J Supercomput (2009) 48: 66–87
DOI 10.1007/s11227-008-0206-0

The super spanning connectivity and super spanning laceability of the enhanced hypercubes

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Published online: 18 April 2008

Abstract A *k-container* $C(\mathbf{u}, \mathbf{v})$ of a graph G is a set of k disjoint paths between $\mathbf{u}$ and $\mathbf{v}$. A k-container $C(\mathbf{u}, \mathbf{v})$ of G is a k^**-container* if it contains all vertices of G. A graph G is k^**-connected* if there exists a k^*-container between any two distinct vertices of G. Therefore, a graph is 1^*-connected (respectively, 2^*-connected) if and only if it is Hamiltonian connected (respectively, Hamiltonian). A graph G is *super spanning connected* if there exists a k^*-container between any two distinct vertices of G for every k with $1 \leq k \leq \kappa(G)$ where $\kappa(G)$ is the connectivity of G. A bipartite graph G is k^**-laceable* if there exists a k^*-container between any two vertices from different partite set of G. A bipartite graph G is *super spanning laceable* if there exists a k^*-container between any two vertices from different partite set of G for every k with $1 \leq k \leq \kappa(G)$. In this paper, we prove that the enhanced hypercube $Q_{n,m}$ is super spanning laceable if m is an odd integer and super spanning connected if otherwise.

Keywords Folded hypercubes · Enhanced hypercubes · Hamiltonian connected · Hamiltonian laceable · Super spanning connected · Super spanning laceable

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1 Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definitions and notations, we basically follow [3]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v})$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are *adjacent* if $(\mathbf{u}, \mathbf{v}) \in E$. The degree $d_G(\mathbf{u})$ of a vertex $\mathbf{u}$ of G is the number of edges incident with $\mathbf{u}$. A *path* is a sequence of vertices represented by $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$ with no repeated vertex and $(\mathbf{v}_i, \mathbf{v}_{i+1})$ is an edge of G for all $0 \le i \le k-1$. We also write the path $P = \langle \mathbf{v}_0, \ldots, \mathbf{v}_k \rangle$ as $\langle \mathbf{v}_0, \ldots, \mathbf{v}_i, Q, \mathbf{v}_j, \ldots, \mathbf{v}_k \rangle$, where Q is a path from $\mathbf{v}_i$ to $\mathbf{v}_j$. We use P^{-1} to denote the path $\langle \mathbf{v}_k, \mathbf{v}_{k-1}, \ldots, \mathbf{v}_1, \mathbf{v}_0 \rangle$. The *length* of a path P, $l(P)$, is the number of edges in P. A path is a *Hamiltonian path* if it contains all vertices of G. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two distinct vertices of G. A *cycle* is a closed path $\langle v_0, v_1, \ldots, v_k, v_0 \rangle$ where $\langle v_0, v_1, \ldots, v_k \rangle$ is a path with $k \ge 2$. A *Hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's theorem [16] that there are k *internal vertex-disjoint paths* joining any two distinct vertices when $k \le \kappa(G)$. A k*-container* of a graph G between $\mathbf{u}$ and $\mathbf{v}$ is a set of k internal vertex-disjoint paths between u and v. Connectivity and container are impotent concepts to measure the fault tolerant of a networks [5, 9].

In this paper, we are interested in some special type of containers. A k-container of G between $\mathbf{u}$ and $\mathbf{v}$ is a k^**-container* if it contains all vertices of G. A graph G is k^**-connected* if there exists a k^*-container between any two distinct vertices. A 1^*-connected graph except K_1 and K_2 is 2^*-connected. Thus, the concept of k^*-connected graph is a hybrid concept of connectivity and Hamiltonicity. The study of k^*-connected graph is motivated by the globally 3^*-connected graphs proposed by Albert, Aldred, and Holton [2]. A globally 3^*-connected graph is a cubic graph that is w^*-connected for all $1 \le w \le 3$. Recently, Lin et al. [12] proved that the pancake graph P_n is w^*-connected for any w with $1 \le w \le n-1$ if and only if $n \ne 3$. The *spanning connectivity* of a graph G, $\kappa^*(G)$, is the largest integer k such that G is w^*-connected for all $1 \le w \le k$ if G is 1^*-connected graph. There are some interesting results of spanning connectivity [8, 13–15]. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. Obviously, the complete graph K_n is super spanning connected. Lin et al. [12] studied the n-dimensional pancake graph P_n is super spanning connected if and only if $n \ne 3$. Tsai et al. [18] studied the recursive circulant graphs $G(2^m, 4)$ is super-connected if and only if $m \ne 2$.

A graph G is *bipartite* if its vertex set can be partitioned into two subsets V_0 and V_1 such that every edge joins vertices of V_0 and V_1. A bipartite graph is k^**-laceable* graph if there exists a k^*-container between any two vertices from different partite sets. Note that a 1^*-laceable graph is also known as *a Hamiltonian laceable graph*. Moreover, a bipartite graph is 2^*-laceable if and only if it is a Hamiltonian graph and all 1^*-laceable graphs except K_1 and K_2 are 2^*-laceable. A Hamiltonian laceable graph G with partition V_0, V_1 is *hyper-Hamiltonian laceable* if we remove any vertex

$\mathbf{v}$ from a partite set, say V_0, there is a Hamiltonian path of $G - \{\mathbf{v}\}$ joining any two vertices in the other partite set V_1. If G is a 1^*-laceable graph, we define the *spanning laceablility* of a bipartite graph G, $\kappa^*(G)$, to be the largest integer k such that G is w^*-laceable for all $1 \le w \le k$. A bipartite graph G is *super spanning laceable* if $\kappa^*(G) = \kappa(G)$. Recently, Chang et al. [4] proved that the hypercube graph Q_n is super spanning laceable. All bipartite hypercube-like graphs are super spanning laceable [14]. The n-dimensional star graph S_n is super spanning laceable if and only if $n \neq 3$ [12].

Graph containers do exist in engineering designed information and telecommunication networks or in biological and neural systems ([1, 9] and their references). The study of w-container and their w^*-container plays a pivotal role in the design and the implementation of parallel routing and efficient information transmission in a large scale networking systems. In biological informatics and neural informatics, the existence of a w^*-container signifies the effects on the signal transduction system and the reactions in metabolic pathways.

Among all interconnection networks proposed in the literature, the hypercubes Q_n is one of the most popular topologies [10]. Let $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ be an n-bit binary strings. The *hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is defined to be the number of i such that $u_i = 1$. The *n-dimensional hypercube* Q_n consists of all n-bit binary strings as its vertices and two vertices $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ and $\mathbf{v} = v_1v_2\cdots v_{n-1}v_n$ are *adjacent* if and only if $\mathbf{u}$ and $\mathbf{v}$ differ by exactly one bit, i.e., $\sum_{i=1}^{n} |u_i - v_i| = 1$. Obviously, Q_n is a bipartite graph with bipartition $W = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$ and $B = \{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$. For convenience, the vertices in W are referred as *even vertices* and the vertices in B are referred as *odd vertices*.

Some variations of hypercubes structures have been reported in the literature, for instance, the *folded hypercubes* FQ_n by El-Amawy and Latifi [6] and *enhanced hypercubes* $Q_{n,m}$ $(2 \le m \le n)$ by Tzeng NF and Wei S [19]. The folded hypercubes FQ_n is obtained from a hypercubes Q_n with add on edges defined by joining any vertex $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ to $\bar{\mathbf{u}} = \bar{u}_1\bar{u}_2\cdots \bar{u}_{n-1}\bar{u}_n$, where $\bar{u}_i = 1 - u_i$ is the complement of u_i. The enhanced hypercube $Q_{n,m}$ is obtained from a hypercubes Q_n with add on edges defined by joining any vertex $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ to $(\mathbf{u})^c = \bar{u}_1\bar{u}_2\cdots\bar{u}_m u_{m+1}u_{m+2}\cdots u_{n-1}u_n$. Obviously, $FQ_n = Q_{n,n}$ and FQ_n and $Q_{n,m}$ are $(n+1)$-regular. Moreover, FQ_n is a bipartite graph if and only if n is odd and $Q_{n,m}$ is a bipartite graph if and only if m is odd.

The rest of the paper is organized as follows. In the next section, we prove some new spanning properties of the hypercubes Q_n. In Sect. 3, we prove that the folded hypercubes FQ_n is super spanning laceable if n is an odd integer and super spanning connected if otherwise. In Sect. 4, we prove that the enhanced hypercubes $Q_{n,m}$ is super spanning laceable if m is an odd integer and super spanning connected if otherwise. In the final section, we give our concluding remark.

2 The super spanning laceability of hypercubes

In this section, we review some known results and prove a new theorem. Let $\mathbf{u} = u_1u_2\cdots u_n$ be a vertex of Q_n. We use $(\mathbf{u})^k = u_1\cdots u_{k-1}\bar{u}_k u_{k+1}\cdots u_{n-1}u_n$ to denote the k-th neighbor of $\mathbf{u}$ and use $(\mathbf{u})_k$ to denote u_k. We set Q^i_{n-1} be the subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_n = i\}$ for $i = 0, 1$. Obviously, Q^i_{n-1} is isomorphic to Q_{n-1} for $i = 0, 1$. It is well known that Q_n is vertex transitive. Furthermore, the permutation on the coordinates of Q_n and the componentwise complement operations are graph isomorphisms. Readers can refer reference [7, 10] for a survey about the properties of hypercubes. We have the following lemmas:

Lemma 1 [11] *Q_n is hyper-Hamiltonian laceable if and only if $n \geq 2$.*

Lemma 2 [4] *Q_n is super spanning laceable for any positive integer n.*

Chang et al. [4] proved that the following *two paths spanning property* of hypercube.

Lemma 3 [4] *Assume that $n \geq 2$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct even vertices of Q_n and $\mathbf{y}_1$ and $\mathbf{y}_2$ be two distinct odd vertices of Q_n. Then there exist two paths P_1 and P_2 of Q_n such that* (1) *P_i joins $\mathbf{x}_i$ and $\mathbf{y}_i$ for $1 \leq i \leq 2$ and* (2) *$P_1 \cup P_2$ spans Q_n.*

Lemma 4 [17] *$Q_n - \{\mathbf{x}, \mathbf{y}\}$ is Hamiltonian laceable if $\mathbf{x}$ is an even vertex, $\mathbf{y}$ is an odd vertex of Q_n, and $n \geq 4$.*

There is another version of Menger theorem on k-connected graphs, called k-fan version. Let G be a graph. Let x be a vertex in G and $S = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ be a set of k vertices not containing $\mathbf{x}$. An $(\mathbf{x}, S)$*-fan* is a set of disjoint paths $\{P_1, P_2, \ldots, P_k\}$ such that P_i is a path joining $\mathbf{x}$ to $\mathbf{y}_i$ for $1 \leq i \leq k$. The k-fan version Menger's theorems states that there exists an $(\mathbf{x}, S)$-fan of G between any vertex $\mathbf{x}$ and any k set S not containing $\mathbf{x}$ with $1 \leq k \leq \kappa(G)$. With this observation, we define a *spanning fan* is a fan that spans G. The following theorem states that there exists a spanning $(\mathbf{x}, S)$-fan, $\{P_1, P_2, \ldots, P_k\}$, of Q_n between any vertex $\mathbf{x}$ and $S = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ with $\mathbf{y}_k$ being the only vertices in $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ in the partite set not containing $\mathbf{x}$ and $1 \leq k \leq n$. The requirement that $\mathbf{y}_k$ is the only vertex in $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ in the partite set not containing $\mathbf{x}$ is needed just because Q_n is a bipartite graph with the same number of vertices in both partite sets.

