non-decreasing order. If $\nabla(G) = s$, then*

$$\sum_{i=1}^{s}(d_i - 1) \geq q - p + 1.$$

As an indication of how this result can be used, we have the following corollary.

Corollary 2.1.2. [3] *If G is a connected graph with p vertices, q edges, and maximum degree d, then*

$$\nabla(G) \geq \frac{q - p + 1}{d - 1}.$$

In the following, we present the results about outerplanar graphs. Bau et al. [2] found formulas of decycling number for maximal outerplanar graphs.

Theorem 2.1.3. [2] *If G is a maximal outerplanar graph of order n,*

then

$$1 \leq \nabla(G) \leq \lfloor \frac{n}{3} \rfloor.$$

In 2002, Fertin, Godard and Raspaud [15] proved the same result by the acyclic coloring argument. They proved the following lemma.

Lemma 2.1.4. [15] *Let $G = (V, E)$ be a graph of order $|V| = N$. If $a(G) \leq k$, then $\nabla(G) \leq \frac{k-2}{k}N$, where $a(G)$ is the acyclic chromatic number of G.*

Lemma 2.1.4 combined with the following theorem in [35] can also get Theorem 2.1.3.

Theorem 2.1.5. [35] *For any outerplanar graph G, $a(G) \leq 3$.*

Similarly, Bordin[4] given the acyclic chromatic number of planar graph.

Theorem 2.1.6. [4] *Every planar graph is acyclically 5-colorable.*

Lemma 2.1.4 combined with Theorem 2.1.6 can obtain the following theorem.

Theorem 2.1.7. [15] *For any planar graph G of order N, $\nabla(G) \leq \frac{3}{5}N$.*

For hypercube, Beineke [3] and Pike [30] gave the results as follows.

Theorem 2.1.8. [3]

(1) $\nabla(Q_3) = 3$.

(2) $\nabla(Q_4) = 6$.

(3) $\nabla(Q_5) = 14$.

(4) $\nabla(Q_6) = 28$.

(5) $\nabla(Q_7) = 56$.

(6) $\nabla(Q_8) = 112$.

Theorem 2.1.9. [30] $\nabla(Q_n) \leq 2^{n-1} - A(n, 4)$ *where $A(n, 4)$ denotes the size of maximum binary code of length n with minimum Hamming distance 4.*

Theorem 2.1.10. [30] $\nabla(Q_n) = 2^{n-1} - A(n, 4)$ *if and only if there exists a minimum decycling set S in Q_n that is also an independent set.*

For circular graphs, Wei et al. [38] provided the following theorems.

Theorem 2.1.11. [38] $\lceil \frac{n+1}{3} \rceil \leq \nabla(C(n,l)) \leq \frac{n}{2}$.

Theorem 2.1.12. [38] $\nabla(C(n,l)) = \lceil \frac{l+1}{2} \rceil$ *where* $l \geq 2$ *and* $n = 2l$.

Theorem 2.1.13. [38]

$$\nabla(C(n,2)) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1 & \textit{if } n \equiv 2 \pmod 6, \\ \lceil \frac{n+1}{3} \rceil & \textit{otherwise} \end{cases}$$

where $n \geq 5$.

$$\nabla(C(n,3)) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1 & \textit{if } n = 3k+2 \textit{ and } k \textit{ is odd,} \\ \lceil \frac{n+1}{3} \rceil & \textit{otherwise} \end{cases}$$

where $n \geq 7$.

$$\nabla(C(n,4)) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1 & \textit{if } n = 3k+2 \textit{ and } k \textit{ is positive integer,} \\ \lceil \frac{n+1}{3} \rceil & \textit{otherwise} \end{cases}$$

where $n \geq 9$.

Theorem 2.1.14. [38] *Suppose* $n = 3k$, $l = 3m - 1$ *and* $(k, m) = 1$ *or* 2 *where* $k \geq 3m$. *Then* $\nabla(C(n,l)) = k + 1 = \lceil \frac{n+1}{3} \rceil$.

Luccio [26] proved the lower and upper bounds of decycling number in both grids and butterflies in 1998.

Theorem 2.1.15. [26] *If* $m, n \geq 2$, *then*

$$\lceil \frac{(m-1)(n-1)+1}{3} \rceil \leq \nabla(P_m \square P_n) \leq \lfloor \frac{mn}{3} + \frac{m+n}{6} + o(m,n) \rfloor.$$

Theorem 2.1.16. [26] *For* k*-dimensional butterfly,*

$$2^{k-1} \lfloor \frac{k+1}{2} \rfloor \leq \nabla(B_k) \leq \lfloor \frac{(k+\frac{1}{3})2^k + \frac{1}{3}}{3} \rfloor.$$

Secondly, Caragiannis, Kaklamanis and Kanellopoulos improved the bounds.

Theorem 2.1.17. [8] $\nabla(P_m \square P_n) \leq \lfloor \frac{mn}{3} - \frac{m+n-5}{6} \rfloor$

Theorem 2.1.18. [8] *For k-dimensional butterfly,*

$$\lceil\frac{(k-1)2^k+1}{3}\rceil \le \nabla(B_k) \le \lfloor\frac{(k+\frac{1}{2})2^k}{3}\rfloor.$$

Subsequently. Chang et al. [9] both improved Luccio's analysis of decycling number in butterflies and exhibited an algorithm which constructed a decycling set in B_k.

Theorem 2.1.19. [9] *For k-dimensional butterfly B_k,*

$$\nabla(B_k) \le \lfloor\frac{(3k+1)2^k+1}{9}\rfloor - \frac{2^k-1}{3}$$

if k is even. Otherwise,

$$\nabla(B_k) \le \lfloor\frac{(3k+1)2^k+1}{9}\rfloor - \frac{2^k-2^{\lceil\frac{k}{2}\rceil}-2^{\lfloor\frac{k}{2}\rfloor+1}}{3}.$$

Finally, Madelaine and Stewart [27] construct new decycling sets in grids so that for certain number of pairs (m,n), the size of decycling set in the grid $P_m \Box P_n$ matches the best lower bound $\left\lceil\frac{(m-1)(n-1)+1}{3}\right\rceil$, and for all other pairs the size of decycling set is at most this lower bound plus 2. We use Table 2.1 to represent Madelaine and Stewart's result.

Theorem 2.1.20. [27]

m \ n	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

Table 2.1: Madelaine and Stewart's result

In Table 2.1, A: $\nabla(P_m \Box P_n) = F_{m,n}$*, B:* $\nabla(P_m \Box P_n) \le F_{m,n}+1$*, C:* $\nabla(P_m \Box P_n) \le F_{m,n}+2$ *where* $F_{m,n} = \left\lceil\frac{(m-1)(n-1)+1}{3}\right\rceil$.

Pike and Zou [31] determined the decycling number of $C_m \Box C_n$ for all m and n. And they also yield a maximum induced tree in $C_m \Box C_n$.

Theorem 2.1.21. [31] *Let $m \geq 3$ and $n \geq 3$ be integers. Then*

$$\nabla(C_m \Box C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \text{if } m = 4, \\ \lceil \frac{3m}{2} \rceil & \text{if } n = 4, \\ \lceil \frac{mn+2}{3} \rceil & \text{otherwise.} \end{cases}$$

Královič et al.[24] determined the decycling number in certain graphs, such as de Bruijn undirected graphs $UB(2, n)$ and Kautz undirected graphs $UK(2, n)$.

Theorem 2.1.22. [24] $\nabla(UB(2, n)) = \lceil \frac{1}{3}(2^n - 2) \rceil$.

Theorem 2.1.23. [24] $\nabla(UK(2, n)) = 2^{n-1}$.

The following theorems show the upper and lower bounds of $UB(d, n)$ and $UK(d, n)$.

Theorem 2.1.24. [44] *For any $d \geq 3$ and $n \geq 1$,*

$$\left\lceil \frac{d^{n+1} - d - \frac{d(d-1)}{2} - d^n + 1}{2d - 1} \right\rceil \leq \nabla(UB(d, n)) \leq d^n\left(1 - \left(\frac{d}{d+1}\right)^{d-1}\right) + \binom{n+d-2}{d-2}.$$

Theorem 2.1.25. [45] *For $d \geq 2$ and $n \geq 3$, the following holds:*

$$\left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d - 1} \right\rceil \leq \nabla(UK(d, n)) \leq d^n - \left(\left\lfloor \frac{d^2}{4} \right\rfloor + 1\right)d^{n-2}.$$

In the following, we present the results on digraphs.

Theorem 2.1.26. [43] *For $d \geq 2$ and $n \geq 1$,*

$$\nabla(B(d, n)) = \begin{cases} \frac{1}{n} \sum\limits_{i|n} d^i \varphi(\frac{n}{i}) & \text{for } 2 \leq n \leq 4, \\ \frac{d^n}{n} + O(nd^{n-4}) & \text{for } n \geq 5, \end{cases}$$

where $i|n$ means i divides n, and $\varphi(i)$ is the Euler totient function.

Theorem 2.1.27. [41] *For $d \geq 2$ and $n \geq 1$,*

$$\nabla(K(d, n)) = \begin{cases} d & \text{for } n = 1, \\ \frac{(\varphi \odot \theta)(n)}{n} + \frac{(\varphi \odot \theta)(n-1)}{n-1} & \text{for } 2 \leq n \leq 7, \\ \frac{d^n}{n} + \frac{d^{n-1}}{n-1} + O(nd^{n-4}) & \text{for } n \geq 8, \end{cases}$$

where $(\varphi \odot \theta)(n) = \sum_{i|n} \varphi(i\theta(n/i))$, $\theta(i) = d^i + (-1)^i d$, $\varphi(1) = 1$ and $\varphi(i) = i\prod_{j=1}^{r}(1 - 1/p_j)$ for $i \geq 2$ and $p_1, p_2, \cdots, p_r$ are the distinct prime factors of i, not equal to 1.

Theorem 2.1.28. [42] *For $d \geq 2$ and $n \geq 1$,*

$$\nabla(G_B(d,n)) \leq \begin{cases} 4 + 7m + \lfloor \frac{5t+3}{8} \rfloor + \lfloor \frac{3t}{4} \rfloor - \lfloor \frac{5t}{8} \rfloor - \lfloor \frac{t+1}{2} \rfloor, \\ \quad m = \lfloor \frac{n}{32} \rfloor, n \equiv t \pmod{32}, \ \textit{for } d = 2. \\ 4 + 6m + \lfloor \frac{7t}{9} \rfloor - \lfloor \frac{t+1}{3} \rfloor, \\ \quad m = \lfloor \frac{n}{18} \rfloor, n \equiv t \pmod{18}, \ \textit{for } d = 3. \\ 2 + t + d(2d-3)m + \lceil \frac{(d+1)t+t-1}{d^2} \rceil - \lceil \frac{t}{d} \rceil, \\ \quad m = \lfloor \frac{n}{d(2d+3)} \rfloor, n \equiv t \pmod{d(2d+3)}, \ \textit{for } d \geq 4. \end{cases}$$

2.2 Relation with Cycle Packing Number

Review that the cycle packing number of a graph G, $\nu(G)$, is the maximum number of vertex-disjoint cycles in G. Therefore, $\nabla(G) \geq \nu(G)$ for every graph G. Dirac and Gallai wondered if there is any inverse relation between $\nabla(G)$ and $\nu(G)$. Define $\nabla(k) = max\{\nabla(G) | \nu(G) = k\}$. Bollobás [7] proved that $\nabla(1) = 3$ and the complete graph of five vertices shows that this bound is sharp. Later, Voss [36] showed that $\nabla(2) = 6$ and $9 \leq \nabla(3) \leq 12$.

