國立交通大學

應用數學系

碩 士 論 文

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Refinements of Kurzweil's Theorem in the Field of Formal Laurent Series

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中 華 民 國 一 〇 三 年 六 月

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A Thesis
Submitted to Department of Applied Mathematics
National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Master
in

Applied Mathematics

June 2014

Hsinchu, Taiwan, Republic of China

中華民國一〇三年六月

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June 20, 2014

前言

在過去十年裡，已經有了許多關於正規 Laurent 級數體下之賦距 Diophantine 逼近的研究，而最近這項研究有了一個有趣的新研究方向——關於 Kurzweil 定理的改良。本論文主要的工作就是總結整理這些不同方向的改良以及提供一些新的貢獻。

其中一項改良是由 Kim、Nakada 和 Natsui 在[6]中所提出，在本文中，我們將指出他們提出的證明當中，有部分細節是可以被改進的;更精確來說，我們可以將其中的單調性條件拿掉，而這可以讓我們重新證明 Kurzweil 定理的其中一個方向。本文另一個主題是關於 Kurzweil 本身在[8]中所提出的在實數體上的另一個改良，我們將證明這個定理在正規 Laurent 級數體下的相似結果並與另一個最近由 Kim、Tan、Wang 與 Xu 在[7]提出的與之相似的改良作比較。

本文的主要架構如下：我們將在第一章介紹一些關於 Diophantine 逼近的背景知識以及本論文的目標。第一節中，我們將回顧有關 Diophantine 逼近的基本性質和介紹我們所用的符號。在第二節，我們將介紹 Diophantine 逼近和賦距 Diophantine 逼近，這可以分成 homogeneous 和 inhomogeneous 的情形。我們收集一些關於這兩個情形的結果，尤其是包含所謂的 double-metric 和 single-metric 兩種情況的 inhomogeneous 情形。最後，我們將在第三節介紹本論文的主要目的。

在第二章，我們將探討 Kim、Nakada 和 Natsui 所提出的改良。我們將在第一節陳述一些引理和呈現他們在改良中的證明。第二節中，我們將改進前一節中的證明，並利用改進後的結果來證明 Kurzweil 定理的其中一個方向。第三節，我們將利用完全不同於第一節的證明方法來證明一個特殊情形。

我們在第三章證明一個與 Kurzweil 在[8]提出之改良相似的結果。因為這個結果與另一個最近由 Kim、Tan、Wang 與 Xu 提出的改良有些異同處，我們將在這個章節的最後讓兩者作些比較。

最後，在第四章，我們將提出一些猜想來對本論文做一個總結。

Preface

The last decade has witnessed a lot of research about metric Diophantine approximation in the field of formal Laurent series, where a recent new and interesting research direction was concerned with refinements of Kurzweil's theorem. The purpose of this thesis is to summarize these refinements and give some new contributions.

One of these refinements was given by Kim, Nakada and Natsui in [6]. In this thesis, we will show that some details of their proofs can be improved. More precisely, we are able to drop the monotonicity condition and this will allow us to reprove one direction of Kurzweil's theorem. Another topic of this thesis will be concerned with another refinement of Kurzweil's theorem which in the real case was obtained by Kurzweil himself in [8]. We will prove an analogue of this theorem in the field of formal Laurent series and compare it with another refinement of a similar flavour which was recently proved by Kim, Tan, Wang and Xu in [7].

An outline of this thesis is as follows. In Chapter 1, we will introduce some background knowledge on Diophantine approximation and explain our aim of this thesis. There are three sections in this chapter. In Section 1.1, we will recall some fundamental properties for formal Laurent series and give some notations. Then, in Section 1.2, we will introduce Diophantine approximation and metric Diophantine approximation. This introduction will be split into homogeneous and inhomogeneous cases. We will collect some results for the two cases, especially the inhomogeneous case which consists of the so-called double-metric and single-metric cases. Finally, we will state the main goal of this thesis in Section 1.3.

In Chapter 2, we will discuss the refinement of Kim, Nakada and Natsui. In Section 2.1, we will state some lemmas and present the proof of their refinement. In Section 2.2, we will give some improvements of the proofs from the previous section and use them to prove one direction of Kurzweil's theorem. In Section 2.3, we will prove a special case of Kim, Nakada and Natsui's refinement with a completely different method as the one in Section 2.1.

In Chapter 3, we will show an analogue of the refinement proved by Kurzweil in [8]. Since this result and another refinement which was recently proved by Kim, Tan, Wang and Xu have some similarities and differences, we will compare them at the end of this chapter.

Finally, in Chapter 4, we will end the thesis with some conjectures.

致 謝 辭

猶記得剛踏入交大校園時內心澎湃的自己，一眨眼竟即將跨出校門。就讀研究所的這兩年，受到了許多人的幫助，最感謝的，莫過於我的指導老師——符麥克教授了，讓我在學習的過程當中，漸漸摸索出自己的方向，並耐心指導我論文的寫作。此外，我也要特別感謝在口試當中給予誠摯建議的兩位口試委員——蕭守仁教授以及楊一帆教授，讓我順利地通過口試。

在研究所的路途上，很開心認識珮柔、妃儀及晏昀等知心的好室友，有你們在的寢室就充滿歡笑，每當我從研究室疲憊地回到宿舍，都能好好放鬆心情。在修課的過程中，感謝同學們幫助，尤其是惠閔、伊婕和伯融，每週的討論作業時間，一起努力合作，互相指教，讓我學習到了許多不同方面的見解。另外，謝謝跟我一起擔任微積分助教的又菱、嬿蓉及家銘，有你們這些夥伴互相合作，才能更有效率地完成任務。也感謝男朋友書誼在我不安時給予的支持與鼓勵，讓我更有動力去面對困難。

在跨出校門的那一刻，等著我的是一個嶄新的路途，希望在未來的道路上，能夠牢牢記住這兩年努力的自己，勇敢面對將來的挑戰。

最後，謹以此文獻給我摯愛的家人。

佳穎　謹誌

Contents

Chapter 1

Introduction

1.1 Fundamental Properties

In the beginning, we will introduce some essential knowledge which is necessary for this thesis. Let $\mathbb{F}_q$ be a finite field of size $q = p^k$ (where p is a prime number and $k \in \mathbb{N}$) and $\mathbb{F}_q[X]$ be the set that contains all polynomials over $\mathbb{F}_q$. We denote by $\mathbb{F}_q(X)$ the quotient set of $\mathbb{F}_q[X]$. With the above notations, define

$$\mathbb{F}_q\left(\left(X^{-1}\right)\right) := \left\{ f = \sum_{i=-\infty}^{N} a_i X^i : a_N \neq 0, a_i \in \mathbb{F}_q \right\} \cup \{0\}$$

to be the set of formal Laurent series. To turn the set $\mathbb{F}_q\left((X^{-1})\right)$ into a field, we define addition and multiplication on it as for polynomials. Then, we have the following proposition that was proved in [9].

Proposition 1.1.1. $(\mathbb{F}_q\left((X^{-1})\right), +, \cdot)$ *is a field.*

Comparing with the real case, $\mathbb{F}_q[X]$, $\mathbb{F}_q(X)$ and $\mathbb{F}_q\left((X^{-1})\right)$ play the roles of integers, rational numbers and real numbers, respectively.

Next, we define a norm for each $f = \sum_{i=-\infty}^{N_0} a_i X^i \in \mathbb{F}_q\left((X^{-1})\right)$ with $a_{N_0} \neq 0$ by $|f| = q^{\deg(f)} = q^{N_0}$ and $|0| = 0$. The following property shows that the norm is non-Archimedean.

Proposition 1.1.2. *For any* $f, g \in \mathbb{F}_q\left((X^{-1})\right)$*, we have*

(1) $|f| = 0 \Leftrightarrow f = 0.$

(2) $|fg| = |f|\,|g|.$

(3) $|f+g| \leq \max\{|f|, |g|\}$.

Proof.

(1) $|f| = 0 \Leftrightarrow \deg(f) = -\infty \Leftrightarrow f = 0$.

(2) $|fg| = q^{\deg(fg)} = q^{\deg(f)+\deg(g)} = q^{\deg(f)}q^{\deg(g)} = |f|\,|g|$.

(3) $$\begin{aligned}|f+g| = q^{\deg(f+g)} = q^{\max\{\deg(f),\deg(g)\}} &= \max\{q^{\deg(f)}, q^{\deg(g)}\} \\ &= \max\{|f|, |g|\}.\end{aligned}$$ ∎

Now, we are going to introduce a set, denoted by $\mathbb{L}$, which plays the role of the unit interval $[0,1)$ in $\mathbb{F}_q((X^{-1}))$. It is the subset of $\mathbb{F}_q((X^{-1}))$ which consists of elements f with $\deg(f) < 0$. In other words,

$$\mathbb{L} := \left\{f \in \mathbb{F}_q((X^{-1})) : |f| < 1\right\}.$$

By restricting the above norm to this set, we get that $\mathbb{L}$ is a compact topological group. Thus, there exists an unique translation-invariant measure called Haar measure which we denote by μ.

Define for all $g \in \mathbb{L}$,

$$B\left(g; \frac{1}{q^k}\right) := \left\{f \in \mathbb{L} : |f - g| < \frac{1}{q^k}\right\} \text{ with } k \geq 1.$$

The following property is a characterization of this measure.

Proposition 1.1.3. *Choose* $b_1, b_2, \ldots, b_k \in \mathbb{F}_q$*,* $g \in \mathbb{L}$ *and* $k \geq 1$*. Then*

$$\mu\left(\left\{f = \sum_{j=1}^{\infty} c_j X^{-j} : c_i = b_i \text{ for } 1 \leq i \leq k\right\}\right) = \frac{1}{q^k},$$

and

$$\mu\left(B\left(g; \frac{1}{q^k}\right)\right) = \frac{1}{q^k}.$$

Proof. Assume that $h = b_1X^{-1} + b_2X^{-2} + \cdots + b_kX^{-k}$. Then, for any $f = \sum_{j=1}^{\infty} c_jX^{-j} \in \mathbb{L}$, we have $f \in B\left(h; q^{-k}\right)$ if and only if $c_i = b_i$ for $1 \leq i \leq k$. Hence,

$$\left\{f = \sum_{j=1}^{\infty} c_j X^{-j} : c_i = b_i \text{ for } 1 \leq i \leq k\right\} = B\left(h; \frac{1}{q^k}\right).$$

Next, we consider the set $\mathbb{L}$ to be the union

$$\bigcup_{b_1,\ldots,b_k \in \mathbb{F}_q} \left\{f = \sum_{j=1}^{\infty} c_j X^{-j} : c_i = b_i \text{ for } 1 \leq i \leq k\right\}.$$

Then we obtain

$$\begin{aligned}1 &= \mu(\mathbb{L}) \\ &= \mu\left(\bigcup_{b_1,\ldots,b_k\in\mathbb{F}_q}\left\{f=\sum_{j=1}^{\infty}c_jX^{-j} : c_i=b_i \text{ for } 1\le i\le k\right\}\right) \\ &= \sum_{b_1,\ldots,b_k\in\mathbb{F}_q}\mu\left(\left\{f=\sum_{j=1}^{\infty}c_jX^{-j} : c_i=b_i \text{ for } 1\le i\le k\right\}\right) \\ &= q^k\,\mu\left(\left\{f=\sum_{j=1}^{\infty}c_jX^{-j} : c_i=b_i \text{ for } 1\le i\le k\right\}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mu\left(B\left(h;\frac{1}{q^k}\right)\right) &= \mu\left(\left\{f=\sum_{j=1}^{\infty}c_jX^{-j} : c_i=b_i \text{ for } 1\le i\le k\right\}\right) \\ &= \frac{1}{q^k}.\end{aligned}$$

Since μ is translation-invariant, we have

$$\mu\left(B\left(g;\frac{1}{q^k}\right)\right)=\mu\left(B\left(h;\frac{1}{q^k}\right)\right)=\frac{1}{q^k}\quad\text{for all } g\in\mathbb{L} \qquad \blacksquare$$

Next, we are going to prove an important property of μ.

