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On the bipanpositionable bipanconnectedness of hypercubes

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ABSTRACT

A bipartite graph G is bipanconnected if, for any two distinct vertices x and y of G, it contains an $[x, y]$-path of length l for each integer l satisfying $d_G(x, y) \le l \le |V(G)| - 1$ and $2|(l - d_G(x, y))$, where $d_G(x, y)$ denotes the distance between vertices x and y in G and $V(G)$ denotes the vertex set of G. We say a bipartite graph G is bipanpositionably bipanconnected if, for any two distinct vertices x and y of G and for any vertex $z \in V(G) - \{x, y\}$, it contains a path $P_{l,k}$ of length l such that x is the beginning vertex of $P_{l,k}$, z is the $(k + 1)$-th vertex of $P_{l,k}$, and y is the ending vertex of $P_{l,k}$ for each integer l satisfying $d_G(x, z) + d_G(y, z) \le l \le |V(G)| - 1$ and $2|(l - d_G(x, z) - d_G(y, z))$ and for each integer k satisfying $d_G(x, z) \le k \le l - d_G(y, z)$ and $2|(k - d_G(x, z))$. In this paper, we prove that an n-cube is bipanpositionably bipanconnected if $n \ge 4$. As a consequence, many properties of hypercubes, such as bipancyclicity, bipanconnectedness, bipanpositionable Hamiltonicity, etc., follow directly from our result.

1. Introduction

In many parallel and distributed computer systems, processors are connected on the basis of interconnection networks. Thus, the interconnection network has been a critical factor affecting the system performance and is widely addressed in the researches [4,5,9,12,18–20]. In this paper, the topological structure of an interconnection network is modeled as a loopless undirected graph in the aspect of network analysis. For the graph definitions and notations, we follow the ones given by Bondy and Murty [3]. A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$ that is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V(G)\}$. Two vertices u and v of G are *adjacent* if $(u, v) \in E(G)$. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint partite sets $V_0(G)$ and $V_1(G)$ such that every edge joins a vertex of $V_0(G)$ and a vertex of $V_1(G)$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *path* P of length k from vertex x to vertex y in graph G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \le i \le k$ if $k \ge 1$. Moreover, a path of length zero from vertex x is denoted by $\langle x\rangle$. For convenience, we write P as $\langle v_1, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$, where $Q = \langle v_i, \ldots, v_j\rangle$. The i-th vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. To emphasize the beginning and ending vertices of P, we call P an $[x, y]$-path. We use $\ell(P)$ to denote the length of P. The *distance* between two distinct vertices u and v of G, denoted by $d_G(u, v)$, is the length of the shortest path between u and v. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A path (or cycle) in a graph G is a *Hamiltonian path* (or *Hamiltonian cycle*) of G if it spans G. A bipartite graph is *Hamiltonian laceable* [16] if there is a Hamiltonian path between any two vertices that are in different partite sets. A Hamiltonian laceable graph G is *hyper-hamiltonian laceable* [10] if, for $i \in \{0, 1\}$ and for any vertex $v \in V_i(G)$, there is a Hamiltonian path of $G - \{v\}$ between any two vertices of $V_{1-i}(G)$.

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A graph G is *pancyclic* [2] if it contains a cycle of length l for each integer l from 3 to $|V(G)|$ inclusive. Since there is no odd cycle in any bipartite graph, Mitchem and Schmeichel [14] defined the *bipancyclicity* for bipartite graphs. A bipartite graph G is *bipancyclic* if it contains cycles of all even lengths from 4 to $|V(G)|$ inclusive. On the other hand, a graph G is said to be *panconnected* [1] if, for any two distinct vertices x and y, it has an $[x, y]$-path of length l for each $d_G(x, y) \leq l \leq |V(G)| - 1$. Obviously, every panconnected graph is pancyclic. Moreover, it is easy to see that any bipartite graph with at least three vertices is not panconnected. Therefore, the concept of bipanconnected graphs was proposed. A bipartite graph G is *bipanconnected* if, for any two different vertices x and y of G, it contains an $[x, y]$-path of length l for each integer l satisfying both $d_G(x, y) \leq l \leq |V(G)| - 1$ and $2|(l - d_G(x, y))$.

A graph G is *panpositionably Hamiltonian* [7] if, for any two distinct vertices x and y of G, it contains a Hamiltonian cycle C such that $d_C(x, y) = k$ for any integer k satisfying $d_G(x, y) \leq k \leq |V(G)|/2$. Recently, Teng et al. [17] studied the panpositionable Hamiltonicity of the arrangement graphs. In contrast, a bipartite graph G is *bipanpositionably Hamiltonian* [7] if, for any two different vertices x and y of G, it has a Hamiltonian cycle C such that $d_C(x, y) = k$ for any integer k satisfying both $d_G(x, y) \leq k \leq |V(G)|/2$ and $2|(k - d_G(x, y))$. In this paper, we further define a property for bipartite graphs. We say a bipartite graph G is *relay-bipanpositionable* between two distinct vertices x and y if, for any vertex $z \in V(G) - \{x, y\}$, it contains an $[x, y]$-path $P_{l,k}$ of length l such that $P_{l,k}(1) = x$, $P_{l,k}(k + 1) = z$, and $P_{l,k}(l + 1) = y$ for each integer l satisfying both $d_G(x, z) + d_G(y, z) \leq l \leq |V(G)| - 1$ and $2|(l - d_G(x, z) - d_G(y, z))$ and for each integer k satisfying both $d_G(x, z) \leq k \leq l - d_G(y, z)$ and $2|(k - d_G(x, z))$. Then a bipartite graph G is said to be *bipanpositionably bipanconnected* if it is relay-bipanpositionable between every pair of distinct vertices.

The hypercube is an attractive underlying network topology for parallel systems [9,19]. For clarity, we use boldface letters to denote n-bit binary strings. Let $\mathbf{u} = b_{n-1} \dots b_i \dots b_0$ be an n-bit binary string. For any i, $0 \leq i \leq n - 1$, we use $(\mathbf{u})^i$ to denote the binary string $b_{n-1} \dots \bar{b}_i \dots b_0$. Moreover, we use $(\mathbf{u})_i$ to denote the bit b_i of $\mathbf{u}$. The *Hamming weight* of $\mathbf{u}$, denoted by $w_H(\mathbf{u})$, is defined as $|\{0 \leq j \leq n - 1 \mid (\mathbf{u})_j = 1\}|$. The *n-dimensional hypercube* (or *n-cube* for short) Q_n consists of 2^n vertices and $n2^{n-1}$ edges. Each vertex corresponds to an n-bit binary string. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $\mathbf{v} = (\mathbf{u})^i$ for some i and we call the edge $(\mathbf{u}, (\mathbf{u})^i)$ an i-dimensional edge. The *Hamming distance* between $\mathbf{u}$ and $\mathbf{v}$, denoted by $h(\mathbf{u}, \mathbf{v})$, is defined to be $|\{0 \leq j \leq n-1 \mid (\mathbf{u})_j \neq (\mathbf{v})_j\}|$. Hence, two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $h(\mathbf{u}, \mathbf{v}) = 1$. Clearly, $d_{Q_n}(\mathbf{u}, \mathbf{v})$ equals $h(\mathbf{u}, \mathbf{v})$, and Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{\mathbf{v} \in V(Q_n) \mid w_H(\mathbf{v}) \text{ is even}\}$ and $V_1(Q_n) = \{\mathbf{v} \in V(Q_n) \mid w_H(\mathbf{v}) \text{ is odd}\}$. Moreover, Q_n is vertex-transitive and edge-transitive [9]. It was proved that Q_n, $n \geq 2$, is bipancyclic [7,15] and bipanconnected [11]. As Kao et al. [7] showed, Q_n is bipanpositionably Hamiltonian if $n \geq 2$. Recently, there are several interesting studies on hypercubes [6,12,13]. In this paper, we are going to prove that Q_n is bipanpositionably bipanconnected if $n \geq 4$. As an immediate consequence, many other properties of hypercubes, such as bipancyclicity, bipanconnectedness, bipanpositionable Hamiltonicity, etc., follow from our result.