Erdös and Pósa [13] proved the following.

Theorem 2.2.1. [13] *There are absolute constants c_1 and c_2 such that*

$$c_1 k \log k \leq \nabla(k) \leq c_2 k \log k.$$

Kloks, Lee and Liu [23] in 2002 conjectured following.

Conjecture 2.2.2. [23] For every planar graph G, $\nabla(G) \leq 2\nu(G)$.

And they also proved the following theorems by greedy algorithm.

Theorem 2.2.3. [23] *Let G be an outerplanar graph. Then $\nabla(G) \leq 2\nu(G)$.*

Theorem 2.2.4. [23] *Let G be a planar graph. Then $\nabla(G) \leq 5\nu(G)$.*

Subsequently, Chen, Fu and Shih [11] improved this bound for planar graphs by discharging method. First, they give a lemma.

Lemma 2.2.5. [11] *Every 2-edge-connected triangle-free planar graph G with minimum degree at least three has either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices.*

Then, they use the following algorithm to prove the main result. The algorithm starts with an empty set $\mathcal{F}$ and goes step by step as follows.

A. Remove all vertices and edges not lying on any cycle. Notice that the resulting graph will be 2-edge-connected. Once no vertex exists, then the process stops and outputs $\mathcal{F}$.

B. Repeatedly remove from the resulting graph 2-vertices (vertices of degree 2) that have nonadjacent neighbors and connect an edge between these two neighbors. Go to the next step.

C. If there is a C_3, then take these three vertices into $\mathcal{F}$ and remove them from the remaining graph, and go back step A. Otherwise, do the next step.

D. Remark that the process enters this step only when all vertices are of degree at least 3 and no C_3 exists. By Lemma 2.2.5, there must be either a C_4 containing a 3-vertex or a C_5 containing at least four 3-vertices. In the former case, take the three vertices other than the 3-vertex into $\mathcal{F}$ and remove them, then go back step A. In the later case, there must be at least two 3-vertices that are nonadjacent in the C_5. Take the other three vertices into $\mathcal{F}$ and remove them, then go back step A.

Theorem 2.2.6. [11] *For every planar graph G, $\nabla(G) \leq 3\nu(G)$.*

Chapter 3

Decycling Number of Graphs

3.1 Outerplanar Graphs

As mentioned in Chapter 2, Theorem 2.2.3, Kloks, Lee and Liu [23] proved that $\nabla(G) \leq 2\nu(G)$ for every outerplanar graph G. Since $\nu(G) \leq \nabla(G)$, it is nature to determine when these bounds are in fact equalities.

An outerplanar graph G is called *lower-extremal* if $\nabla(G) = \nu(G)$ and *upper-extremal* if $\nabla(G) = 2\nu(G)$. In this section, we provide a necessary and sufficient condition for an outerplanar graph being upper-extremal. On the other hand, we provide a sufficient condition for an outerplanar graph being lower-extremal. We find a class $\mathcal{S}$ of outerplanar graphs none of which is lower-extremal and show that if G has no subdivision of S for all $S \in \mathcal{S}$, then G is lower-extremal.

We start by presenting an upper-extremal graph with simplest structure.

Definition 3.1.1. S_k is a graph with vertex set $V = \{0, 1, \cdots, 2k-1\}$ and edge set $E = \{i(i+1) : 0 \leq i \leq 2k-1\} \cup \{i(i+2) : i \text{ is even}\}$ (the indices are under modulo $2k$).

Then $\nabla(S_k) = \lceil \frac{k}{2} \rceil$ and $\nu(S_k) = \lfloor \frac{k}{2} \rfloor$. S_3 is clearly an upper-extremal graph; indeed, its subdivisions are the only 2-edge-connected outerplanar graphs that are upper-extremal and have cycle packing number one. We define the *simplified graph* of a graph G to be the graph obtained from G by continuously deleting vertices of degree one until there is no more degree one vertex and denote it by $\lfloor G \rfloor$.

Let $F(G)$ denote the outer face of an outerplanar G. An edge uv is called a *basic edge* of G if uv and some u, v-path on the boundary of $F(G)$ form the boundary of a face of G. Then, we have

Lemma 3.1.2. *For an outerplanar graph G with $\nu(G) = 1$, G is upper-extremal if and only if $\lfloor G \rfloor$ is an S_3-subdivision.*

Proof. It suffices to prove the necessity. If $\lfloor G \rfloor$ has a cut-vertex v, then v belongs to two blocks of $\lfloor G \rfloor$, say G_1 and G_2, and $\lfloor G \rfloor - v$ has a cycle which is vertex-disjoint with G_1 or G_2. Then $\lfloor G \rfloor$ has two vertex-disjoint cycles, a contradiction. Thus $\lfloor G \rfloor$ is 2-connected. Any two basic edges of $\lfloor G \rfloor$ have a common vertex; otherwise, we can find two vertex-disjoint cycles. This implies that $\lfloor G \rfloor$ has at most three basic edges. Then $\lfloor G \rfloor$ has exactly three basic edges; otherwise we can decycle it by deleting one vertex. Hence it is an S_3-subdivision. ∎

To characterize the upper-extremal graphs, we first define a class of special upper-extremal graphs – S_3-trees. A graph is an *S_3-tree of order t* if it has exactly t vertex-disjoint S_3-subdivisions and every edge not on these S_3-subdivisions belongs to no cycle (see Figure 3.1 for an example).

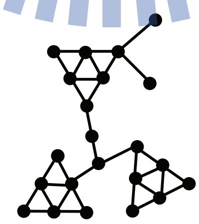


Figure 3.1: An S_3-tree G of order 3, where $\nabla(G) = 6 = 2\nu(G)$.

It is easy to verify that any S_3-tree of order t has exactly t vertex-disjoint cycles, and to decycle an S_3-tree, we have to delete two vertices from each S_3-subdivision. Hence, all S_3-trees are upper-extremal. We will show that there is no other upper-extremal outerplanar graph.

Lemma 3.1.3. *An outerplanar graph G comprised of a connected S_3-tree H of order t and two internally disjoint $v, V(H)$-paths has $t+1$ vertex-disjoint cycles for $v \notin V(H)$.*

Proof. Suppose that $v_1, v_2 \in V(H)$ are the endpoints of these two $v, V(H)$-paths. Let C be the cycle comprised of these two $v, V(H)$-paths and the v_1, v_2-path in H such that C is the boundary of some face of G. Then the intersection (vertex and edge) of C and any S_3-subdivision S in H is either an edge on the boundary of the outer face of S or a vertex of S; otherwise, there would be a subdivision of $K_{2,3}$ or K_4, a contradiction. Hence, we can easily find a cycle in every S_3-subdivision that is vertex-disjoint with C. ∎

Theorem 3.1.4. *An outerplanar graph G is upper-extremal if and only if G is an S_3-tree.*

Proof. It suffices to consider the necessity. We prove it by induction on $\nu(G)$. The statement is clearly true for G if $\nu(G) = 0$. Let G be an upper-extremal graph. Then we can find a maximal induced path P with some endpoints u and v such that uv is an edge of G ($u \neq v$ since G is upper-extremal). Then $G \setminus \{u, v\}$ must be upper-extremal and $\nu(G \setminus \{u, v\}) \leq \nu(G) - 1$. Thus we can assume that $G \setminus \{u, v\}$ is an S_3-tree of order t. Then $\nu(G) \geq t + 1$. Since $\nabla(G) \leq 2t + 2$ and G is upper-extremal, $\nu(G) = t + 1$ and thus $\nabla(G) = 2t + 2$.

Define $G^* := \lfloor G \setminus \{x : x \text{ is on some cycle of } G \setminus \{u, v\}\ \}\rfloor$. Then $\nu(G^*) = 1$. If $\nabla(G^*) = 2$, then by Lemma 3.1.2 G^* is an S_3-tree of order one. This implies that G contains $t+1$ vertex-disjoint S_3-subdivisions. By Lemma 3.1.3, there exists at most one path between any two S_3-subdivisions and thus G is an S_3-tree. Now, we consider w.l.o.g. that $G^* - u$ is acyclic. Let $V^* := V(G^*)$. Then G is a graph comprised of G^*, $\lfloor G \setminus V^* \rfloor$, and some internally disjoint $V^*, V(\lfloor G \setminus V^* \rfloor)$-paths. Notice that there is at most one w, V^*-path if $w \in V(\lfloor G \setminus V^* \rfloor)$ is not on any S_3-subdivision. We classify the vertices in $V^* \setminus V(P)$ into two disjoint sets A and B where A is the union of the vertex sets of components of $G^* - u$ except the one containing v. Let V' be the vertex set of a component of $\lfloor G \setminus V^* \rfloor$. Then each component of $G[A]$ has at most one path to V' and there is at most one B, V'-path; otherwise, by Lemma 3.1.3 $\nu(G) \geq t+2$ (see Figure 3.2 (a)), a contradiction. We consider the following cases.

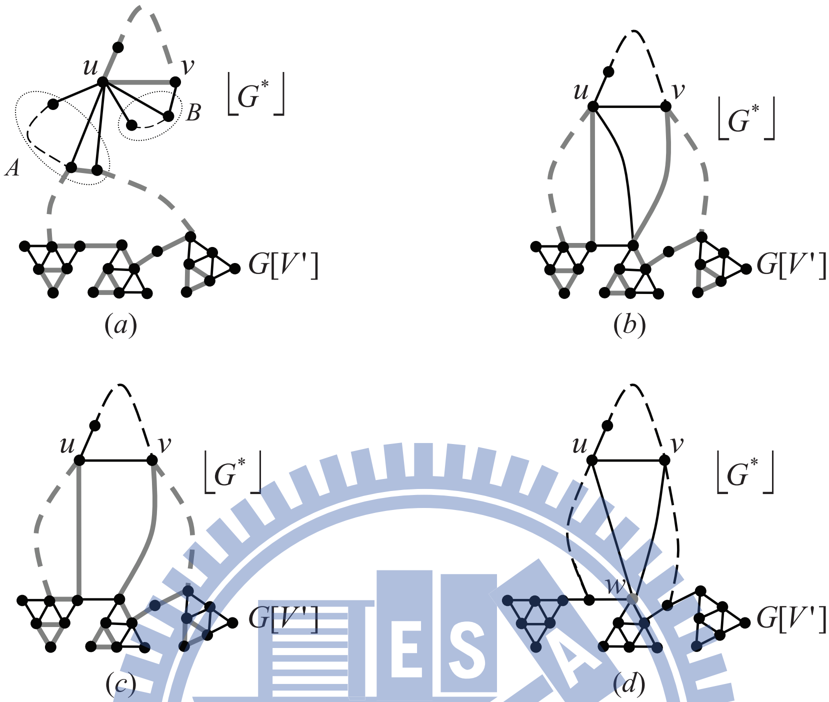


Figure 3.2: Gray edges form some vertex-disjoint cycles.