Proposition 1.1.4. *Each two balls in $\mathbb{L}$ are either disjoint or one is contained in the other.*

Proof. Let $B_1\left(f;q^{-k_1}\right)$ and $B_2\left(g;q^{-k_2}\right)$ be two balls in $\mathbb{L}$. Without loss of generality, we suppose that $k_1>k_2$. If they are not disjoint, then there must exist an element $h\in\mathbb{L}$ such that $h\in B_1\cap B_2$. We have

$$|f-g|=|f-h+h-g|\le\max\{|f-h|,|h-g|\}\le\frac{1}{q^{k_2}}.$$

This implies that $f\in B_2\left(g;q^{-k_2}\right)$.

Now, we want to prove that B_1 is contained in B_2. If there exists $h'\in\mathbb{L}$ such that $h'\in B_1\setminus B_2$, then we get

$$|h'-g|=|h'-f+f-g|\le\max\{|h'-f|,|f-g|\}<\frac{1}{q^{k_2}},$$

which means that $h'\in B_2$. We have a contradiction. Therefore,

$$B_1\left(f;\frac{1}{q^{k_1}}\right)\subset B_2\left(g;\frac{1}{q^{k_2}}\right).$$

The proof is complete. ∎

Similar to the integer part for real numbers, we define for each $f = \sum_{i=-\infty}^{N} a_i X^i \in \mathbb{F}_q\left((X^{-1})\right)$,

$$[f] = \sum_{i=0}^{N} a_i X^i \text{ and } \{f\} = f - [f].$$

We call the former the polynomial part and the latter the fractional part of f.

If we fix an element $f \in \mathbb{F}_q\left((X^{-1})\right)$, then it can be expressed as the sum of its polynomial part and fractional part. That is, $f = [f] + \{f\}$. We can rewrite this as

$$f = A_0 + \frac{1}{g_1}, \quad \text{where } A_0 = [f] \text{ and } g_1 = \frac{1}{\{f\}}.$$

Again, we represent g_1 as the sum of its polynomial part, which we denote by A_1, and the reciprocal of $1/\{g_1\}$. We let $g_2 = 1/\{g_1\}$. Continuing this iterative process an infinite number of times or until g_k is a polynomial for some $k \in \mathbb{N}$, f can be expressed as a unique continued fraction expansion with $A_j = [g_j]$ for $g_j = 1/\{g_{j-1}\}$, $j \geq 2$.

Note that f is *irrational* if and only if the process does not terminate. Thus, for each irrational $f \in \mathbb{L}$, we have

$$f = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots}}},$$

where the A_i's are called *partial quotients*. Moreover, we put

$$\frac{P_k}{Q_k} = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots + \cfrac{1}{A_k}}}}, \ (P_k, Q_k) = 1 \text{ with } P_0 = 0 \text{ and } Q_0 = 1,$$

which are called the *principle convergent* of f with $\deg(Q_k) := n_k$. We have the following property.

Lemma 1.1. *Fix an irrational $f \in \mathbb{L}$. If P_k/Q_k are the principle convergents of f with partial quotients A_k, then for $k \in \mathbb{N}$, we have the recurrence relation*

$$\begin{cases} P_{k+1} = A_{k+1}P_k + P_{k-1} \\ Q_{k+1} = A_{k+1}Q_k + Q_{k-1} \end{cases}, \text{ with initial conditions } P_0 = 0,\ Q_0 = 1,\ P_1 = 1,\ Q_1 = A_1.$$

Consequently,

$$n_{k+1} - n_k = \deg(A_{k+1}),$$

for all $k \in \mathbb{N} \cup \{0\}$.

Lemma 1.2. *For any $k \geq 0$, we have*

$$P_{k+1}Q_k - P_kQ_{k+1} = (-1)^k$$

and

$$|\{Q_k f\}| = \frac{1}{|Q_{k+1}|}.$$

Next, we are going to introduce Diophantine approximation in the field of formal Laurent series.

1.2 Diophantine Approximation in the Field of Formal Laurent Series

The main problem of Diophantine approximation in the field of formal Laurent series is as follows: for $f \in \mathbb{L}$, find a function ψ from $\{q^k : k \in \mathbb{N} \cup \{0\}\}$ into $\{q^k : k \in \mathbb{Z}\}$ such that the Diophantine inequality

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(|Q|)}{|Q|} \text{ with } P, Q \in \mathbb{F}_q[X],\ Q \neq 0, \tag{1.1}$$

has infinitely many solutions P and Q?

The following theorem which is an analogue of Dirichlet's theorem is a representative result in this area.

Theorem 1.1. *Let $f \in \mathbb{L}$. Then,*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2} \textit{ with } P, Q \in \mathbb{F}_q[X],\ Q \neq 0,$$

has infinitely many solutions P and Q.

In the above theorem, $\psi(|Q|)$ was chosen as $1/|Q|$ and this result holds for each $f \in \mathbb{L}$.

Now, we will discuss the subarea called *metric Diophantine approximation in the field of formal Laurent series* which asks for properties that hold for almost all $f \in \mathbb{L}$. We will give some results from this area. Consider the inequality (1.1) for $\psi(|Q|) = q^{-n-l_n}$ with $l_n \geq 1$ and $n = \deg(Q)$. Then, (1.1) becomes

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_n}} \text{ with } P, Q \in \mathbb{F}_q[X],\ Q \neq 0,\ Q \text{ monic},\ n = \deg(Q). \tag{1.2}$$

In [4], Inoue and Nakada investigated the condition that makes (1.2) have infinitely many solutions P and Q for almost all $f \in \mathbb{L}$.

Theorem 1.2 (K. Inoue and H. Nakada [4]). *The inequality (1.2) has infinitely many solutions P and Q for almost all $f \in \mathbb{L}$ if and only if*

$$\sum_{n=0}^{\infty} q^{-l_n} = \infty.$$

Furthermore, Nakada and Natsui found the following result on the asymptotic number of solutions $Q \in \mathbb{F}_q[X]$ in [11].

Theorem 1.3 (H. Nakada and R. Natsui [11]). *Define*

$$\Psi(N) := \sum_{n \leq N} \frac{1}{q^{l_n}}.$$

If l_n satisfies

(i) l_n is non-decreasing and $\sum_{n=0}^{\infty} q^{-l_n} = \infty$,

(ii) there exists a constant $D > 1$ such that $j_{k+1} > Dj_k$ for $k \geq 1$, where

$$j_1 := \min\{n \geq 2 : l_n - l_{n-1} \geq 1\},$$
$$j_k := \min\{n > j_{k-1} : l_n - l_{n-1} \geq 1\}, \;\; \textit{for} \;\; k \geq 2.$$

Then, for almost all $f \in \mathbb{L}$, the number of solutions of (1.2) with $0 \leq \deg(Q) \leq N$ is asymptotic to $\Psi(N)$.

In addition, Fuchs obtained an improvement of the above theorem by dropping the conditions for l_n and adding an error term to this result.

Theorem 1.4 (M. Fuchs [3]). *For almost all $f \in \mathbb{L}$, the number of solutions of (1.2) with $0 \leq \deg(Q) \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} \left(\log \Psi(N)\right)^{2+\epsilon}\right),$$

where $\Psi(N)$ is as above and $\epsilon > 0$ is an arbitrary constant.

So far, what we have discussed is the so-called *homogeneous case*. In this case, the inequlaity (1.1) can be rewritten as

$$|Qf - P| < \psi(|Q|) \;\; \text{with} \;\; P, Q \in \mathbb{F}_q[X].$$

If $\psi(|Q|) \leq 1$, then we get that the degree of $Qf - P$ is less than zero so that P is the polynomial part of Qf. This implies that this inequality can be simplified to

$$|\{Qf\}| < \psi(|Q|) \;\; \text{with} \;\; Q \in \mathbb{F}_q[X].$$

Now, we are going to be concerned with the *metric inhomogeneous Diophantine approximation* problem. For $f,\, g \in \mathbb{L}$, consider the Diophantine inequality

$$|\{Qf\} - g| < \psi(|Q|) \;\text{ with }\; Q \in \mathbb{F}_q[X], \tag{1.3}$$

where ψ is a function from $\{q^k : k \in \mathbb{N} \cup \{0\}\}$ into $\{q^k : k \in \mathbb{Z}\}$. The main question in this area is again the existence of infinitely many solutions to (1.3).

Inhomogeneous Diophantine approximation consists of two cases: *double-metric* and *single-metric*.

Double-metric, as the name suggests, is the case for which f and g are both random. Consider the inequality (1.3) for $\psi(|Q|) = q^{-n-l_n}$ with $l_n \geq 1$ such that

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}} \;\text{ with }\; Q \in \mathbb{F}_q[X],\; Q \text{ monic},\; n = \deg(Q), \tag{1.4}$$

where $f,\, g \in \mathbb{L}$. Ma and Su studied (1.4) for the double-metric case in [10]. The following theorem is their result.

Theorem 1.5 (C. Ma and W.-Y. Su [10]). *The inequality (1.4) has infinitely many solutions $Q \in \mathbb{F}_q[X]$ for almost all $(f, g) \in \mathbb{L}^2$ if and only if*

$$\sum_{n=0}^{\infty} \frac{1}{q^{l_n}} = \infty.$$

Moreover, the asymptotic number of solutions Q was estimated by Fuchs in [3]. He derived a strong law of large numbers with error terms for the number of solutions of (1.4) with $\deg(Q) \leq N$. We have the following result.

Theorem 1.6 (M. Fuchs [3]). *For almost all $(f, g) \in \mathbb{L}^2$, the number of solutions of (1.4) with $0 \leq \deg(Q) \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{\frac{1}{2}} \left(\log \Psi(N)\right)^{\frac{3}{2}+\epsilon} \right),$$

where $\Psi(N) = \sum_{n \leq N} q^{-l_n}$ and $\epsilon > 0$ is an arbitrary constant.

Next, we are going to introduce the two *single-metric* cases:

(1) fix g and choose a random $f \in \mathbb{L}$,

(2) fix f and choose a random $g \in \mathbb{L}$.

As for Case 1, Fuchs in fact showed that Theorem 1.4 holds for any fixed g (not only $g = 0$).

In this research, we will focus on the Case 2. Our main objective of this thesis will be introduced in the next section.

1.3 Kurzweil's Theorem and its Refinements

In this section, we will introduce the main topic of this thesis. Consider the Diophantine inequality for f fixed and g random

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}} \text{ with } Q \in \mathbb{F}_q[X],\ n = \deg(Q), \tag{1.5}$$

where $\{l_n\}$ is a given sequence of positive integers. In [5], Kim and Nakada studied the following problem: for any sequence $\{l_n\}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$, which condition do we need for $f \in \mathbb{L}$ such that (1.5) has infinitely many solutions in Q for almost all $g \in \mathbb{L}$? They found that the condition for f satisfying the above property is that f is *badly approximable* whose definition we give next:

Definition 1.1. *f is **badly approximable** if and only if there exists a constant $c > 0$ such that*

$$|\{Qf\}| > \frac{1}{q^{n+c}},\ n = \deg(Q),$$

for all $Q \in \mathbb{F}_q[X]$, $Q \neq 0$.

Then, we have the following result which is Kurzweil's theorem for formal Laurent series.