Theorem 1 *Assume that $k \leq n$ and $\mathbf{x}$ is a vertex of Q_n. Let $U = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ be a subset of $V(Q_n) - \{\mathbf{x}\}$ with $\mathbf{y}_i \neq \mathbf{y}_j$ for every $i \neq j$ and $\mathbf{y}_k$ is the only vertex in $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ such that $\mathbf{y}_k$ and $\mathbf{x}$ are in different partite set. Then there is a spanning $(\mathbf{x}, U)$-fan of Q_n.*

Proof By Lemma 2, this statement is holds on every Q_n if $k = 1$. Suppose that $k = 2$ and $n \geq 2$. By Lemma 2, there is a Hamiltonian path $P = \langle \mathbf{y}_1, R_1, \mathbf{x}, R_2, \mathbf{y}_2 \rangle$ of Q_n joining $\mathbf{y}_1$ to $\mathbf{y}_2$. We set $P_1 = \langle \mathbf{x}, R_1^{-1}, \mathbf{y}_1 \rangle$ and $P_2 = \langle \mathbf{x}, R_2, \mathbf{y}_2 \rangle$. Then P_1 and P_2 forms the required paths. Thus, we assume that $3 \leq k \leq n$, and this theorem is true for Q_{n-1}. Since Q_n is vertex transitive, we assume that $\mathbf{x} = 0^n$. Thus, $\mathbf{x}$ is an even vertex and $\mathbf{x} \in Q^0_{n-1}$. We have the following cases:

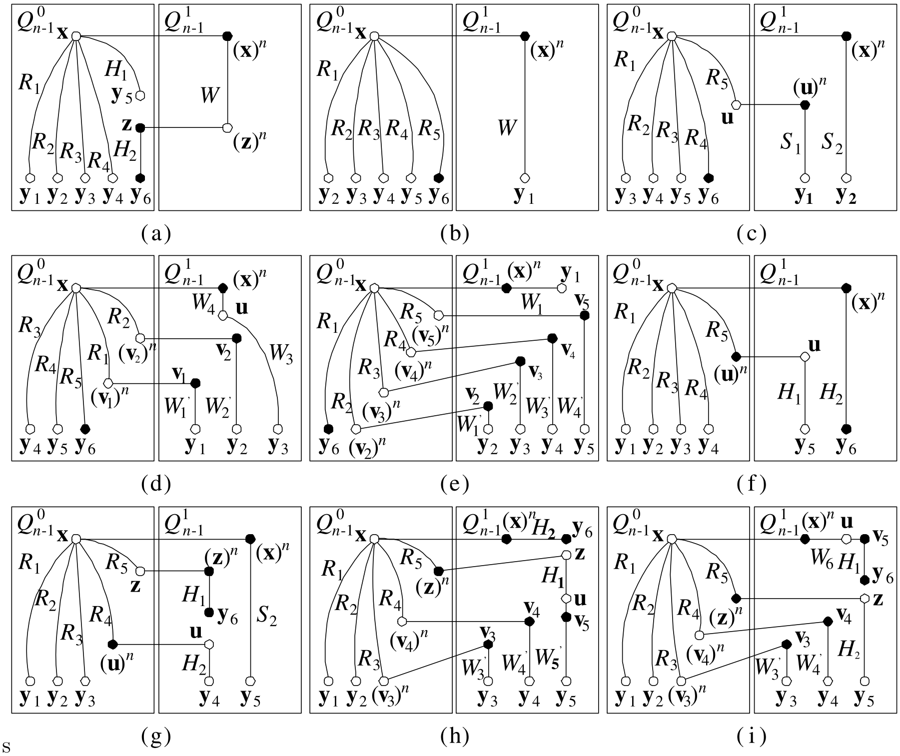


Fig. 1 Illustration for Theorem 1

Case 1: $(\mathbf{y}_k)_i = 0$ for some $1 \le i \le n$. Since Q_n is edge transitive, we assume that $(\mathbf{y}_k)_n = 0$. Thus, $\mathbf{y}_k \in Q^0_{n-1}$. For $0 \le j \le 1$, we set $U_j = \{\mathbf{y}_i \mid \mathbf{y}_i \in Q^j_{n-1}$ for $1 \le i \le k\}$. Without loss of generality, we assume that $U_0 = \{\mathbf{y}_{m+1}, \mathbf{y}_{m+2}, \ldots, \mathbf{y}_k\} \subseteq Q^0_{n-1}$ and $U_1 = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\} \subseteq Q^1_{n-1}$ for some $0 \le m \le k-1$.

Subcase 1.1: $m = 0$. Let $\tilde{U} = U_0 - \{\mathbf{y}_{k-1}\}$. Obviously, $|\tilde{U}| = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U})$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$, of Q^0_{n-1}. Without loss of generality, we assume that $\mathbf{y}_{k-1} \in R_{k-1}$ where R_{k-1} is joining $\mathbf{x}$ to $\mathbf{y}_t$ for some $t \in \{1, 2, \ldots, k-2, k\}$. We can write R_{k-1} as $\langle \mathbf{x}, H_1, \mathbf{y}_{k-1}, \mathbf{z}, H_2, \mathbf{y}_t \rangle$. (Note that $\mathbf{z}$ = $\mathbf{y}_k$ if $l(H_2) = 0$.) By Lemma 2, there is a Hamiltonian path W of Q^1_{n-1} joining $(\mathbf{x})^n$ to $(\mathbf{z})^n$. We set $P_i = R_i$ for every $1 \le i \le k-2$, $P_{k-1} = \langle \mathbf{x}, H_1, \mathbf{y}_{k-1} \rangle$, and $P_k = \langle \mathbf{x}, (\mathbf{x})^n, W, (\mathbf{z})^n, \mathbf{z}, H_2, \mathbf{y}_t \rangle$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a set of required paths of Q_n. See Fig. 1(a) for an illustration for $k = 6$ and $t = 6$.

Subcase 1.2: $m = 1$. Thus, $\mathbf{y}_1 \in Q^1_{n-1}$. By induction, there is a spanning $(\mathbf{x}, U_0)$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$, in Q^0_{n-1} such that R_i joins $\mathbf{x}$ to $\mathbf{y}_{i+1}$ for every $1 \le i \le k-1$. By Lemma 2, there is a Hamiltonian path W of Q^1_{n-1} joining $(\mathbf{x})^n$ to $\mathbf{y}_1$. We set $P_1 = \langle \mathbf{x}, (\mathbf{x})^n, W, \mathbf{y}_1 \rangle$ and $P_i = R_{i-1}$ for every $2 \le i \le k$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(b) for an illustration for $k = 6$.

Subcase 1.3: $m = 2$. We have $\{\mathbf{y}_1, \mathbf{y}_2\} \subseteq Q^1_{n-1}$. Since there are 2^{n-2} even vertices in Q^0_{n-1} and $2^{n-2} - |U_0 \cup \{\mathbf{x}\}| - 1 = 2^{n-2} - (k-2) \geq 2^{n-2} - n + 2 \geq 1$ if $n \geq 3$, we can choose an even vertex $\mathbf{u}$ in $Q^0_{n-1} - (U_0 \cup \{\mathbf{x}\})$. By induction, there is a spanning $(\mathbf{x}, U_0 \cup \{\mathbf{u}\})$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$ of Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{y}_{i+2}$ for every $1 \leq i \leq k-2$ and (2) R_{k-1} joins $\mathbf{x}$ to $\mathbf{u}$. By Lemma 3, there exist two disjoint paths S_1 and S_2 of Q^1_{n-1} such that (1) S_1 joins $(\mathbf{u})^n$ to $\mathbf{y}_1$, (2) S_2 joins $(\mathbf{x})^n$ to $\mathbf{y}_2$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. We set $P_1 = \langle \mathbf{x}, R_{k-1}, \mathbf{u}, (\mathbf{u})^n, S_1, \mathbf{y}_1 \rangle$, $P_2 = \langle \mathbf{x}, (\mathbf{x})^n, S_2, \mathbf{y}_2 \rangle$, and $P_i = R_{i-2}$ for every $3 \leq i \leq k$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(c) for an illustration for $k = 6$.

Subcase 1.4: $3 \leq m \leq k-2$. We have $k \geq 5$. Hence, $n \geq 5$. Since $m \geq 3$ and $k \leq n$, $|U_0 - \{\mathbf{y}_k\}| = k - m - 1 \leq k - 4 \leq n - 4$.

We claim that there exists an even vertex $\mathbf{u}$ in $Q^1_{n-1} - U_1$ such that $(\mathbf{y}_i)^n \notin N_{Q^1_{n-1}}(\mathbf{u})$ for every $m+1 \leq i \leq k-1$. Such claim holds because $(n-1)|U_0 - \{\mathbf{y}_k\}| + |U_1| \leq (n-1)(n-4) + (n-2) \leq (n-1)(n-3) - 1 < 2^{n-2}$ for all $n \geq 5$.

Since $m \leq k-2$ and $k \leq n$, $m+1 \leq n-1$. By induction, there is a spanning $(\mathbf{u}, U_1 \cup \{(\mathbf{x})^n\})$-fan, $\{W_1, W_2, \ldots, W_{m+1}\}$ in Q^1_{n-1} such that (1) W_i joins $\mathbf{u}$ to $\mathbf{y}_i$ for every $1 \leq i \leq m$ and (2) W_{m+1} joins $\mathbf{u}$ to $(\mathbf{x})^n$. We write W_i as $\langle \mathbf{u}, \mathbf{v}_i, W'_i, \mathbf{y}_i \rangle$ for every $1 \leq i \leq m-1$. Since $\mathbf{u}$ is an even vertex in Q^1_{n-1}, $\mathbf{v}_i$ is an odd vertex in Q^1_{n-1} and $(\mathbf{v}_i)^n$ is an even vertex in Q^0_{n-1} for every $1 \leq i \leq m-1$. Let $\tilde{U_0} = U_0 \cup \{(\mathbf{v}_i)^n | 1 \leq i \leq m-1\}$. Obviously, $|\tilde{U_0}| = (k-m) + (m-1) = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U_0})$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$, of Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $(\mathbf{v}_i)^n$ for every $1 \leq i \leq m-1$ and (2) R_i joins $\mathbf{x}$ to $\mathbf{y}_{i+1}$ for every $m \leq i \leq k-1$. We set $P_i = \langle \mathbf{x}, R_i, (\mathbf{v}_i)^n, \mathbf{v}_i, W'_i, \mathbf{y}_i \rangle$ for every $1 \leq i \leq m-1$, $P_m = \langle \mathbf{x}, (\mathbf{x})^n, W^{-1}_{m+1}, \mathbf{u}, W_m, \mathbf{y}_m \rangle$, and $P_i = R_{i-1}$ for every $m+1 \leq i \leq k$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(d) for an illustration for $k = 6$ and $m = 3$.

Subcase 1.5: $m = k-1$ and $k-1 \geq 3$. Let $\tilde{U_1} = (U_1 - \{\mathbf{y}_1\}) \cup \{(\mathbf{x})^n\}$. Obviously, $|\tilde{U_1}| = k-1$. By induction, there is a spanning $(\mathbf{y}_1, \tilde{U_1})$-fan, $\{W_1, W_2, \ldots, W_{k-1}\}$, in Q^1_{n-1} such that (1) W_1 joins $\mathbf{y}_1$ to $(\mathbf{x})^n$ and (2) W_i joins $\mathbf{y}_1$ to $\mathbf{y}_i$ for every $2 \leq i \leq k-1$. We write W_i as $\langle \mathbf{y}_1, \mathbf{v}_i, W'_i, \mathbf{y}_i \rangle$ for every $2 \leq i \leq k-1$. Since $\mathbf{y}_1$ is an even vertex in Q^1_{n-1}, $\mathbf{v}_i$ is an odd vertex in Q^1_{n-1} and $(\mathbf{v}_i)^n$ is an even vertex in Q^0_{n-1} for every $2 \leq i \leq k-1$. Let $\tilde{U_0} = \{\mathbf{y}_k\} \cup \{(\mathbf{v}_i)^n | 2 \leq i \leq k-1\}$. Obviously, $|\tilde{U_0}| = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U_0})$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$, in Q^0_{n-1} such that (1) R_1 joins $\mathbf{x}$ to $\mathbf{y}_k$ and (2) R_i joins $\mathbf{x}$ to $(\mathbf{v}_i)^n$ for every $2 \leq i \leq k-1$. We set $P_1 = \langle \mathbf{x}, (\mathbf{x})^n, W^{-1}_1, \mathbf{y}_1 \rangle$, $P_i = \langle \mathbf{x}, R_i, (\mathbf{v}_i)^n, \mathbf{v}_i, W'_i, \mathbf{y}_i \rangle$ for every $2 \leq i \leq k-1$, and $P_k = R_1$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(e) for an illustration for $k = 6$.