Case 1: *G^* has a cycle containing u but not v.* Then there is at most one v, V'-path; otherwise, $\nu(G) \geq t+2$. For the remaining case we have to deal with is that there is exactly one B, V'-path and one u, V'-path. Let x, y be the endpoints of these two paths in V'. Then at least one of x and y is on an S_3-subdivision in $G[V']$ and thus we can decycle G by deleting u and a minimum decycling set of $G \setminus \{u, v\}$ including it, contradicting the fact that $\nabla(G) = 2t+2$.

Case 2: *Every cycle of G^* contains both u and v.* Then $G^* - v$ is also acyclic. Suppose that $V_u \subseteq V'$ is the set of vertices as the endpoints of some u,V'-paths and $V_v \subseteq V'$ is the set of vertices as the endpoints of some $B \cup \{v\}$,V'-paths. If $\min(|V_u|, |V_v|) \geq 2$ and $\max(|V_u|, |V_v|) \geq 3$, then by Lemma 3.1.3 $\nu(G) \geq t+2$ (see Figure 3.1.3 (b) for an example), a contradiction. Thus $|V_u| = 2 = |V_v|$ or $|V_u| = 1$ or $|V_v| = 1$. If $|V_u| = 1$ (or $|V_v| = 1$), and therefore G can be decycled by deleting v (or u) and a minimum decycling set of $G \setminus \{u, v\}$, contradicting that $\nabla(G) = 2t + 2$. It remains to consider that $|V_u| = 2 = |V_v|$. If $V_u \cap V_v = \emptyset$, then $\nu(G) \geq t + 2$ (see Figure 3.2 (c) for an example), a contradiction. Suppose that $V_u \cap V_v = \{w\}$. Then w must be on some S_3-subdivision. Therefore, we can decycle G by deleting u and a minimum decycling set of $G \setminus \{u, v\}$ with w included (see Figure 3.2 (d) for an example), again a contradiction. ∎

To prove that a property is sufficient for a graph being lower-extremal, we will use induction. In order to facilitate the proof of the induction step, we need a hereditary graph property. A graph property is called *monotone* if it is closed under removal of vertices. We provide the following general result that is applicable to all graphs.

Lemma 3.1.5. *Suppose that a 2-connected graph is lower-extremal provided that it satisfies a monotone property $\mathcal{P}$. Then G is lower-extremal if G satisfies $\mathcal{P}$.*

Proof. We prove the statement by induction on $|G|$. The statement is true for graphs with $\nu(G) = 0$ or $|V(G)| = 1$. For a graph G of connectivity one, let G_1 be a leaf block of G and v be the cut-vertex of G in $V(G_1)$. Let $G_2 = G \setminus V(G_1 - v)$. Then $\nu(G)$ is either $\nu(G_1) + \nu(G_2)$ or $\nu(G_1) + \nu(G_2) - 1$, and $\nabla(G) \leq \nabla(G_1) + \nabla(G_2)$. Thus suppose to the contrary that $\nabla(G) > \nu(G)$. Then $\nu(G) = \nu(G_1) + \nu(G_2) - 1$ and $\nabla(G) = \nabla(G_1) + \nabla(G_2)$. The first equality shows that every maximum set of vertex-disjoint cycles of G_i must contain a cycle with v for $i = 1, 2$, and thus $\nu(G_i - v) < \nu(G_i)$ for $i = 1, 2$. The second equality shows that v does not belong to any minimum decycling set of G^* where $G^* = G_1$ or G_2 and thus $\nabla(G^* - v) = \nabla(G^*)$. Thus by the monotonicity of $\mathcal{P}$ and the induction hypothesis, $\nu(G^* - v) = \nabla(G^* - v) = \nabla(G^*) = \nu(G^*)$, a contradiction. ∎

To introduce a sufficient condition for a graph being lower-extremal, we first classify all edges of an outerplanar graph. For a 2-connected outerplanar graph G, let $E_0(G)$ and $E_1(G)$ be the set of edges on the boundary of $F(G)$ and the set of basic edges of G, respectively. For $k \geq 2$, define $E_k(G)$ to be the set of basic edges of $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$. For an edge $uv \in E_k(G)$, we use $C(uv)$ to denote a cycle generated by uv and a u, v-path on the boundary of $F(G)$ such that the cycle is the boundary of a face of $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$. We also call it a *basic cycle* of the graph $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$ generated from edge uv.

Lemma 3.1.6. *If G is a 2-connected outerplanar graph with no S_k-subdivision for all odd number k, then G is lower-extremal.*

Proof. We prove the statement by induction on $|E(G)|$. It is easy to verify that the statement is true for graphs with at most three edges. It suffices to prove that there exists a 2-connected subgraph G' of G that has fewer number of edges and no S_k-subdivision for all odd number k and satisfies $\nabla(G) \leq \nabla(G')$ (then $\nabla(G) \leq \nabla(G') = \nu(G') \leq \nu(G)$).

The statement is clearly true for G with $|E_2(G)| = 0$. Suppose $|E_2(G)| \geq 1$ (and thus $|E_1(G)| \geq 1$). Take an edge $e = xy \in E_2(G)$ and a basic cycle $C(e)$ of $G \setminus E_1(G)$. Let $E \subseteq E_1(G)$ be the set of edges with both endpoints on $C(e)$. We consider the following cases.

Case 1: *E induces an x,y-path of G, say $xv_1v_2\cdots v_ty$.* Here, t must be even since G contains an S_{t+2}-subdivision. Let D be a minimum decycling set of $G - e$. If D contains x or y, then D is also a decycling set of G and thus $\nabla(G) \leq \nabla(G - e)$. Suppose $x, y \notin D$. W.l.o.g., we can assume that $D \cap C(e)$ contains only vertices of degree larger than two. Then $|D \cap C(e)| \geq (t + 2)/2$. Let $D' = (D \setminus C(e)) \cup \{x, v_2, v_4, \cdots, v_t\}$. Then D' is a decycling set of G of size at most $\nabla(G - e)$. Thus, $\nabla(G) \leq \nabla(G - e)$.

Case 2: *E generates a maximal path that contains none of x and y, say $v_1v_2\cdots v_t$.* We let G' to denote $G \setminus V(C(e) - x - y)$ if $E = \{v_iv_{i+1} : i = 1, \cdots, t - 1\}$ and $G \setminus \{v_iv_{i+1} : i = 1, \cdots, t - 1\}$ otherwise. Then G' is clearly 2-connected. Thus we have $\nabla(G) \leq \nabla(G') + \lfloor \frac{t}{2} \rfloor = \nu(G') + \lfloor \frac{t}{2} \rfloor = \nu(G)$.

Case 3: *E induces at most two components which are paths as $xv_1v_2\cdots v_t$ and $yu_1u_2\cdots u_{t'}$.* Suppose t (or t') is odd. Let D be a minimum decycling set of $G-e$. Similar to the argument in Case 1, suppose that $x, y \notin D$. Then $|D \cap \{v_1, v_2, \cdots, v_t\}| \geq (t+1)/2$ and thus $(D \setminus \{v_1, v_2, \cdots, v_t\}) \cup \{x, v_2, v_4, \cdots, v_{t-1}\}$ is a decycling set of G. Hence $\nabla(G) \leq \nabla(G-e)$. It remains to consider that t and t' are even. Let $G' = G \setminus (V(C(e)-x-y))$ and

D be a minimum decycling set of G'. Then $D \cup \{v_1, v_3, \cdots, v_{t-1}\} \cup \{u_1, u_3, \cdots, u_{t'-1}\}$ is a decycling set of G of size $\nabla(G') + (t+t')/2$. Since $G[V(C(e))]$ has $(t+t')/2$ vertex-disjoint cycles that do not contain x and y, $\nabla(G) \leq \nabla(G') + (t+t')/2 = \nu(G') + (t+t')/2 \leq \nu(G)$. This concludes the proof. ∎

The property of being without S_k-subdivision is monotone. Therefore, by Lemma 3.1.5 and Lemma 3.1.6, we have

Theorem 3.1.7. *For an outerplanar graph G, if G has no S_k-subdivision for all odd number k, then G is lower-extremal.*

We remark here that the results obtained in this section have been included in a joint work with Chang and Fu [10].

3.2 $P_m \square P_n$

Reviewing that the decycling number of the grid $P_m \square P_n$ shown by Luccio is at most $\left\lfloor \frac{mn}{3} + \frac{m+n}{6} + o(m,n) \right\rfloor$ and at least $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$ [26]. Subsequently, in [8], Caragiannis, Kaklamanis and Kanellopoulos improved the upper bound. They showed that the decycling number of the grid $P_m \square P_n$ is at most $\left\lfloor \frac{mn}{3} - \frac{m+n-5}{6} \right\rfloor$. Finally, Madelaine and Stewart [27] construct new decycling sets in grids so that for certain number of pairs (m,n), the size of decycling set in the grid $P_m \square P_n$ matches the best lower bound $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$, and for all other pairs the size of decycling set is at most this lower bound plus 2.

In this section, we further improve both the lower and upper bounds of $\nabla(P_m \square P_n)$ for several classes of (m,n) such that for more (m,n) the decycling number of $P_m \square P_n$ matches the lower bound and for all others it differs from the known lower bound by at most 1.

Theorem 2.1.15 showed $\nabla(P_m \square P_n) \geq \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

For convenience, we use $F_{m,n}$ and $f_{m,n}$ to denote $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$ and $\frac{(m-1)(n-1)+1}{3}$ respectively. The following proposition is implicit in the proof of Theorem 2.1.15.

Proposition 3.2.1. If $m \geq 5$ and $f_{m,n}$ is an integer, then each decycling set S of size $f_{m,n}$ satisfies the following two properties:

(1) S contains exactly one vertex of degree 3 and contains no vertex of degree 2; and

(2) S induces a subgraph of $P_m \Box P_n$ with no edges.

Now, we have a result on the lower bound of $\nabla(P_m \Box P_n)$.