Theorem 1.7 (D. H. Kim and H. Nakada [5])**.** *f is badly approximable if and only if (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$ and all sequence $\{l_n\}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$.*

In the sequel, we are going to discuss some refinements of Kurzweil's theorem in the field of formal Laurent series. Kim, Nakada and Natsui [6] investigated the condition for f when adding the additional requirement to $\{l_n\}$ that l_n is non-decreasing. Note that this set of f contains all elements that are badly approximable. They obtained some partial results even though they did not find the exact set for f. In order to state their results, we need some notations. Define

$$\Omega := \left\{ l_n \geq 1 : \text{ non-decreasing and } \sum_{n=0}^{\infty} \frac{1}{q^{l_n}} = \infty \right\}$$

and

$$W_\Omega := \{f \in \mathbb{L} : \forall l_n \in \Omega,\ (1.5) \text{ has infinitely many solutions } Q \text{ for almost all } g \in \mathbb{L}\}.$$

Then, the following results were proved by Kim, Nakada and Natsui in [6].

Proposition 1.3.1 (D. H. Kim, H. Nakada and R. Natsui [6]). *Let P_k/Q_k be the principle convergents of f with $\deg(Q_k) = n_k$. If $\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty$, then we have $f \notin W_\Omega$.*

Note that $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$ is satisfied for almost every $f \in \mathbb{L}$.

The second result proved in [6] is as folows:

Proposition 1.3.2 (D. H. Kim, H. Nakada and R. Natsui [6]). *If there exists a positive integer C such that $n_k \le Ck$ for all $k \in \mathbb{N}$, then we have $f \in W_\Omega$.*

Note that Proposition 1.3.2 is only a sufficient but not necessary condition. We will prove Proposition 1.3.1, Proposition 1.3.2 and improve some details of this result in the next chapter.

Kim, Tan, Wang and Xu put forward another refinement from a new viewpoint in [7]. Consider the inequality which is different from (1.5)

$$|\{Qf\} - g| < \frac{1}{q^{l_n}},\ \deg(Q) = n, \tag{1.6}$$

where $f,\ g,\ Q$ and l_n are as above. For $s \ge 1$, we define

$$U_s := \left\{ f \in \mathbb{L} : \exists c > 0 \text{ such that } |\{Qf\}| > \frac{1}{q^{sn+c}},\ \forall Q \in \mathbb{F}_q[X] \text{ with } n = \deg(Q) \right\}.$$

It is obvious that U_s is the set of badly approximable elements when $s = 1$.

The main goal of [7] was to search for the set of $\{l_n\}$ such that $f \in U_s$ is a necessary and sufficient condition for (1.6) having infinitely many solutions Q for almost all $g \in \mathbb{L}$ whenever $\{l_n\}$ belongs to this set. The set is defined as

$$\Omega_s := \left\{ l_n \ge 1 : \sum_{n=0}^{\infty} q^{n-sl_n} = \infty \right\}.$$

Then, we have the following theorem.

Theorem 1.8 (D. H. Kim, B. Tan, B. Wang and J. Xu [7]). *Define*

$$W_s := \{ f \in \mathbb{L} : \forall l_n \in \Omega_s,\ \text{(1.6) has infinitely many solutions } Q \text{ for almost all } g \in \mathbb{L} \}.$$

Then, we have $W_s = U_s$.

In addition, Kurzweil also gave a refinement of the same flavour for the real case in [8]. One of the main goals of this thesis is to obtain an analogue of this result in the field of formal Laurent series. We first consider a non-negative and non-decreasing sequence $\{r_n\}$ which fulfils the following conditions:

(1) $n - r_n$ is non-increasing,

(2) $r_n \geq 2n$, for all $n \in \mathbb{N}$.

Define

$$U_{\{r_n\}} := \left\{ f \in \mathbb{L} : \exists c > 0 \text{ such that } \left| f - \frac{P}{Q} \right| > q^{-r_n + c}, \forall P,\, Q \in \mathbb{F}_q[X] \text{ with } \deg(Q) = n \right\}.$$

Our goal is similar as the one in Theorem 1.8. However, the set of $\{l_n\}$ now becomes of course different. More precisely, we define a set $\Omega_{\{r_n\}}$ as follows: the sequence $\{l_n\}$ belongs to $\Omega_{\{r_n\}}$ if

(1) l_n is non-decreasing, and

(2) there exists an increasing sequence of non-negative integers $t_1 < t_2 < t_3 < ...$, and a function $\delta(n)$ which is non-decreasing with $\delta(n) \to \infty$ as $n \to \infty$ such that

$$t_{i+1} > r_{t_i + \delta(t_i)} - t_i,$$

and

$$\sum_{i \geq 1} q^{t_i - l_{r_{t_i + \delta(t_i)} - t_i}} = \infty.$$

Then, we have the following result.

Theorem 1.9. *Define the set*

$$W_{\{r_n\}} := \left\{ f \in \mathbb{L} : \forall l_n \in \Omega_{\{r_n\}},\ \text{(1.6) has infinitely many solutions } Q \text{ for almost all } g \in \mathbb{L} \right\}.$$

Then, we have $W_{\{r_n\}} = U_{\{r_n\}}$.

Note that $U_s = U_{\{r_n\}}$ when we take $r_n = (s+1)n$. In Chapter 3, we are going to prove Theorem 1.9 and compare the sets Ω_s and $\Omega_{\{r_n\}}$ which have some similarities and differences when $r_n = (s+1)n$.

Chapter 2

The Refinement of Kim, Nakada and Natsui

In this chapter, we will prove Proposition 1.3.1 and Proposition 1.3.2 from the introduction and give some improvements.

2.1 Proof of Proposition 1.3.1 and Proposition 1.3.2

From now on, we assume that f is irrational. In order to prove the two propositions, we need the following lemmas.

Lemma 2.1. *$\{\{Qf\} : Q \in \mathbb{F}_q[X]\}$ is dense in $\mathbb{L}$.*

Lemma 2.2 (0-1 law)**.** *Let E be a measurable set contained in $\mathbb{L}$. If E is invariant under the action $\langle \cdot + \{Qf\}\rangle$ for all $Q \in \mathbb{F}_q[X]$, then we have*

$$\mu(E) = 0 \ or \ 1.$$

The proofs of Lemma 2.1 and Lemma 2.2 are in [1].

From the second lemma, we obtain the following result.

Lemma 2.3. *Define the set*

$$E := \{g \in \mathbb{L} : |\{Qf\} - g| < \psi(|Q|) \text{ with } Q \in \mathbb{F}_q[X] \text{ has infinitely many solutions}\}.$$

Then, E is invariant under the action $\langle \cdot + \{Qf\}\rangle$ for all $Q \in \mathbb{F}_q[X]$. Consequently, we have $\mu(E) = 0 \ or \ 1$.

Proof. Let $g \in E$. If we fix a polynomial Q', then we can find infinitely many Q with $\deg(Q - Q') > 0$ such that

$$|\{(Q - Q')f\} - g| = |\{Qf\} - (g + \{Q'f\})| < \psi(|Q - Q'|) = \psi(|Q|).$$

This means $g + \{Q'f\} \in E$. That is, $E + \{Q'f\} \subseteq E$. On the other hand, since

$$|\{(Q + Q')f\} - g| = |\{Qf\} - (g - \{Q'f\})| < \psi(|Q|)$$

has infinitely many solutions, we have $g - \{Q'f\} \in E$. Then, $g = g - \{Q'f\} + \{Q'f\} \in E + \{Q'f\}$, which implies that $E \subseteq E + \{Q'f\}$. Hence, we get $E = E + \{Q'f\}$. Consequently, E is invariant under the action $\langle \cdot + \{Qf\} \rangle$ for all $Q \in \mathbb{F}_q[X]$ so that $\mu(E) = 0$ or 1 by Lemma 2.2. ∎

From now on, we let $\psi(|Q|) = q^{-n-l_n}$ throughout this chapter. (In the next chapter, we will discuss the inequality with $\psi(|Q|) = q^{-l_n}$.)

Next, we state the *Borel-Cantelli Lemma* which will be used below.

Lemma 2.4 (Borel-Cantelli Lemma)**.** *Let F_n be a sequence of events in a probability space. Then,*

(i) *If $\sum_{n=1}^{\infty} \mu(F_n) < \infty$, then $\mu(\limsup_{n\to\infty} F_n) = 0$.*

(ii) *If $\sum_{n=1}^{\infty} \mu(F_n) = \infty$, then*

$$\mu\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} F_n\right) \geq \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N}\mu(F_n)\right)^2}{\sum_{n=1}^{N}\sum_{m=1}^{N}\mu(F_n \cap F_m)} \text{ for all } N \in \mathbb{N}.$$

Moreover, we have the right hand side is positive if there exists $K > 0$ such that $\mu(F_i \cap F_j) \leq K\mu(F_i)\mu(F_j)$, $\forall i, j \in \mathbb{N}$.

Fix $f \in \mathbb{L}$. Let P_k/Q_k be the principle convergents of f with $\deg(Q_k) = n_k$. Then, we have the following result that was proved by Kim and Nakada in [5].

Lemma 2.5 (D. H. Kim and H. Nakada [5])**.** (i) *For each $Q \in \mathbb{F}_q[X]$ with $\deg(Q) < n_{k+1}$, there exists a unique decomposition $Q = B_1Q_0 + B_2Q_1 + \cdots + B_{k+1}Q_k$ with $B_i \in \mathbb{F}_q[X]$ and $\deg(B_i) < n_i - n_{i-1}$.*

(ii) *For each nonzero $Q \in \mathbb{F}_q[X]$ with $\deg(Q) < n_{k+1}$, we have $|\{Qf\}| \geq q^{-n_{k+1}}$. Moreover,*

$$|\{Qf\}| = \frac{1}{q^s} \text{ for } 0 < s \leq n_{k+1},$$

if and only if $B_i = 0$ for $1 \leq i \leq m$ with $n_m < s \leq n_{m+1}$ and $\deg(B_{m+1}) = n_{m+1} - s$ in the decomposition of Q in (i).

Next, we are going to discuss two lemmas from [6]. Since we will revisit the proof in the next section, we will give them as well.

Let $\{l_n\}$ be a non-decreasing sequence. Define for $n_k \leq n < n_{k+1}$, $l_n^* := \max\{n_{k+1} - n, l_n\}$. Let $B_Q = B(\{Qf\}; q^{-n-l_n^*})$ with $n = \deg(Q)$. Note that B_Q is a ball whose radius is not greater than $q^{-n_{k+1}}$. We denote by F_k the union of B_Q with $n_k \leq \deg(Q) < n_{k+1}$. That is,

$$F_k = \bigcup_{n_k \leq n < n_{k+1}} \bigcup_{\deg(Q)=n} B_Q.$$

Note that the B_Q's are disjoint. Thus,

$$\mu(F_k) = \sum_{n_k \leq n < n_{k+1}} (q-1)q^n.\frac{1}{q^{n+l_n^*}} = (q-1) \sum_{n_k \leq n < n_{k+1}} \frac{1}{q^{l_n^*}}. \tag{2.1}$$

Lemma 2.6 (D. H. Kim, H. Nakada and R. Natsui [6]). *Let* $Q \in \mathbb{F}_q[X]$ *with* $\deg(Q) = n$ *and* $n_k \leq n < n_{k+1}$. *Then, for* $m \geq 1$,

$$\mu(B_Q \cap F_{k+m}) = \begin{cases} \mu(B_Q)\mu(F_{k+m}), & \text{if } n + l_n^* < n_{k+m}, \\ q^{-n_{k+m}} \left(\mu(F_{k+m}) - \displaystyle\sum_{n_{k+m+1}+n_{k+m}-n-l_n^* \leq s < n_{k+m+1}} (q-1) q^{-l_s^*} \right), & \\ & \text{if } n_{k+m} \leq n + l_n^* < n_{k+m+1}, \\ 0, & \text{if } n + l_n^* \geq n_{k+m+1}. \end{cases}$$

Proof. Let Q' be a polynomial with $\deg(Q') = n'$ and $n_{k+m} \leq n' < n_{k+m+1}$ such that $B(\{Qf\}; q^{-n-l_n^*}) \cap B(\{Q'f\}; q^{-n'-l_{n'}^*}) \neq \emptyset$. Since $n + l_n^* < n' + l_{n'}^*$, we have $B_{Q'} \subset B_Q$ by Proposition 1.1.4.

