國 立 交 通 大 學 理 學 院 應 用 數 學 系

碩 士 論 文

廣義彼德森圖的 α-控制數

α-Domination of Generalized Petersen Graph

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摘要

一個圖 G 的控制數是圖論中最重要的一個不變量，在很多文獻中都有相當不錯的研究成果。但是，控制的概念可做更進一步的探討，而 α-控制數的研究就是其中的一種延伸研究。對於任意 α 大於 0 且小或等於 1 時，存在一集合 S 包含於點集合 V 中，如果對於所有在點集合 V 中卻不屬於 S 中的點 v，點 v 在 S 中的鄰居數大或等於點 v 的鄰居數乘上 α 倍，我們就稱 S 是 α-控制集並表示成 $\gamma_\alpha(G)$。

因為我們已知對於度數為 3 的正則圖，當 α 大於 0 且小或等於 1/3 時，$\gamma_\alpha(G) = \gamma(G)$; 而當 α 大於 2/3 且小或等於 1 時，$\gamma_\alpha(G) = \alpha_0(G)$; 所以在此篇論文中，我們討論在 $\frac{1}{3} < \alpha \leq \frac{2}{3}$ 時，廣義彼德森圖的 α-控制數，並獲得一些具體成果。

Abstract

Let $G = (V, E)$ be a graph with n vertices, m edges and no isolated vertices. For some α with $0 < \alpha \leq 1$ and a set $S \subseteq V$, we say that S is $\alpha-$*dominating* if for all $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The size of a smallest such S is called the $\alpha -$ *domination number* of G denoted by $\gamma_\alpha(G)$.

For positive integers n and k, the generalized Petersen graph $P(n, k)$ is the graph with vertex set $V = \{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\}$ and the edge set $E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i \in \mathbb{Z}_n\}$ where addition is modulo n. Clearly, $P(n, k)$ is a 3-regular graph.

In this thesis, we study $\gamma_\alpha(P(n, k))$. Since for 3-regular graphs $\gamma_\alpha(G) = \gamma(G)$(domination number of G), provided $0 < \alpha \leq \frac{1}{3}$ and $\gamma_\alpha(G) = \alpha_0(G)$(vertex cover number of G) provided $\frac{2}{3} < \alpha \leq 1$, we shall focus on the case $\frac{1}{3} < \alpha \leq \frac{2}{3}$. As a consequence, the exact values of $\gamma_\alpha(P(n, k))$ are obtained for certain n and k.

誌謝

在這兩年的研究期間，最要感謝的人是我的兩位指導教授傅恆霖老師以及史青林老師，每當我遇到困難時，都能不厭其煩的指導我，細心地給予協助，使我能順利完成學業。

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最後，謝謝最親愛的家人和朋友們，不論我遭遇任何挫折都在我身邊給予我鼓勵與支持，有了你們的陪伴與支持才使我更加堅強，我也才能夠有今天的成就，獻上最深的感謝。

Contents

Chapter 1

Introduction and Preliminaries

A puzzle mentioned by David Woolbright [9] involves occupying each cell of a 6×6 array with a guard or a prisoner, subject only to the constraint that every prisoner is adjacent to at least as many guards as prisoners (where adjacency is vertical, horizontal or diagonal).

This puzzle generalizes to a graph invariant in the following way. We may say that the $Woolbrightnumber$ of a graph G is the size of a smallest set of vertices with the property that every vertex not in S has at least as many neighbors in S as neighbors not in S.

1.1 Basic Notations

Graph terminology not presented here can be found in Chartrand and Lesniak [4].

The (open) neighborhood $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v. The closed neighborhood $N[v]$ of v is $N(v) \cup \{v\}$. For any set $S \subseteq V$, the neighborhood $N(S)$ of S is defined as $\cup_{v \in S} \mathrm{N}(v)$, and the closed neighborhood $N[S]$ of S is $N(S) \cup S$. Generalizing the Woolbright number, we introduce the concept of α-domination: For any α with $0 < \alpha \leq 1$ and a set $S \subseteq V$, we say that S is α-dominating if for all $v \in V - S$, $|N(v) \cap S| \geq \alpha |N(v)|$. The size of the smallest such S is called the $\alpha-$*domination number* and is denoted by $\gamma_\alpha(G)$. Thus the Woolbright number of a graph G is $\gamma_{\frac{1}{2}}(G)$. The size of the largest minimal such set S is called the *upper* α-domination number and is denoted by $\Gamma_\alpha(G)$.

Recall that a set $S \subseteq V$ is said to dominate a graph if every vertex in the graph is either in S or is adjacent to a vertex in S. Stated in the current context, we might say that for every vertex $v \in V - S$, $|N(v) \cap S| \geq 1$. The size of a smallest such set is called the *domination number* and is denoted by $\gamma(G)$. The size of a largest minimal dominating set is called the upper domination number and is denoted by $\Gamma(G)$. Since a smallest α-dominating set is a dominating set, it is immediate to see that $\gamma(G) \leq \gamma_\alpha(G)$ for all G and for every α.