Theorem 3.2.2. *If $m \geq 5$, mn is even and $f_{m,n}$ is an integer, then $\nabla(P_m \Box P_n) \geq f_{m,n} + 1 = F_{m,n} + 1$.*

Proof. Suppose not. Assume that $\nabla(P_m \Box P_n) = f_{m,n} = F_{m,n}$ and S is a decycling set with size $f_{m,n}$. By Proposition 3.2.1, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$. Since S is a decycling set and induces no edges in $P_m \Box P_n$, $v_{m-1,2} \in S$ and $v_{m-1,3} \notin S$. For otherwise, we have a 4-cycle $(v_{m-1,1}, v_{m-1,2}, v_{m,2}, v_{m,1})$ or $v_{m-1,2}, v_{m-1,3}$ is an edge in $(P_m \Box P_n)[S]$. Following this observation, we conclude that S contains $v_{m-1,2}, v_{m-1,4}, \cdots, v_{m-1,n-1}$ since S has no other vertices on the boundary of $P_m \Box P_n$. Hence, $n-1$ is even and n is odd. Similarly, $v_{m-3,n-1}, v_{m-5,n-1}, \cdots, v_{2,n-1}$ are contained in S and therefore, m is also odd. This contradicts to the assumption and we have the proof. ∎

Corollary 3.2.3. *For $m \geq 5$, if $m \equiv 0$ (mod 6) and $n \equiv 2$ (mod 3) or $(m,n) \equiv (3,2)$ (mod 6), $\nabla(P_m \Box P_n) \geq F_{m,n} + 1$.*

Proof. By direct checking, $f_{m,n}$ is an integer and $m \cdot n$ is even. ∎

Using this fact, we can estimate $\nabla(P_m \Box P_n)$ for more pairs (m,n) by using the Theorem 2.1.20 which was obtained by Madelaine and Stewart.

Now, combining Theorem 2.1.20 with Corollary 3.2.3, we have

Theorem 3.2.4. *For $m \geq 5$, if $(m,n) \equiv (0,2), (0,5), (3,2), (2,0), (5,0), (2,3)$ (mod 6), then $\nabla(P_m \Box P_n) = F_{m,n} + 1$.*

In what follows, we prove that for cases in class "C" mentioned in Table 2.1 $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$ for $m \geq 6$. Before we go any further, we need to introduce a couple of new notations. We shall use $P_m \Box P_r \mid P_m \Box P_k$ to represent that $P_m \Box P_{r+k-1}$ can be separated into $P_m \Box P_r$ and $P_m \Box P_k$ with a common vertical path P_m (see Figure 3.3(a)). Similarly, we use $\frac{P_r \Box P_n}{P_k \Box P_n}$ to represent that $P_{r+k-1} \Box P_n$ can be separated into $P_r \Box P_n$ and $P_k \Box P_n$ and they overlap a horizontal path P_n (see Figure 3.3(b) for an example).

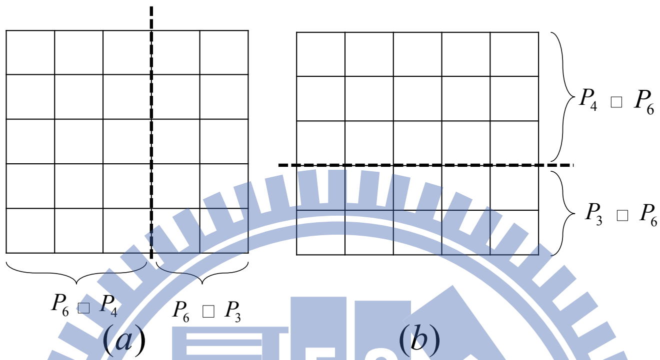


Figure 3.3: (a)$P_6 \Box P_6 = P_6 \Box P_4 \mid P_6 \Box P_3$; (b)$P_6 \Box P_6 = \frac{P_4 \Box P_6}{P_3 \Box P_6}$

In order to prove the main theorem, we need the following three smaller cases.

Lemma 3.2.5. *For* $(m, n) = \{(6, 6), (6, 8), (8, 8)\}$, $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$.

Proof. Beineke and Vandell [3] have already proved the first two cases. By direct checking, the third one is also true. For clearness, we include a decycling set of $P_8 \Box P_8$ in Figure 3.4. ∎

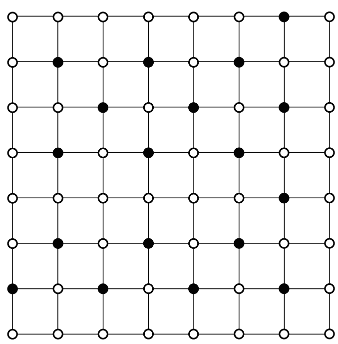


Figure 3.4: A decycling set of $P_8 \Box P_8$.

Lemma 3.2.6. [3] *If G and H are homeomorphic graphs, then $\nabla(G) = \nabla(H)$.*

Theorem 3.2.7. *For $m, n \geq 6$, $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$.*

Proof. By Theorem 2.1.20, Lemma 3.2.5 and the symmetry of the graph, it suffices to consider the following 2 cases.

Case 1. $m \equiv 5 \pmod 6$ and $n \equiv 5 \pmod 6$.

Let $X_{6k+5,6r+5} = \{v_{i,j} : i \text{ and } j \text{ are even}, 1 \leq i \leq 6k+5, 1 \leq j \leq 6r+5\}$. Then $P_{6k+5} \Box P_{6r+5} \setminus X_{6k+5,6r+5}$ is homeomorphic to the graph $P_{3k+3} \Box P_{3r+3}$. By Lemma 3.2.6, for $k, r \geq 0, \nabla(P_{6k+5} \Box P_{6r+5}) \leq (3k+2)(3r+2) + \lceil \frac{(3k+2)(3r+2)+1}{3} \rceil + 1 = F_{6k+5,6r+5} + 1$.

Case 2. $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$.

First, we can find a decycling set of $P_9 \Box P_{11}$ directly. (See Figure 3.5, $\nabla(P_9 \Box P_{11}) \leq 28 = F_{9,11} + 1$.) Then, we partition this case into 3 subcases and apply the case $m \equiv 1 \pmod 3$ in [27] to solve the following.

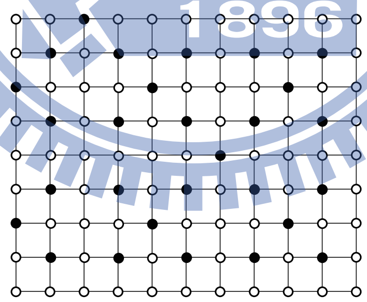


Figure 3.5: Decycling set (black vertices) of $P_9 \Box P_{11}$.

Subcase 2.1. $m = 9$ and $n \equiv 5 \pmod 6$.

Separate $P_9 \Box P_{6k+5}$ into $P_9 \Box P_{6(k-1)+1} \mid P_9 \Box P_{11}$. We can find a set of vertices $X_{9,6(k-1)+1}$ in $P_9 \Box P_{6(k-1)+1}$ by using Madelaine and Stewart's method [27].

Define $X_{9,6(k-1)+1}$

$$
\begin{aligned}
&= \{v_{i,j} : 5 \le i \le 7, i \text{ is odd}, 3 \le j \le 6(k-1)+1, j \equiv 3,5 \pmod 6\} \\
&\bigcup \{v_{i,j} : 5 \le i \le 8, i \text{ is even}, 2 \le j \le 6(k-1), j \equiv 0,2 \pmod 6\} \\
&\bigcup \{v_{5,j} : 2 \le j \le 6(k-1)+1, j \equiv 1 \pmod 6\} \\
&\bigcup \{v_{8,j} : 2 \le j \le 6(k-1)+1, j \equiv 4 \pmod 6\} \\
&\bigcup \{v_{2,j} : 2 \le j \le 6(k-1), j \text{ is even}\} \\
&\bigcup \{v_{3,j} : 3 \le j \le 6(k-1)+1, j \text{ is odd}\} \\
&\bigcup \{v_{4,2}\}.
\end{aligned}
$$

And we find $X_{9,11}$ in $P_9 \square P_{11}$ by letting $X_{9,11}$

$$
\begin{aligned}
&= \{v_{i,j} : 2 \le i \le 8, i \text{ is even}, 6(k-1)+1 \le j \le 6k+5, j \text{ is even}\} \\
&\bigcup \{v_{3,j}, v_{7,j} : j = 6(k-1)+1, 6(k-1)+5, 6k+3\} \\
&\bigcup \{v_{1,6(k-1)+3}, v_{5,6k+1}\}.
\end{aligned}
$$

Define $X_{9,6k+5} = X_{9,6(k-1)+1} \cup X_{9,11}$. The set $X_{9,17}$ is shown in Figure 3.6.

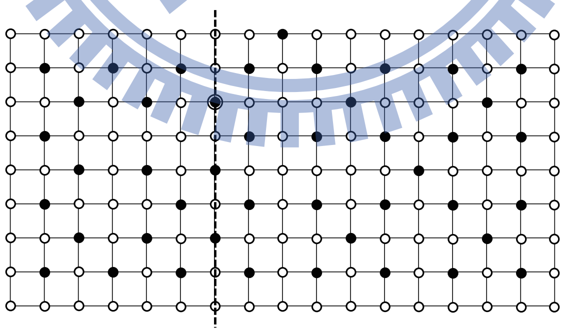


Figure 3.6: Decycling set of $P_9 \square P_{17}$.

We claim that $X_{9,6k+5}$ is a decycling set. Observe that if there is a cycle in $P_9 \square P_{6k+5} \setminus X_{9,6k+5}$, then the cycle must use the perimeter vertices of $P_9 \square P_{6(k-1)+1}$ excluding $\{v_{i,6k-5} : 3 \le 7\}$ and a $(v_{2,6k-5}, v_{8,6k-5})$ -path in $P_9 \square P_{11} \setminus X_{9,11}$. However, there is no $(v_{2,6k-5}, v_{8,6k-5})$ -path in $P_9 \square P_{11} \setminus X_{9,11}$. Hence, $X_{9,6k+5}$ is a decycling set of $P_9 \square P_{6k+5}$. Since $v_{3,6(k-1)+1}$ belongs to both

$X_{9,6(k-1)+1}$ and $X_{9,11}$, the size of $X_{9,6k+5}$ is

$$\left\lceil \frac{8\cdot 6(k-1)+1}{3} \right\rceil + 28 - 1 = \left\lceil \frac{8(6k+4)+1}{3} \right\rceil + 1.$$

Subcase 2.2. $m \equiv 3 \pmod 6$ and $n = 11$.