In the case $n + l_n^* \geq n_{k+m+1}$, we obtain $|\{(Q - Q')f\}| < q^{-n-l_n^*} \leq q^{-n_{k+m+1}}$. Conversely, since $\deg(Q - Q') < n_{k+m+1}$, we get $|\{(Q - Q')f\}| \geq q^{-n_{k+m+1}}$ by Lemma 2.5 (**ii**), a contradiction. Hence, B_Q does not contain $B_{Q'}$ in this case. Then,

$$B\left(\{Qf\}; \frac{1}{q^{n+l_n^*}}\right) \cap B\left(\{Q'f\}; \frac{1}{q^{n'+l_{n'}^*}}\right) = \emptyset.$$

Therefore,

$$\mu(B_Q \cap F_{k+m}) = 0.$$

Now, we consider the case $n + l_n^* < n_{k+m+1}$. Choose $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $n_{k+i} \leq n + l_n^* < n_{k+i+1}$. Then,

$$|\{(Q - Q')f\}| < \frac{1}{q^{n+l_n^*}} \leq \frac{1}{q^{n_{k+i}}}.$$

By Lemma 2.5, we have

$$Q' = Q + B_{k+i+1}Q_{k+i} + \cdots + B_{k+m+1}Q_{k+m},$$

where $B_{k+j+1} \in \mathbb{F}_q[X]$ with $\deg(B_{k+j+1}) < n_{k+j+1} - n_{k+j}$ for $i \le j \le m$ and $B_{k+m+1} \neq 0$.

If $B_{k+i+1} \neq 0$, then we use the Lemma 2.5 (**ii**) again to obtain

$$|\{(Q - Q')f\}| = \frac{q^{\deg(B_{k+i+1})}}{q^{n_{k+i+1}}}.$$

Thus, we have $|\{(Q - Q')f\}| < q^{-n-l_n^*}$ if and only if $\deg(B_{k+i+1}) < n_{k+i+1} - n - l_n^*$.

If $B_{k+i+1} = 0$, then there exists $r \in \mathbb{N}$ with $i < r \le m$ such that

$$|\{(Q - Q')f\}| = \frac{q^{\deg(B_{k+r+1})}}{q^{n_{k+r+1}}} < \frac{1}{q^{n_{k+r}}} \le \frac{1}{q^{n_{k+i+1}}} < \frac{1}{q^{n+l_n^*}}.$$

Therefore, we have

$$B_Q \cap F_{k+m} = \bigcup_{\substack{\deg(B_{k+i+1}) < n_{k+i+1} - n - l_n^* \\ \deg(B_{k+j+1}) < n_{k+j+1} - n_{k+j}, i<j<m \\ 0 \le \deg(B_{k+m+1}) < n_{k+m+1} - n_{k+m}}} B_{Q+B_{k+i+1}Q_{k+i}+\cdots+B_{k+m+1}Q_{k+m}},$$

where the union is disjoint. By the definition of B_Q, we have

$$\mu(B_{Q+B_{k+i+1}Q_{k+i}+\cdots+B_{k+m+1}Q_{k+m}}) = \frac{1}{q^{n_{k+m}+t+l^*_{n_{k+m}+t}}},$$

where $t = \deg(B_{k+m+1})$.

If $i < m$ (i.e., $n + l_n^* < n_{k+m}$), then the number of $Q' = Q + B_{k+i+1}Q_{k+i} + \cdots + B_{k+m+1}Q_{k+m}$ with $\deg(Q') = n_{k+m} + t$ such that $|\{(Q - Q')f\}| < q^{-n-l_n^*}$ is

$$q^{n_{k+i+1}-n-l_n^*} \cdot q^{n_{k+i+2}-n_{k+i+1}} \cdots q^{n_{k+m}-n_{k+m-1}} \cdot (q-1)q^t = (q-1)q^{n_{k+m}-n-l_n^*+t}.$$

Hence,

$$\begin{aligned}
&\mu(B_Q \cap F_{k+m}) \\
&= \sum_{0 \le t < n_{k+m+1}-n_{k+m}} \frac{\sharp\{\deg(Q') = n_{k+m} + t : |\{(Q - Q')f\}| < q^{-n-l_n^*}\}}{q^{n_{k+m}+t+l^*_{n_{k+m}+t}}} \\
&= \sum_{0 \le t < n_{k+m+1}-n_{k+m}} \frac{(q-1)q^{n_{k+m}-n-l_n^*+t}}{q^{n_{k+m}+t+l^*_{n_{k+m}+t}}} \\
&= q^{-n-l_n^*} \sum_{0 \le t < n_{k+m+1}-n_{k+m}} \frac{(q-1)}{q^{l^*_{n_{k+m}+t}}} \\
&= \mu(B_Q)\mu(F_{k+m}).
\end{aligned}$$

If $i = m$ (i.e., $n_{k+m} \leq n + l_n^* < n_{k+m+1}$), then

$$B_Q \cap F_{k+m} = \bigcup_{0 \leq \deg(B_{k+m+1}) < n_{k+m+1} - n - l_n^*} B_{Q + B_{k+m+1} Q_{k+m}},$$

and

$$\begin{aligned}
&\mu(B_Q \cap F_{k+m}) \\
&= \sum_{0 \leq t < n_{k+m+1} - n - l_n^*} \frac{\sharp\{\deg(Q') = n_{k+m} + t : |\{(Q - Q')f\}| < q^{-n-l_n^*}\}}{q^{n_{k+m}+t+l^*_{n_{k+m}+t}}} \\
&= \sum_{0 \leq t < n_{k+m+1} - n - l_n^*} \frac{(q-1)q^t}{q^{n_{k+m}+t+l^*_{n_{k+m}+t}}} \\
&= \frac{1}{q^{n_{k+m}}} \sum_{0 \leq t < n_{k+m+1} - n - l_n^*} \frac{q-1}{q^{l^*_{n_{k+m}+t}}} \\
&= \frac{1}{q^{n_{k+m}}} \cdot \left(\mu(F_{k+m}) - \sum_{n_{k+m+1} - n - l_n^* \leq t < n_{k+m+1} - n_{k+m}} \frac{q-1}{q^{l^*_{n_{k+m}+t}}} \right),
\end{aligned}$$

where the last equality is by (2.1). ∎

Applying the above lemma, we have the following result.

Lemma 2.7 (D. H. Kim, H. Nakada and R. Natsui [6])**.** *For any* $k \geq 0$ *and* $m \geq 1$*, we have*

$$|\mu(F_k \cap F_{k+m}) - \mu(F_k)\mu(F_{k+m})| \leq \frac{1}{q^{m-1}} \mu(F_{k+m}).$$

Proof. By Lemma 2.6, we have

$$\mu(B_Q \cap F_{k+m}) \leq \begin{cases} \mu(B_Q)\mu(F_{k+m}), & \text{if } n + l_n^* < n_{k+m}, \\ \frac{1}{q^{n_{k+m}}} \mu(F_{k+m}), & \text{if } n_{k+m} \leq n + l_n^* < n_{k+m+1}, \\ 0, & \text{if } n + l_n^* \geq n_{k+m+1}. \end{cases}$$

This implies that

$$\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) \leq \frac{1}{q^{n_{k+m}}} \mu(F_{k+m}).$$

On the other hand,

$$\mu(B_Q \cap F_{k+m}) \geq \begin{cases} \mu(B_Q)\mu(F_{k+m}), & \text{if } n + l_n^* < n_{k+m}, \\ 0, & \text{if } n + l_n^* \geq n_{k+m}. \end{cases}$$

Now, we observe that:

1. For $n + l_n^* < n_{k+m}$, we have

$$\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) \geq 0 \geq -\frac{\mu(F_{k+m})}{q^{n_{k+m}}}.$$

2. For $n + l_n^* \geq n_{k+m}$, we have

$$\begin{aligned}\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) &\geq -\mu(B_Q)\mu(F_{k+m}) \\ &= -\frac{\mu(F_{k+m})}{q^{n+l_n^*}} \geq -\frac{\mu(F_{k+m})}{q^{n_{k+m}}}.\end{aligned}$$

Therefore, for all $m \geq 1$, we have

$$|\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m})| \leq \frac{\mu(F_{k+m})}{q^{n_{k+m}}}.$$

Notice that

$$\mu(F_k \cap F_{k+m}) - \mu(F_k)\mu(F_{k+m}) = \sum_{n_k \leq \deg(Q) < n_{k+1}} \left(\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m})\right).$$

Thus,

$$\begin{aligned}|\mu(F_k \cap F_{k+m}) - \mu(F_k)\mu(F_{k+m})| &\leq \sum_{n_k \leq \deg(Q) < n_{k+1}} \frac{\mu(F_{k+m})}{q^{n_{k+m}}} \\ &= (q^{n_{k+1}} - q^{n_k})\frac{\mu(F_{k+m})}{q^{n_{k+m}}} \\ &\leq \frac{\mu(F_{k+m})}{q^{n_{k+m}-n_{k+1}}} \leq \frac{\mu(F_{k+m})}{q^{m-1}}.\end{aligned}$$

The proof is complete. ∎

By the above lemma and Lemma 2.4, we have the following result.

Lemma 2.8 (D. H. Kim, H. Nakada and R. Natsui [6]).

$$\sum_{n=0}^{\infty} \frac{1}{q^{l_n^*}} = \infty \quad \left(i.e., \sum_{k=0}^{\infty} \mu(F_k) = \infty \right)$$

if and only if $|\{Qf\} - g| < q^{-n-l_n^*}$ *has infinitely many solutions* Q *with* $\deg(Q) = n$ *for almost all* $g \in \mathbb{L}$.

This result implies the following two theorems that were proved in [6].

Theorem 2.1 (D. H. Kim, H. Nakada and R. Natsui [6]). *Let* $\{l_n\}$ *be a non-decreasing sequence. Then, (1.5) has infinitely many solutions* Q *for almost all* $g \in \mathbb{L}$ *if and only if*

$$f \in \left\{ f \in \mathbb{L} : \sum_{n=0}^{\infty} \frac{1}{q^{l_n^*}} = \infty, \ \textit{where } l_n^* = \max\{n_{k+1} - n, l_n\} \textit{ for } n_k \leq n < n_{k+1} \right\}.$$

Theorem 2.2 (D. H. Kim, H. Nakada and R. Natsui [6]). *Let $\{l_n\}$ be a non-decreasing sequence. Then, (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$ if and only if*

$$f \in \left\{ f \in \mathbb{L} : \sum_{k=0}^{\infty} \frac{\min\{l_{n_k}, n_{k+1} - n_k\}}{q^{l_{n_k}}} = \infty \right\}.$$

Now, we are going to recall Proposition 1.3.1, Proposition 1.3.2 and provide the proofs of them via Theorem 2.1 and Theorem 2.2.

Recall the sets

$$\Omega := \left\{ l_n \geq 1 : \text{ non-decreasing and } \sum_{n=0}^{\infty} \frac{1}{q^{l_n}} = \infty \right\}$$

and

$$W_\Omega := \{ f \in \mathbb{L} : \forall l_n \in \Omega, \text{ (1.5) has infinitely many solutions } Q \text{ for almost all } g \in \mathbb{L} \}.$$

We have the following results.