For any graph G and for any α, with $0 < \alpha \leq 1$, if S is any set of minimum size which α-dominates G, we will call S a γ_α-set. Similarly, if S is a set of minimum size which dominates G, we call S a γ-set.

The following known results for special graphs are straightforward.

Proposition 1.1.1. [5] If P_n is a path with n vertices, then

$$\gamma_\alpha(P_n) = \lceil n/3 \rceil, \text{ if } 0 < \alpha \leq \tfrac{1}{2},$$
$$\gamma_\alpha(P_n) = \lfloor n/2 \rfloor, \text{ if } \tfrac{1}{2} < \alpha \leq 1.$$

Proposition 1.1.2. [5] If C_n is a cycle with n vertices, then

$$\gamma_\alpha(C_n) = \lceil n/3 \rceil, \text{ if } 0 < \alpha \leq \tfrac{1}{2},$$
$$\gamma_\alpha(C_n) = \lceil n/2 \rceil, \text{ if } \tfrac{1}{2} < \alpha \leq 1.$$

Proposition 1.1.3. [5] If K_n is a complete graph with n vertices, then

$$\gamma_\alpha(K_n) = \lceil \alpha(n-1) \rceil.$$

We have already mentioned that for any graph G the standard domination parameter $\gamma(G)$ is a lower bound for $\gamma_\alpha(G)$. An upper bound is found by examining the vertex cover number. The *vertex cover number* $\alpha_0(G)$ is the size of a smallest set of vertices S such that every edge has at least one endvertex in S. Clearly, $\gamma_1(G) = \alpha_0(G)$.

A *generalized Petersen graph* $P(n, k)$ is the graph with vertices $\{u_0, u_1, \ldots, u_{n-1}\}$ and $\{v_0, v_1, \ldots, v_{n-1}\}$ and edges u_iu_{i+1} , u_iv_i and v_iv_{i+k} where the addition is modulo n.

1.2 Preliminaries

Next sufficient conditions are examined to guarantee that the parameter $\gamma_\alpha(G)$ equals its upper or lower bound. In any graph G, we will denote the maximum (minimum) degree of a vertex by $\Delta(G)$ (respectively, $\delta(G)$).

Proposition 1.2.1. [5] If G has maximum degree $\Delta(G)$, then the following holds: if $0 < \alpha \leq 1/\Delta(G)$, then $\gamma_\alpha(G) = \gamma(G)$.

Proposition 1.2.2. [5] Let G be a graph. If $1 \geq \alpha > (\Delta(G) - 1)/\Delta(G)$, then $\gamma_\alpha(G) = \alpha_0(G)$.

Theorem 1.2.3. [5] If $0 < \alpha < 1$, then for any graph G, $\gamma_\alpha(G) + \gamma_{1-\alpha}(G) \leq n$.

Chapter 2

Known Results

In this chapter, we introduce several known results related to α-domination, domination number and vertex cover number.

2.1 Bounds on α-domination

Let S be an α-dominating set with $|S| = \gamma_\alpha(G)$. Let M be the set of edges between S and $V - S$. Counting the edges from S to $V - S$, we see that $|M| \leq \Sigma_{v \in S} deg(v)$. Further, counting the number of edges from $V-S$ to S, we see that $|M| \geq \Sigma_{v \in V-S} \alpha deg(v)$. Combining these it is clear that

$\Delta(G)|S| \geq \Sigma_{v \in S} deg(v) \geq \Sigma_{v \in V-S} \alpha deg(v) \geq \alpha\delta(G)|V - S|$ .... (1)

The following proposition can be obtain by using (1).

Proposition 2.1.1. [5] For any graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$,

$$\gamma_\alpha(G) \geq \frac{\alpha\delta(G)n}{\Delta(G) + \alpha\delta(G)}$$

The following upper bound for $\gamma_\alpha(G)$ can be obtained by Theorem 1.2.3 .

Proposition 2.1.2. [5] For any graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$,

$$\gamma_\alpha(G) \leq \frac{\Delta(G)n}{\Delta(G) + (1-\alpha)\delta(G)}.$$

The following corollary is straightforward.

Corollary 2.1.3. [5] For any tree T and for any α with $0 < \alpha \leq 1$.

$$\frac{\alpha n}{\Delta(T) + \alpha} \leq \gamma_\alpha(T) \leq \frac{\Delta(T)n}{\Delta(T) + 1 - \alpha}$$

Proposition 2.1.4. [5] For any graph G with m edges ,

$$\gamma_\alpha(G) \geq \frac{2\alpha m}{(\alpha+1)\Delta(G)}$$

Proposition 2.1.5. [5] For any graph G with maximum degree $\Delta(G)$ and m edges,

$$\gamma_\alpha(G) \leq \frac{(2-\alpha)\Delta(G)n - (2-2\alpha)m}{(2-\alpha)\Delta(G)}$$

Next, we will consider bounds for regular graphs. If every vertex of a graph G has degree k, we say that G is k-regular. Clearly, if G is k-regular, then $\gamma_{1/k}(G) = \gamma_\alpha(G)$. In a k-regular graph, the number of edges $m = kn/2$ and $\Delta(G) = \delta(G) = k$. The next corollary follows from Propositions 2.1.4 and 2.1.5.