Similar to **Subcase 2.1**, we let $P_{6k+3} \square P_{11} = \frac{P_{6(k-1)+1} \square P_{11}}{P_9 \square P_{11}}$ and let $X_{6(k-1)+1,11}$

$$\begin{aligned}
&= \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 0,2 \pmod 6, 2 \le j \le 7, j \text{ is even}\} \\
&\bigcup \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 3,5 \pmod 6, 2 \le j \le 7, j \text{ is odd}\} \\
&\bigcup \{v_{i,7} : 2 \le i \le 6(k-1)+1, i \equiv 1 \pmod 6\} \\
&\bigcup \{v_{i,2} : 2 \le i \le 6(k-1)+1, i \equiv 4 \pmod 6\} \\
&\bigcup \{v_{i,10} : 1 \le i \le 6(k-1), i \text{ is even}\} \\
&\bigcup \{v_{i,9} : 3 \le i \le 6(k-1)+1, i \text{ is odd}\} \\
&\bigcup \{v_{2,8}\}.
\end{aligned}$$

We use a different construction to find $X_{9,11}$ in $P_9 \square P_{11}$, where $X_{9,11} = \{v_{i,j} : 6(k-1)+1 \le i \le 6k+3, i \text{ is even}, 1 \le j \le 11, j \text{ is even}\} \bigcup \{v_{6k-5,9}, v_{6k-3,3}, v_{6k-3,5}, v_{6k-1,1}, v_{6k-1,9}, v_{6k+1,3}, v_{6k+1,7}, v_{6k+3,9}\}$.

Define $X_{6k+3,11} = X_{6(k-1)+1,11} \cup X_{9,11}$. The construction of $X_{15,11}$ can be visualized as in Figure 3.7. The argument is similar to Subcase 3.1 which yields that $X_{6k+3,11}$ is a decycling set of $P_{6k+3} \square P_{11}$. Since $v_{6(k-1)+1,9}$ belongs to both $X_{6(k-1)+1,11}$ and $X_{9,11}$, the size of $X_{6k+3,11}$ is

$$\left\lceil \frac{6(k-1)10+1}{3} \right\rceil + 28 - 1 = \left\lceil \frac{(6k+2)10+1}{3} \right\rceil + 1.$$

Subcase 2.3. $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$ and $m > 9, n > 11$.

Let $P_{6k+3} \square P_{6r+5}$ be $\frac{P_{6(k-1)+1} \square P_{6r+5}}{P_9 \square P_{6(r-1)+1} | P_9 \square P_{11}}$. We note that the labeling of each vertex in the following is the same as the labeling used in the original grid. Now, define

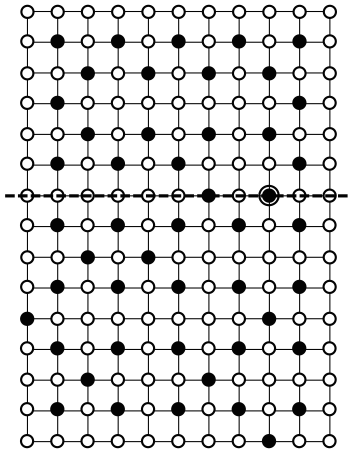


Figure 3.7: Decycling set of $P_{15}\Box P_{11}$.

$X_{6(k-1)+1,6r+5}$ in $P_{6(k-1)+1}\Box P_{6r+5}$ as

$$
\begin{aligned}
& \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 0,2 \pmod 6, 2 \le j \le 6r+1, j \text{ even}\} \\
\bigcup\ & \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 3,5 \pmod 6, 2 \le j \le 6r+1, j \text{ odd}\} \\
\bigcup\ & \{v_{i,6r+1} : 2 \le i \le 6(k-1)+1, i \equiv 1 \pmod 6\} \\
\bigcup\ & \{v_{i,2} : 2 \le i \le 6(k-1)+1, i \equiv 4 \pmod 6\} \\
\bigcup\ & \{v_{i,6r+4} : 1 \le i \le 6(k-1), i \text{ even}\} \\
\bigcup\ & \{v_{i,6r+3} : 3 \le i \le 6(k-1)+1, i \text{ odd}\} \\
\bigcup\ & \{v_{2,6r+2}\}.
\end{aligned}
$$

Define $X_{9,6(r-1)+1}$ in $P_9\Box P_{6(r-1)+1}$ as following. $X_{9,6(r-1)+1}$

$$
\begin{aligned}
= \ & \{v_{i,j} : 6k-1 \le i \le 6k+1, i \text{ odd}, 3 \le j \le 6r-5, j \equiv 3,5 \pmod 6\} \\
\bigcup\ & \{v_{i,j} : 6k-1 \le i \le 6k+2, i \text{ even}, 2 \le j \le 6r-6, j \equiv 0,2 \pmod 6\} \\
\bigcup\ & \{v_{6(k-1)+5,j} : 2 \le j \le 6(r-1)+1, j \equiv 1 \pmod 6\} \\
\bigcup\ & \{v_{6k+2,j} : 2 \le j \le 6(r-1)+1, j \equiv 4 \pmod 6\} \\
\bigcup\ & \{v_{6(k-1)+2,j} : 2 \le j \le 6(r-1), j \text{ even}\} \\
\bigcup\ & \{v_{6(k-1)+3,j} : 3 \le j \le 6(r-1)+1, j \text{ odd}\} \\
\bigcup\ & \{v_{6(k-1)+4,2}\}.
\end{aligned}
$$

Define $X_{9,11}$ in $P_9 \square P_{11}$ as the following Figure 3.8, the size of $X_{9,11}$ is 30.

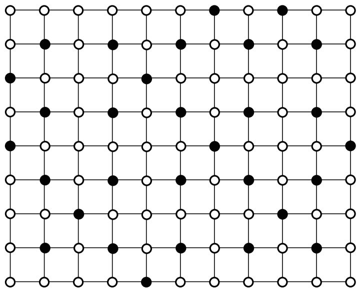


Figure 3.8: Decycling set of $P_9 \square P_{11}$ (Different from Figure 3.5).

Define $X_{6k+3,6r+5} = X_{6(k-1)+1,6r+5} \cup X_{9,6(r-1)+1} \cup X_{9,11}$. The construction is illustrated for $P_{15} \square P_{17}$ in Figure 3.9.

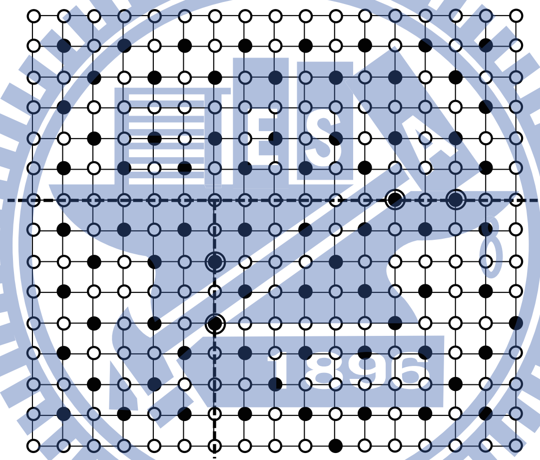


Figure 3.9: Decycling set of $P_{15} \square P_{17}$.

We claim that $X_{6k+3,6r+5}$ is a decycling set. Observe that if there is a cycle in $P_{6k+3} \square P_{6r+5} \setminus X_{6k+3,6r+5}$ then the cycle must use the perimeter vertices of $P_{6(k-1)+1} \square P_{6r+5}$ excluding $\{v_{6(k-1)+1,6r+j} : j = 1, 2, 3\}$ and a $(v_{6(k-1)+1,6r}, v_{6(k-1)+1,6r+4})$ -path in $(P_9 \square P_{6(r-1)+1} \mid P_9 \square P_{11}) \setminus (X_{9,6(r-1)+1} \cup X_{9,11})$. By directly checking, there is no path from the right boundary of $P_9 \square P_{11}$ to the left boundary of $P_9 \square P_{11}$. There is no $(v_{6(k-1)+1,6r}, v_{6(k-1)+1,6r+4})$ -path in $(P_9 \square P_{6(r-1)+1} \mid P_9 \square P_{11}) \setminus (X_{9,6(r-1)+1} \cup X_{9,11})$. Hence $X_{6k+3,6r+5}$ is a decycling set of $P_{6k+3} \square P_{6r+5}$. Since $v_{6(k-1)+1,6r+1}, v_{6(k-1)+1,6r+3} \in X_{9,11} \cap X_{6(k-1)+1,6r+5}$ and $v_{6(k-1)+3,6(r-1)+1}, v_{6(k-1)+5,6(r-1)+1} \in X_{9,11} \cap X_{9,6(r-1)+1}$, the size of $X_{6k+3,6r+5}$

is $\left\lceil \frac{6(k-1)(6r+4)+1}{3} \right\rceil + \left\lceil \frac{8 \cdot 6(r-1)+1}{3} \right\rceil + 30 - 4 = \left\lceil \frac{(6k+2)(6r+4)+1}{3} \right\rceil + 1.$

We complete the proof. ∎

We use Table 3.1 to represent the improvement of Madelaine and Stewart's results.

m \ n	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

: increasing the lower bound in this paper

: decreasing the upper bound in this paper

Table 3.1: Improvement of known results [27]

In Table 3.1, A: $\nabla(P_m \Box P_n) = F_{m,n}$, B: $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$, C: $\nabla(P_m \Box P_n) \leq F_{m,n} + 2$.

Again, we remark that the results obtained in this section have been included in a joint work with Fu and Shih which is to appear in Discrete Math., Alg. and Appl. [25].

Chapter 4

Decycling Number of Digraphs

In this chapter, we study $\nabla(G_K(d,n))$ and $\nabla(G_B(d,n))$ for $n \geq d \geq 2$.

4.1 Generalized Kautz Digraphs

First, we presents a systematic approach of finding a decycling set in a digraph. It is the key idea in the following.

Lemma 4.1.1. *Let S be a set of vertices in a digraph G. Then S is a decycling set of G if and only if we can find a sequence of subsets of $V(G)$, $S = S_0, S_1, \cdots, S_t = V(G)$ such that*

(1) $S_i \subseteq S_{i+1}$; and

(2) $N^+(S_{i+1} \setminus S_i) \subseteq S_i$ for $i = 0, 1, \cdots, t-1$.

Proof. First, we prove the necessity. Since S is a decycling set, $G - S$ is acyclic. Thus, there exists at least one vertex v that $d^+_{G-S}(v) = 0$. Now, we can partition $V(G \setminus S)$ into $V_1, V_2, \cdots, V_t$ by the following construction. For convenience, we denote $G_0 = G[V(G)\setminus S]$. Define $V_i = \{v \in V(G_{i-1}) | d^+_{G_{i-1}}(v) = 0\}$ where $G_i = G[V(G) \setminus (S \cup \cup_{j=1}^{i-1})(V_j)]$ for $i = 1, 2, \cdots, t$. Let $S_0 = S$, $S_1 = S_0 \cup V_1$, $S_i = S_{i-1} \cup V_i$ for $i = 1, 2, \cdots, t$. It can be easily checked that $S_i \subseteq S_{i+1}$ and $N^+(S_{i+1} \setminus S_i) \subseteq S_i$ for $i = 0, 1, \cdots, t-1$.