Proposition 2.1.1 (D. H. Kim, H. Nakada and R. Natsui [6]). *Let P_k/Q_k be the principle convergents of f with $\deg(Q_k) = n_k$. If $\sum_{k=0}^{\infty} \frac{1}{n_k} < \infty$, then we have $f \notin W_\Omega$.*

Proof. Choose

$$l_n = \begin{cases} \lfloor \log_q (n \log_q n) \rfloor, & \text{if } n \geq q, \\ 1, & \text{if } n < q. \end{cases}$$

Then, we have

$$\sum_{n=0}^{\infty} \frac{1}{q^{l_n}} \geq \sum_{n=q}^{\infty} \frac{1}{n \log_q n} = \infty.$$

Let n_{k^*} be the smallest positive integer that is larger or equal to q. Then,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\min\{l_{n_k}, n_{k+1} - n_k\}}{q^{l_{n_k}}} &\leq \sum_{k=0}^{\infty} \frac{l_{n_k}}{q^{l_{n_k}}} \leq \sum_{k=0}^{k^*-1} \frac{l_{n_k}}{q^{l_{n_k}}} + \sum_{k \geq k^*} \frac{\log_q n_k + \log_q(\log_q n_k)}{q^{-1} \cdot n_k \log_q n_k} \\ &\leq \sum_{k=0}^{k^*-1} \frac{l_{n_k}}{q^{l_{n_k}}} + \sum_{k=k^*}^{\infty} \frac{2q}{n_k} < \infty. \end{aligned}$$

By Theorem 2.2, the proof is complete. ∎

Proposition 2.1.2 (D. H. Kim, H. Nakada and R. Natsui [6]). *If there exist a positive integer C such that $n_k \leq Ck$ for all $k \in \mathbb{N}$, then we have $f \in W_\Omega$.*

Proof. Let $\{l_n\} \in \Omega$. Then,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\min\{l_{n_k}, n_{k+1} - n_k\}}{q^{l_{n_k}}} &\geq \sum_{k=0}^{\infty} \frac{1}{q^{l_{n_k}}} \geq \sum_{k=0}^{\infty} \frac{1}{q^{l_{Ck}}} \\ &\geq \frac{1}{C} \sum_{k=0}^{\infty} \left(\frac{1}{q^{l_{Ck}}} + \frac{1}{q^{l_{Ck+1}}} + \cdots + \frac{1}{q^{l_{C(k+1)-1}}} \right) \\ &\geq \frac{1}{C} \sum_{n=0}^{\infty} \frac{1}{q^{l_n}} = \infty.\end{aligned}$$

Since $\{l_n\} \in \Omega$ is arbitrary, we have $f \in W_\Omega$ by applying Theorem 2.2. ∎

In the next section, we will give some improvements of this refinement and prove them.

2.2 Theorem 2.1 without the Monotonicity Condition

In this section, we will improve Lemma 2.6 and Lemma 2.7 by dropping the monotonicity condition that $\{l_n\}$ is non-decreasing and use these improvements to prove one direction of Theorem 2.1 without the monotonicity condition. Moreover, we will show that this theorem also implies one direction of Kurzweil's theorem.

If we remove the condition that $\{l_n\}$ is non-decreasing, then the main difference in Lemma 2.6 is that: $q^{-n'-l^*_{n'}}$, the radius of $B_{Q'}$, might be greater than $q^{-n-l^*_n}$. We will show the following modification of Lemma 2.6.

Lemma 2.9. *Let $Q \in \mathbb{F}_q[X]$ with $\deg(Q) = n$ and $n_k \leq n < n_{k+1}$. Then, for $m \geq 1$,*

$$\mu(B_Q \cap F_{k+m}) = \begin{cases} \mu(B_Q)\mu(F_{k+m}), & \text{if } n + l^*_n < n_{k+m}, \\ q^{-n_{k+m}} \left(\mu(F_{k+m}) - \displaystyle\sum_{n_{k+m+1}+n_{k+m}-n-l^*_n \leq s < n_{k+m+1}} (q-1) q^{-l^*_s} \right), & \\ & \text{if } n_{k+m} \leq n + l^*_n < n_{k+m+1}, \\ 0 \text{ or } \mu(B_Q), & \text{if } n + l^*_n \geq n_{k+m+1}. \end{cases}$$

Proof. Let Q' be a polynomial with $\deg(Q') = n'$ and $n_{k+m} \leq n' < n_{k+m+1}$ such that $B(\{Qf\}; q^{-n-l^*_n}) \cap B(\{Q'f\}; q^{-n'-l^*_{n'}}) \neq \emptyset$.

In the case $n + l^*_n < n_{k+m+1}$, we have $n' + l^*_{n'} \geq n_{k+m+1} > n + l^*_n$ according to the definition of $l^*_{n'}$. Thus, the method in this case is the same as before so that we only have to consider the case $n + l^*_n \geq n_{k+m+1}$.

In the case $n + l^*_n \geq n_{k+m+1}$, we have the following two cases:

1. If $n+l_n^* < n'+l_{n'}^*$, then $|\{(Q-Q')f\}| < q^{-n-l_n^*} \le q^{-n_{k+m+1}}$. Since the degree of $Q-Q'$ is less than n_{k+m+1}, we have $|\{(Q-Q')f\}| \ge q^{-n_{k+m+1}}$ by Lemma 2.5 (**ii**). This implies that this case cannot happen, i.e., such a Q' does not exist.

2. If $n+l_n^* \ge n'+l_{n'}^*$, then
$$B\left(\{Qf\}; \frac{1}{q^{n+l_n^*}}\right) \subseteq B\left(\{Q'f\}; \frac{1}{q^{n'+l_{n'}^*}}\right),$$
and from the disjointness of the balls in F_{k+m}, we get
$$\mu(B_Q \cap F_{k+m}) = \mu(B_Q).$$

Overall this implies that either $\mu(B_Q \cap F_{k+m}) = 0$ or $\mu(B_Q)$ in the case $n + l_n^* \ge n_{k+m+1}$ as claimed. ∎

By the above lemma, Lemma 2.7 still holds when dropping the monotonicity condition and the proof is almost the same as before.

Lemma 2.10. *For any $k \ge 0$ and $m \ge 1$, we have*
$$|\mu(F_k \cap F_{k+m}) - \mu(F_k)\mu(F_{k+m})| \le \frac{1}{q^{m-1}}\mu(F_{k+m}).$$

Proof. As in Lemma 2.7, our first goal is to show that
$$|\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m})| \le \frac{\mu(F_{k+m})}{q^{n_{k+m}}} \tag{2.2}$$

Since the case $n+l_n^* < n_{k+m+1}$ can be proved in the same way as before, we focus on the case $n+l_n^* \ge n_{k+m+1}$.

Here, we first show that
$$\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) \le \frac{\mu(F_{k+m})}{q^{n_{k+m}}}.$$

There are two cases:

(**i**) If $\mu(B_Q \cap F_{k+m}) = 0$, then the claim is trivial.

(**ii**) If $\mu(B_Q \cap F_{k+m}) = \mu(B_Q)$, then $B_Q \subseteq F_{k+m}$ which means that $B_Q \subseteq B_{Q'}$ with $n_{k+m} \le n' = \deg(Q') < n_{k+m+1}$. Observe that
$$\mu(B_Q) = \frac{1}{q^{n+l_n^*}} \le \frac{1}{q^{n'+l_{n'}^*}} \le \frac{1}{q^{n_{k+m}+l_{n'}^*}} \le \frac{1}{q^{n_{k+m}}}\frac{\mu(F_{k+m})}{q-1} \le \frac{\mu(F_{k+m})}{q^{n_{k+m}}}.$$
Consequently,
$$\begin{aligned}\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) &= \mu(B_Q) - \mu(B_Q)\mu(F_{k+m}) \\ &\le \frac{1}{q^{n_{k+m}}}\mu(F_{k+m}).\end{aligned}$$

Thus, the claim is established for this case as well.

Next, since $\mu(B_Q \cap F_{k+m}) \geq 0$ for $n + l_n^* \geq n_{k+m}$, we have

$$\mu(B_Q \cap F_{k+m}) - \mu(B_Q)\mu(F_{k+m}) \geq -\frac{\mu(F_{k+m})}{q^{n_{k+m}}}.$$

This concludes the proof of (2.2). The remaining proof follows along the same lines as in Lemma 2.7. ∎

This lemma together with Lemma 2.4 shows that Lemma 2.8 holds without the monotonicity condition.

Now, we are going to prove that one direction of Theorem 2.1 also holds without the monotonicity condition.

Theorem 2.3. *Let $\{l_n\}$ be a sequence. Define*

$$U := \left\{ f \in \mathbb{L} : \sum_{n=0}^{\infty} \frac{1}{q^{l_n^*}} = \infty, \ \text{where } l_n^* = \max\{n_{k+1} - n, l_n\} \text{ for } n_k \leq n < n_{k+1} \right\}.$$

If $f \in U$, then (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$.

Proof. Let $f \in U$, we have $\sum_{n=0}^{\infty} q^{-l_n^*} = \infty$. Consequently,

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n^*}}$$

has infinitely many solutions Q for almost all $g \in \mathbb{L}$ by Lemma 2.8.

Next, by the definition of l_n^*, we have $q^{-n-l_n^*} \leq q^{-n-l_n}$. Thus, (1.5) has also infinitely many solutions Q for almost all $g \in \mathbb{L}$. ∎

We will show that this theorem implies one direction of Kurzweil's theorem, namely, the direction that if f is badly approximable and $\{l_n\}$ is a sequence with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$, then (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$.

Proof. Let f be badly approximable. Then, there exists a c such that $n_{k+1} - n_k \leq c$ for all k. Fix a sequence $\{l_n\}$ with $\sum_{n=0}^{\infty} q^{-l_n} = \infty$. Then, we consider the following cases:

- **(i)** $l_n \geq c$ for all large n. Then $l_n^* = \max\{n_{k+1} - n, l_n\} = l_n$ for all large n. Thus,we have $\sum_{n=0}^{\infty} q^{-l_n^*} = \sum_{n=0}^{\infty} q^{-l_n} = \infty$. The claim now follows from Theorem 2.3.
- **(ii)** $l_n < c$ for infinitely many n. Set $\widetilde{l_n} = \max\{l_n, c\}$. Then $\sum_{n=1}^{\infty} q^{-\widetilde{l_n}} = \infty$. Applying part (i) gives that $|\{Qf\} - g| < q^{-n-\widetilde{l_n}}$ has infinitely many solutions Q for almost all $g \in \mathbb{L}$. However, since $\widetilde{l_n} \geq l_n$, the same holds for (1.5) as well.

This concludes this proof. ∎

In the next section, we will prove a special case of Theorem 2.2. The purpose for doing so is two-fold: first we will prove our result with a completely different method as the one used in Section 2.1 and second this is a kind of warm-up for Chapter 3 where the same method will be applied.

2.3 A Special Case of Theorem 2.2

The main purpose in this section is to discuss the existence of infinitely many solutions to (1.5) when the series $\sum_{k=0}^{\infty} q^{-l_{n_k}}$ diverges. In order to prove this, we need the following lemma.

Lemma 2.11. *Let $g \in \mathbb{L}$. Then, the number of $\{Qf\}$ with $\deg(Q) < n_{k+1}$ belonging to $B(g, q^{-d})$ is at most $\max\{q^{n_{k+1}-d}, 1\}$.*

Proof. Let Q, Q' be two different polynomials with $\deg(Q), \deg(Q') < n_{k+1}$. Then, by Lemma 2.5, we have

$$|\{Qf\} - \{Q'f\}| \geq \frac{1}{q^{n_{k+1}}}.$$

This means that the distance between two points $\{Qf\}, \{Q'f\}$ with $\deg(Q), \deg(Q') < n_{k+1}$ is at least $q^{-n_{k+1}}$. Now, consider two cases:

1. If $q^{-n_{k+1}} \geq q^{-d}$, then there is at most one point in $B(g, q^{-d})$.
2. If $q^{-n_{k+1}} < q^{-d}$, then the number of points in $B(g, q^{-d})$ is at most $q^{n_{k+1}-d}$.