Case 2: $(\mathbf{y}_k)_i = 1$ for every $1 \leq i \leq n$. Obviously, n is odd with $n \geq 3$ and $\mathbf{y}_k \in Q^1_{n-1}$. Since Q_n is edge transitive, we assume that $U_0 = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\} \subseteq Q^0_{n-1}$ and $U_1 = \{\mathbf{y}_{m+1}, \mathbf{y}_{m+2}, \ldots, \mathbf{y}_k\} \subseteq Q^1_{n-1}$ for some $1 \leq m \leq k-2$.

Subcase 2.1: $m = k-2$. We have $\{\mathbf{y}_{k-1}, \mathbf{y}_k\} \subseteq Q^1_{n-1}$. Let H be a Hamiltonian path of Q^1_{n-1} joining $\mathbf{y}_{k-1}$ to $\mathbf{y}_k$. We write H as $\langle \mathbf{y}_{k-1}, H_1, \mathbf{u}, (\mathbf{x})^n, H_2, \mathbf{y}_k \rangle$. Since

$(\mathbf{x})^n$ is an odd vertex, $\mathbf{u}$ is an even vertex and $(\mathbf{u})^n$ is an odd vertex in Q^0_{n-1}. (Note that $\mathbf{y}_{k-1} = \mathbf{u}$ if $l(H_1) = 0$ or $(\mathbf{x})^n = \mathbf{y}_k$ if $l(H_2) = 0$.) By induction, there is a spanning $(\mathbf{x}, U_0 \cup \{(\mathbf{u})^n\})$-fan, $\{R_1, R_2, \dots, R_{k-1}\}$ in Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{y}_i$ for $1 \le i \le k-2$ and (2) R_{k-1} joins $\mathbf{x}$ to $(\mathbf{u})^n$. We set $P_i = R_i$ for $1 \le i \le k-2$, $P_{k-1} = \langle \mathbf{x}, R_{k-1}, (\mathbf{u})^n, \mathbf{u}, H_1^{-1}, \mathbf{y}_{k-1} \rangle$, and $P_k = \langle \mathbf{x}, (\mathbf{x})^n, H_2, \mathbf{y}_k \rangle$. Then $\{P_1, P_2, \dots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(f) for an illustration for $k = 6$.

Subcase 2.2: $m = k-3$. We have $n \ge 5$ and $\{\mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k\} \subseteq Q^1_{n-1}$. Since $m+1 \le n-2 < 2^{n-2}$, we can pick an even vertex $\mathbf{z} \in Q^0_{n-1} - (\{\mathbf{y}_i \mid 1 \le i \le k-3\} \cup \{\mathbf{x}\})$. By Lemma 3, there exist two disjoint paths S_1 and S_2 of Q^1_{n-1} such that (1) S_1 joins $(\mathbf{z})^n$ to $\mathbf{y}_{k-2}$, (2) S_2 joins $(\mathbf{x})^n$ to $\mathbf{y}_{k-1}$, and (3) $S_1 \cup S_2$ spans Q^1_{n-1}. Obviously, $\mathbf{y}_k \in S_i$ for some $1 \le i \le 2$.

Subcase 2.2.1: $\mathbf{y}_k \in S_1$. We write S_1 as $\langle (\mathbf{z})^n, H_1, \mathbf{y}_k, \mathbf{u}, H_2, \mathbf{y}_{k-2} \rangle$. Obviously, $\mathbf{u}$ is an even vertex and $(\mathbf{u})^n$ is an odd vertex in Q^0_{n-1}. Let $\tilde{U}_0 = U_0 \cup \{\mathbf{z}, (\mathbf{u})^n\}$. Obviously, $|\tilde{U}_0| = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U}_0)$-fan, $\{R_1, R_2, \dots, R_{k-1}\}$, in Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{y}_i$ for $1 \le i \le k-3$, (2) R_{k-2} joins $\mathbf{x}$ to $\mathbf{z}$, and (3) R_{k-1} joins $\mathbf{x}$ to $(\mathbf{u})^n$. We set $P_i = R_i$ for $1 \le i \le k-3$, $P_{k-2} = \langle \mathbf{x}, R_{k-1}, (\mathbf{u})^n, \mathbf{u}, H_2, \mathbf{y}_{k-2} \rangle$, $P_{k-1} = \langle \mathbf{x}, (\mathbf{x})^n, S_2, \mathbf{y}_{k-1} \rangle$, and $P_k = \langle \mathbf{x}, R_{k-2}, \mathbf{z}, (\mathbf{z})^n, H_1, \mathbf{y}_k \rangle$. Then $\{P_1, P_2, \dots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(g) for an illustration for $k = 6$.

Subcase 2.2.2: $\mathbf{y}_k \in S_2$. Similar to Subcase 2.2.1, there is a spanning $(\mathbf{x}, U)$-fan of Q_n.

Subcase 2.3: $1 \le m \le k-4$. We have $k \ge 5$. Moreover, $n \ge 5$. Since $m \le k-4$ and $k \le n$, $|U_0| = m \le k-4 \le n-4$.

We claim that there exists an even vertex $\mathbf{u}$ in $Q^1_{n-1} - U_1$ such that $(\mathbf{y}_i)^n \notin N_{Q^1_{n-1}}(\mathbf{u})$ for every $1 \le i \le m$. Such claim holds because $(n-1)|U_0| + |U_1 - \{\mathbf{y}_k\}| = (n-1)m + (k-m) - 1 = (n-2)m + k - 1 \le (n-2)(n-4) + n - 1 = (n-1)(n-4) + 3 < 2^{n-2}$ for all $n \ge 5$.

Let $\tilde{U}_1 = (U_1 - \{\mathbf{y}_k\}) \cup \{(\mathbf{x})^n\}$. Obviously, $|\tilde{U}_1| = k-m$. By induction, there is a spanning $(\mathbf{u}, \tilde{U}_1)$-fan, $\{W_{m+1}, W_{m+2}, \dots, W_k\}$, in Q^1_{n-1} joining $\mathbf{u}$ to $\tilde{U}_1$ such that (1) W_i joins $\mathbf{u}$ to $\mathbf{y}_i$ for every $m+1 \le i \le k-1$ and (2) W_k joins $\mathbf{u}$ to $(\mathbf{x})^n$. We write W_i as $\langle \mathbf{u}, \mathbf{v}_i, W_i', \mathbf{y}_i \rangle$ for every $m+1 \le i \le k-1$. Since $\mathbf{u}$ is an even vertex in Q^1_{n-1}, $\mathbf{v}_i$ is an odd vertex in Q^1_{n-1} and $(\mathbf{v}_i)^n$ is an even vertex in Q^0_{n-1} for every $m+1 \le i \le k-2$.

Subcase 2.3.1: $\mathbf{y}_k \in W_k$. We write W_k as $\langle \mathbf{u}, H_1, \mathbf{z}, \mathbf{y}_k, H_2, (\mathbf{x})^n \rangle$. Since $\mathbf{y}_k$ is an odd vertex in Q^1_{n-1}, $\mathbf{z}$ is an even vertex in Q^1_{n-1}, and $(\mathbf{z})^n$ is an odd vertex in Q^0_{n-1}. Let $\tilde{U}_0 = U_0 \cup \{(\mathbf{v}_i)^n | m+1 \le i \le k-2\} \cup \{(\mathbf{z})^n\}$. Obviously, $|\tilde{U}_0| = m + (k-m-2) + 1 = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U}_0)$-fan, $\{R_1, R_2, \dots, R_{k-1}\}$, in Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{y}_i$ for $1 \le i \le m$, (2) R_i joins $\mathbf{x}$ to $(\mathbf{v}_i)^n$ for every $m+1 \le i \le k-2$, and (3) R_{k-1} joins $\mathbf{x}$ to $(\mathbf{z})^n$. We set $P_i = R_i$ for every $1 \le i \le m$, $P_i = \langle \mathbf{x}, R_i, (\mathbf{v}_i)^n, \mathbf{v}_i, W_i', \mathbf{y}_i \rangle$ for every $m+1 \le i \le k-2$, $P_{k-1} = \langle \mathbf{x}, R_{k-1}, (\mathbf{z})^n, \mathbf{z}, H_1^{-1}, \mathbf{u}, \mathbf{v}_{k-1}, W_{k-1}', \mathbf{y}_{k-1} \rangle$, and $P_k = \langle \mathbf{x}, (\mathbf{x})^n, H_2^{-1}, \mathbf{y}_k \rangle$.

Then $\{P_1, P_2, \ldots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(h) for an illustration for $k = 6$ and $m = 2$.

Subcase 2.3.2: $\mathbf{y}_k \in W_i$ for some $1 \le i \le k-1$. Without loss of generality, we assume that $\mathbf{y}_k \in W_{k-1}$. We write W_{k-1} as $\langle \mathbf{u}, \mathbf{v}_{k-1}, H_1, \mathbf{y}_k, \mathbf{z}, H_2, \mathbf{y}_{k-1}\rangle$. Since $\mathbf{y}_k$ is an odd vertex in Q^1_{n-1}, $\mathbf{z}$ is an even vertex in Q^1_{n-1} and $(\mathbf{z})^n$ is an odd vertex in Q^0_{n-1}. Let $\tilde{U_0} = U_0 \cup \{(\mathbf{v}_i)^n | m+1 \le i \le k-2\} \cup \{(\mathbf{z})^n\}$. Obviously, $|\tilde{U_0}| = m + (k-m-2) + 1 = k-1$. By induction, there is a spanning $(\mathbf{x}, \tilde{U_0})$-fan, $\{R_1, R_2, \ldots, R_{k-1}\}$, in Q^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{y}_i$ for every $1 \le i \le m$, (2) R_i joins $\mathbf{x}$ to $(\mathbf{v}_i)^n$ for every $m+1 \le i \le k-2$, and (3) R_{k-1} joins $\mathbf{x}$ to $(\mathbf{z})^n$. We set $P_i = R_i$ for every $1 \le i \le m$, $P_i = \langle \mathbf{x}, R_i, (\mathbf{v}_i)^n, \mathbf{v}_i, W'_i, \mathbf{y}_i\rangle$ for every $m+1 \le i \le k-2$, $P_{k-1} = \langle \mathbf{x}, R_{k-1}, (\mathbf{z})^n, \mathbf{z}, H_2, \mathbf{y}_{k-1}\rangle$, and $P_k = \langle \mathbf{x}, (\mathbf{x})^n, W_k^{-1}, \mathbf{u}, \mathbf{v}_{k-1}, H_1, \mathbf{y}_k\rangle$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a spanning $(\mathbf{x}, U)$-fan of Q_n. See Fig. 1(i) for an illustration for $k = 6$ and $m = 2$. □

3 The super spanning properties of folded hypercubes

Let $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ be a vertex of FQ_n. The *c-neighbor* of $\mathbf{u}$ in FQ_n, $(\mathbf{u})^c$, is $\bar{u}_1\bar{u}_2\cdots\bar{u}_n$. Note that $(\mathbf{u})^c$ and $\mathbf{u}$ are of the same parity if and only if n is an even integer. Let $E^c = \{(u_1u_2\cdots u_n, \bar{u}_1\bar{u}_2\cdots\bar{u}_n) \mid u_1u_2\cdots u_n \in V(FQ_n)\}$. By definition, the n-dimensional folded hypercube FQ_n is obtained from Q_n by adding E^c. Let f be a function on $V(FQ_n)$ defined by $f(\mathbf{u}) = \mathbf{u}$ if $(\mathbf{u})_n = 0$ and $f(\mathbf{u}) = ((\mathbf{u})^c)^n$ if otherwise. The following theorem can be proved easily.