Subsequently, we consider the sufficiency. Suppose not. Assume that there exists a directed cycle $C = (x_0, x_1, \cdots, x_k)$ in $G - S$. Since $S_t = V(G)$, $x_i \in S_j \setminus S_{j-1}$ for $i \in \{0, 1, \cdots, k\}$ and $j \in \{1, 2, \cdots, t\}$. ($x_i \notin S_0$ for all i, otherwise C does not exist.) Let

$m = \min\{j | x_i \in S_j \setminus S_{j-1}, i = 0, 1, \cdots, k\}$ and $x_l \in S_m \setminus S_{m-1}$. Since $x_{l+1} \in N^+(x_l)$, $x_{l+1} \in S_{m-1}$ by *(2)*. This contradicts the assumption that m is minimum. Therefore, there is no directed cycle in G. We complete the proof. ∎

Now, we are ready to deal with $\nabla(G_K(d,n))$. Recall that the generalized Kautz digraph $G_K(d,n)$ is defined as follows:

$$\begin{cases} V(G_K(d,n)) = \{0,1,2,\cdots,n-1\}; \\ A(G_K(d,n)) = \{(x,y) | y \equiv -dx - i \pmod{n}, 1 \le i \le d\}. \end{cases}$$

By definition, for each $\alpha \in V(G_K(d,n))$, the set of out-neighbors of α in $V(G_K(d,n))$ is $\{-d\alpha - i \pmod{n}, 1 \le i \le d\}$, denoted by $N^+(\alpha)$. Subsequently, for $S \subseteq V(G_K(d,n))$, we let $N^+(S) = \bigcup_{v \in S} N^+(v)$. Then, it is easy to check $N^+([a,b]) = \{-db-d, -db-d+1, \cdots, -da-1\} \pmod{n}$. For example, if $d = 3$ and $n = 10$, then $N^+(\{2\}) = \{-9,-8,-7\} = [1,3]$ and $N^+([1,4]) = \{-15,-14,\cdots,-4\} = [0,9]$.

Now we consider the decycling set of $G_K(d,n)$ for $n \ge d \ge 2$.

Theorem 4.1.2. *Let $n = (d+1)m + t$, where $0 \le t \le d$ and $S_0 = \bigcup_{i=1}^{d} A_i$ where*

$$A_1 = [0, m];$$

$$A_2 = [\lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor, 2m+1]; \textit{and}$$

$$A_i = [\lfloor \frac{(i-1)n}{d} \rfloor + \lfloor \frac{(i-1)n}{d^3} \rfloor, im + (i-1)], \textit{ for } i = 3, 4, \cdots d.$$

Then S_0 is a decycling set of $G_K(d,n)$.

Proof. It suffices to construct a sequence satisfying the conditions in Lemma 4.1.1.

Step 1. Let $S_1 = S_0 \cup W_1 \cup X_1 \cup Y_1$, where $W_1 = [m+1, \lfloor \frac{n}{d} \rfloor - 1]$, $X_1 = [n - \lfloor \frac{n}{d^2} \rfloor, n-1]$ and $Y_1 = [\lfloor \frac{n}{d} \rfloor, \lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor - 1]$.

It's routine to check

$N^+(W_1) = [n - d\lfloor \frac{n}{d} \rfloor, m - (d-t) - 1] \subseteq S_0$,

$N^+(X_1) = [0, d\lfloor \frac{n}{d^2} \rfloor - 1] \subseteq S_0 \cup W_1$ and

$N^+(Y_1) = [2n - d\lfloor \frac{n}{d} \rfloor - d\lfloor \frac{n}{d^3} \rfloor, 2n - d\lfloor \frac{n}{d} \rfloor - 1] \subseteq S_0 \cup W_1 \cup X_1$.

Now, we have $S_1 = [0, 2m+1] \cup [n - \lfloor \frac{n}{d^2} \rfloor, n-1] \cup \bigcup_{i=3}^{d} A_i$.

Step 2. Now, we add more vertices to S_1.

Let $S_2 = S_1 \cup W_2 \cup X_2 \cup Y_2$, where $W_2 = [2m+2, \lfloor \frac{2n}{d} \rfloor - 1]$, $X_2 = [n - \lfloor \frac{2n}{d^2} \rfloor, n - \lfloor \frac{n}{d^2} \rfloor - 1]$ and $Y_2 = [\lfloor \frac{2n}{d} \rfloor, \lfloor \frac{2n}{d} \rfloor + \lfloor \frac{2n}{d^3} \rfloor - 1]$.

It's easy to check

$N^+(W_2) = [2n - d\lfloor \frac{2n}{d} \rfloor, 2m - 2(d-t) - 1] \subseteq S_1$,

$N^+(X_2) = [d\lfloor \frac{n}{d^2} \rfloor, d\lfloor \frac{2n}{d^2} \rfloor - 1] \subseteq S_1 \cup W_2$ and

$N^+(Y_2) = [3n - d\lfloor \frac{2n}{d} \rfloor - d\lfloor \frac{2n}{d^3} \rfloor, 3n - d\lfloor \frac{2n}{d} \rfloor - 1] \subseteq S_1 \cup W_2 \cup X_2$.

After this step, $S_2 = [0, 3m+2] \cup [n - \lfloor \frac{2n}{d^2} \rfloor, n-1] \cup \bigcup_{i=4}^{d} A_i$.

Step k. For $d \geq k \geq 3$, let $S_k = S_{k-1} \cup W_k \cup X_k \cup Y_k$, where $S_{k-1} = [0, km + (k-1)] \cup [n - \lfloor \frac{(k-1)n}{d^2} \rfloor, n-1] \cup \bigcup_{i=k+1}^{d} A_i$, $W_k = [km+k, \lfloor \frac{kn}{d} \rfloor - 1]$, $X_k = [n - \lfloor \frac{kn}{d^2} \rfloor, n - \lfloor \frac{(k-1)n}{d^2} \rfloor - 1]$ and $Y_k = [\lfloor \frac{kn}{d} \rfloor, \lfloor \frac{kn}{d} \rfloor + \lfloor \frac{kn}{d^3} \rfloor - 1]$.

We can check that

$N^+(W_k) = [kn - d\lfloor \frac{kn}{d} \rfloor, km - k(d-t) - 1] \subseteq S_{k-1}$,

$N^+(X_k) = [d\lfloor \frac{(k-1)n}{d^2} \rfloor, d\lfloor \frac{kn}{d^2} \rfloor - 1] \subseteq S_{k-1} \cup W_k$ and

$N^+(Y_k) = [(k+1)n - d\lfloor \frac{kn}{d} \rfloor - d\lfloor \frac{kn}{d^3} \rfloor, (k+1)n - d\lfloor \frac{kn}{d} \rfloor - 1] \subseteq S_{k-1} \cup W_k \cup X_k$.

$S_k = [0, (k+1)m + k] \cup [n - \lfloor \frac{kn}{d^2} \rfloor, n-1] \cup \bigcup_{i=k+2}^{d} A_i$.

This concludes the proof.

■

Corollary 4.1.3. *Let $d \geq 2$ and $n \equiv t \pmod{d+1}$. Then $\nabla(G_K(d,n)) \leq (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d-t+5) - 2$.*

Proof. Let $n = (d+1)m + t$. Then by Theorem 4.1.2,

$$
\begin{aligned}
\nabla(G_K(d,n)) &\le \sum_{i=1}^{d} |A_i| \\
&= (m+1) + [(2m+1) - (\lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor) + 1] + \cdots \\
&+ [(km + (k-1)) - (\lfloor \frac{(k-1)n}{d} \rfloor + \lfloor \frac{(k-1)n}{d^3} \rfloor) + 1] + \cdots \\
&+ [(dm + (d-1)) - (\lfloor \frac{(d-1)n}{d} \rfloor + \lfloor \frac{(d-1)n}{d^3} \rfloor) + 1].
\end{aligned}
$$

By the facts, $\frac{kn}{d} - 1 \le \lfloor \frac{kn}{d} \rfloor \le \frac{kn}{d}$ and $\frac{kn}{d^3} - 1 \le \lfloor \frac{kn}{d^3} \rfloor \le \frac{kn}{d^3}$, we conclude that $\nabla(G_K(d,n)) \le (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d - t + 5) - 2$. ∎

When d is smaller we can get a better bound by refining the decycling set.

Theorem 4.1.4. *Let $n = 36m + t$, where $0 \le t \le 35$ and $S_0 = \bigcup_{i=1}^{3} A_i$ where*

$$A_1 = [28m - 1, n - 1];$$

$$A_2 = [24m, 24m + t - 1]; \textit{ and}$$

$$A_3 = [12m, 12m + t].$$

Then S_0 is a decycling set of $G_K(2, n)$.

Proof. It suffices to construct a sequence satisfying the conditions in Lemma 4.1.1.

(1) Let $S_1 = W_1 \cup S_0$, where $W_1 = [0, 4m - 1]$.

$N^+(W_1) = [28m + t, n - 1] \subseteq S_0$.

Now, we have $S_1 = [0, 4m - 1] \cup A_1 \cup A_2 \cup A_3$.

(2) Let $S_2 = W_2 \cup S_1$, where $W_2 = [16m + t, 22m - 1]$.

$N^+(W_2) = [28m + 2t, n - 1] \cup [0, 4m - t - 1] \subseteq S_1$.

Now, $S_2 = [0, 4m - 1] \cup [16m + t, 22m - 1] \cup A_1 \cup A_2 \cup A_3$.

(3) Let $S_3 = W_3 \cup S_2$, where $W_3 = [7m + t, 10m - 1]$.

$N^+(W_3) = [16m + t, 22m - t - 1] \subseteq S_2$.

Therefore, $S_3 = [0, 4m - 1] \cup [7m + t, 10m - 1] \cup [16m + t, 22m - 1] \cup A_1 \cup A_2 \cup A_3$.

(4) Let $S_4 = W_4 \cup S_3$, where $W_4 = [25m + t, 28m - 1]$.

$N^+(W_4) = [16m + 2t, 22m - 1] \subseteq S_3$.

Hence, $S_4 = [0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 22m-1] \cup [25m+t, n-1] \cup A_2 \cup A_3$.

(5) Let $k = \lceil \log_2 m \rceil$, then $k \geq \log_2 m$, $2^k \geq m$ and let $S_5 = S_{5k} \cup \{24m-1\} \cup \{24m+t\}$, where S_{5k} is defined as follows.

(5-1) Let $S_{51} = S_4 \cup L_{51} \cup R_{51}$, where $L_{51} = [22m, 24m - 2^{k-1} - 1]$ and $R_{51} = [24m + 2^{k-1} + t, 25m + t - 1]$.

$N^+(L_{51}) = [24m + 2^k + 2t, 28m + 2t - 1] \subseteq S_4$, since $24m + 2^k + 2t \geq 25m + t$.

$N^+(R_{51}) = [22m, 24m - 2^k - 1] \subseteq S_4 \cup L_{51}$.

$S_{51} = [0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-2}-1] \cup [24m+2^{k-1}+t, n-1] \cup A_2 \cup A_3$.

(5-2) Let $S_{52} = S_{51} \cup L_{52} \cup R_{52}$, where $L_{52} = [24m - 2^{k-1}, 24m - 2^{k-2} - 1]$ and $R_{52} = [24m + 2^{k-2} + t, 24m + 2^{k-1} + t - 1]$.