Hence, the number of $\{Qf\}$ with $\deg(Q) < n_{k+1}$ belonging to $B(g, q^{-d})$ is at most $\max\{q^{n_{k+1}-d}, 1\}$. ∎

Applying the above lemma, we can prove the following result.

Lemma 2.12. *Let $\{l_n\}$ be a non-decreasing sequence with $\sum_{k=0}^{\infty} q^{-l_{n_k}} = \infty$. Then, for all $N \geq 0$, we have*

$$\mu\left(\bigcup_{k=N}^{\infty} F_k\right) > \frac{1}{q^c}, \text{ for all } c \geq 2. \tag{2.3}$$

Proof. Assume that (2.3) is false. Then, there exists $N_0 \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{k=N_0}^{K} F_k\right) \leq \frac{1}{q^c}, \text{ for all } K \geq N_0. \tag{2.4}$$

Define a set

$$L_{n_{K+1}-1} := \left\{ \deg(Q) = n_{_{K+1}} - 1 : \{Qf\} \in \bigcup_{k=N_0}^{K} F_k \backslash \bigcup_{k=N_0}^{K-1} F_k \right\}.$$

First, we estimate the number of elements in $L_{n_{K+1}-1}$. Let

$$\begin{aligned} \bigcup_{k=N_0}^{K-1} F_k &= \bigcup_{k=N_0}^{K-1} \bigcup_{n_k \le n < n_{k+1}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{n+l_n^*}}\right) \\ &= \bigcup_i B\left(\{Q_i f\}; \frac{1}{q^{d_i}}\right), \end{aligned}$$

where $B(\{Q_i f\}; q^{-d_i})$ are disjoint for all i. By (2.4), we obtain

$$\begin{aligned} \frac{1}{q^c} \ge \mu\left(\bigcup_{k=N_0}^{K-1} F_k\right) &= \mu\left(\bigcup_i B\left(\{Q_i f\}; \frac{1}{q^{d_i}}\right)\right) \\ &= \sum_i \mu\left(B\left(\{Q_i f\}; \frac{1}{q^{d_i}}\right)\right) = \sum_i \frac{1}{q^{d_i}}. \end{aligned}$$

Using Lemma 2.11, the number of $\{Qf\}$ with $\deg(Q) < n_{K+1}$ belonging to $\bigcup_i B(\{Q_i f\}; q^{-d_i})$ is at most $\sum_i \max\{q^{n_{K+1}-d_i}, 1\} = \max\{q^{n_{K+1}} \sum_i q^{-d_i}, q^{n_K}\} \le q^{n_{K+1}-1}$. Thus, the number of elements in $L_{n_{K+1}-1}$ is at least

$$q^{n_{K+1}-1}(q-1) - q^{n_{K+1}-1} = q^{n_{K+1}-1}(q-2).$$

Next, we claim that

$$\bigcup_{Q \in L_{n_{K+1}-1}} B\left(\{Qf\}; \frac{1}{q^{n_{K+1}-1+l^*_{n_{K+1}-1}}}\right) \subset \bigcup_{k=N_0}^{K} F_k \backslash \bigcup_{k=N_0}^{K-1} F_k. \tag{2.5}$$

In order to show this, fix $Q_1 \in L_{n_{K+1}-1}$. Suppose there exists a polynomial Q_2 with $\deg(Q_2) = u < n_K$ and $B(\{Q_1 f\}; q^{-n_{K+1}+1-l^*_{n_{K+1}-1}}) \cap B(\{Q_2 f\}; q^{-u-l^*_u}) \ne \emptyset$. We know that $\{Q_1 f\}$ does not belong to B_{Q_2}. Hence, we have

$$B\left(\{Q_2 f\}; \frac{1}{q^{u+l_u^*}}\right) \subset B\left(\{Q_1 f\}; \frac{1}{q^{n_{K+1}-1+l^*_{n_{K+1}-1}}}\right).$$

Then, we have

$$|\{Q_1 f\} - \{Q_2 f\}| < \frac{1}{q^{n_{K+1}-1+l^*_{n_{K+1}-1}}}.$$

By Lemma 2.11, the number of $\{Qf\}$ with $\deg(Q) < n_{K+1}$ belonging to B_{Q_1} is at most $\max\{q^{n_{K+1}-n_{K+1}+1-l^*_{n_{K+1}-1}}, 1\} = 1$. Thus, we get $\{Q_1 f\} = \{Q_2 f\}$, a contradiction. Consequently, (2.5) holds.

Note that any two balls appearing on the left side of (2.5) are disjoint.

By (2.5), we obtain

$$\begin{aligned}
\mu\left(\bigcup_{k=N_0}^{K} F_k\right) &\geq \mu\left(\bigcup_{k=N_0}^{K-1} F_k\right) + \mu\left(\bigcup_{Q\in L_{n_{K+1}-1}} B\left(\{Qf\}; \frac{1}{q^{n_{K+1}-1+l^*_{n_{K+1}-1}}}\right)\right) \\
&\geq \mu\left(\bigcup_{k=N_0}^{K-1} F_k\right) + q^{n_{K+1}-1}(q-2)\frac{1}{q^{n_{K+1}-1+l^*_{n_{K+1}-1}}} \\
&= \mu\left(\bigcup_{k=N_0}^{K-1} F_k\right) + (q-2)\frac{1}{q^{l^*_{n_{K+1}-1}}} \\
&\geq \mu\left(\bigcup_{k=N_0}^{K-1} F_k\right) + (q-2)\frac{1}{q^{l_{n_{K+1}}}} \\
&\geq \mu\left(\bigcup_{k=N_0}^{K-2} F_k\right) + \frac{(q-2)}{q^{l_{n_K}}} + \frac{(q-2)}{q^{l_{n_{K+1}}}} \\
&\geq \cdots \geq (q-2)\sum_{s=N_0+2}^{K+1} \frac{1}{q^{l_{n_s}}}.
\end{aligned}$$

As the series $\sum q^{-l_{n_k}}$ diverges, we have a contradiction for K large enough. ∎

Note that the proof of the above lemma only works for $q > 2$. However, the case $q = 2$ can be proved in the same way as in [1].

Now, we are going to prove the main result of this section.

Theorem 2.4. *Let $\{l_n\}$ be a non-decreasing sequence. Then, (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$ if*

$$f \in \left\{ f \in \mathbb{L} : \sum_{n=0}^{\infty} \frac{1}{q^{l_{n_k}}} = \infty \right\}.$$

Proof. Let $\{l_n\}$ be a non-decreasing sequence and $\sum_{n=0}^{\infty} q^{-l_{n_k}} = \infty$. Our goal is to prove that

$$\mu\left(\bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} F_k\right) = 1.$$

By Lemma 2.12, we have

$$\mu\left(\bigcup_{k=N}^{\infty} F_k\right) > \frac{1}{q^c} > 0, \text{ for all } N.$$

This implies

$$\mu\left(\bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} F_k\right) > 0.$$

By Lemma 2.3, the proof is complete. ∎

Chapter 3

Proof of Theorem 1.9

In Chapter 1, we have mentioned that Kurzweil also gave a refinement of his Theorem 1.7 for the real case in [8]. In this chapter, we will prove Theorem 1.9 which is an analogue of this refinement and compare the set of $\{l_n\}$ with the refinement in Theorem 1.8 which was proved by Kim, Tan, Wang and Xu.

For the sake of convenience, we recall some notation of Theorem 1.9. First recall that $\{r_n\}$ is a sequence which is assumed to be non-negative and non-decreasing. Moreover, $\{r_n\}$ satisfies that $n-r_n$ is non-increasing and $r_n \geq 2n$, for all $n \in \mathbb{N}$. Next, the definition of $\Omega_{\{r_n\}}$ was as follows: the sequence $\{l_n\}$ belongs to $\Omega_{\{r_n\}}$ if

(1) l_n is non-decreasing, and

(2) there exists an increasing sequence of non-negative integers $t_1 < t_2 < t_3 < ...$ and a function $\delta(n)$ which is non-decreasing with $\delta(n) \to \infty$ as $n \to \infty$ such that

$$t_{i+1} > r_{t_i+\delta(t_i)} - t_i,$$

and

$$\sum_{i\geq 1} q^{t_i - l_{r_{t_i+\delta(t_i)}-t_i}} = \infty.$$

Finally, recall the sets

$$U_{\{r_n\}} := \left\{ f \in \mathbb{L} : \exists c > 0 \text{ such that } \left| f - \frac{P}{Q} \right| > q^{-r_{n+c}}, \forall P, Q \in \mathbb{F}_q[X] \text{ with } \deg(Q) = n \right\}.$$

and

$$W_{\{r_n\}} := \left\{ f \in \mathbb{L} : \forall l_n \in \Omega_{\{r_n\}}, \text{ (1.6) has infinitely many solutions } Q \text{ for almost all } g \in \mathbb{L} \right\}.$$

The goal is to show that $U_{\{r_n\}} = W_{\{r_n\}}$.

In order to show this, we prove the following lemma.

Lemma 3.1. *Fix a non-negative integer n and non-negative integers t, k. If we choose a fraction R/S with $|f - R/S| < q^{-t}$, then we have*

$$\mu\left(\bigcup_{\deg(Q)\le n} B\left(\{Qf\}; \frac{1}{q^k}\right)\right) \le \max\{q^{s-k},\ q^{s+n-t}\} \text{ with } s = \deg(S).$$

Proof. From the inequality $|f - R/S| < q^{-t}$, we have

$$\left|Qf - \frac{R'}{S}\right| < q^{n-t},$$

where $R' = QR$ and $\deg(Q) \le n$. Consider the following cases:

1. If $n \le t$, then $|Qf - R'/S| < 1$. Let $R' = US + V$ with $\deg(V) < \deg(S)$, we obtain

$$\begin{aligned}\left|Qf - \frac{R'}{S}\right| &= \left|Qf - U - \frac{V}{S}\right| \\ &= \left|\{Qf\} - \frac{V}{S}\right| < q^{n-t}.\end{aligned}$$

   Hence, for $g \in B\left(\{Qf\}; q^{-k}\right)$,

$$\begin{aligned}\left|g - \frac{V}{S}\right| &= \left|g - \{Qf\} + \{Qf\} - \frac{V}{S}\right| \\ &< \max\{q^{-k}, q^{n-t}\}.\end{aligned}$$

   This implies that

$$B\left(\{Qf\}; q^{-k}\right) \subseteq B\left(\frac{V}{S}; \max\{q^{-k}, q^{n-t}\}\right),$$

   and consequently

$$\bigcup_{\deg(Q)\le n} B\left(\{Qf\}; q^{-k}\right) \subseteq \bigcup_{\deg(V)<\deg(S)} B\left(\frac{V}{S}; \max\{q^{-k}, q^{n-t}\}\right).$$

   Thus,

$$\begin{aligned}\mu\left(\bigcup_{\deg(Q)\le n} B\left(\{Qf\}; \frac{1}{q^k}\right)\right) &\le \max\{q^{-k},\ q^{n-t}\}\cdot q^s \\ &= \max\{q^{s-k},\ q^{s+n-t}\}.\end{aligned}$$

2. If $n > t$, then the conclusion still holds since $q^{s+n-t} > 1$. ∎

The following proposition is one direction of Theorem 1.9.