Theorem 2 *The function f is an isomorphism of FQ_n into itself.*

Let FQ^j_{n-1} be the subgraph of FQ_n induced by $\{\mathbf{v} \in V(FQ_n) \mid (\mathbf{v})_n = j\}$ for $0 \le j \le 1$. Obviously, FQ^j_{n-1} is isomorphic to Q_{n-1} for $0 \le j \le 1$.

Lemma 5 *Let $\mathbf{x}$ be an even vertex and $\mathbf{y}$ be an odd vertex of FQ_n for any positive integer $n \ge 2$. Then there exists a k^*-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \le k \le n+1$.*

Proof Since FQ_2 is isomorphic to the complete graph K_4, this statement holds for $n = 2$. Suppose that $n \ge 3$. Since Q_n is a spanning subgraph of FQ_n, by Lemma 2, there exists a k^*-container between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \le k \le n$. Thus, we only need to construct an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. Since FQ_n is vertex transitive, we assume that $\mathbf{x} = 0^n \in V(FQ^0_{n-1})$.

Case 1: $\mathbf{y} \in FQ^0_{n-1}$. We have the following cases:

Subcase 1.1: $n = 3$. Without loss of generality, we assume that $\mathbf{y} = 100$. We set $P_1 = \langle 000, 001, 101, 100\rangle$, $P_2 = \langle 000, 010, 110, 100\rangle$, $P_3 = \langle 000, 100\rangle$, and $P_4 = \langle 000, 111, 011, 100\rangle$. Then $\{P_1, P_2, P_3, P_4\}$ forms a 4^*-container of FQ_3 between $\mathbf{x}$ and $\mathbf{y}$.

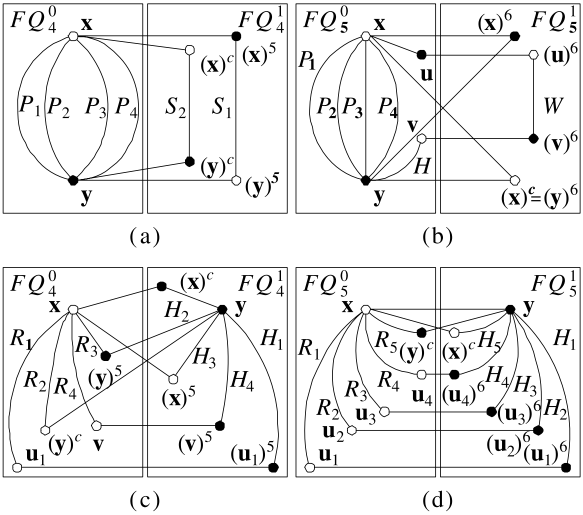


Fig. 2 Illustration for Lemma 5

Subcase 1.2: $n \geq 4$. Since FQ^0_{n-1} is isomorphic to Q_{n-1}, by Lemma 2, there is an $(n-1)^*$-container $\{P_1, P_2, \ldots, P_{n-1}\}$ of FQ^0_{n-1} between $\mathbf{x}$ and $\mathbf{y}$.

Subcase 1.2.1: $(\mathbf{x})^c \neq (\mathbf{y})^n$. Obviously, $(\mathbf{x})^c$ and $(\mathbf{y})^c$ are of different parity. Since FQ^1_{n-1} is isomorphic to Q_{n-1}, by Lemma 3, there exist two disjoint paths S_1 and S_2 of FQ^1_{n-1} such that (1) S_1 joins $(\mathbf{x})^n$ to $(\mathbf{y})^n$, (2) S_2 joins $(\mathbf{x})^c$ to $(\mathbf{y})^c$, and (3) $S_1 \cup S_2$ spans FQ^1_{n-1}. We set $P_n = \langle \mathbf{x}, (\mathbf{x})^n, S_1, (\mathbf{y})^n, \mathbf{y} \rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^c, S_2, (\mathbf{y})^c, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 2(a) for illustration for $n = 5$.

Subcase 1.2.2: $(\mathbf{x})^c = (\mathbf{y})^n$. Then $(\mathbf{y})^c = (\mathbf{x})^n$ and n is even.

Suppose that $n = 4$. We have $\mathbf{x} = 0000$ and $\mathbf{y} = 1110$. We set $P_1 = \langle 0000, 0001, 1110 \rangle$, $P_2 = \langle 0000, 0010, 0110, 1110 \rangle$, $P_3 = \langle 0000, 0100, 0101, 0111, 0011, 1011, 1001, 1101, 1100, 1110 \rangle$, $P_4 = \langle 0000, 1000, 1010, 1110 \rangle$, and $P_5 = \langle 0000, 1111, 1110 \rangle$. Then $\{P_1, P_2, P_3, P_4, P_5\}$ forms a 5^*-container of FQ_4 between $\mathbf{x}$ and $\mathbf{y}$.

Since $2^{n-1} - 2 \geq 3(n-1)$ for $n \geq 6$, there is one path P_i in $\{P_1, P_2, \ldots, P_{n-1}\}$ such that $I(P_i) \geq 3$. Without loss of generality, we may assume that $I(P_{n-1}) \geq 3$. We write P_{n-1} as $\langle \mathbf{x}, \mathbf{u}, \mathbf{v}, H, \mathbf{y} \rangle$ where $\mathbf{u}$ is an odd vertex and $\mathbf{v}$ is an even vertex. By Lemma 4, there is a Hamiltonian path W of $Q^1_{n-1} - \{(\mathbf{x})^n, (\mathbf{y})^n\}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We set $P'_{n-1} = \langle \mathbf{x}, \mathbf{u}, (\mathbf{u})^n, W, (\mathbf{v})^n, \mathbf{v}, H, \mathbf{y} \rangle$, $P_n = \langle \mathbf{x}, (\mathbf{x})^n = (\mathbf{y})^c, \mathbf{y} \rangle$, and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^c = (\mathbf{y})^n, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n-2}, P'_{n-1}, P_n, P_{n+1}\}$ forms an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 2(b) for illustration for $n = 6$.

Case 2: $\mathbf{y} \in FQ^1_{n-1}$. We have the following cases:

Subcase 2.1: n is odd and $\mathbf{y} \in \{(\mathbf{x})^n, (\mathbf{x})^c\}$. By Theorem 2, we only consider that $\mathbf{y} = (\mathbf{x})^c$. By Lemma 2, there is an n^*-container $\{P_1, P_2, \ldots, P_n\}$ of Q_n between $\mathbf{x}$ and $\mathbf{y}$. We set $P_{n+1} = \langle \mathbf{x}, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$.

Subcase 2.2: n is odd and $\mathbf{y} \notin \{(\mathbf{x})^c, (\mathbf{x})^n\}$. Since $\mathbf{y} \in FQ^1_{n-1}$ and $\mathbf{y}$ is an odd vertex, we have $\mathbf{y} = (\mathbf{x})^c$ or $\mathbf{y} = (\mathbf{x})^n$ if $n = 3$. Thus, $n \geq 5$. Since there are 2^{n-2} even vertices in FQ^0_{n-1} and $2^{n-2} \geq n-1$ for $n \geq 5$, we can choose $(n-4)$ distinct even vertices $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-4}$ in $FQ^0_{n-1} - \{\mathbf{x}, (\mathbf{y})^c, (\mathbf{y})^n\}$ such that $(\mathbf{u}_i)^n \neq (\mathbf{x})^c$ for $1 \leq i \leq n-4$. Let $\mathbf{v}$ be an odd vertex of FQ^0_{n-1} and let $U_0 = \{\mathbf{u}_i | 1 \leq i \leq n-4\} \cup \{(\mathbf{y})^c, (\mathbf{y})^n, \mathbf{v}\}$. Obviously, $|U_0| = n-1$. By Theorem 1, there is a spanning $(\mathbf{x}, U_0)$-fan, $\{R_1, R_2, \ldots, R_{n-1}\}$, in FQ^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{u}_i$ for $1 \leq i \leq n-4$, (2) R_{n-3} joins $\mathbf{x}$ to $(\mathbf{y})^c$, (3) R_{n-2} joins $\mathbf{x}$ to $(\mathbf{y})^n$, and (4) R_{n-1} joins $\mathbf{x}$ to $\mathbf{v}$. Let $U_1 = \{(\mathbf{u}_i)^n | 1 \leq i \leq n-4\} \cup \{(\mathbf{x})^c, (\mathbf{x})^n, (\mathbf{v})^n\}$. Obviously, $|U_1| = n-1$. By Theorem 1, there is a spanning $(\mathbf{y}, U_1)$-fan, $\{H_1, H_2, \ldots, H_{n-1}\}$, in FQ^1_{n-1} such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \leq i \leq n-4$, (2) H_{n-3} joins $(\mathbf{x})^c$ to $\mathbf{y}$, (3) H_{n-2} joins $(\mathbf{x})^n$ to $\mathbf{y}$, and (4) H_{n-1} joins $(\mathbf{v})^n$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y} \rangle$ for $1 \leq i \leq n-4$, $P_{n-3} = \langle \mathbf{x}, R_{n-3}, (\mathbf{y})^c, \mathbf{y} \rangle$, $P_{n-2} = \langle \mathbf{x}, R_{n-2}, (\mathbf{y})^n, \mathbf{y} \rangle$, $P_{n-1} = \langle \mathbf{x}, R_{n-1}, \mathbf{v}, (\mathbf{v})^n, H_{n-1}, \mathbf{y} \rangle$, $P_n = \langle \mathbf{x}, (\mathbf{x})^c, H_{n-3}, \mathbf{y} \rangle$, and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, H_{n-2}, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 2(c) for illustration for $n = 5$.

Subcase 2.3: n is even with $n \geq 4$ and $\mathbf{y} = (\mathbf{x})^n$. Since there are 2^{n-2} even vertices in FQ^0_{n-1} and $2^{n-2} \geq n-1$ for $n \geq 4$, we can choose $(n-2)$ distinct even vertices $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-2}$ in $FQ^0_{n-1} - \{\mathbf{x}\}$. Let $U_0 = \{\mathbf{u}_i | 1 \leq i \leq n-2\} \cup \{(\mathbf{y})^c\}$. Obviously, $|U_0| = n-1$. By Theorem 1, there is a spanning $(\mathbf{x}, U_0)$-fan, $\{R_1, R_2, \ldots, R_{n-1}\}$, in FQ^0_{n-1} such that (1) R_i joins $\mathbf{x}$ to $\mathbf{u}_i$ for $1 \leq i \leq n-2$ and (2) R_{n-1} joins $\mathbf{x}$ to $(\mathbf{y})^c$. Let $U_1 = \{(\mathbf{u}_i)^n | 1 \leq i \leq n-2\} \cup \{(\mathbf{x})^c, \}$. Obviously, $|U_1| = n-1$. By Theorem 1, there is a spanning $(\mathbf{y}, U_1)$-fan, $\{H_1, H_2, \ldots, H_{n-1}\}$, in FQ^1_{n-1} such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \leq i \leq n-2$ and (2) H_{n-1} joins $(\mathbf{x})^c$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y} \rangle$ for $1 \leq i \leq n-2$, $P_{n-1} = \langle \mathbf{x}, R_{n-1}, (\mathbf{y})^c, \mathbf{y} \rangle$, $P_n = \langle \mathbf{x}, (\mathbf{x})^c, H_{n-1}, \mathbf{y} \rangle$, and $P_{n+1} = \langle \mathbf{x}, \mathbf{y} = (\mathbf{x})^n \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 2(d) for illustration for $n = 6$.