$N^+(L_{52}) = [24m + 2^{k-1} + 2t, 24m + 2^k + 2t - 1] \subseteq S_{51}$ and

$N^+(R_{52}) = [24m - 2^k, 24m - 2^{k-1} - 1] \subseteq S_{51} \cup L_{52}$.

$S_{52} = [0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-2}-1] \cup [24m+2^{k-2}+t, n-1] \cup A_2 \cup A_3$.

Continuing in this way, we have $S_{5(i+1)} = S_{5i} \cup L_{5(i+1)} \cup R_{5(i+1)}$ where $L_{5(i+1)} = [24m - 2^{k-i}, 24m - 2^{k-i-1} - 1]$, $R_{5(i+1)} = [24m + 2^{k-i-1} + t, 24m + 2^{k-i} + t - 1]$ for $i = 2, 3, \cdots, k-1$, and $N^+(L_{5(i+1)}) \subseteq S_{5i}$, $N^+(R_{5(i+1)}) \subseteq S_{5i} \cup L_{5(i+1)}$.

Since $N^+(\{24m-1\}) \subseteq S_{5k}$ and $N^+(\{24m+t\}) \subseteq S_{5k} \cup \{24m-1\}$.

We have $S_5 = [0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, n-1] \cup A_3$.

(6) Let $S_6 = W_6 \cup S_5$, where $W_6 = [4m, 7m + t - 1]$.

$N^+(W_6) = [22m - t, 28m + t - 1] \subseteq S_5$.

Hence, $S_6 = [0, 10m-1] \cup [16m+t, n-1] \cup A_3$.

(7) Let $S_7 = W_7 \cup S_6$, where $W_7 = [13m + t, 16m + t - 1]$.

$N^+(W_7) = [4m - t, 10m - t - 1] \subseteq S_6$.

Hence, $S_7 = [0, 10m - 1] \cup [13m + t, n - 1] \cup A_3$.

(8) Let $k = \log_2 m$ and $S_8 = S_{8(k+1)}$, where $S_{8(k+1)}$ is defined as follows.

(8-1) Let $S_{81} = S_7 \cup L_{81} \cup R_{81}$, where $L_{81} = [10m, 12m - 2^k - 1]$ and $R_{81} = [12m + 2^k + t, 13m + t - 1]$.

$N^+(L_{81}) = [12m + 2^{k+1} + t, 16m + t - 1] \subseteq S_7$, since $12m + 2^{k+1} + t \geq 13m + t$.

$N^+(R_{81}) = [10m - t, 12m - 2^{k+1} - t - 1] \subseteq S_7 \cup L_{81}$.

Now, $S_{81} = [0, 12m - 2^k - 1] \cup [12m + 2^k + t, n - 1] \cup A_3$.

(8-2) Let $S_{82} = S_{81} \cup L_{82} \cup R_{82}$, where $L_{82} = [12m - 2^k, 12m - 2^{k-1} - 1]$ and $R_{82} = [12m + 2^{k-1} + t, 12m + 2^k + t - 1]$.

$N^+(L_{82}) = [12m + 2^k + t, 12m + 2^{k+1} + t - 1] \subseteq S_{81}$ and

$N^+(R_{82}) = [12m - 2^{k+1} - t, 12m - 2^k - t - 1] \subseteq S_{81} \cup L_{82}$.

Consequently, $S_{82} = [0, 12m - 2^{k-1} - 1] \cup [12m + 2^{k-1} + t, n - 1] \cup A_3$.

Continuing in this way, we have $S_{8(i+1)} = S_{8i} \cup L_{8(i+1)} \cup R_{8(i+1)}$ where $L_{8(i+1)} = [12m - 2^{k-i+1}, 12m - 2^{k-i} - 1]$, $R_{8(i+1)} = [12m + 2^{k-i} + t, 12m + 2^{k-i+1} + t - 1]$ for $i = 2, 3, \cdots, k-1$, and $N^+(L_{8(i+1)}) \subseteq S_{8i}$, $N^+(R_{8(i+1)}) \subseteq S_{8i} \cup L_{8(i+1)}$.

Finally, we have $S_8 = [0, n - 1]$. This completes the proof.

∎

Corollary 4.1.5. *For $n \geq 2$ and $n \equiv t \pmod{36}$, $\nabla(G_K(2, n)) \leq \frac{2}{9}n + 3t + 1$.*

Proof. Let $n = 36m + t$. Then by Theorem 4.1.4

$$\nabla(G_K(2, n)) \leq \sum_{i=1}^{3} |A_i| = 8m + 3t + 1 \leq \frac{2}{9}n + 3t + 1.$$

∎

Theorem 4.1.6. *Let $n = 36m + t$, where $0 \le t \le 35$ and $S_0 = \bigcup_{i=1}^{4} A_i$ where*

$$A_1 = [0, 6m + t];$$
$$A_2 = [12m + \lceil \frac{t}{3} \rceil, 18m + \lceil \frac{t}{2} \rceil];$$
$$A_3 = [9m - 1, 9m + \lfloor \frac{t}{3} \rfloor]; \text{ and}$$
$$A_4 = [27m, 27m + \lfloor \frac{3t}{4} \rfloor].$$

Then S_0 is a decycling set of $G_K(3, n)$.

Proof. It suffices to construct a sequence satisfying the conditions in Lemma 4.1.1.

(1) Let $S_1 = W_1 \cup S_0$, where $W_1 = [10m, 12m + \lfloor \frac{t}{3} \rfloor - 1]$.

$N^+(W_1) = [t - 3\lfloor \frac{t}{3} \rfloor, 6m + t - 1] \subseteq S_0$.

Now, we have $S_1 = A_1 \cup [10m, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4$.

(2) Let $S_2 = W_2 \cup S_1$, where $W_2 = [6m + t + 1, 8m - 1]$.

$N^+(W_2) = [12m + t, 18m - 2t - 4] \subseteq S_1$.

Now, $S_2 = [0, 8m - 1] \cup [10m, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4$.

(3) Let $k = \lceil \log_3 m + t \rceil$ and $S_3 = S_{3(k-1)}$, where $S_{3(k-1)}$ is defined as follows.

(3-0) Let $S_{30} = S_2 \cup R_{30} \cup L_{30}$, where $R_{30} = [9m + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{t}{3} \rfloor + 1, 10m - 1]$ and $L_{30} = [8m, 8m + \lfloor \frac{2m}{3} \rfloor - 1]]$.

$N^+(R_{30}) = [6m + t, 9m - 3\lfloor \frac{m}{3} \rfloor + t - \lfloor \frac{t}{3} \rfloor - 4] \subseteq S_2$.

$N^+(L_{30}) = [12m - 3\lfloor \frac{2m}{3} \rfloor + t, 12m + t - 1] \subseteq S_2 \cup R_{30}$.

Hence, $S_{30} = [0, 8m + \lfloor \frac{2m}{3} \rfloor - 1] \cup [9m + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{t}{3} \rfloor + 1, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4$.

(3-1) Let $S_{31} = S_{30} \cup R_{31} \cup L_{31}$, where $R_{31} = [9m + 3^{k-2} + \lfloor \frac{t}{3} \rfloor + 1, 9m + 3^{k-1}]$ and $L_{31} = [9m - 3^{k-1} - 1, 9m - 3^{k-2} - 1]$.

$N^+(R_{31}) = [9m - 3^k + t - 3, 9m - 3^{k-1} + t - 3\lfloor \frac{t}{3} \rfloor - 4] \subseteq S_{30}$.

$N^+(L_{31}) = [9m + 3^{k-1} + t, 9m + 3^k + t + 2] \subseteq S_{30} \cup R_{31}$.

$S_{31} = [0, 9m - 3^{k-2} - 1] \cup [9m + 3^{k-2} + \lfloor \frac{t}{3} \rfloor + 1, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4$.

(3-2) Let $S_{32} = S_{31} \cup R_{32} \cup L_{32}$, where $R_{32} = [9m+3^{k-3}+\lfloor \frac{t}{3} \rfloor+1, 9m+3^{k-2}+\lfloor \frac{t}{3} \rfloor]$ and $L_{32} = [9m - 3^{k-2}, 9m - 3^{k-3} - 1]$.

$N^+(R_{32}) = [9m - 3^{k-1} + t - 3\lfloor \frac{t}{3} \rfloor - 3, 9m - 3^{k-2} + t - 3\lfloor \frac{t}{3} \rfloor - 4] \subseteq S_{31}$ and

$N^+(L_{32}) = [9m + 3^{k-2} + t, 9m + 3^{k-1} + t - 1] \subseteq S_{31} \cup R_{32}$.

Then, $S_{32} = [0, 9m - 3^{k-3} - 1] \cup [9m + 3^{k-3} + \lfloor \frac{t}{3} \rfloor + 1, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4$.

Continuing in this way, we have $S_{3i} = S_{3(i-1)} \cup R_{3i} \cup L_{3i}$ where $R_{3i} = [9m+3^{k-i-1}+\lfloor \frac{t}{3} \rfloor+1, 9m+3^{k-i}+\lfloor \frac{t}{3} \rfloor]$, $L_{3i} = [9m-3^{k-i}, 9m-3^{k-i-1}-1]$ for $i = 3, \cdots, k-1$ and $N^+(R_{3i}) \subseteq S_{3(i-1)}$, $N^+(L_{3i}) \subseteq S_{3(i-1)} \cup R_{3i}$.

Now, we have $S_3 = [0, 18m + \lceil \frac{t}{2} \rceil] \cup A_4$.

(4) Let $S_4 = W_4 \cup S_3$, where $W_4 = [18m + \lceil \frac{t}{2} \rceil + 1, 20m]$.

$N^+(W_4) = [12m + 2t - 3, 18m + 2t - 3\lceil \frac{t}{2} \rceil - 4] \subseteq S_3$.

Hence, $S_4 = [0, 20m] \cup A_4$.

(5) Let $S_5 = W_5 \cup S_4$, where $W_5 = [n - \lfloor \frac{20m}{3} \rfloor, n - 1]$.

$N^+(W_5) = [0, 3\lfloor \frac{20m}{3} \rfloor - 1] \subseteq S_4$.

Hence, $S_5 = [0, 20m] \cup A_4 \cup [n - \lfloor \frac{20m}{3} \rfloor, n - 1]$.

(6) Let $S_6 = W_6 \cup S_5$, where $W_6 = [20m + 1, 26m - 1]$.

$N^+(W_6) = [30m + 3t, n - 1] \cup [0, 12m + 2t - 4] \subseteq S_5$.

Hence, $S_6 = [0, 26m - 1] \cup A_4 \cup [n - \lfloor \frac{20m}{3} \rfloor, n - 1]$.

(7) Let $S_7 = W_7 \cup S_6$, where $W_7 = [28m + t, n - \lfloor \frac{20m}{3} \rfloor - 1]$.