Proposition 3.1. *If $f \notin U_{\{r_n\}}$, then there exists a sequence $\{l_n\} \in \Omega_{\{r_n\}}$ such that (1.6) has only finitely many solutions Q with $\deg(Q) = n$.*

Proof. If $f \notin U_{\{r_n\}}$, then there exist a non-decreasing sequence $\{c_n\}$ tending to infinity and a sequence (P_k, Q_k) such that

$$\left|f - \frac{P_k}{Q_k}\right| \leq \frac{1}{q^{r_{n_k+c_{n_k}}}}, \text{ with } \deg(Q_k) = n_k.$$

Let us choose a sequence $\{d_n\}$ satisfies the following conditions:

(i) $c_n \geq d_n \geq 0$ and $n \geq d_n$ for all $n \geq 0$,

(ii) d_n is non-decreasing and $d_n \to \infty$ as $n \to \infty$,

(iii) $c_n - d_n$ is non-decreasing and $c_n - d_n \to \infty$ as $n \to \infty$,

(iv) $n - d_n \to \infty$ as $n \to \infty$.

Define

$$\delta_1(n) = c_{n-d_n} - d_{n-d_n},$$

and

$$\delta(n) = \inf_{n \leq k < \infty} \delta_1(n).$$

From the definition of $\delta_1(n)$, we have $\delta(n)$ is non-decreasing and $\delta(n) \to \infty$ as $n \to \infty$. Moreover,

$$\delta(n + d_n) \leq \delta_1(n + d_n) = c_{n+d_n-d_{n+d_n}} - d_{n+d_n-d_{n+d_n}} \leq c_n - d_n.$$

Now, select a subsequence (P_{k_i}, Q_{k_i}) of (P_k, Q_k) with $\deg(Q_{k_i}) = n_{k_i}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{q^{d_{n_{k_i}}}} < \infty$$

and

$$n_{k_{i+1}} > r_{n_{k_i}+c_{n_{k_i}}} - n_{k_i}. \tag{3.1}$$

Define $t_i = n_{k_i} + d_{n_{k_i}}$, $i \in \mathbb{N}$. Since $n - r_n$ is non-increasing and $c_n \geq d_n + \delta(n + d_n)$, we obtain

$$r_{t_i+\delta(t_i)} - r_{n_{k_i}+c_{n_{k_i}}} - t_i = -n_{k_i} - c_{n_{k_i}} + \delta(t_i) + n_{k_i} + c_{n_{k_i}} - r_{n_{k_i}+c_{n_{k_i}}} - (t_i + \delta(t_i) - r_{t_i+\delta(t_i)})$$

$$\leq -n_{k_i} - d_{n_{k_i}}. \tag{3.2}$$

From this and (3.1), we get

$$r_{t_i+\delta(t_i)} - t_i \leq r_{n_{k_i}+c_{n_{k_i}}} - n_{k_i} - d_{n_{k_i}} < n_{k_{i+1}} < t_{i+1}.$$

Next, we define

$l_j = n_{k_1} + d_{n_{k_1}}$, for $0 \leq j \leq r_{t_1+\delta(t_1)} - t_1$,

$l_j = n_{k_i} + d_{n_{k_i}}$, for $r_{t_{i-1}+\delta(t_{i-1})} - t_{i-1} < j \leq r_{t_i+\delta(t_i)} - t_i$.

By the above definition of $\{l_j\}$, we have

$$\sum_{i\geq 1} q^{t_i - l_{r_{t_i+\delta(t_i)}-t_i}} = \infty.$$

This implies that $\{l_j\} \in \Omega_{\{r_n\}}$.

By Lemma 3.1 and (3.2), we can estimate the measure of the union of the following balls

$$\begin{aligned}
&\mu\left(\bigcup_{n=1+r_{t_{i-1}+\delta(t_{i-1})}-t_{i-1}}^{r_{t_i+\delta(t_i)}-t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l_n}}\right)\right)\\
&\leq \max\left\{q^{n_{k_i}-n_{k_i}-d_{n_{k_i}}}, q^{n_{k_i}+r_{t_i+\delta(t_i)}-t_i-r_{n_{k_i}+c_{n_{k_i}}}}\right\}\\
&\leq \frac{1}{q^{d_{n_{k_i}}}}.
\end{aligned}$$

Hence,

$$\sum_{i=2}^{\infty} \mu\left(\bigcup_{n=1+r_{t_{i-1}+\delta(t_{i-1})}-t_{i-1}}^{r_{t_i+\delta(t_i)}-t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l_n}}\right)\right) \leq \sum_{i=2}^{\infty} \frac{1}{q^{d_{n_{k_i}}}} < \infty.$$

The proof is complete. ∎

The converse inclusion $U_{\{r_n\}} \subseteq W_{\{r_n\}}$ is a consequence of the following result.

Proposition 3.2. *If $f \in U_{\{r_n\}}$ and $\{l_n\} \in \Omega_{\{r_n\}}$, then for all $k \geq 0$, we have*

$$\mu\left(\bigcup_{n=k}^{\infty} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{l_n}}\right)\right) > \frac{1}{q^m} \tag{3.3}$$

for all constants $m \geq 2$.

Proof. Let us fix a sequence $\{l_n\} \in \Omega_{\{r_n\}}$. Choose the fuction $\delta(n)$ and the sequence t_i according to the definition of $\Omega_{\{r_n\}}$.

Put $\delta'(n) = \lfloor \delta(n)/2 \rfloor$ and

$$l'_0 = l'_1 = \cdots = l'_{r_{t_1+\delta'(t_1)}-t_1} = l_{r_{t_1+\delta(t_1)}-t_1},$$

$l'_n = l_{r_{t_s+\delta(t_s)}-t_s}$ for $r_{t_{s-1}+\delta'(t_{s-1})} - t_{s-1} < n \leq r_{t_s+\delta'(t_s)} - t_s$, $s \geq 2$.

Assume that (3.3) is false. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right) \leq \frac{1}{q^m}, \text{ for all } N \geq k_0. \tag{3.4}$$

Let P_i/Q_i be the principle convergent of f. Since $f \in U_{\{r_n\}}$, there exists a nonnegative integer c such that

$$\left|f - \frac{P_i}{Q_i}\right| = \frac{1}{|Q_i||Q_{i+1}|} > q^{-r_{n_i+c}}, \text{ with } n_i = \deg(Q_i).$$

This implies that

$$n_{i+1} < r_{n_i+c} - n_i. \tag{3.5}$$

Let us fix an integer s_0 fulfilling the conditions

$$\delta'(t_{s_0}) > \max\{c, m\} \text{ and } t_{s_0} > n_0. \tag{3.6}$$

Then, we choose P_{i_s}/Q_{i_s} as the subsequence of P_i/Q_i with $\deg(Q_{i_s}) = n_{i_s}$ whose indices forms a sequence $\{i_s\}$ defined by

$$n_{i_s-1} < t_s \leq n_{i_s},\ s \geq s_0. \tag{3.7}$$

Obviously, $i_s > 1$ for $s \geq s_0$.

Since $r_n \geq 2n$, we have

$$r_{t_i+\delta'(t_i)} - t_i = t_i + 2\delta'(t_i) + r_{t_i+\delta(t_i)} - 2(t_i + \delta(t_i)) \geq t_i + 2\delta'(t_i).$$

Hence, we can define a set

$$L_{s+1} := \left\{k_0 \leq \deg(Q) < n_{i_{s+1}} : \{Qf\} \in \bigcup_{n=k_0}^{n_{i_{s+1}-1}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) \right.$$
$$\left. \setminus \bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right\}$$

for $s \geq s_1$, where the integer s_1 fulfills the conditions $s_1 \geq s_0$ and $t_{s_1} > k_0$.

Next, we want to estimate the number of elements in L_{s+1}. In order to do this, we need to find the number of elements $\{Qf\}$ with $\deg(Q) \leq n_{i_{s+1}}$ contained in a fixed ball with radius q^{-d}. Since

$$f = \frac{P_{i_{s+1}}}{Q_{i_{s+1}}} + R, \text{ with } |R| < \frac{1}{|Q_{i_{s+1}}|^2},$$

we have

$$\{Qf\} = \frac{P'}{Q_{i_{s+1}}} + R', \tag{3.8}$$

where $|P'| < |Q_{i_{s+1}}|$ and $|R'| < 1/|Q_{i_{s+1}}|$ for all $Q \in \mathbb{F}_q[X]$ with $\deg(Q) \leq n_{i_{s+1}}$. By (3.8), we know that $\{Qf\}$ is contained in a ball of the form $B(\frac{P'}{Q_{i_{s+1}}}, q^{-n_{i_{s+1}}})$ and all these balls are disjoint. Then, the number of $\{Qf\}$ with $\deg(Q) \leq n_{i_{s+1}}$ belonging to $B(g, q^{-d})$ is

$$\max\left\{1, \frac{q^{-d}}{q^{-n_{i_{s+1}}}}\right\} = \max\left\{1, q^{n_{i_{s+1}}-d}\right\}. \tag{3.9}$$

Now, we are going to estimate the number of elements in L_{s+1}. Let

$$\bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) = \bigcup_j B\left(\{Q_jf\}; \frac{1}{q^{d_j}}\right),$$

where $B(\{Q_jf\}; q^{-d_j})$ are disjoint, for all j. By (3.4), we get

$$\begin{aligned}\frac{1}{q^m} &\geq \mu\left(\bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right)\\ &= \mu\left(\bigcup_j B\left(\{Q_jf\}; \frac{1}{q^{d_j}}\right)\right)\\ &= \sum_j \mu\left(B\left(\{Q_jf\}; \frac{1}{q^{d_j}}\right)\right) = \sum_j \frac{1}{q^{d_j}}.\end{aligned}$$

Using (3.9), the number of $\{Qf\}$ with $\deg(Q) \leq n_{i_{s+1}}$ belonging to $\bigcup_j B\left(\{Q_jf\}, q^{-d_j}\right)$ is at most

$$\begin{aligned}\sum_j \max\{q^{n_{i_{s+1}}-d_j}, 1\} &= \max\left\{q^{n_{i_{s+1}}}\sum_j q^{-d_j}, \sum_j 1\right\}\\ &\leq \max\left\{q^{n_{i_{s+1}}-m}, q^{r_{t_s+\delta'(t_s)}-t_s+1}\right\}\end{aligned} \tag{3.10}$$

Since $n - r_n$ is non-increasing,

$$r_{n+\delta(n)} - \delta(n) \geq r_{n+\delta'(n)} - \delta'(n).$$

Adding $\delta(n)$ to both sides, we obtain

$$r_{n+\delta(n)} \geq r_{n+\delta'(n)} - \delta'(n) + \delta(n) \geq r_{n+\delta'(n)} - \frac{\delta(n)}{2} + \delta(n) \geq r_{n+\delta'(n)} + \delta'(n). \quad (3.11)$$

By (3.6), (3.7), (3.11) and the definition of t_i, we have

$$\begin{aligned} r_{t_s+\delta'(t_s)} - t_s &\leq r_{t_s+\delta(t_s)} - t_s - \delta'(t_s) \\ &\leq r_{t_s+\delta(t_s)} - t_s - m \\ &\leq t_{s+1} - m \\ &\leq n_{i_{s+1}} - m. \end{aligned}$$

This implies that (3.10) is less than $q^{n_{i_{s+1}}-m+1}$. Hence, the number of elements in L_{s+1} is at least $q^{n_{i_{s+1}}} - q^{n_{i_{s+1}}-m+1}$.