2. Preliminaries

Obviously, Q_2 is not only bipanconnected but also bipanpositionably bipanconnected. It is easy to see that Q_3 has no $[000, 011]$-paths P of length six such that $P(4) = 001$. Hence, Q_3 is not relay-bipanpositionable between vertices 000 and 011. It is noticed that vertices 000 and 011 are in the same partite set of Q_3. However, we can show that Q_3 is relay-bipanpositionable between every two vertices in different partite sets.

Lemma 1. *The 3-cube Q_3 is relay-bipanpositionable between every two vertices in different partite sets.*

Proof. Let $\mathbf{x} \in V_0(Q_3)$ and $\mathbf{y} \in V_1(Q_3)$. Without loss of generality, we suppose that $\mathbf{z} \in V_0(Q_3) - \{\mathbf{x}\}$. Since Q_3 is vertex-transitive, we can assume that $\mathbf{x} = 000$. Hence, we have $\mathbf{z} \in \{011, 101, 110\}$ and $\mathbf{y} \in \{001, 010, 100, 111\}$. Since Q_3 is edge-transitive, we only consider the case that $\mathbf{y} \in \{001, 111\}$. We list all the required $[\mathbf{x}, \mathbf{y}]$-paths obtained by brute force in Table 1. □

The following lemma shows that the relay-bipanpositionability between every two vertices in different partite sets of Q_n implies the bipanconnectedness of Q_n.

Lemma 2. *Suppose that the n-cube Q_n, $n \geq 2$, is relay-bipanpositionable between every two vertices in different partite sets. Then Q_n is bipanconnected.*

Proof. Let $\mathbf{e} = 0^n$ and $\mathbf{v} \in V(Q_n) - \{\mathbf{e}\}$. Since Q_n is vertex-transitive, we only concern the paths between $\mathbf{e}$ and $\mathbf{v}$.

Case 1: Suppose that $\mathbf{v} \in V_1(Q_n)$. Let i be an integer of $\{0, 1, \dots, n - 1\}$ with $(\mathbf{e})_i \neq (\mathbf{v})_i$ and let j be an integer of $\{0, 1, \dots, n - 1\} - \{i\}$. We set $\mathbf{w}$ to be $(\mathbf{v})^i$ if $\mathbf{e} \neq (\mathbf{v})^i$ and set $\mathbf{w}$ to be $(\mathbf{v})^j$ if $\mathbf{e} = (\mathbf{v})^i$. Hence we have $d_{Q_n}(\mathbf{v}, \mathbf{w}) = 1$. Since Q_n is relay-bipanpositionable between any two vertices in different partite sets, it has an $[\mathbf{e}, \mathbf{v}]$-path P of length l such that $P(1) = \mathbf{e}$, $P(l) = \mathbf{w}$, and $P(l + 1) = \mathbf{v}$ for any odd integer l from $d_{Q_n}(\mathbf{e}, \mathbf{w}) + 1$ to $2^n - 1$ inclusive. If $\mathbf{e} \neq (\mathbf{v})^i$, then we have $d_{Q_n}(\mathbf{e}, \mathbf{w}) + 1 = d_{Q_n}(\mathbf{e}, \mathbf{v})$. Otherwise, we have $d_{Q_n}(\mathbf{e}, \mathbf{w}) + 1 = 3$. Thus, Q_n has an $[\mathbf{e}, \mathbf{v}]$-path of any odd length from $d_{Q_n}(\mathbf{e}, \mathbf{v})$ to $2^n - 1$.

Case 2: Suppose that $\mathbf{v} \in V_0(Q_n)$. Let k be any integer of $\{0, 1, \dots, n-1\}$. Obviously Q_n is relay-bipanpositionable between $\mathbf{e}$ and $(\mathbf{v})^k$ because $\mathbf{e}$ and $(\mathbf{v})^k$ belong to the different partite sets of Q_n. Hence, Q_n has an $[\mathbf{e}, (\mathbf{v})^k]$-path P of length l such that $P(1) = \mathbf{e}$, $P(l) = \mathbf{v}$, and $P(l + 1) = (\mathbf{v})^k$ for any odd integer l from $d_{Q_n}(\mathbf{e}, \mathbf{v}) + 1$ to $2^n - 1$ inclusive. For convenience, path P can be written as $\langle \mathbf{e}, P', \mathbf{v}, (\mathbf{v})^k \rangle$, where P' is an $[\mathbf{e}, \mathbf{v}]$-path of length $l - 1$. Clearly Q_n has an $[\mathbf{e}, \mathbf{v}]$-path of any even length in the range from $d_{Q_n}(\mathbf{e}, \mathbf{v})$ to $2^n - 2$.

In summary, Q_n is bipanconnected. □

Table 1
Q_3 is relay-bipanpositionable between every two vertices in different partite sets.

$\mathbf{y}$	$\mathbf{z}$	Paths between $\mathbf{x} = 000$ and $\mathbf{y}$	$\mathbf{y}$	$\mathbf{z}$	Paths between $\mathbf{x} = 000$ and $\mathbf{y}$
001	011	$\langle \mathbf{x} = 000, 010, 011 = \mathbf{z}, 001 = \mathbf{y} \rangle$	111	011	$\langle \mathbf{x} = 000, 001, 011 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 110, 111, 011 = \mathbf{z}, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 100, 101, 001, 011 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 110, 100, 101, 111, 011 = \mathbf{z}, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 010, 110, 100, 101, 001, 011 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 011 = \mathbf{z}, 111, 101, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 001, 011 = \mathbf{z}, 010, 110, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 110, 010, 011 = \mathbf{z}, 111, 101, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 100, 101, 001, 011 = \mathbf{z}, 010, 110, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 011 = \mathbf{z}, 111, 110, 100, 101, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 001, 011 = \mathbf{z}, 010, 110, 100, 101, 111 = \mathbf{y} \rangle$
	101	$\langle \mathbf{x} = 000, 100, 101 = \mathbf{z}, 001 = \mathbf{y} \rangle$		101	$\langle \mathbf{x} = 000, 100, 101 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 110, 111, 101 = \mathbf{z}, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 010, 110, 100, 101 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 110, 010, 011, 111, 101 = \mathbf{z}, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 001, 011, 010, 110, 100, 101 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 101 = \mathbf{z}, 111, 011, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 100, 101 = \mathbf{z}, 001, 011, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 110, 100, 101 = \mathbf{z}, 111, 011, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 010, 110, 100, 101 = \mathbf{z}, 001, 011, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 101 = \mathbf{z}, 111, 110, 010, 011, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 001, 101 = \mathbf{z}, 100, 110, 010, 011, 111 = \mathbf{y} \rangle$
	110	$\langle \mathbf{x} = 000, 100, 110 = \mathbf{z}, 010, 011, 001 = \mathbf{y} \rangle$		110	$\langle \mathbf{x} = 000, 100, 110 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 100, 101, 111, 110 = \mathbf{z}, 010, 011, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 001, 101, 100, 110 = \mathbf{z}, 111 = \mathbf{y} \rangle$
		$\langle \mathbf{x} = 000, 010, 110 = \mathbf{z}, 100, 101, 111, 011, 001 = \mathbf{y} \rangle$			$\langle \mathbf{x} = 000, 100, 101, 001, 011, 010, 110 = \mathbf{z}, 111 = \mathbf{y} \rangle$
					$\langle \mathbf{x} = 000, 100, 110 = \mathbf{z}, 010, 011, 111 = \mathbf{y} \rangle$
					$\langle \mathbf{x} = 000, 001, 101, 100, 110 = \mathbf{z}, 010, 011, 111 = \mathbf{y} \rangle$
					$\langle \mathbf{x} = 000, 100, 110 = \mathbf{z}, 010, 011, 001, 101, 111 = \mathbf{y} \rangle$

Two paths P_1 and P_2 are *vertex-disjoint* if $V(P_1) \cap V(P_2) = \emptyset$. Let Q_n^i, $i \in \{0, 1\}$, be the subgraph of Q_n induced by $\{\mathbf{u} \in V(Q_n) \mid (\mathbf{u})_{n-1} = i\}$. Obviously, Q_n^i is isomorphic to Q_{n-1}. Then we can prove the following lemma.