Corollary 2.1.6. [5] For a k-regular graph G, and for any α with $0 < \alpha \leq 1$,

$$\frac{\alpha n}{1+\alpha} \leq \gamma_\alpha(G) \leq \frac{n}{2-\alpha}.$$

Lettng $\alpha = i/k$ where i is an integer with $1 < i \leq k$, the following lower bound can bee obtained.

Corollary 2.1.7. [5] If G is a k-regular graph and i is an integer with $1 < i \leq k$, then

$$\gamma_{i/k}(G) \geq \lceil [i/(i+k)]n \rceil$$

2.2 Bounds on domination number and vertex cover number

Domination numbers for graphs and associated concepts have been studied for many years and there is an extensive literature on the subject. In general, determining the domination number (and most of its variations) is an NP-complete problem.

We present some bounds for $\gamma(P(n,k))$.

Theorem 2.2.1. [3] For each odd integer $n \geq 3$, $\gamma(P(n,k)) \leq \lceil 3n/5 \rceil$.

Theorem 2.2.2. [8] If $n \geq 3$, we have

$$\gamma(P(n,1)) = \begin{cases} \frac{n}{2} + 1\,,\ if\ n \equiv 2\,(mod\,4) \\ \left\lceil \frac{n}{2} \right\rceil\,,\ otherwise\,. \end{cases}.$$

Proposition 2.2.3. [8] If k is an even number greater than 2 and $n > 2k$, then $\gamma(P(n,k)) \leq \frac{5n}{9} + O(k)$.

We present some bounds for $\alpha_0(P(n,k))$ and exact values of $\alpha_0(P(n,k))$ for some n and k. First, we introduce some lower bounds and upper bounds.

Proposition 2.2.4. [1] If n is odd then we have $\alpha_0(P(n,k)) \geq n + \frac{(n,k)+1}{2}$, where (n,k) is the greatest common divisor of n and k.

Corollary 2.2.5. [1] For all odd n, we have $\alpha_0(P(n,k)) \geq n+1$.

Theorem 2.2.6. [1] If both n and k are odd, then $\alpha_0(P(n,k)) \leq n + \frac{k+1}{2}$.

Next, we introduce exact values of $\alpha_0(P(n,k))$ for some n and k.

Theorem 2.2.7. [2] For all n, $\alpha_0(P(n,2)) = n + \lceil \frac{n}{5} \rceil$.

Proposition 2.2.8. [1] $\alpha_0(P(n,k)) = n$, if and only if n is even and k is odd.

Theorem 2.2.9. [1] $\alpha_0(P(n,k)) = n+1$ if and only if n is odd and $k=1$, or $(n,k)=(5,2)$.

In [5], Dunbar et al, introduced α-domination, discussed bounds for $\gamma_{1/2}(G)$ for the Kings graph, and gave bounds on $\gamma_\alpha(G)$ for a general $\alpha, 0<\alpha\leq 1$. Furthermore, they showed that the problem of deciding whether $\gamma_\alpha(G)\leq k$ is NP-complete.

In this thesis, we discuss α-domination for generalized Petersen graph.

Chapter 3

Main Results

Throughout this chapter, the generalized Petersen graph $P(n,k)$ is the graph with vertex set $V = \{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\}$ and the edge set $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i \in \mathbb{Z}_n\}$ where addition is modulo n.

3.1 Exact values of α-domination number

Since for 3-regular graphs G, $\gamma_\alpha(G) = \gamma(G)$ provided $0 < \alpha \leq \frac{1}{3}$ and $\gamma_\alpha(G) = \alpha_0(G)$ provided $\frac{2}{3} < \alpha \leq 1$, we consider $\frac{1}{3} < \alpha \leq \frac{2}{3}$ in what follows.
We start with the graphs $P(n,1)$.

Proposition 3.1.1. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(n,1)) = n$.

Proof. Let $G = P(n,1)$, and let S be a α-dominating set. Now, let

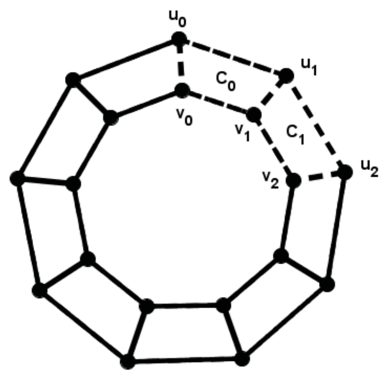
$$S = \{u_0, v_1, u_2, \ldots, v_{n-2}, u_{n-1}\}$$

provided n is odd and

$$S = \{u_0, v_1, u_2, \ldots, u_{n-2}, v_{n-1}\}$$

provided n is even. By direct checking, S is a α-dominating set of G in respective cases, we conclude that $\gamma_\alpha(G) \leq n$.