Subcase 2.4: n is even with $n \geq 4$ and $\mathbf{y} \neq (\mathbf{x})^n$. Since there are 2^{n-2} even vertices in FQ^0_{n-1} and $2^{n-2} \geq n-1$ for $n \geq 4$, we can choose $(n-3)$ distinct even vertices $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_{n-3}$ in $FQ^0_{n-1} - \{\mathbf{x}, (\mathbf{y})^n\}$ such that $(\mathbf{u}_i)^n \neq (\mathbf{x})^n$ for $1 \leq i \leq n-3$. Let $U_0 = \{\mathbf{u}_i | 1 \leq i \leq n-3\} \cup \{(\mathbf{y})^n, (\mathbf{y})^c\}$ and let $U_1 = \{(\mathbf{u}_i)^n | 1 \leq i \leq n-3\} \cup \{(\mathbf{x})^n, (\mathbf{x})^c\}$. Obviously, $|U_0| = |U_1| = n-1$. Similar to Subcase 2.2, there is an $(n+1)^*$-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$. □

Theorem 3 *FQ_n is super spanning laceable if n is an odd integer and FQ_n is super spanning connected if n is an even integer.*

Proof Since FQ_1 is isomorphic to Q_1, this statement holds for $n = 1$. By Lemma 5, this statement holds if n is odd and $n \geq 3$. Thus, we assume that n is even. Let $\mathbf{x}$ and $\mathbf{y}$ be any two different vertices of FQ_n. We need to find a k^*-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$ for $1 \leq k \leq n+1$. Without loss of generality, we assume that $\mathbf{x}$ is an even vertex. By Lemma 5, this statement holds if $\mathbf{y}$ is an odd vertex. Thus, we assume that $\mathbf{y}$ is an even vertex. Without loss of generality, we assume that $(\mathbf{x})_n = 0$ and $(\mathbf{y})_n = 1$. Let f be the function on $V(FQ_n)$ defined by $f(\mathbf{u}) = \mathbf{u}$ if $(\mathbf{u})_n = 0$ and $f(\mathbf{u}) = ((\mathbf{u})^c)^n$ if otherwise. By Theorem 2, f is an isomorphism from FQ_n into itself. In other words, we still get FQ_n if we relabel all the vertices $\mathbf{u}$ with $f(\mathbf{u})$. However, $f(\mathbf{x}) = \mathbf{x}$ is an even vertex and $f(\mathbf{y}) = ((\mathbf{y})^c)^n$ is an odd vertex. By Lemma 5, there exists a k^*-container of FQ_n between $f(\mathbf{x})$ and $f(\mathbf{y})$ for every $1 \le k \le n+1$. Thus, there exists a k^*-container of FQ_n between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \le k \le n+1$. This theorem is proved. □

4 The super spanning properties of enhanced hypercubes

Let $\mathbf{u} = u_1u_2\cdots u_{n-1}u_n$ be a vertex of $Q_{n,m}$. Similar to before, *c-neighbor* of $\mathbf{u}$ in $Q_{n,m}$, $(\mathbf{u})^c$, is $\bar{u}_1\bar{u}_2\cdots\bar{u}_m u_{m+1}u_{m+2}\cdots u_{n-1}u_n$. Note that $(\mathbf{u})^c$ and $\mathbf{u}$ are of the same parity if and only if m is even. Let $E^c = \{(u_1u_2\cdots u_n, \bar{u}_1\bar{u}_2\cdots\bar{u}_m u_{m+1}u_{m+2}\cdots u_{n-1}u_n) | u_1u_2\cdots u_n \in V(Q_{n,m})\}$. By definition, the n-dimensional enhanced hypercube $Q_{n,m}$ is obtained from Q_n by adding E^c. Obviously, $Q_{n,m}$ is FQ_n if $m = n$. We use $Q^j_{n,m}$ to denote the subgraph of $Q_{n,m}$ induced by $\{\mathbf{v} \in V(Q_{n,m}) \mid (\mathbf{v})_n = j\}$ for $0 \le j \le 1$. Moreover, we use $Q^{ij}_{n,m}$ to denote the subgraph of $Q_{n,m}$ induced by $\{\mathbf{v} \in V(Q_{n,m}) | (\mathbf{v})_{n-1} = i \text{ and } (\mathbf{v})_n = j\}$ for $0 \le i, j \le 1$.

Lemma 6 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be any two distinct vertices of* $Q^j_{n,m}$ *with* $n - m \ge 1$ *for some* j. *Suppose that there is a* k^**-container of* $Q^j_{n,m}$ *between* $\mathbf{x}$ *and* $\mathbf{y}$ *and there is an* 1^**-container of* $Q^{1-j}_{n,m}$ *between* $(\mathbf{x})^n$ *and* $(\mathbf{y})^n$. *Then there is a* $(k+1)^*$*-container of* $Q_{n,m}$ *between* $\mathbf{x}$ *and* $\mathbf{y}$.

Proof Let $\{P_1, P_2, \ldots, P_k\}$ be a k^*-container of $Q^j_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$ and W be a Hamiltonian path of $Q^{1-j}_{n,m}$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^n$. Set $P_{k+1} = \langle \mathbf{x}, (\mathbf{x})^n, W, (\mathbf{y})^n, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{k+1}\}$ forms a $(k+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. □

Lemma 7 *Let* $\mathbf{x}$ *be an even vertex and* $\mathbf{y}$ *be an odd vertex of* $Q_{n,n-1}$ *for any positive integer* $n \ge 3$. *Then there exists a* k^**-container of* $Q_{n,n-1}$ *between* $\mathbf{x}$ *and* $\mathbf{y}$ *for every* $1 \le k \le n+1$.

Proof Since Q_n is a spanning subgraph of $Q_{n,n-1}$, by Lemma 2, there exists a k^*-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \le k \le n$. Thus, we only need to construct an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. Without loss of generality, we assume that $\mathbf{x} \in Q^{00}_{n,n-1}$. We have the following cases:

Case 1: $\mathbf{y} \in Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$. Since $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1} = Q^0_{n,n-1}$ is isomorphic to FQ_{n-1}, by Lemma 5, there exists an n^*-container of $Q^0_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. Since $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1} = Q^1_{n,n-1}$ is isomorphic to FQ_{n-1}, by Lemma 5, there exists a Hamiltonian path of $Q^1_{n,n-1}$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^n$. By Lemma 6, there exists an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$.

Case 2: $\mathbf{y} \in Q^{01}_{n,n-1}$. Suppose that $n = 3$. We have $\mathbf{x} = 000$ and $\mathbf{y} = 001$. We set $P_1 = \langle 000, 001 \rangle$, $P_2 = \langle 000, 010, 011, 001 \rangle$, $P_3 = \langle 000, 100, 101, 001 \rangle$, and $P_4 = \langle 000, 110, 111, 001 \rangle$. Then $\{P_1, P_2, P_3, P_4\}$ forms a 4^*-container of $Q_{3,2}$ between $\mathbf{x}$ and $\mathbf{y}$.

Now, we consider $n \geq 4$. Since $Q^{00}_{n,n-1} \cup Q^{01}_{n,n-1}$ is isomorphic to Q_{n-1}, by Lemma 2, there exists an $(n-1)^*$-container $\{P_1, P_2, \ldots, P_{n-1}\}$ of $Q^{00}_{n,n-1} \cup Q^{01}_{n,n-1}$ joining $\mathbf{x}$ to $\mathbf{y}$. Obviously, $(\mathbf{x})^c$ and $(\mathbf{y})^c$ are different parity. Note that $(\mathbf{x})^{n-1}$ is an odd vertex and $(\mathbf{y})^{n-1}$ is an even vertex. By Lemma 3, there exist two disjoint paths S_1 and S_2 of $Q^{10}_{n,n-1} \cup Q^{11}_{n,n-1}$ such that (1) S_1 joins $(\mathbf{x})^{n-1}$ to $(\mathbf{y})^{n-1}$, (2) S_2 joins $(\mathbf{x})^c$ to $(\mathbf{y})^c$, and (3) $S_1 \cup S_2$ spans $Q^{10}_{n,n-1} \cup Q^{11}_{n,n-1}$. We set $P_n = \langle \mathbf{x}, (\mathbf{x})^{n-1}, S_1, (\mathbf{y})^{n-1}, \mathbf{y} \rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^c, S_2, (\mathbf{y})^c, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 3(a) for illustration.

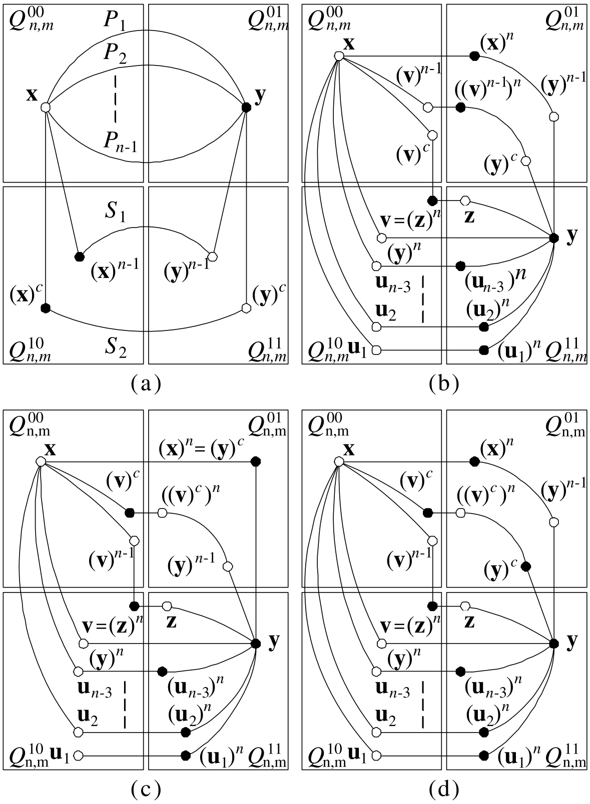
Case 3: $\mathbf{y} \in Q^{11}_{n,n-1}$. Suppose that $n = 3$. We have $\mathbf{x} = 000$ and $\mathbf{y} = 111$. We set $P_1 = \langle 000, 100, 101, 111 \rangle$, $P_2 = \langle 000, 010, 011, 111 \rangle$, $P_3 = \langle 000, 001, 111 \rangle$, and $P_4 = \langle 000, 110, 111 \rangle$. Then $\{P_1, P_2, P_3, P_4\}$ forms a 4^*-container of $Q_{3,2}$ between $\mathbf{x}$ and $\mathbf{y}$.

Now, we consider $n \geq 4$. Since $\mathbf{y}$ is adjacent to $(n-2)$ even vertices in $Q^{11}_{n,n-1}$, we can choose an even vertex $\mathbf{z} \in Q^{11}_{n,n-1}$ which is a neighbor of $\mathbf{y}$ such that $(\mathbf{z})^n \neq (\mathbf{x})^c$ and $(\mathbf{z})^n \neq (\mathbf{x})^{n-1}$. Let $\mathbf{v} = (\mathbf{z})^n$. Obviously, $\mathbf{v}$ is an odd vertex. Since $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1} = Q^{0}_{n,n-1}$ is isomorphic to FQ_{n-1}, by Lemma 5, there exists an n^*-container $\{R_1, R_2, \ldots, R_n\}$ of $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{v}$. Since $\mathbf{v}$ is adjacent to n vertices in $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$, by relabeling, we can write R_i as $\langle \mathbf{x}, R'_i, \mathbf{u}_i, \mathbf{v} \rangle$ for $1 \leq i \leq n-3$, write R_{n-2} as $\langle \mathbf{x}, R'_{n-2}, (\mathbf{y})^n, \mathbf{v} \rangle$, write R_{n-1} as $\langle \mathbf{x}, R'_{n-1}, (\mathbf{v})^c, \mathbf{v} \rangle$, and write R_n as $\langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, \mathbf{v} \rangle$. Let $A = \{(\mathbf{u}_i)^n \mid 1 \leq i \leq n-3\}$. Obviously, A is a set of $(n-3)$ odd vertices of $Q^{11}_{n,n-1}$. Since $Q^{11}_{n,n-1}$ is isomorphic to Q_{n-2}, by Theorem 1, there is a spanning $(\mathbf{y}, A \cup \{\mathbf{z}\})$-fan, $\{H_1, H_2, \ldots, H_{n-2}\}$ in $Q^{11}_{n,n-1}$ such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \leq i \leq n-3$ and (2) H_{n-2} joins $\mathbf{z}$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y} \rangle$ for $1 \leq i \leq n-3$ and $P_{n-2} = \langle \mathbf{x}, R'_{n-2}, (\mathbf{y})^n, \mathbf{y} \rangle$.