Lemma 3. *Suppose that an n-cube Q_n is bipanconnected if $n \geq 2$. Let $(\mathbf{w}_1, \mathbf{b}_1)$ and $(\mathbf{w}_2, \mathbf{b}_2)$ be two vertex-disjoint edges of Q_n such that $\{\mathbf{w}_2, \mathbf{b}_2\} \cap \{(\mathbf{w}_1)^0, (\mathbf{w}_1)^1, (\mathbf{w}_1)^2, (\mathbf{b}_1)^0, (\mathbf{b}_1)^1, (\mathbf{b}_1)^2\} \neq \emptyset$ if $n = 3$. For each even integer l from 2 to $2^n - 2$ and for each odd integer k from 1 to $l - 1$, Q_n has two vertex-disjoint paths $S_{l,k}^{(1)}$ and $S_{l,k}^{(2)}$ such that $S_{l,k}^{(1)}$ is a $[\mathbf{w}_1, \mathbf{b}_1]$-path of length k and $S_{l,k}^{(2)}$ is a $[\mathbf{w}_2, \mathbf{b}_2]$-path of length $l - k$.*

Proof. We claim that Q_n can be partitioned along some dimension in such a way that $(\mathbf{w}_1, \mathbf{b}_1)$ and $(\mathbf{w}_2, \mathbf{b}_2)$ are located on different subcubes. Without loss of generality, we assume that $\mathbf{w}_1, \mathbf{w}_2 \in V_0(Q_n)$ and $\mathbf{b}_1, \mathbf{b}_2 \in V_1(Q_n)$. Moreover, we assume that $h(\mathbf{w}_1, \mathbf{w}_2) = \max\{h(\mathbf{w}_1, \mathbf{w}_2), h(\mathbf{b}_1, \mathbf{b}_2)\}$. Let $(\mathbf{w}_1, \mathbf{b}_1)$ be an i_1-dimensional edge and $(\mathbf{w}_2, \mathbf{b}_2)$ be an i_2-dimensional edge. If $|\{0, 1, \ldots, n-1\} - \{i_1, i_2\}| + h(\mathbf{w}_1, \mathbf{w}_2) \geq n + 1$, then there exists an integer $i \in \{0 \leq j \leq n-1 \mid (\mathbf{w}_1)_j \neq (\mathbf{w}_2)_j\}$ such that $i \notin \{i_1, i_2\}$. Suppose that $|\{0, 1, \ldots, n-1\} - \{i_1, i_2\}| + h(\mathbf{w}_1, \mathbf{w}_2) = n$. Thus, we have $i_1 \neq i_2$ and $h(\mathbf{w}_1, \mathbf{w}_2) = 2$. For convenience, let $\{j_1, j_2\} = \{0 \leq j \leq n-1 \mid (\mathbf{w}_1)_j \neq (\mathbf{w}_2)_j\}$. Clearly we have $\{j_1, j_2\} - \{i_1, i_2\} \neq \emptyset$. Let i be any integer of $\{j_1, j_2\} - \{i_1, i_2\}$. Then we partition Q_n along dimension i so that $(\mathbf{w}_1, \mathbf{b}_1)$ and $(\mathbf{w}_2, \mathbf{b}_2)$ are located on different subcubes. Since Q_n is edge-transitive, we assume that $i = n - 1$. Without loss of generality, we assume that $(\mathbf{w}_1, \mathbf{b}_1) \in E(Q_n^0)$ and $(\mathbf{w}_2, \mathbf{b}_2) \in E(Q_n^1)$.

This lemma can be proved by induction on n. First of all, the result is trivial for $n = 2$. When $n = 3$, let $\mathbf{x} = (\mathbf{w}_1)^2$, $\mathbf{y} = (\mathbf{b}_1)^2$, and $\{\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}\} = V(Q_3^1)$. See Fig. 1(a) for illustration. Since $\{\mathbf{w}_2, \mathbf{b}_2\} \cap \{(\mathbf{w}_1)^0, (\mathbf{w}_1)^1, (\mathbf{w}_1)^2, (\mathbf{b}_1)^0, (\mathbf{b}_1)^1, (\mathbf{b}_1)^2\} \neq \emptyset$, we have $(\mathbf{w}_2, \mathbf{b}_2) \in \{(\mathbf{u}, \mathbf{x}), (\mathbf{x}, \mathbf{y}), (\mathbf{v}, \mathbf{y})\}$. Then it is easy to see that there are two vertex-disjoint paths $S_{l,k}^{(1)}$ and $S_{l,k}^{(2)}$ in Q_3 such that $S_{l,k}^{(1)}$ is a $[\mathbf{w}_1, \mathbf{b}_1]$-path of length k and $S_{l,k}^{(2)}$ is a $[\mathbf{w}_2, \mathbf{b}_2]$-path of length $l - k$ for any $l \in \{2, 4, 6\}$ and for any odd integer k from 1 to $l - 1$.

As the inductive hypothesis, we suppose that the result is true for Q_{n-1}, $n \geq 4$. Obviously, at least one of k and $l - k$ is less than 2^{n-1}. By symmetry, we only consider the case that $1 \leq k \leq 2^{n-1} - 1$. Then we distinguish the following two cases.

Case 1: Suppose that $k \leq 2^{n-1} - 3$. Since Q_m, $m \leq n - 1$, is supposed to be bipanconnected, Q_n^1 has a $[\mathbf{w}_2, \mathbf{b}_2]$-path R_r of length r for each odd integer r from 1 to $2^{n-1} - 1$. Let $\tilde{r} = 2^{n-1} - 1$ and let $A = \{(R_{\tilde{r}}(i), R_{\tilde{r}}(i+1)) \mid 1 \leq i \leq \tilde{r}$ and $i \equiv 1 \pmod 2\}$. Since $|A| = 2^{n-2} > 3$ for $n = 4$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq \tilde{r}$, such that $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath} + 1))^{n-1}\} \cap \{\mathbf{w}_1, \mathbf{b}_1\} = \emptyset$ and $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath} + 1))^{n-1}\} \cap \{(\mathbf{w}_1)^0, (\mathbf{w}_1)^1, (\mathbf{w}_1)^2, (\mathbf{b}_1)^0, (\mathbf{b}_1)^1, (\mathbf{b}_1)^2\} \neq \emptyset$ if $n = 4$. Since $|A| = 2^{n-2} > 7$ for $n \geq 5$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq \tilde{r}$, such that $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath} + 1))^{n-1}\} \cap \{\mathbf{w}_1, \mathbf{b}_1\} = \emptyset$ if $n \geq 5$. For convenience, let $\mathbf{u} = R_{\tilde{r}}(\hat{\imath})$ and $\mathbf{v} = R_{\tilde{r}}(\hat{\imath} + 1)$. Thus, $R_{\tilde{r}}$ can be written as $\langle \mathbf{w}_2, R', \mathbf{u}, \mathbf{v}, R'', \mathbf{b}_2 \rangle$. By the inductive hypothesis, Q_n^0 has two vertex-disjoint paths $T_{p,q}^{(1)}$ and $T_{p,q}^{(2)}$ such that $T_{p,q}^{(1)}$ is a $[\mathbf{w}_1, \mathbf{b}_1]$-path of length q and $T_{p,q}^{(2)}$ is a $[(\mathbf{u})^{n-1}, (\mathbf{v})^{n-1}]$-path of length $p - q$ for any even integer p satisfying $2 \leq p \leq 2^{n-1} - 2$ and for any odd integer q satisfying $1 \leq q \leq p - 1$. Then we set $S_{l,k}^{(1)} = T_{2^{n-1}-2,k}^{(1)}$ and $S_{l,k}^{(2)} = R_{l-k}$ if $l - k \leq 2^{n-1} - 1$ (See Fig. 1(b)); we set $S_{l,k}^{(1)} = T_{l-2^{n-1},k}^{(1)}$ and $S_{l,k}^{(2)} = \langle \mathbf{w}_2, R', \mathbf{u}, (\mathbf{u})^{n-1}, T_{l-2^{n-1},k}^{(2)}, (\mathbf{v})^{n-1}, \mathbf{v}, R'', \mathbf{b}_2 \rangle$ if $l - k \geq 2^{n-1} + 1$ (See Fig. 1(c)).