On the other hand, let $C_i =< u_i, v_i, u_{i+1}, v_{i+1} >_R$, $i = 0, 1, \ldots, n-1$. Clearly, C_i is a 4-cycle and there must exist at least two vertices in S. Since each vertex is in exactly two C_i's, this implies that $\gamma_\alpha(G) \geq n$.



P(n , 1)

Figure 3.1: Example

Proposition 3.1.2. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(5s, 5t+2)) = 4s$, for each $s \geq 0$ and $0 \leq t < s$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(G) \geq 4s$. Now, it suffices to claim $\gamma_\alpha(G) \leq 4s$. This follows from the selection defined below :

$$S = \{u_i, v_j \mid i \equiv 0\,,\, 2\,(mod\ 5); j \equiv 3\,,\, 4\,(mod\ 5)\}\,.$$

It is easy to see $|S| = 4s$. Now we need to show that S is a α-dominating set of G, that is, for each vertex $v \in V(G)$ and $v \notin S$, $|N(v) \cap S| \geq 2$. Consider $u_r \notin S$. Clearly, $r \equiv 1, 3, 4\ (mod\ 5)$. If $r \equiv 1\ (mod\ 5)$, then $(r-1) \equiv 0\ (mod\ 5)$ and $(r+1) \equiv 2\ (mod\ 5)$. It follows that $\{u_{r-1}, u_{r+1}\} \subseteq N(u_r) \cap S$. If $r \equiv 3\ (mod\ 5)$, then $(r-1) \equiv 2\ (mod\ 5)$. It follows that $\{u_{r-1}, v_r\} \subseteq N(u_r) \cap S$. If $r \equiv 4\ (mod\ 5)$, then $(r+1) \equiv 0\ (mod\ 5)$. It follows that $\{u_{r+1}, v_r\} \subseteq N(u_r) \cap S$. Consider $v_m \notin S$. Clearly, $m \equiv 0, 1, 2\ (mod\ 5)$. If $m \equiv 0\ (mod\ 5)$, then $(m-2) \equiv 3\ (mod\ 5)$. It follows that $\{v_{m-2}, u_m\} \subseteq N(v_m) \cap S$.

If $m \equiv 1 \ (mod\ 5)$, then $(m+2) \equiv 3 \ (mod\ 5)$ and $(m-2) \equiv 4 \ (mod\ 5)$. It follows that $\{v_{m+2}, v_{m-2}\} \subseteq N(v_m) \cap S$. If $m \equiv 2 \ (mod\ 5)$, then $(m+2) \equiv 4 \ (mod\ 5)$. It follows that $\{v_{m+2}, u_m\} \subseteq N(v_m) \cap S$. Therefore, we conclude that $\gamma_\alpha(G) \leq 4s$ and we have the proof. ∎

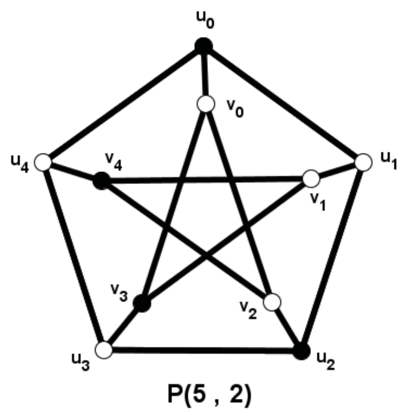


Figure 3.2: Example for $\gamma_\alpha(P(5s, 2))$.

Proposition 3.1.3. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(5s+3, 2)) = 4s + 3$, for each $s > 0$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(G) \geq 4s + 3$. Now, it suffices to claim $\gamma_\alpha(G) \leq 4s + 3$. This follows from the selection defined below :

$$\begin{aligned} S &= \{u_i, v_j \mid i \equiv 0\,,\,2\,(mod\ 5), 0 \leq i \leq 5s-1; j \equiv 3\,,\,4\,(mod\ 5), 0 \leq j \leq 5s-1\} \\ &\cup \{v_{n-2}, v_{n-1}, u_{n-3}\}. \end{aligned}$$

It is easy to see $|S| = 4s + 3$. Now we need to show that S is a α-dominating set of G, that is, for each vertex $v \in V(G)$ and $v \notin S$, $|N(v) \cap S| \geq 2$. Consider $u_r \notin S$, for $0 \leq r \leq 5s-1$. Clearly, $r \equiv 1, 3, 4 \ (mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(u_r) \cap S| \geq 2$. Consider $v_m \notin S$, for $0 \leq m \leq 5s-1$. Clearly, $m \equiv 0, 1, 2 \ (mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(v_m) \cap S| \geq 2$. Consider $u_r \notin S$, $r = n-2, n-1$. If $r = n-2$, then $r - 1 = n - 3$.