$N^+(W_7) = [3\lfloor \frac{20m}{3} \rfloor, 24m - 1] \subseteq S_6$.

Hence, $S_7 = [0, 26m - 1] \cup A_4 \cup [28m + t, n - 1]$.

(8) Let $k = \lceil \log_3 m + t \rceil$ and $S_8 = S_{8(k+1)} \cup \{27m - 1\}$, where $S_{8(k+1)}$ is defined as follows.

Let $S_{81} = S_7 \cup R_{81} \cup L_{81}$, where $R_{81} = [27m + 3^k + t, 27m + 3^{k+1}]$ and $L_{81} = [27m - 3^{k+1} - 1, 27m - 3^k - 1]$.

$N^+(R_{81}) = [27m - 3^{k+2} + 3t - 3, 27m - 3^{k+1} - 1] \subseteq S_7$, since $27m - 3^{k+1} - 1 \leq 26m - 1$.

$N^+(L_{81}) = [27m + 3^{k+1} + 3t, 27m + 3^{k+2} + 3t + 2] \subseteq S_7 \cup R_{81}$.

Now, $S_{81} = [0, 27m - 3^k - 1] \cup [27m + 3^k + t, n - 1] \cup A_4$.

Continuing in this way, we have $S_{8i} = S_{8(i-1)} \cup R_{8i} \cup L_{8i}$ where $R_{8i} = [27m + 3^{k+1-i} + t, 27m + 3^{k+2-i} + t - 1]$, $L_{8i} = [27m - 3^{k+2-i} - 1, 27m - 3^{k+1-i} - 1]$ for $i = 2, 3, \cdots, k+1$, and $N^+(R_{8i}) \subseteq S_{8(i-1)}$, $N^+(L_{8i}) \subseteq S_{8(i-1)} \cup R_{8i}$. It's easy to check $N^+(\{27m - 1\}) = [27m + 3t, 27m + 3t + 2] \subseteq S_{8(k+1)}$.

We have $S_8 = [0, 27m + \lfloor \frac{3t}{4} \rfloor] \cup [27m + t + 1, n - 1]$.

(9) Let $S_9 = W_9 \cup S_8$, where $W_9 = [27m + \lfloor \frac{3t}{4} \rfloor + 1, 27m + t]$.

$N^+(W_9) = [27m - 3, 27m + 3t - 3\lfloor \frac{3t}{4} \rfloor - 4] \subseteq S_8$.

Now, $S_9 = V(G_K(3, n))$. This concludes the proof.

∎

Corollary 4.1.7. *For $n \geq 2$ and $n \equiv t \pmod{36}$, $\nabla(G_K(3, n)) \leq \frac{n}{3} + \frac{9}{4}t + 6$.*

Proof. Let $n = 36m + t$. Then by Theorem 4.1.6,

$$\nabla(G_K(3, n)) \leq \sum_{i=1}^{4} |A_i| = 12m + t + \lceil \frac{t}{2} \rceil + \lfloor \frac{3t}{4} \rfloor + 5 \leq 12m + \frac{9}{4}t + 6 \leq \frac{n}{3} + \frac{9}{4}t + 6.$$

∎

4.2 Generalized de Bruijn Digraphs

In this section, we give an upper bound that improves the best known result. Recall that the the generalized de Bruijn digraph $G_B(d, n)$ is defined by congruence equations as follows: $V(G_B(d, n)) = \{0, 1, 2, \cdots, n-1\}$ and $A(G_B(d, n)) = \{(x, y) | y \equiv dx + i \pmod{n}, 0 \leq i < d\}$. By definition, for each $\alpha \in V(G_B(d, n))$, the set of out-neighbors of α in $V(G_B(d, n))$ is $\{d\alpha + i \pmod{n}, 0 \leq i < d\}$ denoted by $N^+(\alpha)$. That is easy to check

$N^+([a,b]) = \{da, da+1 \cdots, db+(d-1)\} \pmod{n}$. For example, if $d = 3$ and $n = 10$, then $N^+(\{2\}) = \{6,7,8\} = [6,8]$ and $N^+([1,3]) = \{3,4,\cdots,11\} = [0,1] \cup [3,9]$.

Now we consider the decycling set of $G_B(d,n)$ for $n \geq d \geq 2$.

Theorem 4.2.1. *For $n \geq d \geq 2$ and $S_0 = \bigcup_{i=1}^{d} A_i$ where*

$$A_1 = [0, \lfloor \frac{n}{d} \rfloor];$$
$$A_i = [\lfloor \frac{(i-1)n}{d} \rfloor + \lfloor \frac{(i-1)n}{d^2} \rfloor, \lfloor \frac{in}{d} \rfloor], \ \textit{for } i = 2,3,\cdots d-1; \textit{and}$$
$$A_d = [\lfloor \frac{(d-1)n}{d} \rfloor + \lfloor \frac{(d-1)n}{d^2} \rfloor, n-1].$$

Then S_0 is a decycling set of $G_B(d,n)$.

Proof. It suffices to construct a sequence satisfying the conditions in Lemma 4.1.1.

Step 1. Let $S_1 = S_0 \cup W_1$, where $W_1 = [\lfloor \frac{n}{d} \rfloor + 1, \lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^2} \rfloor - 1]$.

It is routine to check $N^+[W_1] = [d\lfloor \frac{n}{d} \rfloor + d, d\lfloor \frac{n}{d} \rfloor + d\lfloor \frac{n}{d^2} \rfloor - 1] \subseteq S_0$.

Now, we have $S_1 = [0, \lfloor \frac{2n}{d} \rfloor] \cup \bigcup_{i=3}^{d} A_i$.

Step 2. Find S_2.

Let $S_2 = S_1 \cup W_2$, where $W_2 = [\lfloor \frac{2n}{d} \rfloor + 1, \lfloor \frac{2n}{d} \rfloor + \lfloor \frac{2n}{d^2} \rfloor - 1]$.

It's easy to check $N^+(W_2) = [d\lfloor \frac{2n}{d} \rfloor + d, d\lfloor \frac{2n}{d} \rfloor + d\lfloor \frac{2n}{d^2} \rfloor - 1] \subseteq S_1$.

After this step, $S_2 = [0, \lfloor \frac{3n}{d} \rfloor] \cup \bigcup_{i=4}^{d} A_i$.

Step k. For $d \geq k \geq 3$, let $S_k = S_{k-1} \cup W_k$, where $S_{k-1} = [0, \lfloor \frac{kn}{d} \rfloor] \cup \bigcup_{i=k+1}^{d} A_i$ and $W_k = [\lfloor \frac{kn}{d} \rfloor + 1, \lfloor \frac{kn}{d} \rfloor + \lfloor \frac{kn}{d^2} \rfloor - 1]$.

We can check $N^+(W_k) = [d\lfloor \frac{kn}{d} \rfloor + d, d\lfloor \frac{kn}{d} \rfloor + d\lfloor \frac{kn}{d^2} \rfloor - 1] \subseteq S_{k-1}$.

$S_k = [0, \lfloor \frac{k+1}{n} \rfloor] \cup \bigcup_{i=k+2}^{d} A_i$.

This concludes the proof.

∎

Proposition 4.2.2. For $G_B(d,n), d \geq 2$, we have $\nabla(G_B(d,n)) \leq (\frac{d+1}{2d})n + 2(d-1)$.

Proof. By Theorem 4.2.1,

$$\begin{aligned}\nabla(G_B(d,n)) \leq \sum_{i=1}^{d} |A_i| = (\lfloor\frac{n}{d}\rfloor + 1) + [\lfloor\frac{2n}{d}\rfloor - (\lfloor\frac{n}{d}\rfloor + \lfloor\frac{n}{d^2}\rfloor) + 1] + \cdots \\ + [\lfloor\frac{kn}{d}\rfloor - (\lfloor\frac{(k-1)n}{d}\rfloor + \lfloor\frac{(k-1)n}{d^2}\rfloor) + 1] + \cdots \\ + [(n-1)) - (\lfloor\frac{(d-1)n}{d}\rfloor + \lfloor\frac{(d-1)n}{d^2}\rfloor) + 1].\end{aligned}$$

By the facts that $\frac{kn}{d^2} - 1 \leq \lfloor\frac{kn}{d^2}\rfloor \leq \frac{kn}{d^2}$, we conclude that $\nabla(G_B(d,n)) \leq (\frac{d+1}{2d})n + 2(d-1)$.■

By considering the order of n with respect to d, the upper bound we obtained asymptotically approaches $\frac{d+1}{2d}n$, which is better (smaller) than $\frac{2d-3}{2d+3}n$ for $d \geq 6$ obtained by Xu et al. [42].

Chapter 5

Conclusion and Remarks

The problem of finding the decycling number has been extensively studied and has been proved to be NP-complete for general graphs, even for elementary graphs. In this thesis, we provide the following results.

First, we provide a necessary and sufficient condition for an outerplanar graphs been upper-extremal, and given a sufficient condition for an outerplanar graph been lower-extremal. We find a class $\mathcal{S}$ of outerplanar graphs none of which is lower-extremal and show that if G has no subdivision of S for all $S \in \mathcal{S}$, then G is lower-extremal.

Second, we improve both the lower and upper bounds of $\nabla(P_m \Box P_n)$ for several classes of (m, n) such that for more (m, n) the decycling number of $P_m \Box P_n$ matches the lower bound and for all others it differs from the known lower bound by at most 1.

Finally, we give a systematic approach of finding a decycling set in a digraph. We give the bound generalized Kautz digraphs. And improve the best known bound of generalized be Bruijn digraphs.

Continuing our work in this thesis, we shall focus on the followings.

Problem 1. For every planar graph G, prove that $\nabla(G) \leq 2\nu(G)$ (Conjecture 2.2.2).

Problem 2. Determine the $\nabla(P_m \Box P_n)$ for unsettled (m, n).

Problem 3. For a directed graph G, find the general lower bound of $\nabla(G)$.

In this thesis, we only consider the decycling number of graphs on unweighted version. The weighted version is looking for a minimum-weight set of vertices so that the remaining graph is acyclic. The problem is known to be NP-complete [20]. In contrast, the problem of finding a minimum-weight set of edges containing at least one edge of any cycle is equivalent to finding a maximum spanning tree, which has been shown solvable in polynomial time. These two problems motivate us to consider a new version of decycling set, namely *total decycling set* of graphs .

Let $G = (V, E)$ be a graph, $w : V(G) \cup E(G) \to \mathbb{R}^+ \cup \{\infty\}$ be a weight function on $V(G) \cup E(G)$. The total decycling set S of G is a subset of $V \cup E$ such that $G - S$ is acyclic. The weight of total decycling set is $\sum_{x \in V \cup E} w(x)$ and a minimum total decycling set of a weighted graph is a total decycling set of G of minimum weight. The minimum weight of a total decycling set of G is the *total decycling number* of G, denoted by $\nabla_T(G)$.

Problem 4. Determine $\nabla_T(G)$ for any weighted graph G.

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