Case 2: Suppose that $k = 2^{n-1} - 1$. Since Q_{n-1} is bipanconnected, Q_n^1 has a $[\mathbf{w}_2, \mathbf{b}_2]$-path R_r of length r for each odd integer r from 1 to $2^{n-1} - 1$. Similarly, Q_n^0 has a $[\mathbf{w}_1, \mathbf{b}_1]$-path H of length $2^{n-1} - 1$. Then we set $S_{l,k}^{(1)} = H$ and $S_{l,k}^{(2)} = R_{l-k}$. See Fig. 1(d).

Therefore, the proof is completed. □

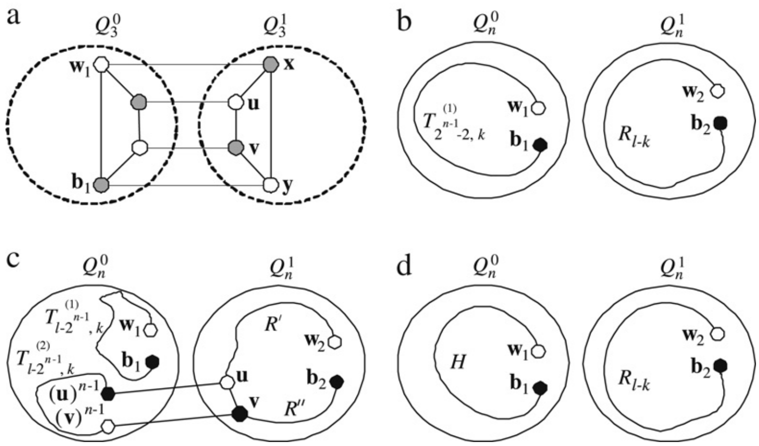

Fig. 1. Illustrations for Lemma 3. (a) $n = 3$; (b) $k \le 2^{n-1} - 3$ and $1 \le l - k \le 2^{n-1} - 1$; (c) $k \le 2^{n-1} - 3$ and $l - k \ge 2^{n-1} + 1$; (d) $k = 2^{n-1} - 1$.

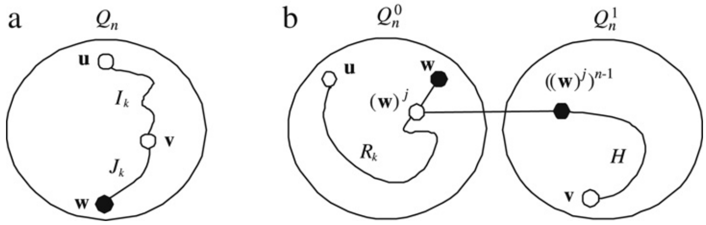

Fig. 2. Illustrations for Lemma 4.

A bipanconnected graph G is *hyper-bipanconnected* if, for any vertex $w \in V_i(G)$ ($i \in \{0, 1\}$) and for any two distinct vertices u and v of $V_{1-i}(G)$, $G - \{w\}$ has a $[u, v]$-path of length l for any even integer l ranging from $d_G(u, v)$ to $|V(G)| - 2$ inclusive.

Lemma 4. *Suppose that the n-cube Q_n, $n \ge 2$, is relay-bipanpositionable between every two vertices in different partite sets. Then Q_n is hyper-bipanconnected.*

Proof. The result is trivial for $n = 2$. In what follows, we consider the case that $n \ge 3$. Since Q_n is relay-bipanpositionable between every two vertices in different partite sets, Lemma 2 ensures that Q_n is bipanconnected. Hence, we only concern the paths between every pair of distinct vertices in the same partite set. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of $V_i(Q_n)$ and let $\mathbf{w}$ be any vertex of $V_{1-i}(Q_n)$ for some $i \in \{0, 1\}$. Then we have to show that $Q_n - \{\mathbf{w}\}$ has $[\mathbf{u}, \mathbf{v}]$-paths of all possible lengths. Since Q_n is relay-bipanpositionable between every two vertices in different partite sets, it has a $[\mathbf{u}, \mathbf{w}]$-path P_k such that $P_k(1) = \mathbf{u}$, $P_k(k + 1) = \mathbf{v}$, and $d_{P_k}(\mathbf{w}, \mathbf{v}) = d_{Q_n}(\mathbf{w}, \mathbf{v})$ for each even integer k satisfying $d_{Q_n}(\mathbf{u}, \mathbf{v}) \le k \le 2^n - d_{Q_n}(\mathbf{w}, \mathbf{v}) - 1$. For clarity, we can write P_k as $\langle \mathbf{u}, I_k, \mathbf{v}, J_k, \mathbf{w} \rangle$, where I_k is a $[\mathbf{u}, \mathbf{v}]$-path of length k and J_k is some shortest path between $\mathbf{v}$ and $\mathbf{w}$. That is, $Q_n - \{\mathbf{w}\}$ has $[\mathbf{u}, \mathbf{v}]$-paths of even lengths in a range from $d_{Q_n}(\mathbf{u}, \mathbf{v})$ to $2^n - d_{Q_n}(\mathbf{w}, \mathbf{v}) - 1$. See Fig. 2(a).

The $[\mathbf{u}, \mathbf{v}]$-paths of lengths greater than $2^n - d_{Q_n}(\mathbf{w}, \mathbf{v}) - 1$ can be constructed as follows. Since Q_n is edge-transitive, we can assume that $(\mathbf{u})_{n-1} \ne (\mathbf{v})_{n-1}$. Thus, we can partition Q_n into Q_n^0 and Q_n^1 in such a way that $\mathbf{u}$ and $\mathbf{v}$ are located on different subcubes. Without loss of generality, we assume that $\mathbf{u}, \mathbf{w} \in V(Q_n^0)$ and $\mathbf{v} \in V(Q_n^1)$. Let j be an integer of $\{0, 1, \dots, n - 2\}$ with $(\mathbf{w})^j \ne \mathbf{u}$. Since Q_{n-1} is relay-bipanpositionable between every two vertices in different partite sets, Q_n^0 has a $[\mathbf{u}, \mathbf{w}]$-path T_k such that $T_k(1) = \mathbf{u}$, $T_k(k + 1) = (\mathbf{w})^j$, and $d_{T_k}(\mathbf{w}, (\mathbf{w})^j) = 1$ for each even integer k satisfying $d_{Q_n}(\mathbf{u}, (\mathbf{w})^j) \le k \le 2^{n-1} - 2$. Thus, we can write T_k as $\langle \mathbf{u}, R_k, (\mathbf{w})^j, \mathbf{w} \rangle$, where R_k is a $[\mathbf{u}, (\mathbf{w})^j]$-path of length k. By Lemma 2, Q_{n-1} is bipanconnected. Accordingly, Q_n^1 has a $[((\mathbf{w})^j)^{n-1}, \mathbf{v}]$-path H of length $2^{n-1} - 1$. Then $\langle \mathbf{u}, R_k, (\mathbf{w})^j, ((\mathbf{w})^j)^{n-1}, H, \mathbf{v} \rangle$ turns out to be a $[\mathbf{u}, \mathbf{v}]$-path of length $2^{n-1} + k$. See Fig. 2(b). Obviously, we have $2^{n-1} + d_{Q_n}(\mathbf{u}, (\mathbf{w})^j) \le 2^{n-1} + k \le 2^n - 2$. Since $2^{n-1} + d_{Q_n}(\mathbf{u}, (\mathbf{w})^j) \le 2^{n-1} + n - 1$ and $(2^n - d_{Q_n}(\mathbf{w}, \mathbf{v}) - 1) + 2 \ge (2^n - n - 1) + 2 = 2^n - n + 1 \ge 2^{n-1} + n - 1$ for $n \ge 3$, we have $2^{n-1} + d_{Q_n}(\mathbf{u}, (\mathbf{w})^j) \le 2^n - d_{Q_n}(\mathbf{w}, \mathbf{v}) + 1$ if $n \ge 3$. Hence, all possible lengths have been concerned and the proof is completed. □