It follows that $\{u_{r-1}, v_r\} \subseteq N(u_r) \cap S$. If $r = n-1$, then $(r+1) \equiv 0$ $(mod\ 5)$. It follows that $\{u_{r+1}, v_r\} \subseteq N(u_r) \cap S$. Consider $v_m \notin S$, $m = n-3$. If $m = n-3$, then $(m-1) \equiv 4$ $(mod\ 5)$. It follows that $\{v_{m-1}, u_m\} \subseteq N(v_m) \cap S$. Therefore, we conclude that $\gamma_\alpha(G) \leq 4s+3$ and we have the proof. ∎

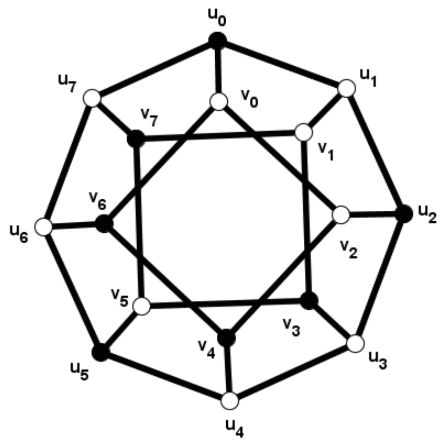


Figure 3.3: Example for $\gamma_\alpha(P(5s+3, 2))$.

Proposition 3.1.4. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(5s+4, 2)) = 4s+4$, for each $s > 0$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(G) \geq 4s+4$. Now, it suffices to claim $\gamma_\alpha(G) \leq 4s+4$. This follows from the selection defined below :

$$\begin{aligned} S &= \{u_i, v_j \mid i \equiv 0\,,\, 2\,(mod\ 5), 0 \leq i \leq 5s-1; j \equiv 3\,,\, 4\,(mod\ 5), 0 \leq j \leq 5s-1\} \\ &\cup \{v_{n-2}, v_{n-1}, u_{n-4}, u_{n-3}\}. \end{aligned}$$

It is easy to see $|S| = 4s+4$. Now we need to show that S is a α-dominating set of G, that is, for each vertex $v \in V(G)$ and $v \notin S$, $|N(v) \cap S| \geq 2$. Consider $u_r \notin S$, for $0 \leq r \leq 5s-1$. Clearly, $r \equiv 1, 3, 4$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(u_r) \cap S| \geq 2$. Consider $v_m \notin S$, for $0 \leq m \leq 5s-1$. Clearly, $m \equiv 0, 1, 2$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have

$|N(v_m) \cap S| \geq 2$. Consider $u_r \notin S$, $r = n-2, n-1$. If $r = n-2$, then $r-1 = n-3$. It follows that $\{u_{r-1}, v_r\} \subseteq N(u_r) \cap S$. If $r = n-1$, then $(r+1) \equiv 0 \ (mod\ 5)$. It follows that $\{u_{r+1}, v_r\} \subseteq N(u_r) \cap S$. Consider $v_m \notin S$, $m = n-3, n-4$. If $m = n-3$, then $(m-2) \equiv 4 \ (mod\ 5)$. It follows that $\{v_{m-2}, u_m\} \subseteq N(v_m) \cap S$. If $m = n-4$, then $(m-2) \equiv 3 \ (mod\ 5)$. It follows that $\{v_{m-2}, u_m\} \subseteq N(v_m) \cap S$. Therefore, we conclude that $\gamma_\alpha(G) \leq 4s+4$ and we have the proof. ∎

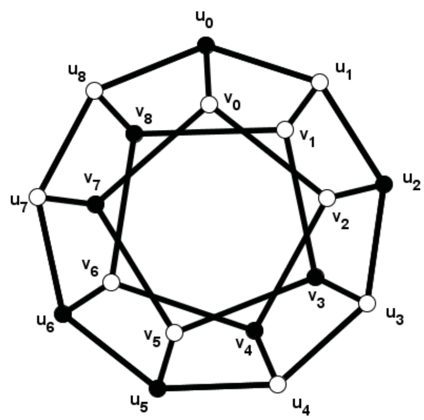


Figure 3.4: Example for $\gamma_\alpha(P(5s+4, 2))$.

3.2 Bounds of α-domination number

Proposition 3.2.1. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $4s+1 \leq \gamma_\alpha(P(5s+1, 2)) \leq 4s+2$, for each $s > 0$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(G) \geq 4s+1$. Now, it suffices to claim $\gamma_\alpha(G) \leq 4s+2$. This follows from the selection defined below :

$$S = \{u_i, v_j \mid i \equiv 0\,,\, 2\,(mod\ 5), 0 \leq i \leq 5s-1; j \equiv 3\,,\, 4\,(mod\ 5), 0 \leq j \leq 5s-1\} \cup \{v_{n-1}, u_{n-1}\}.$$

It is easy to see $|S| = 4s + 2$. Now we need to show that S is a α-dominating set of G, that is, for each vertex $v \in V(G)$ and $v \notin S$, $|N(v) \cap S| \geq 2$. Consider $u_r \notin S$, for $0 \leq r \leq 5s - 1$. Clearly, $r \equiv 1, 3, 4$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(u_r) \cap S| \geq 2$. Consider $v_m \notin S$, for $0 \leq m \leq 5s - 1$. Clearly, $m \equiv 0, 1, 2$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(v_m) \cap S| \geq 2$. Therefore, we conclude that $\gamma_\alpha(G) \leq 4s + 2$ and we have the proof.∎