Suppose that $(n-1)$ is an odd integer. We set $P_{n-1} = \langle \mathbf{x}, R'_{n-1}, (\mathbf{v})^c, \mathbf{v}, \mathbf{z}, H_{n-2}, \mathbf{y} \rangle$. Since $Q^{01}_{n,n-1}$ is isomorphic to Q_{n-2}, by Lemma 3, there exist two disjoint paths S_1 and S_2 of $Q^{01}_{n,n-1}$ such that (1) S_1 joins $((\mathbf{v})^{n-1})^n$ to $(\mathbf{y})^c$, (2) S_2 joins $(\mathbf{x})^n$ to $(\mathbf{y})^{n-1}$, and (3) $S_1 \cup S_2$ spans $Q^{01}_{n,n-1}$. Let $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, S_1, (\mathbf{y})^c, \mathbf{y} \rangle$, and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, S_2, (\mathbf{y})^{n-1}, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 3(b) for illustration.

Suppose that $(n-1)$ is an even integer. We set $P_{n-1} = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, \mathbf{v}, \mathbf{z}, H_{n-2}, \mathbf{y} \rangle$. Suppose that $(\mathbf{y})^c = (\mathbf{x})^n$. By Lemma 1, there exists a Hamiltonian path S of $Q^{01}_{n,n-1} - \{(\mathbf{x})^n\}$ joining $((\mathbf{v})^c)^n$ to $(\mathbf{y})^{n-1}$. Set $P_n = \langle \mathbf{x}, R'_{n-1}, (\mathbf{v})^c, ((\mathbf{v})^c)^n, S, (\mathbf{y})^{n-1}, \mathbf{y} \rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n = (\mathbf{y})^c, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 3(c) for illustration. Thus, we assume that $(\mathbf{y})^c \neq (\mathbf{x})^n$. By Lemma 3, there exist two disjoint paths S_1 and S_2 of $Q^{01}_{n,n-1}$ such that (1) S_1 joins $((\mathbf{v})^c)^n$ to $(\mathbf{y})^c$, (2) S_2 joins $(\mathbf{x})^n$ to $(\mathbf{y})^{n-1}$, and (3) $S_1 \cup S_2$ spans $Q^{01}_{n,n-1}$. Let $P_n = \langle \mathbf{x}, R'_{n-1}, (\mathbf{v})^c, ((\mathbf{v})^c)^n, S_1, (\mathbf{y})^c, \mathbf{y} \rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, S_2, (\mathbf{y})^{n-1}, \mathbf{y} \rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 3(d) for illustration. □

Fig. 3 Illustration for Lemma 7



Lemma 8 *Let* $\mathbf{x}$ *be an even vertex and* $\mathbf{y}$ *be an odd vertex of* $Q_{n,m}$ *for any two positive integers* $n \geq m \geq 2$. *Then there exists a* k^**-container of* $Q_{n,m}$ *between* $\mathbf{x}$ *and* $\mathbf{y}$ *for every* $1 \leq k \leq n+1$.

Proof Since $Q_{2,2}$ is isomorphic to complete graph K_4, this statement holds for $n = 2$. Suppose that $n \geq 3$.

Since Q_n is a spanning subgraph of $Q_{n,m}$, by Lemma 2, there exists a k^*-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \leq k \leq n$. Thus, we only need to construct an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. Without loss of generality, we assume that $\mathbf{x} \in Q^{00}_{n,m}$. We prove our claim by induction on $t = n - m$. The induction bases are $t = 0$ and 1. By Lemma 5, our claim holds for $t = 0$. With Lemma 7, our claim holds for $t = 1$. Consider $t \geq 2$ and assume that our claim holds for $(t-1)$. We have the following cases:

Case 1: $\mathbf{y} \in Q^{00}_{n,m} \cup Q^{10}_{n,m}$. Since $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ is isomorphic to $Q_{n-1,m}$, by induction, there exists an n^*-container of $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. Since $Q^{01}_{n,m} \cup Q^{11}_{n,m}$ is isomorphic to $Q_{n-1,m}$, by induction, there is a Hamiltonian path of $Q^{01}_{n,m} \cup Q^{11}_{n,m}$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^n$. Thus, by Lemma 6, there exists an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$.

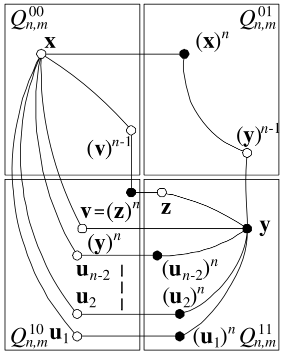


Fig. 4 Illustration for Lemma 8

Case 2: $\mathbf{y} \in Q_{n,m}^{01}$. Note that $Q_{n,m}^{01}$ and $Q_{n,m}^{10}$ are symmetric with respect to $Q_{n,m}$ and $Q_{n,m}^{00} \cup Q_{n,m}^{01}$ is isomorphic to $Q_{n-1,m}$. Similar to Case 1, there is an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$.

Case 3: $\mathbf{y} \in Q_{n,m}^{11}$. Since $\mathbf{y}$ is adjacent to $(n-1)$ vertices in $Q_{n,m}^{11}$, we can choose a neighbor $\mathbf{z}$ of $\mathbf{y}$ in $Q_{n,m}^{11}$ such that $\mathbf{z} \neq (\mathbf{y})^c$ and $(\mathbf{z})^n \neq (\mathbf{x})^{n-1}$. Let $\mathbf{v} = (\mathbf{z})^n$. Obviously, $\mathbf{v}$ is an odd vertex. Since $Q_{n,m}^{00} \cup Q_{n,m}^{10}$ is isomorphic to $Q_{n-1,m}$, by induction, there exists an n^*-container $\{R_1, R_2, \dots, R_n\}$ of $Q_{n,m}^{00} \cup Q_{n,m}^{10}$ joining $\mathbf{x}$ to $\mathbf{v}$. Since $\mathbf{v}$ is adjacent to n vertices in $Q_{n,m}^{00} \cup Q_{n,m}^{10}$, by relabeling, we can write R_i as $\langle \mathbf{x}, R_i', \mathbf{u}_i, \mathbf{v}\rangle$ for $1 \le i \le n-2$, write R_{n-1} as $\langle \mathbf{x}, R_{n-1}', (\mathbf{y})^n, \mathbf{v}\rangle$, and write R_n as $\langle \mathbf{x}, R_n', (\mathbf{v})^{n-1}, \mathbf{v}\rangle$. Since $Q_{n,m}^{11}$ is isomorphic to $Q_{n-2,m}$, by induction, there exists an $(n-1)^*$-container $\{H_1, H_2, \dots, H_{n-1}\}$ of $Q_{n,m}^{11}$ joining $\mathbf{z}$ to $\mathbf{y}$. Since $\mathbf{y}$ is adjacent to $(n-1)$ vertices in $Q_{n,m}^{11}$ and $(\mathbf{z}, \mathbf{y}) \in E(Q_{n,m}^{11})$, one of these paths is $\langle \mathbf{z}, \mathbf{y}\rangle$. Without loss of generality, we assume that $H_i = \langle \mathbf{z}, (\mathbf{u}_i)^n, H_i', \mathbf{y}\rangle$ for $1 \le i \le n-2$ and $H_{n-1} = \langle \mathbf{z}, \mathbf{y}\rangle$. We set $P_i = \langle \mathbf{x}, R_i', \mathbf{u}_i, (\mathbf{u}_i)^n, H_i', \mathbf{y}\rangle$ for $1 \le i \le n-2$, $P_{n-1} = \langle \mathbf{x}, R_{n-1}', (\mathbf{y})^n, \mathbf{y}\rangle$, and $P_n = \langle \mathbf{x}, R_n', (\mathbf{v})^{n-1}, \mathbf{v}, \mathbf{z}, \mathbf{y}\rangle$. Since $Q_{n,m}^{01}$ is isomorphic to $Q_{n-2,m}$, by induction, there exists a Hamiltonian path W in $Q_{n,m}^{01}$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^{n-1}$. We set $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, W, (\mathbf{y})^{n-1}, \mathbf{y}\rangle$. Then $\{P_1, P_2, \dots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 4 for illustration. □

Lemma 9 $Q_{n,n-1}$ *is* 1^**-connected and* 2^**-connected if* n *is an odd integer with* $n \ge 3$.

Proof Since any 1^*-connected graph with more than 3 vertices is 2^*-connected. Thus, we only need to show $Q_{n,n-1}$ is 1^*-connected. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct vertices of $Q_{n,n-1}$. Without loss of generality, we assume that $\mathbf{x} \in Q_{n,n-1}^0$.

Suppose that $\mathbf{y} \in Q_{n,n-1}^0$. By Theorem 3, there exists a Hamiltonian path $R = \langle \mathbf{x}, \mathbf{v}, R', \mathbf{y}\rangle$ in $Q_{n,n-1}^0$ joining $\mathbf{x}$ to $\mathbf{y}$ and there exists a Hamiltonian path H in $Q_{n,n-1}^1$ joining $(\mathbf{x})^n$ to $(\mathbf{v})^n$. Set $P = \langle \mathbf{x}, (\mathbf{x})^n, H, (\mathbf{v})^n, \mathbf{v}, R', \mathbf{y}\rangle$. Thus, P forms a Hamiltonian path in $Q_{n,n-1}$ joining $\mathbf{x}$ to $\mathbf{y}$. See Fig. 5(a) for illustration.

Suppose that $\mathbf{y} \in Q_{n,n-1}^1$. Note that there are $(2^{n-1}-1)$ vertices in $Q_{n,n-1}^0 - \{\mathbf{x}\}$ and $2^{n-1}-1 \ge 3$ for $n \ge 3$. We can pick a vertex $\mathbf{z}$ in $Q_{n,n-1}^0$ such that $(\mathbf{z})^n \neq \mathbf{y}$. By

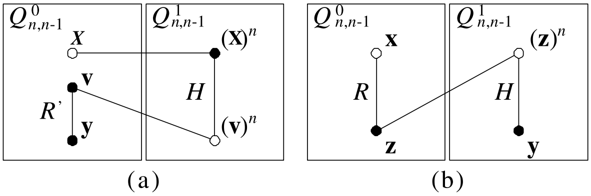


Fig. 5 Illustration for Lemma 9

Theorem 3, there exists a Hamiltonian path R in $Q^0_{n,n-1}$ joining $\mathbf{x}$ to $\mathbf{z}$ and there exists a Hamiltonian path H in $Q^1_{n,n-1}$ joining $(\mathbf{z})^n$ to $\mathbf{y}$. Set $P = \langle \mathbf{x}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{y} \rangle$. Thus, P forms a Hamiltonian path in $Q_{n,n-1}$ joining $\mathbf{x}$ to $\mathbf{y}$. See Fig. 5(b) for illustration. □

Lemma 10 *$Q_{3,2}$ is super spanning connected.*

Proof Let $\mathbf{x}$ and $\mathbf{y}$ be any two different vertices of $Q_{3,2}$. By Lemma 9, $Q_{3,2}$ is 1^*-connected and 2^*-connected. Hence, we need to construct a 3^*-container and a 4^*-container between $\mathbf{x}$ and $\mathbf{y}$. Without loss of generality, we assume that $\mathbf{x} = 000$. By Lemma 7, this statement holds if $\mathbf{y}$ is an odd vertex. Thus, we assume that $\mathbf{y}$ is an even vertex. We list all possible cases as follows:

$\mathbf{y}$	3^*-container	4^*-container
110	⟨000, 010, 110⟩ ⟨000, 100, 110⟩ ⟨000, 001, 011, 101, 111, 110⟩	⟨000, 010, 110⟩ ⟨000, 100, 110⟩ ⟨000, 001, 011, 101, 111, 110⟩ ⟨000, 110⟩
011	⟨000, 010, 011⟩ ⟨000, 100, 101, 001, 011⟩ ⟨000, 110, 111, 011⟩	⟨000, 001, 011⟩ ⟨000, 010, 011⟩ ⟨000, 100, 101, 011⟩ ⟨000, 110, 111, 011⟩
101	⟨000, 001, 011, 101⟩ ⟨000, 010, 110, 111, 101⟩ ⟨000, 100, 101⟩	⟨000, 001, 101⟩ ⟨000, 010, 011, 101⟩ ⟨000, 100, 101⟩ ⟨000, 110, 111, 101⟩

□

Lemma 11 *Suppose that $n \geq 3$ is an odd integer. Let $\mathbf{x}$ and $\mathbf{y}$ be any two different even vertices of $Q_{n,n-1}$. Then there exists a k^*-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \leq k \leq n+1$.*

Proof By Lemma 10, this statement holds for $Q_{3,2}$. Thus, we assume that $n \geq 5$. By Lemma 9, $Q_{n,n-1}$ is 1^*-connected and 2^*-connected. Thus, we need to construct a k^*-container between $\mathbf{x}$ and $\mathbf{y}$ for every $3 \leq k \leq n+1$. Without loss of generality, we assume that $\mathbf{x} \in Q^{00}_{n,n-1}$.