Lemma 5. *Suppose that* $\mathbf{x}$ *is any vertex of* Q_3 *and* $(\mathbf{w}, \mathbf{b})$ *is any edge of* $Q_3 - \{\mathbf{x}\}$. *Then* $Q_3 - \{\mathbf{x}\}$ *has a* $[\mathbf{w}, \mathbf{b}]$*-path of length l for each* $l \in \{1, 3, 5\}$.

Proof. Since Q_3 is vertex-transitive, we assume that $\mathbf{x} = 000$. It is easy to see that any edge $(\mathbf{w}, \mathbf{b}) \in E(Q_3 - \{000\})$ lies on a cycle of length four. Therefore, $Q_3 - \{000\}$ has $[\mathbf{w}, \mathbf{b}]$-paths of lengths one and three. The $[\mathbf{w}, \mathbf{b}]$-path of length five is listed in Table 2. □

Table 2
Paths of length five between all pairs of adjacent vertices in $Q_3 - \{000\}$.

$(\mathbf{w}, \mathbf{b})$	$[\mathbf{w}, \mathbf{b}]$-path of length five
(110,100)	⟨110, 111, 011, 001, 101, 100⟩
(101,111)	⟨101, 100, 110, 010, 011, 111⟩
(011,001)	⟨011, 111, 110, 100, 101, 001⟩
(101,100)	⟨101, 111, 011, 010, 110, 100⟩
(110,111)	⟨110, 100, 101, 001, 011, 111⟩
(011,010)	⟨011, 111, 101, 100, 110, 010⟩
(101,001)	⟨101, 100, 110, 111, 011, 001⟩
(110,010)	⟨110, 100, 101, 111, 011, 010⟩
(011,111)	⟨011, 001, 101, 100, 110, 111⟩

3. Bipanpositionable bipanconnectedness

Applying Lemmas 1–3, we are able to prove the following theorem.

Theorem 1. *The n-cube Q_n is relay-bipanpositionable between every two vertices in different partite sets if $n \geq 2$.*

Proof. The result is trivial for $n = 2$. We prove this theorem by induction for $n \geq 3$. The induction basis follows from Lemma 1. As the inductive hypothesis, we assume that Q_{n-1}, $n \geq 4$, is relay-bipanpositionable between every two vertices in different partite sets. Let $\mathbf{x}$ and $\mathbf{y}$ be any two vertices in different partite sets of Q_n. We have to show that for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x}, \mathbf{y}\}$, Q_n contains an $[\mathbf{x}, \mathbf{y}]$-path $P_{l,k}$ of length l such that $P_{l,k}(1) = \mathbf{x}$, $P_{l,k}(k+1) = \mathbf{z}$, and $P_{l,k}(l+1) = \mathbf{y}$ for each odd integer l from $d_{Q_n}(\mathbf{x}, \mathbf{z}) + d_{Q_n}(\mathbf{y}, \mathbf{z})$ to $2^n - 1$ and for each integer k satisfying both $d_{Q_n}(\mathbf{x}, \mathbf{z}) \leq k \leq l - d_{Q_n}(\mathbf{y}, \mathbf{z})$ and $2|(k - d_{Q_n}(\mathbf{x}, \mathbf{z}))$. For convenience, we write $P_{l,k} = \langle \mathbf{x}, P_1, \mathbf{z}, P_2, \mathbf{y} \rangle$ with $\ell(P_1) = k$ and $\ell(P_2) = l - k$. Since Q_n is vertex-transitive, we can assume that $\mathbf{x}, \mathbf{z} \in V_0(Q_n)$ and $\mathbf{y} \in V_1(Q_n)$. Since $\mathbf{x}$ and $\mathbf{z}$ are in the same partite set of Q_n, we have $d_{Q_n}(\mathbf{x}, \mathbf{z}) \geq 2$. Obviously, there exists an integer a of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{x})_a \neq (\mathbf{z})_a$ and $(\mathbf{z})^a \neq \mathbf{y}$. By symmetry, we assume that $a = n - 1$. Thus, Q_n can be partitioned into Q_n^0 and Q_n^1 so that $\mathbf{x}$ and $\mathbf{z}$ are on different subcubes. Without loss of generality, we assume that $\mathbf{x}$ is on Q_n^0.

Case 1: Suppose that $\mathbf{y}$ is on Q_n^1. Based on the inductive hypothesis, Lemma 2 ensures that Q_n^0 and Q_n^1 are bipanconnected. Let j be an integer of $\{0, 1, \ldots, n-2\}$ such that $(\mathbf{x})_j \neq (\mathbf{z})_j$. Then we consider the following subcases.

Subcase 1.1: Suppose that $d_{Q_n}(\mathbf{x}, \mathbf{z}) = 2$. Therefore, we have $((\mathbf{x})^j)^{n-1} = \mathbf{z}$. First, we consider the case that $\ell(P_1) = k \leq 2^{n-1}$. Since Q_n^1 is bipanconnected, it has a $[\mathbf{z}, \mathbf{y}]$-path R_r of length r for every odd integer r satisfying $d_{Q_n}(\mathbf{y}, \mathbf{z}) \leq r \leq 2^{n-1} - 1$. Let $\tilde{r} = 2^{n-1} - 1$ and $A = \{(R_{\tilde{r}}(i), R_{\tilde{r}}(i+1)) \mid 1 \leq i \leq \tilde{r} \text{ and } i \equiv 1 \pmod 2\}$. Since $|A| = 2^{n-2} > 3$ for $n = 4$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq \tilde{r}$, such that $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath}+1))^{n-1}\} \cap \{\mathbf{x}, (\mathbf{x})^j\} = \emptyset$ and $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath}+1))^{n-1}\} \cap \{(\mathbf{x})^0, (\mathbf{x})^1, (\mathbf{x})^2, ((\mathbf{x})^j)^0, ((\mathbf{x})^j)^1, ((\mathbf{x})^j)^2\} \neq \emptyset$ if $n = 4$. Since $|A| = 2^{n-2} > 7$ for $n \geq 5$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq \tilde{r}$, such that $\{(R_{\tilde{r}}(\hat{\imath}))^{n-1}, (R_{\tilde{r}}(\hat{\imath}+1))^{n-1}\} \cap \{\mathbf{x}, (\mathbf{x})^j\} = \emptyset$ if $n \geq 5$. For convenience, let $\mathbf{w} = R_{\tilde{r}}(\hat{\imath})$ and $\mathbf{b} = R_{\tilde{r}}(\hat{\imath}+1)$. Hence, path $R_{\tilde{r}}$ can be written as $\langle \mathbf{z}, R', \mathbf{w}, \mathbf{b}, R'', \mathbf{y} \rangle$. For each even integer p from 2 to $2^{n-1} - 2$ and for each odd integer q from 1 to $p - 1$, Lemma 3 ensures that Q_n^0 has two vertex-disjoint paths $S_{p,q}^{(1)}$ and $S_{p,q}^{(2)}$ such that $S_{p,q}^{(1)}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length q and $S_{p,q}^{(2)}$ is a $[(\mathbf{w})^{n-1}, (\mathbf{b})^{n-1}]$-path of length $p - q$. Therefore, we set $P_1 = \langle \mathbf{x}, S^{(1)}_{2^{n-1}-2,k-1}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1} = \mathbf{z} \rangle$ and $P_2 = R_{l-k}$ if $\ell(P_1) = k \leq 2^{n-1} - 2$ and $\ell(P_2) = l - k \leq 2^{n-1} - 1$ (See Fig. 3(a)); we set $P_1 = \langle \mathbf{x}, S^{(1)}_{l-2^{n-1}-1,k-1}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1} = \mathbf{z} \rangle$ and $P_2 = \langle \mathbf{z}, R', \mathbf{w}, (\mathbf{w})^{n-1}, S^{(2)}_{l-2^{n-1}-1,k-1}, (\mathbf{b})^{n-1}, \mathbf{b}, R'', \mathbf{y} \rangle$ if $\ell(P_1) = k \leq 2^{n-1} - 2$ and $\ell(P_2) = l - k \geq 2^{n-1} + 1$ (See Fig. 3(b)). Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{x})^j]$-path H of length $2^{n-1} - 1$. Thus, we set $P_1 = \langle \mathbf{x}, H, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1} = \mathbf{z} \rangle$ and $P_2 = R_{l-k}$ if $\ell(P_1) = k = 2^{n-1}$ (See Fig. 3(c)). As a result, P_1 is indeed an $[\mathbf{x}, \mathbf{z}]$-path of length k and P_2 is indeed an $[\mathbf{z}, \mathbf{y}]$-path of length $l - k$.