Proposition 3.2.2. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $4s + 2 \leq \gamma_\alpha(P(5s + 2, 2)) \leq 4s + 3$, for each $s > 0$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(G) \geq 4s + 2$. Now, it suffices to claim $\gamma_\alpha(G) \leq 4s + 3$. This follows from the selection defined below :

$$\begin{aligned} S &= \{u_i, v_j \mid i \equiv 0\,,\, 2\,(mod\ 5), 0 \leq i \leq 5s - 1; j \equiv 3\,,\, 4\,(mod\ 5), 0 \leq j \leq 5s - 1\} \\ &\cup \{v_{n-2}, v_{n-1}, u_{n-2}\}. \end{aligned}$$

It is easy to see $|S| = 4s + 3$. Now we need to show that S is a α-dominating set of G, that is, for each vertex $v \in V(G)$ and $v \notin S$, $|N(v) \cap S| \geq 2$. Consider $u_r \notin S$, for $0 \leq r \leq 5s - 1$. Clearly, $r \equiv 1, 3, 4$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(u_r) \cap S| \geq 2$. Consider $v_m \notin S$, for $0 \leq m \leq 5s - 1$. Clearly, $m \equiv 0, 1, 2$ $(mod\ 5)$. By the similar argument as Proposition 3.1.2, we have $|N(v_m) \cap S| \geq 2$. Consider $u_r \notin S$, $r = n - 1$. If $r = n - 1$, then $r - 1 = n - 2$. It follows that $\{u_{r-1}, v_r\} \subseteq N(u_r) \cap S$. Therefore, we conclude that $\gamma_\alpha(G) \leq 4s + 3$ and we have the proof. ∎

Proposition 3.2.3. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(6, 2)) = 6$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(P(6, 2)) \geq 4s + 1 = 5$. First, we will show that $\gamma_\alpha(P(6, 2)) \geq 6$. Let $G = P(6, 2)$, and let C_1 and C_2 be subgraphs of

G induced by $\{u_0, u_1, \cdots, u_5\}$ and $\{v_0, v_1, \cdots, v_5\}$, respectively. Let S be the α-dominating set of G. Observe that if $\{u_i, u_{i+1}, u_{i+2}\} \cap S = \phi$, for some i, then $|N(u_{i+1}) \cap S| < \alpha|N(u_{i+1})|$. Similarly, if $\{v_j, v_{j+2}, v_{j+4}\} \cap S = \phi$, for some j, then $|N(v_{j+2}) \cap S| < \alpha|N(v_{j+2})|$. Therefore, we have three cases to consider.

Case 1. $V(C_1) \bigcap S = \{u_r, u_{r+3}\}$, for some $r \in \{0, 1, 2\}$.

Consider u_{r+1}, its neighbor v_{r+1} must belong to S. Otherwise, $|N(u_{r+1}) \cap S| = \{u_r\}$. Thus, $|N(u_{r+1}) \cap S| < \alpha|N(u_{r+1})|$. Hence, $v_{r+1} \in S$. Similarly, $v_{r+2}, v_{r+4}, v_{r+5} \in S$ (see Figure 3.5). This implies that $|S| \geq 6$.

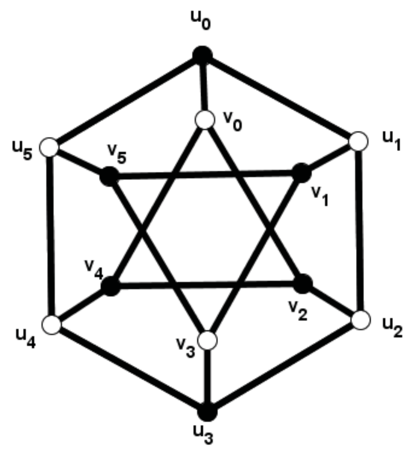


Figure 3.5: $\{u_0, u_3\} \subseteq S$

Case 2. $V(C_2) \bigcap S = \{v_m, v_{m+1}\}$, for some $m \in \{0, 1, 2, 3, 4, 5\}$.

Consider v_{m+2}, its neighbor u_{m+2} must belong to S. Otherwise, $|N(v_{m+2}) \cap S| = \{v_m\}$. Thus, $|N(v_{m+2}) \cap S| < \alpha|N(v_{m+2})|$. Hence, $u_{m+2} \in S$. Similarly, $u_{m+3}, u_{m+4}, u_{m+5} \in S$ (see Figure 3.6). This implies that $|S| \geq 6$.

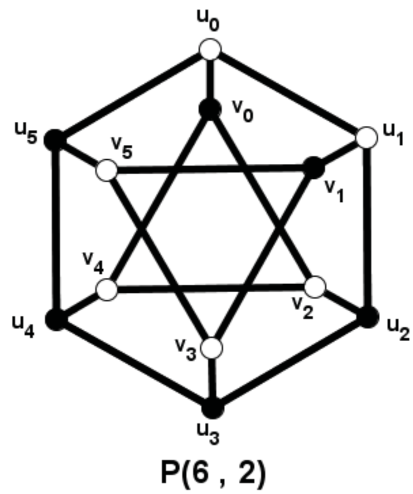


Figure 3.6: $\{v_0, v_1\} \subseteq S$

Case 3. $V(C_2) \bigcap S = \{v_m, v_{m+3}\}$, for some $m \in \{0, 1, 2\}$.