Case 1: $(\mathbf{y})_n = 0$. Since $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ is isomorphic to FQ_{n-1}, by Theorem 3, there exists a $(k-1)^*$-container of $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $2 \le k-1 \le n$. By Lemma 6, there is a k^*-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $3 \le k \le n+1$.

Case 2: $(\mathbf{y})_n = 1$. Since $\mathbf{y}$ is an even vertex, $|\{i \mid i \ne n \text{ and } (\mathbf{y})_i = 1\}|$ is odd. Without loss of generality, we assume that $(\mathbf{y})_{n-1} = 1$. Thus, $\mathbf{y} \in Q^{11}_{n,n-1}$. We have the following cases:

Subcase 2.1: $n \le k \le n+1$. Since $\mathbf{y}$ is adjacent to $(n-2)$ vertices in $Q^{11}_{n,n-1}$, we can choose a neighbor $\mathbf{z}$ of $\mathbf{y}$ in $Q^{11}_{n,n-1}$ such that $(\mathbf{z})^n \ne (\mathbf{x})^{n-1}$. Let $\mathbf{v} = (\mathbf{z})^n$. Obviously, $\mathbf{v}$ is an even vertex. By Theorem 3, there exists an n^*-container $\{R_1, R_2, \ldots, R_n\}$ of $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{v}$. Since $\mathbf{v}$ is adjacent to n vertices in $Q^{00}_{n,m} \cup Q^{10}_{n,m}$, by relabeling, we can write R_i as $\langle \mathbf{x}, R'_i, \mathbf{u}_i, \mathbf{v}\rangle$ for $1 \le i \le n-3$, write R_{n-2} as $\langle \mathbf{x}, R'_{n-2}, (\mathbf{v})^c, \mathbf{v}\rangle$, write R_{n-1} as $\langle \mathbf{x}, R'_{n-1}, (\mathbf{y})^n, \mathbf{v}\rangle$, and write R_n as $\langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, \mathbf{v}\rangle$. Let $A = \{(\mathbf{u}_i)^n | 1 \le i \le n-3\}$. Obviously, A is a set of $(n-3)$ even vertices of $Q^{11}_{n,n-1}$.

Subcase 2.1.1: $k = n+1$. Since Q_{n-2} is a spanning sbgraph of $Q^{11}_{n,n-1}$, by Theorem 1, there is a spanning $(\mathbf{y}, A \cup \{\mathbf{z}\})$-fan, $\{H_1, H_2, \ldots, H_{n-2}\}$, in $Q^{11}_{n,n-1}$ such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \le i \le n-3$ and (2) H_{n-2} joins $\mathbf{z}$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y}\rangle$ for $1 \le i \le n-3$, $P_{n-2} = \langle \mathbf{x}, R'_{n-2}, (\mathbf{v})^c, \mathbf{v}, \mathbf{z}, H_{n-2}, \mathbf{y}\rangle$, and $P_{n-1} = \langle \mathbf{x}, R'_{n-1}, (\mathbf{y})^n, \mathbf{y}\rangle$.

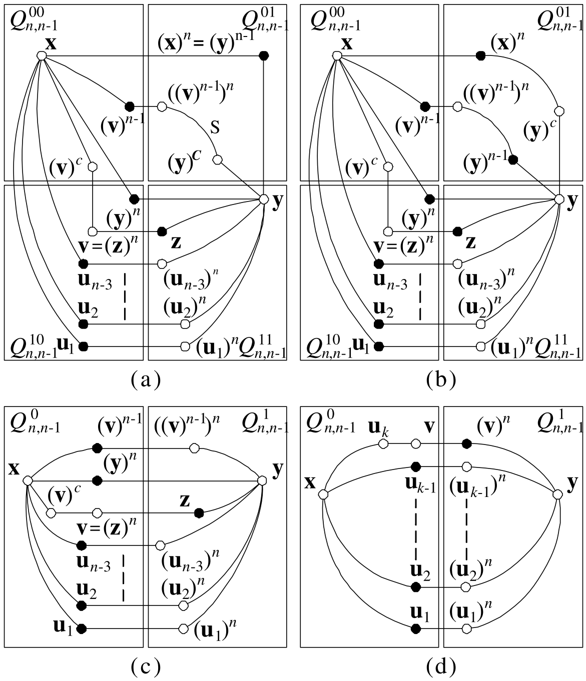
Suppose that $(\mathbf{y})^{n-1} = (\mathbf{x})^n$. Note that Q_{n-2} is a spanning subgraph of $Q^{01}_{n,n-1}$. By Lemma 1, there exists a Hamiltonian path S of $Q^{01}_{n,n-1} - \{(\mathbf{x})^n\}$ joining $((\mathbf{v})^{n-1})^n$ to $(\mathbf{y})^c$. We set $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, S, (\mathbf{y})^c, \mathbf{y}\rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n = (\mathbf{y})^{n-1}, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 6(a) for illustration.

Now, we consider $(\mathbf{y})^{n-1} \ne (\mathbf{x})^n$. Since Q_{n-2} is a spanning subgraph of $Q^{01}_{n,n-1}$, by Lemma 3, there exist two disjoint paths S_1 and S_2 of $Q^{01}_{n,n-1}$ such that (1) S_1 joins $((\mathbf{v})^{n-1})^n$ to $(\mathbf{y})^{n-1}$, (2) S_2 joins $(\mathbf{x})^n$ to $(\mathbf{y})^c$, and (3) $S_1 \cup S_2$ spans $Q^{01}_{n,n-1}$. Let $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, S_1, (\mathbf{y})^{n-1}, \mathbf{y}\rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, S_2, (\mathbf{y})^c, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 6(b) for illustration.

Subcase 2.1.2: $k = n$. Obviously, $A \cup \{((\mathbf{v})^{n-1})^n\}$ is a set of $(n-2)$ even vertices of $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$. Since Q_{n-1} is a spanning subgraph of $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$, by Theorem 1, there is a spanning $(\mathbf{y}, A \cup \{\mathbf{z}, ((\mathbf{v})^{n-1})^n\})$-fan, $\{H_1, H_2, \ldots, H_{n-1}\}$, in $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$ such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \le i \le n-3$, (2) H_{n-2} joins $\mathbf{z}$ to $\mathbf{y}$, and (3) H_{n-1} joins $((\mathbf{v})^{n-1})^n$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y}\rangle$ for $1 \le i \le n-3$, $P_{n-2} = \langle \mathbf{x}, R'_{n-2}, (\mathbf{v})^c, \mathbf{v}, \mathbf{z}, H_{n-2}, \mathbf{y}\rangle$, $P_{n-1} = \langle \mathbf{x}, R'_{n-1}, (\mathbf{y})^n, \mathbf{y}\rangle$, and $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, H_{n-1}, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_n\}$ forms an n^*-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 6(c) for illustration.

Subcase 2.2: $3 \le k \le n-1$. Let $\mathbf{v}$ be an even vertex of $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ such that $\mathbf{v} \ne \mathbf{x}$ and $(\mathbf{y})^n$ is not neighbor of $\mathbf{v}$. Since $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ is isomorphic to FQ_{n-1},

Fig. 6 Illustration for Lemma 11



by Theorem 3, there exists a k^*-container $\{R_1, R_2, \ldots, R_k\}$ of $Q^{00}_{n,n-1} \cup Q^{10}_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{v}$. We write $R_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, \mathbf{v}\rangle$ for $1 \le i \le k$. Let $A = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Since $k \ge 3$, at most one vertex of A is an even vertex. Without loss of generality, we assume that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ is a set of $(k-1)$ odd vertices. Obviously, $\{(\mathbf{u}_i)^n \mid 1 \le i \le k-1\}$ is a set of $(k-1)$ even vertices of $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$. Since Q_{n-1} is a spanning subgraph of $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$, by Theorem 1, there is a spanning $(\mathbf{y}, \{(\mathbf{u}_i)^n \mid 1 \le i \le k-1\} \cup \{(\mathbf{v})^n\})$-fan, $\{H_1, H_2, \ldots, H_k\}$, of $Q^{01}_{n,n-1} \cup Q^{11}_{n,n-1}$ such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \le i \le k-1$ and (2) H_k joins $(\mathbf{v})^n$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y}\rangle$ for $1 \le i \le k-1$ and $P_k = \langle \mathbf{x}, R'_k, \mathbf{u}_k, \mathbf{v}, (\mathbf{v})^n, H_k, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a k^*-container of $Q_{n,n-1}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 6(d) for illustration. □

Lemma 12 *Suppose that $n \ge 3$ and m is even. Let $\mathbf{x}$ and $\mathbf{y}$ be any two different even vertices of $Q_{n,m}$. Then there exists a Hamiltonian path P of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$.*

Proof For the fixed number m, we prove this statement by induction on $t = n - m$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ be any two different even vertices of $Q_{n,m}$. By Lemma 10, this statement holds for $t = 1$. Consider $t \ge 2$ and assume that our claim holds for $(t-1)$. Without loss of generality, we assume that $\mathbf{x} \in Q^0_{n,m}$.

Suppose that $\mathbf{y} \in Q^0_{n,m}$. Since $Q^0_{n,m}$ is isomorphic to $Q_{n-1,m}$, by induction, there exists a Hamiltonian path $R = \langle \mathbf{x}, \mathbf{v}, R', \mathbf{y}\rangle$ in $Q^0_{n,m}$ joining $\mathbf{x}$ to $\mathbf{y}$ and there exists a Hamiltonian path H in $Q^1_{n,m}$ joining $(\mathbf{x})^n$ to $(\mathbf{v})^n$. Set $P = \langle \mathbf{x}, (\mathbf{x})^n, H, (\mathbf{v})^n, \mathbf{v}, R', \mathbf{y}\rangle$. Thus, P forms a Hamiltonian path in $Q_{n,m}$ joining $\mathbf{x}$ to $\mathbf{y}$.