Next, we claim that

$$\bigcup_{Q\in L_{s+1}} B\left(\{Qf\}; \frac{1}{q^{l'_{n_{i_{s+1}}}}}\right) \subset \bigcup_{n=k_0}^{n_{i_{s+1}}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) \setminus \bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right). \quad (3.12)$$

In order to show this, fix $Q_1 \in L_{s+1}$. Suppose there exists a polynomial Q_2 with $\deg(Q) = u$ and $k_0 \leq u \leq r_{t_s+\delta'(t_s)} - t_s$ such that

$$B(\{Q_1 f\}; q^{-l'_{n_{i_{s+1}}}}) \cap B(\{Q_2 f\}; q^{-l'_u}) \neq \emptyset.$$

We know that $\{Q_1 f\}$ does not belong to $B(\{Q_2 f\}; q^{-l'_u})$. Hence,

$$B\left(\{Q_2 f\}; \frac{1}{q^{l'_u}}\right) \subset B\left(\{Q_1 f\}; \frac{1}{q^{l'_{n_{i_{s+1}}}}}\right).$$

Then, we get $q^{-l'_u} < q^{-l'_{n_{i_{s+1}}}}$. On the other hand, since l'_n is non-decreasing and $\deg(Q_2) = u < n_{i_{s+1}}$, we have $q^{-l'_u} \geq q^{-l'_{n_{i_{s+1}}}}$, a contradiction. Consequently, (3.12) holds.

Now, we consider two cases:

1. If $q^{-n_{i_{s+1}}} \leq q^{-l'_{n_{i_{s+1}}}}$ with $s \geq s_1$, then we have

$$\bigcup_{Q\in L_{s+1}} B\left(\{Qf\}; \frac{1}{q^{n_{i_{s+1}}}}\right) \subset \bigcup_{Q\in L_{s+1}} B\left(\{Qf\}; \frac{1}{q^{l'_{n_{i_{s+1}}}}}\right). \quad (3.13)$$

Since $|\{Q_1 f\} - \{Q_2 f\}| = |\{(Q_1 - Q_2) f\}| \geq q^{-n_{i_{s+1}}}$ for $Q_1, Q_2 \in L_{s+1}$ with $Q_1 \neq Q_2$, we get $B\left(\{Q_1 f\}; q^{-n_{i_{s+1}}}\right) \cap B\left(\{Q_2 f\}; q^{-n_{i_{s+1}}}\right) = \emptyset$. Thus, any two balls in $\bigcup_{Q \in L_{s+1}} B\left(\{Qf\}, q^{-n_{i_{s+1}}}\right)$ are disjoint. By (3.12) and (3.13), we obtain

$$
\begin{aligned}
&\mu\left(\bigcup_{n=k_0}^{n_{i_{s+1}-1}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) \setminus \bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right) \\
&\geq \mu\left(\bigcup_{Q \in L_{s+1}} B\left(\{Qf\}; \frac{1}{q^{n_{i_{s+1}}}}\right)\right) \\
&= \sum_{Q \in L_{s+1}} \frac{1}{q^{n_{i_{s+1}}}} \\
&\geq \left(q^{n_{i_{s+1}}} - q^{n_{i_{s+1}}-m+1}\right) \frac{1}{q^{n_{i_{s+1}}}} \\
&= (1 - q^{-m+1}),
\end{aligned}
$$

which when iterated yields a contradiction.

2. If $q^{-n_{i_{s+1}}} > q^{-l'_{n_{i_{s+1}}}}$ with $s \geq s_1$, then any two balls in $\bigcup_{Q \in L_{s+1}} B\left(\{Qf\}; q^{-l'_{n_{i_{s+1}}}}\right)$ are disjoint.

   Thus, we obtain

$$
\begin{aligned}
&\mu\left(\bigcup_{n=k_0}^{n_{i_{s+1}-1}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) \setminus \bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right) \\
&\geq \sum_{Q \in L_{s+1}} \mu\left(B\left(\{Qf\}; \frac{1}{q^{l'_{n_{i_{s+1}}}}}\right)\right) \\
&= \sum_{Q \in L_{s+1}} \frac{1}{q^{l'_{n_{i_{s+1}}}}} \\
&\geq \left(q^{n_{i_{s+1}}} - q^{n_{i_{s+1}}-m+1}\right) \frac{1}{q^{l'_{n_{i_{s+1}}}}}.
\end{aligned}
$$

   Applying (3.5), (3.6), (3.7) and the property that $n - r_n$ is non-increasing, we have

$$
n_{i_s} < r_{n_{i_s-1}+c} - n_{i_s-1} \leq r_{t_s+c} - t_s \leq r_{t_s+\delta'(t_s)} - t_s - \delta'(t_s) + c \leq r_{t_s+\delta'(t_s)} - t_s. \tag{3.14}
$$

   By (3.14) and the definition of l'_n,

$$
\frac{1}{q^{l'_{n_{i_{s+1}}}}} \geq \frac{1}{q^{l_{r_{t_{s+1}+\delta(t_{s+1})}-t_{s+1}}}}.
$$

Using (3.7), we get

$$q^{n_{i_{s+1}}} - q^{n_{i_{s+1}}-m+1} \geq q^{n_{i_{s+1}}}(1-q^{-m+1}) \geq q^{t_{s+1}}(1-q^{-m+1}).$$

Therefore,

$$\mu\left(\bigcup_{n=k_0}^{n_{i_{s+1}-1}} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right) \setminus \bigcup_{n=k_0}^{r_{t_s+\delta'(t_s)}-t_s} \bigcup_{\deg(Q)=n} B\left(\{Qf\}; \frac{1}{q^{l'_n}}\right)\right)$$
$$\geq (1-q^{-m+1}) \cdot \frac{q^{t_{s+1}}}{q^{l_{r_{t_{s+1}+\delta(t_{s+1})}}-t_{s+1}}}.$$

As the series $\sum_s q^{t_{s+1}-l_{r_{t_{s+1}+\delta(t_{s+1})}}-t_{s+1}}$ diverges, we have a contradiction again by iteration.

Hence, the proof is complete ∎

Finally, Proposition 3.1 and Proposition 3.2 imply Theorem 1.9.

Now, we are going to compare Theorem 1.9 with Theorem 1.8. As we have mentioned at the end of Section 1.3, the sets $U_{\{r_n\}}$ and U_s are the same when $r_n = (s+1)n$. Thus, we should discuss the relationship between $\Omega_{\{r_n\}}$ and Ω_s.

Recall the set

$$\Omega_s := \left\{ l_n \geq 1 : \sum_{n=0}^{\infty} q^{n-sl_n} = \infty \right\}.$$

When $s = 1$ (i.e., $r_n = 2n$), it is obvious that $\Omega_{\{2n\}} \subseteq \Omega_1$.

On the other hand, the following two results show that the two sets $\Omega_{\{r_n\}}$ and Ω_s are not contained in each other when $s > 1$.

Proposition 3.3. *Let $s > 1$. Choose $l_n = \left\lfloor \frac{n+\log_q n+\log_q(\log_q n)}{s} \right\rfloor$. Then, we have $\{l_n\} \in \Omega_s \backslash \Omega_{\{(s+1)n\}}$.*

Proof. Clearly, $\{l_n\} \in \Omega_s$. Assume that $\{l_n\} \in \Omega_{\{(s+1)n\}}$. Then, there exists $\{t_n\}$ and $\delta(n)$ from the definition of $\Omega_{\{(s+1)n\}}$. Since $t_{i+1} > st_i + (s+1)\delta(t_i) > st_i$ for all $i \geq 1$, we have $t_n > s^{n-1}$ for $n \geq 2$.

Thus,

$$\begin{aligned}
\infty &= \sum_{n\geq 2} q^{t_n - l_{st_n+(s+1)\delta(t_n)}} \\
&\leq \sum_{n\geq 2} q^{t_n - \frac{st_n+\log_q st_n}{s}+1} \\
&\leq q\sum_{n\geq 2} q^{-\frac{n\log_q s}{s}} = q\sum_{n\geq 2} s^{-\frac{n}{s}} < \infty,
\end{aligned}$$

a contradiction. ∎

Proposition 3.4. *For $s > 1$, we have $\Omega_{\{(s+1)n\}} \backslash \Omega_s \neq \emptyset$.*

Proof. Choose any $\{t_i\}$ as in the definition of $\Omega_{\{(s+1)n\}}$ where $\delta(t_i) = \left\lfloor \frac{s-1}{2(s+1)} \log_q i \right\rfloor$.

For $st_{i-1} + (s+1)\delta(t_{i-1}) < n \leq st_i + (s+1)\delta(t_i)$, define $l_n = t_i + \lfloor \log_q i \rfloor$.

Then,

$$\sum_{i \geq 1} q^{t_i - l_{st_i + (s+1)\delta(t_i)}} = \sum_{i \geq 1} q^{-\lfloor \log_q i \rfloor} \geq \sum_{i \geq 1} q^{-\log_q i} = \sum_{i \geq 1} \frac{1}{i} = \infty.$$

Thus, $\{l_n\} \in \Omega_{\{(s+1)n\}}$.

However,

$$\begin{aligned}
\sum_{n \geq 0} q^{n - sl_n} &\leq \sum_{i \geq 1} \sum_{st_{i-1}+(s+1)\delta(t_{i-1}) < n \leq st_i + (s+1)\delta(t_i)} q^{n - sl_n} \\
&= \sum_{i \geq 1} q^{-s(t_i + \lfloor \log_q i \rfloor)} \cdot \sum_{st_{i-1}+(s+1)\delta(t_{i-1}) < n \leq st_i + (s+1)\delta(t_i)} q^n \\
&\leq \frac{q}{q-1} \sum_{i \geq 1} q^{-s\lfloor \log_q i \rfloor + (s+1)\delta(t_i)} \\
&\leq \frac{q}{q-1} \sum_{i \geq 1} q^{-s\lfloor \log_q i \rfloor + \frac{s-1}{2}\lfloor \log_q i \rfloor} \\
&\leq \frac{q}{q-1} \sum_{i \geq 1} \frac{1}{i^{\frac{s+1}{2}}} < \infty.
\end{aligned}$$

Hence, $\{l_n\}$ is not in Ω_s, which implies $\Omega_{\{(s+1)n\}} \backslash \Omega_s \neq \emptyset$. ∎

Chapter 4

Conclusion

We conclude the thesis with some conjectures.

In Section 2.2, we gave some improvements and proved that one direction of Theorem 2.1 still holds even when dropping the monotonicity condition on $\{l_n\}$. In fact, we conjecture that the converse direction also holds if we remove the monotonicity condition. Thus, we have the following conjecture.

Conjecture 4.1. *Let $\{l_n\}$ be a sequence. Define*

$$U := \left\{ f \in \mathbb{L} : \sum_{n=0}^{\infty} \frac{1}{q^{l_n^*}} = \infty, \ \textit{where } l_n^* = \max\{n_{k+1} - n, l_n\} \textit{ for } n_k \leq n < n_{k+1} \right\}.$$

Then, $f \in U$ if and only if (1.5) has infinitely many solutions Q for almost all $g \in \mathbb{L}$

If the above conjecture is true, then this would allow us to prove Kurzweil's theorem in a particular easy manner.

In Chapter 3, we have proved Theorem 1.9 and compared the sets $\Omega_{\{r_n\}}$ with Ω_s when $r_n = (s+1)n$. Note that the approximation functions of Theorem 1.8 and Theorem 1.9 are of the same form q^{-l_n} (in contrast to the other theorems, the fuction does not tend to 0 as n tends to infinity). The sequence $\{l_n\}$ in $\Omega_{\{r_n\}}$ is assumed to be non-decreasing. An interesting question is whether or not one can improve Theorem 1.9 by dropping the monotonicity condition on $\{l_n\}$? If yes, then what can be said about the relation between the sets $\Omega_{\{r_n\}}$ and Ω_s when $r_n = (s+1)n$?

Overall, there are still interesting questions left concerning inhomogeneous Diophantine approximation in the field of formal Laurent series.

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