Next, we consider the case that $\ell(P_1) = k \geq 2^{n-1} + 2$. Let $\mathbf{w}$ be a vertex of Q_n^1 with $d_{Q_n}(\mathbf{w}, \mathbf{z}) = 2$. Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{w})^{n-1}]$-path H of length $2^{n-1} - 1$. By the inductive hypothesis, Q_n^1 is relay-bipanpositionable between every two vertices in different partite sets; thus, Q_n^1 has a $[\mathbf{w}, \mathbf{y}]$-path J of length $l - 2^{n-1}$ such that $J(1) = \mathbf{w}$, $J(k - 2^{n-1} + 1) = \mathbf{z}$, and $J(l - 2^{n-1} + 1) = \mathbf{y}$. For clarity, path J can be written as $J = \langle \mathbf{w}, J'_{k-2^{n-1}}, \mathbf{z}, J''_{l-k}, \mathbf{y} \rangle$, where $J'_{k-2^{n-1}}$ is a $[\mathbf{w}, \mathbf{z}]$-path of length $k - 2^{n-1}$ and J''_{l-k} is a $[\mathbf{z}, \mathbf{y}]$-path of length $l - k$. Then we set $P_1 = \langle \mathbf{x}, H, (\mathbf{w})^{n-1}, \mathbf{w}, J'_{k-2^{n-1}}, \mathbf{z} \rangle$ and $P_2 = J''_{l-k}$ (See Fig. 3(d)). As a consequence, P_1 is indeed an $[\mathbf{x}, \mathbf{z}]$-path of length k and P_2 is indeed an $[\mathbf{z}, \mathbf{y}]$-path of length $l - k$. Obviously, $\ell(P_2) = l - k$ can be any odd integer from $d_{Q_n}(\mathbf{y}, \mathbf{z})$ to $2^n - \ell(P_1) - 1$.

Subcase 1.2: Suppose that $d_{Q_n}(\mathbf{x}, \mathbf{z}) > 2$. By the inductive hypothesis, Q_n^1 is relay-bipanpositionable between two arbitrary vertices in different partite sets. Hence, Q_n^1 has an $[((\mathbf{x})^j)^{n-1}, \mathbf{y}]$-path $H_{s,t}$ of length s such that $H_{s,t}(1) = ((\mathbf{x})^j)^{n-1}$, $H_{s,t}(t+1) = \mathbf{z}$, and $H_{s,t}(s+1) = \mathbf{y}$ for any odd integer s from $d_{Q_n}(((\mathbf{x})^j)^{n-1}, \mathbf{z}) + d_{Q_n}(\mathbf{y}, \mathbf{z}) = d_{Q_n}(\mathbf{x}, \mathbf{z}) + d_{Q_n}(\mathbf{y}, \mathbf{z}) - 2$ to $2^{n-1} - 1$ and for any even integer t from $d_{Q_n}(((\mathbf{x})^j)^{n-1}, \mathbf{z}) = d_{Q_n}(\mathbf{x}, \mathbf{z}) - 2$ to $2^{n-1} - d_{Q_n}(\mathbf{y}, \mathbf{z}) - 1$. For clarity, path $H_{s,t}$ can be written as $\langle ((\mathbf{x})^j)^{n-1}, H^{(1)}_{s,t}, \mathbf{z}, H^{(2)}_{s,t}, \mathbf{y} \rangle$, where $H^{(1)}_{s,t}$ is an $[((\mathbf{x})^j)^{n-1}, \mathbf{z}]$-path of length t and $H^{(2)}_{s,t}$ is a $[\mathbf{z}, \mathbf{y}]$-path of length $s - t$.

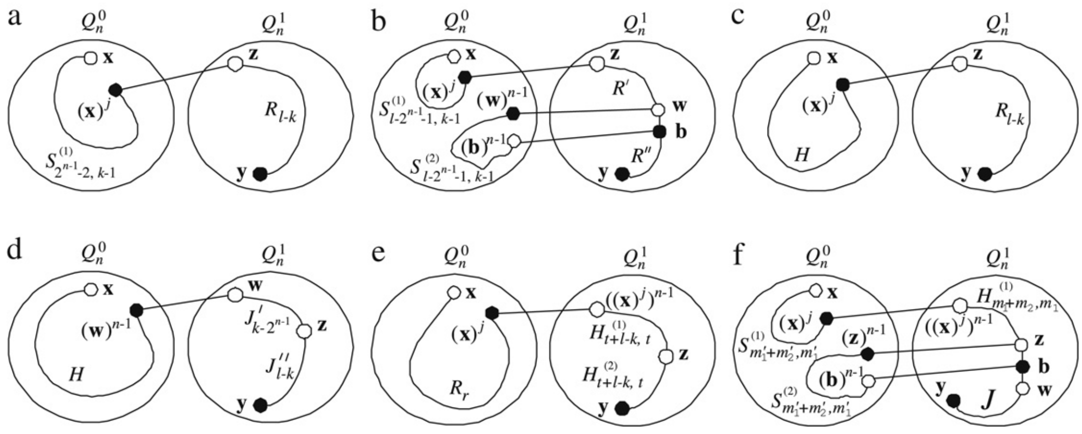

Fig. 3. Case 1 of Theorem 1.

First, we consider the case that $\ell(P_2) = l-k \leq 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z}) + 1$. Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{x})^j]$-path R_r of length r for every odd integer r satisfying $1 \leq r \leq 2^{n-1} - 1$. Then we set P_1 to be the path $\langle \mathbf{x}, R_r, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, H^{(1)}_{t+l-k,t}, \mathbf{z}\rangle$ with $r + t = k - 1$. We set P_2 to be the path $H^{(2)}_{t+l-k,t}$. See Fig. 3(e).