Suppose that $\mathbf{y} \in Q^1_{n,m}$. We pick an even vertex $\mathbf{z}$ in $Q^0_{n,m}$ such that $\mathbf{z} \neq \mathbf{x}$. By induction, there exists a Hamiltonian path R in $Q^0_{n,m}$ joining $\mathbf{x}$ to $\mathbf{z}$. Obviously, $(\mathbf{z})^n$ is an odd vertex of $Q^1_{n,m}$. Since Q_{n-1} is a spanning subgraph of $Q^1_{n,m}$, by Lemma 2, there exists a Hamiltonian path H in $Q^1_{n,m}$ joining $(\mathbf{z})^n$ to $\mathbf{y}$. Set $P = \langle \mathbf{x}, R, \mathbf{z}, (\mathbf{z})^n, H, \mathbf{y}\rangle$. Thus, P forms a Hamiltonian path in $Q_{n,m}$ joining $\mathbf{x}$ to $\mathbf{y}$. □

Lemma 13 *Suppose that $n \geq 3$ and m is even. Let $\mathbf{x}$ and $\mathbf{y}$ be any two different even vertices of $Q_{n,m}$. Then there exists a k^*-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \leq k \leq n+1$.*

Proof By Lemma 10, this statement holds for $n = 3$. Suppose that $n \geq 4$. We claim that there exists a k^*-container between $\mathbf{x}$ and $\mathbf{y}$ for every $1 \leq k \leq n+1$. By Lemma 12, this statement holds for $k = 1$. Note that Q_n is a spanning subgraph of $Q_{n,m}$ and Q_n is Hamiltonian. So, this statement holds for $k = 2$. We claim that there exists a k^*-container between $\mathbf{x}$ and $\mathbf{y}$ for every $3 \leq k \leq n+1$. Without loss of generality, we assume that $\mathbf{x} \in Q^{00}_{n,m}$. We prove our claim by induction on $t = n - m$. The induction bases are $t = 0$ and 1. By Theorem 3, our claim holds for $t = 0$. With Lemma 11, this statement holds for $t = 1$. Consider $t \geq 2$ and assume that this statement holds for $(t-1)$. We have the following cases.

Case 1: $\mathbf{y} \in Q^{00}_{n,m} \cup Q^{10}_{n,m}$. Since $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ is isomorphic to $Q_{n-1,m}$, by induction, there exists a $(k-1)^*$-container $\{P_1, P_2, \ldots, P_{k-1}\}$ of $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $2 \leq k-1 \leq n$. By Lemma 6, there exists a k^*-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$.

Case 2: $\mathbf{y} \in Q^{01}_{n,m}$. Note that $Q^{01}_{n,m}$ and $Q^{10}_{n,m}$ are symmetric with respect to $Q_{n,m}$ and $Q^{00}_{n,m} \cup Q^{01}_{n,m}$ is isomorphic to $Q_{n-1,m}$. Similar to Case 1, there is a k^*-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$ for every $3 \leq k \leq n+1$.

Case 3: $\mathbf{y} \in Q^{11}_{n,m}$.

Subcase 3.1: $3 \leq k \leq n$. Let $\mathbf{v}$ be an even vertex of $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ such that $\mathbf{v} \neq \mathbf{x}$ and $(\mathbf{y})^n$ is not a neighbor of $\mathbf{v}$. By induction, there exists a k^*-container $\{R_1, R_2, \ldots, R_k\}$ of $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ between $\mathbf{x}$ and $\mathbf{v}$. We write $R_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, \mathbf{v}\rangle$ for $1 \leq i \leq k$. Let $A = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Since $k \geq 3$, at most one vertex of A is an even vertex. Without loss of generality, we assume that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ is a set of $(k-1)$ odd vertices. Obviously, $\{(\mathbf{u}_i)^n \mid 1 \leq i \leq k-1\}$ is a set of $(k-1)$ even vertices of $Q^{01}_{n,m} \cup Q^{11}_{n,m}$. By Theorem 1, there is a spanning $(\mathbf{y}, \{(\mathbf{u}_i)^n \mid 1 \leq i \leq k-1\} \cup \{(\mathbf{v})^n\})$-fan, $\{H_1, H_2, \ldots, H_k\}$, of $Q^{01}_{n,m} \cup Q^{11}_{n,m}$ such that (1) H_i joins $(\mathbf{u}_i)^n$ to $\mathbf{y}$ for $1 \leq i \leq k-1$ and (2) H_k joins $(\mathbf{v})^n$ to $\mathbf{y}$. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H_i, \mathbf{y}\rangle$ for $1 \leq i \leq k-1$ and $P_k = \langle \mathbf{x}, R'_k, \mathbf{u}_k, \mathbf{v}, (\mathbf{v})^n, H_k, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_k\}$ forms a k^*-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$.

Subcase 3.2: $k = n+1$. Since $\mathbf{y}$ is adjacent to $(n-1)$ vertices in $Q^{11}_{n,m}$, we can choose a neighbor $\mathbf{z}$ of $\mathbf{y}$ in $Q^{11}_{n,m}$ such that $\mathbf{z} \neq (\mathbf{y})^c$ and $(\mathbf{z})^n \neq (\mathbf{x})^{n-1}$. Let $\mathbf{v} = (\mathbf{z})^n$. Obviously, both $\mathbf{v}$ and $((\mathbf{v})^{n-1})^n$ are even vertices. By induction, there exists an n^*-container $\{R_1, R_2, \ldots, R_n\}$ of $Q^{00}_{n,m} \cup Q^{10}_{n,m}$ between $\mathbf{x}$ and $\mathbf{v}$. Since $\mathbf{v}$ is adjacent to n vertices in $Q^{00}_{n,m} \cup Q^{10}_{n,m}$, by relabeling, we can write R_i as $\langle \mathbf{x}, R'_i, \mathbf{u}_i, \mathbf{v}\rangle$ for

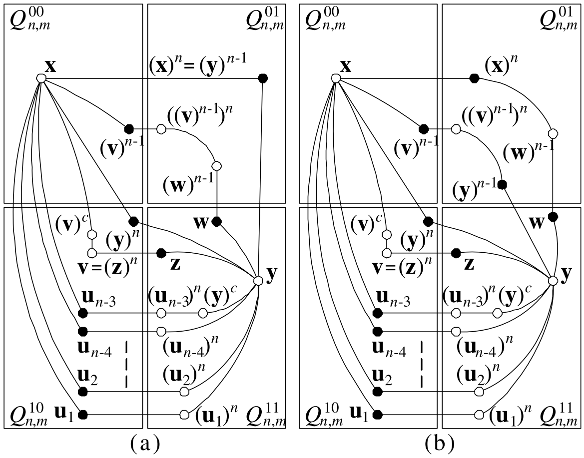


Fig. 7 Illustration for Lemma 13

$1 \le i \le n-3$, write R_{n-2} as $\langle \mathbf{x}, R'_{n-2}, (\mathbf{v})^c, \mathbf{v}\rangle$, write R_{n-1} as $\langle \mathbf{x}, R'_{n-1}, (\mathbf{y})^n, \mathbf{v}\rangle$, and write R_n as $\langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, \mathbf{v}\rangle$. Let $A = \{(\mathbf{u}_i)^n \mid 1 \le i \le n-3\}$. Obviously, A is a set of $(n-3)$ even vertices of $Q^{11}_{n,m}$.

By Lemma 2, there exists an $(n-2)^*$-container $\{H_1, H_2, \ldots, H_{n-2}\}$ of $Q^{11}_{n,m}$ between $\mathbf{z}$ and $\mathbf{y}$. Since $\mathbf{y}$ is adjacent to $(n-1)$ vertices in $Q^{11}_{n,m}$ and $(\mathbf{z}, \mathbf{y}) \in E(Q^{11}_{n,m})$, one of these paths is $\langle \mathbf{z}, \mathbf{y}\rangle$ and $(\mathbf{y})^c \in H_i$ for some $1 \le i \le n-2$. Without loss of generality, we assume that $(\mathbf{y})^c \in H_{n-3}$. We can write H_i as $\langle \mathbf{z}, (\mathbf{u}_i)^n, H'_i, \mathbf{y}\rangle$ for $1 \le i \le n-4$, write H_{n-3} as $\langle \mathbf{z}, (\mathbf{u}_{n-3})^n, H'_{n-3}, (\mathbf{y})^c, \mathbf{w}, H''_{n-3}, \mathbf{y}\rangle$, and write H_{n-2} as $\langle \mathbf{z}, \mathbf{y}\rangle$. Obviously, $\mathbf{w}$ is an odd vertex. We set $P_i = \langle \mathbf{x}, R'_i, \mathbf{u}_i, (\mathbf{u}_i)^n, H'_i, \mathbf{y}\rangle$ for $1 \le i \le n-4$, $P_{n-3} = \langle \mathbf{x}, R'_{n-3}, \mathbf{u}_{n-3}, (\mathbf{u}_{n-3})^n, H'_{n-3}, (\mathbf{y})^c, \mathbf{y}\rangle$, $P_{n-2} = \langle \mathbf{x}, R'_{n-2}, (\mathbf{v})^c, \mathbf{v}, \mathbf{z}, H_{n-2}, \mathbf{y}\rangle$, and $P_{n-1} = \langle \mathbf{x}, R'_{n-1}, (\mathbf{y})^n, \mathbf{y}\rangle$.

Suppose that $(\mathbf{y})^{n-1} = (\mathbf{x})^n$. By Lemma 1, there exists a Hamiltonian path S of $Q^{01}_{n,m} - \{(\mathbf{x})^n\}$ joining $((\mathbf{v})^{n-1})^n$ to $(\mathbf{w})^{n-1}$. Set $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, S, (\mathbf{w})^{n-1}, \mathbf{w}, H''_{n-3}, \mathbf{y}\rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n = (\mathbf{y})^{n-1}, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 7(a) for illustration.

Now, we consider $(\mathbf{y})^{n-1} \ne (\mathbf{x})^n$. By Lemma 3, there exist two disjoint paths S_1 and S_2 of $Q^{11}_{m,n}$ such that (1) S_1 joins $((\mathbf{v})^{n-1})^n$ to $(\mathbf{y})^{n-1}$, (2) S_2 joins $(\mathbf{x})^n$ to $(\mathbf{w})^{n-1}$, and (3) $S_1 \cup S_2$ spans $Q^{01}_{n,n-1}$. Set $P_n = \langle \mathbf{x}, R'_n, (\mathbf{v})^{n-1}, ((\mathbf{v})^{n-1})^n, S_1, (\mathbf{y})^{n-1}, \mathbf{y}\rangle$ and $P_{n+1} = \langle \mathbf{x}, (\mathbf{x})^n, S_2, (\mathbf{w})^{n-1}, \mathbf{w}, H''_{n-3}, \mathbf{y}\rangle$. Then $\{P_1, P_2, \ldots, P_{n+1}\}$ forms an $(n+1)^*$-container of $Q_{n,m}$ between $\mathbf{x}$ and $\mathbf{y}$. See Fig. 7(b) for illustration. □

With Lemma 8 and Lemma 13, we have the following theorem.

Theorem 4 *The enhanced hypercube $Q_{n,m}$ is super spanning laceable if m is an odd integer and $Q_{n,m}$ is super spanning connected if m is an even integer.*

Proof Since $Q_{2,2}$ is isomorphic to complete graph K_4. Obviously, this theorem holds for $n = 2$. By Lemma 8, this theorem holds if $n \ge 3$ and m is an odd integer. Thus, we suppose that $n \ge 3$ and m is an even integer. Let $\mathbf{x}$ and $\mathbf{y}$ be any two different vertices of $Q_{n,m}$. We need to find a k^*-container of $Q_{n,m}$ between **x** and **y** for every $1 \le k \le n+1$. Without loss of generality, we assume that **x** is an even vertex. By Lemma 8, this theorem holds if **y** is an odd vertex. By Lemma 13, this theorem holds if **y** is an even vertex. Thus, this theorem is proved. □

5 Conclusion

With Theorem 1, we can easily prove again that Q_n is super spanning laceable in [4]. Let **x** and **y** be any two vertices in the different partite set of Q_n. Assume $1 \le k \le n$. Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{k-1}$ be any $(k-1)$ neighbor of **y** with $\mathbf{y}_i \ne \mathbf{x}$ for $1 \le i \le k-1$. Let $\mathbf{y}_k = \mathbf{y}$ and $S = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$. By Theorem 1, there exists a spanning $(\mathbf{x}, S)$-fan $\{R_1, R_2, \ldots, R_k\}$ such that R_i is a path joining **x** to $\mathbf{y}_i$. Then we set $P_i = \langle \mathbf{x}, R_i, \mathbf{y}_i, \mathbf{y} \rangle$ for $1 \le i \le k-1$ and $P_k = R_k$. Obviously, $\{P_1, P_2, \ldots, P_k\}$ forms a k^*-container between **x** and **y**. Thus, the existence of spanning k-fan implies the existence of spanning k-container. However, the converse is not correct. Thus, there is a lot of work to be done on spanning k-fan.

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