Consider v_{m+1} its neighbor u_{m+1} must belong to S. Otherwise, $|N(v_{m+1}) \cap S| = \{v_{m+3}\}$. Thus, $|N(v_{m+1}) \cap S| < \alpha|N(v_{m+1})|$. Hence, $u_{m+1} \in S$. Similarly, $u_{m+2}, u_{m+4}, u_{m+5} \in S$ (see Figure 3.7). This implies that $|S| \geq 6$.

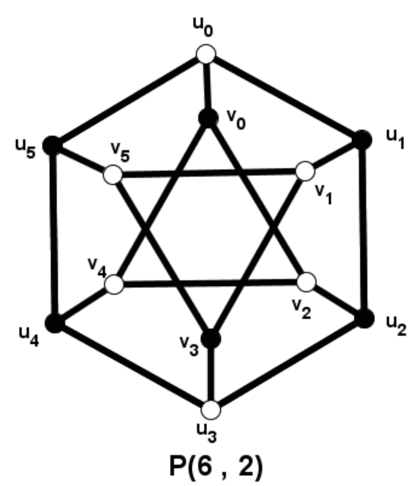


Figure 3.7: $\{v_0, v_3\} \subseteq S$

We can let $S = \{u_1, u_2, u_4, u_5, v_0, v_3\}$. It is easy to see that S is a α-dominating set of G(see Figure 3.7). Therefore, we conclude $\gamma_\alpha(P(6, 2)) = 6$. ∎

Proposition 3.2.4. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(7,2)) = 7$.

Proof. By Corollary 2.1.7, we have $\gamma_\alpha(P(7,2)) \geq 4s + 2 = 6$. First, we will prove that $\gamma_\alpha(P(7,2)) \geq 7$. Let $G = P(7,2)$, and let C_1 and C_2 be subgraphs of G induced by $\{u_0, u_1, \cdots, u_6\}$ and $\{v_0, v_1, \cdots, v_6\}$, respectively. Let S be the α-dominating set of G. Observe that if $\{u_i, u_{i+1}, u_{i+2}\} \cap S = \phi$, for some i, then $|N(u_{i+1}) \cap S| < \alpha|N(u_{i+1})|$. Note that $C_2 = v_0, v_2, v_4, v_6, v_1, v_3, v_5, v_0$, if $\{v_j, v_{j+2}, v_{j+4}\} \cap S = \phi$, for some j, then $|N(v_{j+2}) \cap S| < \alpha|N(v_{j+2})|$. Therefore, we have four cases to consider.

Case 1. $V(C_1) \bigcap S = \{u_r, u_{r+1}, u_{r+4}\}$, for some $r \in \{0, 1, \cdots, 6\}$.

Consider u_{r+2} its neighbor v_{r+2} must belong to S. Otherwise, $|N(u_{r+2}) \cap S| = \{u_{r+1}\}$. Thus, $|N(u_{r+2}) \cap S| < \alpha|N(u_{r+2})|$. Hence, $v_{r+2} \in S$. Similarly, $v_{r+3}, v_{r+5}, v_{r+6} \in S$ (see Figure 3.8). This implies that $|S| \geq 7$.

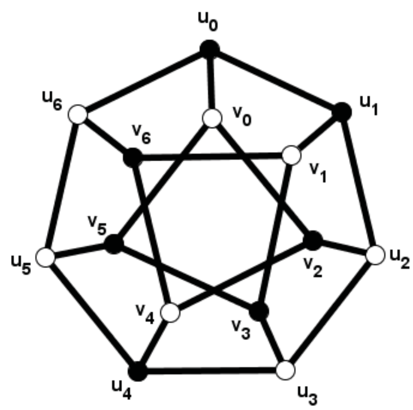


Figure 3.8: $\{u_0, u_1, u_4\} \subseteq S$

Case 2. $V(C_1) \bigcap S = \{u_r, u_{r+2}, u_{r+4}\}$, for some $r \in \{0, 1, \cdots, 6\}$.

Consider u_{r+5} its neighbor v_{r+5} must belong to S. Otherwise, $|N(u_{r+5}) \cap S| = \{u_{r+4}\}$. Thus, $|N(u_{r+5}) \cap S| < \alpha|N(u_{r+5})|$. Hence, $v_{r+5} \in S$. Similarly, $v_{r+6} \in S$. If

$v_{r+1} \notin S$, then $v_{r+3} \in S$ because $u_{r+1} \notin S$. Also, since $\{v_r, v_{r+2}, v_{r+4}\} \cap S = \phi$, then $v_{r+2} \in S$ (see Figure 3.9). This implies that $|S| \geq 7$.