Next, we consider the case that $\ell(P_2) = l - k \geq 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z}) + 3$. For convenience, let $m_1 = d_{Q_n}(\mathbf{x}, \mathbf{z}) - 2$ and $m_2 = 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z}) + 1$. Since $m_2 \geq 2^{n-1} - n + 1 \geq 5$ for $n \geq 4$, path $H^{(2)}_{m_1+m_2,m_1}$ can be written as $\langle \mathbf{z}, \mathbf{b}, \mathbf{w}, J, \mathbf{y}\rangle$, where $\mathbf{b}$ is some vertex adjacent to $\mathbf{z}$, $\mathbf{w}$ is some vertex adjacent to $\mathbf{b}$, and J is a $[\mathbf{w}, \mathbf{y}]$-path. Since $d_{Q_n}(\mathbf{x}, \mathbf{z}) > 2$, we have $\{\mathbf{x}, (\mathbf{x})^j\} \cap \{(\mathbf{z})^{n-1}, (\mathbf{b})^{n-1}, (\mathbf{w})^{n-1}\} = \emptyset$. Let $m_1' = k - m_1 - 1$ and $m_2' = l - k - m_2 - 1$.

I. When $n \geq 5$, Lemma 3 ensures that Q_n^0 has two vertex-disjoint paths $S^{(1)}_{m_1'+m_2',m_1'}$ and $S^{(2)}_{m_1'+m_2',m_1'}$ such that $S^{(1)}_{m_1'+m_2',m_1'}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length m_1' and $S^{(2)}_{m_1'+m_2',m_1'}$ is a $[(\mathbf{z})^{n-1}, (\mathbf{b})^{n-1}]$-path of length m_2'. Then we set $P_1 = \langle \mathbf{x}, S^{(1)}_{m_1'+m_2',m_1'}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, H^{(1)}_{m_1+m_2,m_1}, \mathbf{z}\rangle$ and $P_2 = \langle \mathbf{z}, (\mathbf{z})^{n-1}, S^{(2)}_{m_1'+m_2',m_1'}, (\mathbf{b})^{n-1}, \mathbf{b}, \mathbf{w}, J, \mathbf{y}\rangle$. See Fig. 3(f).

II. When $n = 4$, let $A = \{(\mathbf{x})^0, (\mathbf{x})^1, (\mathbf{x})^2, ((\mathbf{x})^j)^0, ((\mathbf{x})^j)^1, ((\mathbf{x})^j)^2\} - \{\mathbf{x}, (\mathbf{x})^j\}$. Then we have $\{(\mathbf{z})^{n-1}, (\mathbf{b})^{n-1}\} \cap A \neq \emptyset$ or $\{(\mathbf{b})^{n-1}, (\mathbf{w})^{n-1}\} \cap A \neq \emptyset$. If $\{(\mathbf{z})^{n-1}, (\mathbf{b})^{n-1}\} \cap A \neq \emptyset$, the desired path can be obtained as discussed for the case that $n \geq 5$. Otherwise, Lemma 3 ensures that Q_n^0 has two vertex-disjoint paths $T^{(1)}_{m_1'+m_2',m_1'}$ and $T^{(2)}_{m_1'+m_2',m_1'}$ such that $T^{(1)}_{m_1'+m_2',m_1'}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length m_1' and $T^{(2)}_{m_1'+m_2',m_1'}$ is a $[(\mathbf{b})^{n-1}, (\mathbf{w})^{n-1}]$-path of length m_2'. Then the desired path can be formed similarly.

Case 2: Suppose that $\mathbf{y}$ is on Q_n^0. Recall that $(\mathbf{z})^{n-1} \neq \mathbf{y}$. By the inductive hypothesis, Q_n^0 has an $[\mathbf{x}, \mathbf{y}]$-path $H_{s,t}$ of length s such that $H_{s,t}(1) = \mathbf{x}$, $H_{s,t}(t+1) = (\mathbf{z})^{n-1}$, and $H_{s,t}(s+1) = \mathbf{y}$ for any odd integer s from $d_{Q_n}(\mathbf{x}, (\mathbf{z})^{n-1}) + d_{Q_n}(\mathbf{y}, (\mathbf{z})^{n-1}) = d_{Q_n}(\mathbf{x}, \mathbf{z}) + d_{Q_n}(\mathbf{y}, \mathbf{z}) - 2$ to $2^{n-1} - 1$ and for any odd integer t from $d_{Q_n}(\mathbf{x}, (\mathbf{z})^{n-1}) = d_{Q_n}(\mathbf{x}, \mathbf{z}) - 1$ to $2^{n-1} - d_{Q_n}(\mathbf{y}, (\mathbf{z})^{n-1}) - 1 = 2^{n-1} - d_{Q_n}(\mathbf{y}, \mathbf{z})$. For clarity, path $H_{s,t}$ can be written as $\langle \mathbf{x}, H^{(1)}_{s,t}, (\mathbf{z})^{n-1}, H^{(2)}_{s,t}, \mathbf{y}\rangle$, where $H^{(1)}_{s,t}$ is an $[\mathbf{x}, (\mathbf{z})^{n-1}]$-path of length t and $H^{(2)}_{s,t}$ is a $[(\mathbf{z})^{n-1}, \mathbf{y}]$-path of length $s - t$.

First, we consider the case that $\ell(P_2) = l-k \leq 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z}) + 1$. For convenience, let $m = l - k - 1$. Obviously, $H^{(1)}_{t+m,t}$ can be written as $\langle \mathbf{x}, J_{t-1}, \mathbf{w}_t, (\mathbf{z})^{n-1}\rangle$, where $\mathbf{w}_t$ is some vertex adjacent to $(\mathbf{z})^{n-1}$ and J_{t-1} is an $[\mathbf{x}, \mathbf{w}_t]$-path of length $t - 1$. Since Q_n^1 is bipanconnected, it has a $[(\mathbf{w}_t)^{n-1}, \mathbf{z}]$-path R_r of length r for every odd integer r satisfying $1 \leq r \leq 2^{n-1} - 1$. Then we set P_1 to be the path $\langle \mathbf{x}, J_{t-1}, \mathbf{w}_t, (\mathbf{w}_t)^{n-1}, R_r, \mathbf{z}\rangle$, where t and r are two integers satisfying $t + r = k$. We set P_2 to be the path $\langle \mathbf{z}, (\mathbf{z})^{n-1}, H^{(2)}_{t+m,t}, \mathbf{y}\rangle$. See Fig. 4(a).

Next, we consider the case that $\ell(P_2) = l - k \geq 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z}) + 3$. For convenience, let $m_1 = d_{Q_n}(\mathbf{x}, \mathbf{z}) - 1$, $m_2 = 2^{n-1} - d_{Q_n}(\mathbf{x}, \mathbf{z})$, and $A = \{(H^{(2)}_{m_1+m_2,m_1}(i), H^{(2)}_{m_1+m_2,m_1}(i+1)) \mid 2 \leq i \leq m_2 \text{ and } i \equiv 0 \pmod 2\}$. Moreover, we write $H^{(1)}_{m_1+m_2,m_1}$ as $\langle \mathbf{x}, J, \mathbf{w}, (\mathbf{z})^{n-1}\rangle$, where $\mathbf{w}$ is some vertex adjacent to $(\mathbf{z})^{n-1}$ and J is an $[\mathbf{x}, \mathbf{w}]$-path. Clearly, we have $\{(H^{(2)}_{m_1+m_2,m_1}(i))^{n-1}, (H^{(2)}_{m_1+m_2,m_1}(i+1))^{n-1}\} \cap \{(\mathbf{w})^{n-1}, \mathbf{z}\} = \emptyset$ for any $2 \leq i \leq m_2$. Since $m_2 \geq 2^{n-1} - n$, we have $|A| = \lceil m_2/2 \rceil \geq 2^{n-2} - \lfloor n/2 \rfloor = 2$ for $n = 4$. Hence, there exists an even integer $\hat{\imath}$, $2 \leq \hat{\imath} \leq m_2$, such that $\{(H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}))^{n-1}, (H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1))^{n-1}\} \cap \{(\mathbf{w})^{n-1}, \mathbf{z}\} = \emptyset$ and $\{(H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}))^{n-1}, (H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1))^{n-1}\} \cap \{(\mathbf{z})^0, (\mathbf{z})^1, (\mathbf{z})^2, ((\mathbf{w})^{n-1})^0, ((\mathbf{w})^{n-1})^1, ((\mathbf{w})^{n-1})^2\} \neq \emptyset$ if $n = 4$. For $n \geq 5$, let $\hat{\imath}$ be any even integer of $\{2, \ldots, m_2\}$ such that $\{(H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}))^{n-1}, (H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1))^{n-1}\} \cap \{(\mathbf{w})^{n-1}, \mathbf{z}\} = \emptyset$. For convenience, let $\mathbf{u} = H^{(2)}_{m_1+m_2,m_1}(\hat{\imath})$ and $\mathbf{v} = H^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1)$. Accordingly, $H^{(2)}_{m_1+m_2,m_1}$ can be represented as $\langle (\mathbf{z})^{n-1}, I', \mathbf{u}, \mathbf{v}, I'', \mathbf{y}\rangle$. Let $m_1' = k - m_1$ and $m_2' =$