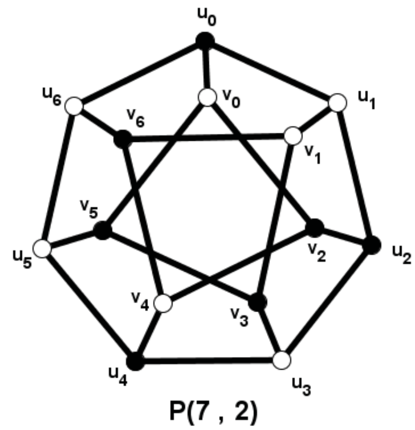


Figure 3.9: $\{u_0, u_2, u_4\} \subseteq S$

Case 3. $V(C_2) \bigcap S = \{v_m, v_{m+1}, v_{m+2}\}$, for some $m \in \{0, 1, \cdots, 6\}$.

Consider v_{m+3}, its neighbor u_{m+3} must belong to S. Otherwise, $|N(v_{m+3}) \cap S| = \{v_{m+1}\}$. Thus, $|N(v_{m+3}) \cap S| < \alpha |N(v_{m+3})|$. Hence, $u_{m+3} \in S$. Similarly, $u_{m+4}, u_{m+5}, u_{m+6} \in S$. Also, since $\{u_m, u_{m+1}, u_{m+2}\} \cap S = \phi$, then $u_{m+1} \in S$ (see Figure 3.10). This implies that $|S| \geq 8$.

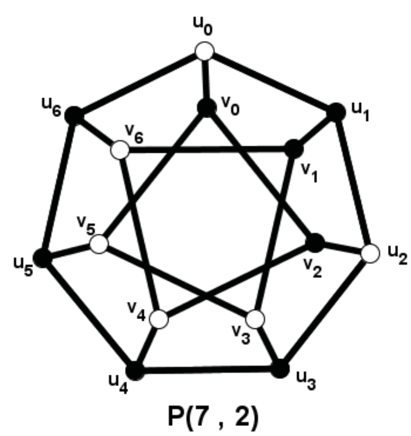


Figure 3.10: $\{v_0, v_1, v_2\} \subseteq S$

Case 4. $V(C_2) \bigcap S = \{v_m, v_{m+1}, v_{m+4}\}$, for some $m \in \{0, 1, \cdots, 6\}$.

Consider v_{m+3}, its neighbor u_{m+3} must belong to S. Otherwise, $|N(v_{m+3}) \cap S| = \{v_{m+1}\}$. Thus, $|N(v_{m+3}) \cap S| < \alpha |N(v_{m+3})|$. Hence, $u_{m+3} \in S$. Similarly, $u_{m+5} \in S$. If $u_{m+2} \notin S$, then $u_{m+1} \in S$ because $v_{m+2} \notin S$. Similarly, if $u_{m+6} \notin S$, then $u_m \in S$ because $v_{m+6} \notin S$ (see Figure 3.11). This implies that $|S| \geq 7$.

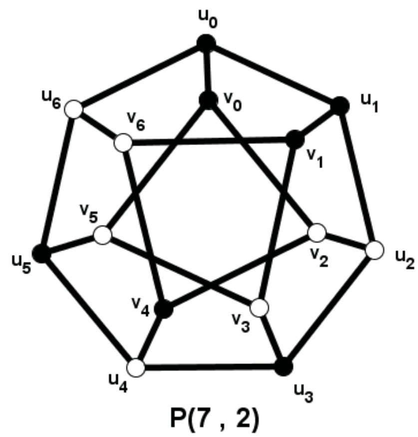


Figure 3.11: $\{v_0, v_1, v_4\} \subseteq S$

We can let $S = \{u_0, u_1, u_3, u_5, v_0, v_1, v_4\}$. It is easy to see that S is an α-dominating set of G (see Figure 3.11). Therefore, we conclude $\gamma_\alpha(P(7, 2)) = 7$. ∎

We find $P(5s, 5t + 2)$ is isomorphic to $P(5s, 5(s - t - 1) + 3)$. The following corollary are immediate.

Corollary 3.2.5. *If* $\frac{1}{3} < \alpha \leq \frac{2}{3}$, *then* $\gamma_\alpha(P(5s, 5t + 3)) = 4s$, *for each* $s \geq 0$ *and* $0 \leq t \leq s$.

3.3 Concluding Remark

In this thesis, we are focusing on finding the α-domination number of a generalized Petersen Graph $P(n, k)$ where $1 \leq k \leq 2$. By a careful arguments, we are able to obtain the exact values for $k = 1$, and $k = 2$ where $n \equiv 0, 3, 4 \ (mod\ 5)$ respectively. But, for the other cases, only bounds are provided. The difficulty comes from determining a sharp lower bound. We do believe the upper bounds we obtained for these cases are in fact their lower bounds, but no able to prove it at this moment. Therefore, we have the following two conjectures.

Conjecture 3.3.1. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(5s+1, 2)) = 4s + 2$.

Conjecture 3.3.2. If $\frac{1}{3} < \alpha \leq \frac{2}{3}$, then $\gamma_\alpha(P(5s+2, 2)) = 4s + 3$.

For future research, we shall focus on solving the remaining cases where $k = 2$ and larger k where $k \leq \lfloor \frac{n}{2} \rfloor$.

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