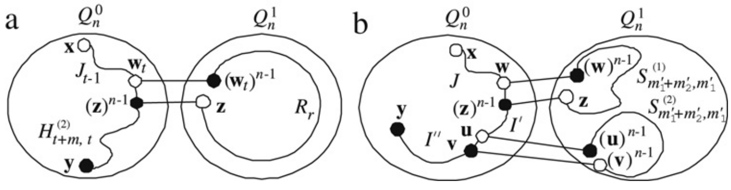

Fig. 4. Case 2 of Theorem 1.

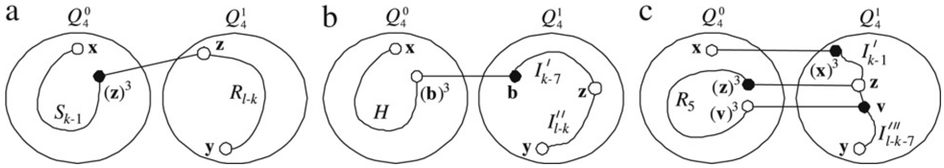

Fig. 5. Case 1 of Lemma 6.

$l-k-m_2-2$. By Lemma 3, Q_n^1 has two vertex-disjoint paths $S^{(1)}_{m_1'+m_2',m_1'}$ and $S^{(2)}_{m_1'+m_2',m_1'}$ such that $S^{(1)}_{m_1'+m_2',m_1'}$ is a $[(\mathbf{w})^{n-1},\mathbf{z}]$-path of length m_1' and $S^{(2)}_{m_1'+m_2',m_1'}$ is a $[(\mathbf{u})^{n-1},(\mathbf{v})^{n-1}]$-path of length m_2'. Then we set $P_1=\langle \mathbf{x},J,\mathbf{w},(\mathbf{w})^{n-1},S^{(1)}_{m_1'+m_2',m_1'},\mathbf{z}\rangle$ and $P_2=\langle \mathbf{z},(\mathbf{z})^{n-1},I',\mathbf{u},(\mathbf{u})^{n-1},S^{(2)}_{m_1'+m_2',m_1'},(\mathbf{v})^{n-1},\mathbf{v},I'',\mathbf{y}\rangle$. See Fig. 4(b).

In summary, P_1 is indeed an $[\mathbf{x},\mathbf{z}]$-path of length k and P_2 is indeed an $[\mathbf{z},\mathbf{y}]$-path of length $l-k$. Hence, the proof is completed. □

Lemma 6. *The 4-cube Q_4 is relay-bipanpositionable between every pair of distinct vertices in the same partite set.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be any two distinct vertices in the same partite set of Q_4 and let $\mathbf{z}$ be any vertex of $V(Q_4)-\{\mathbf{x},\mathbf{y}\}$. We have to construct an $[\mathbf{x},\mathbf{y}]$-path $P_{l,k}$ of length l such that $P_{l,k}(1)=\mathbf{x}$, $P_{l,k}(k+1)=\mathbf{z}$, and $P_{l,k}(l+1)=\mathbf{y}$ for any even integer l from $d_{Q_4}(\mathbf{x},\mathbf{z})+d_{Q_4}(\mathbf{y},\mathbf{z})$ to 14 and for any integer k satisfying $d_{Q_4}(\mathbf{x},\mathbf{z})\le k\le 14-d_{Q_4}(\mathbf{y},\mathbf{z})$ and $2|(k-d_{Q_4}(\mathbf{x},\mathbf{z}))$. For convenience, we write $P_{l,k}=\langle \mathbf{x},P_1,\mathbf{z},P_2,\mathbf{y}\rangle$ with $\ell(P_1)=k$ and $\ell(P_2)=l-k$. Without loss of generality, we assume that $\mathbf{x},\mathbf{y}\in V_0(Q_4)$. Then we distinguish the following two cases.

Case 1: Suppose that $\mathbf{z}\in V_0(Q_4)$. Since Q_4 is edge-transitive, we assume that $(\mathbf{x})_3\ne(\mathbf{y})_3$. Without loss of generality, we assume that $\mathbf{x}\in V(Q_4^0)$ and $\mathbf{y},\mathbf{z}\in V(Q_4^1)$.

First, we consider the case that $\ell(P_1)=k\le 8$ and $\ell(P_2)=l-k\le 6$. By Lemma 2 and Theorem 1, Q_3 is bipanconnected. Therefore, Q_4^0 has an $[\mathbf{x},(\mathbf{z})^3]$-path S_p of length p for each odd integer p from $d_{Q_4}(\mathbf{x},(\mathbf{z})^3)=d_{Q_4}(\mathbf{x},\mathbf{z})-1$ to 7. Similarly, Q_4^1 has a $[\mathbf{z},\mathbf{y}]$-path R_r of length r for each even integer r from $d_{Q_4}(\mathbf{y},\mathbf{z})$ to 6. Then we set $P_1=\langle \mathbf{x},S_{k-1},(\mathbf{z})^3,\mathbf{z}\rangle$ and $P_2=R_{l-k}$. As a result, $P_{l,k}=\langle \mathbf{x},P_1,\mathbf{z},P_2,\mathbf{y}\rangle$ is indeed an $[\mathbf{x},\mathbf{y}]$-path of length l such that $P_{l,k}(k+1)=\mathbf{z}$. See Fig. 5(a).

Next we consider the case that $\ell(P_1)=k\ge 10$. Let $\mathbf{b}$ be a vertex of Q_4^1 such that $d_{Q_4}(\mathbf{b},\mathbf{z})=1$ and $(\mathbf{b})^3\ne\mathbf{x}$. Since Q_4^0 is bipanconnected, it has an $[\mathbf{x},(\mathbf{b})^3]$-path H of length six. By Theorem 1, Q_3 is relay-bipanpositionable between any two vertices in different partite sets. Therefore, Q_4^1 has a $[\mathbf{b},\mathbf{y}]$-path $I_{l-7,k-7}$ of length $l-7$ such that $I_{l-7,k-7}(1)=\mathbf{b}$, $I_{l-7,k-7}(k-6)=\mathbf{z}$, and $I_{l-7,k-7}(l-6)=\mathbf{y}$. For convenience, we write $I_{l-7,k-7}$ as $\langle \mathbf{b},I'_{k-7},\mathbf{z},I''_{l-k},\mathbf{y}\rangle$, where I'_{k-7} is a $[\mathbf{b},\mathbf{z}]$-path of length $k-7$ and I''_{l-k} is a $[\mathbf{z},\mathbf{y}]$-path of length $l-k$. Then we set $P_1=\langle \mathbf{x},H,(\mathbf{b})^3,\mathbf{b},I'_{k-7},\mathbf{z}\rangle$ and $P_2=I''_{l-k}$. See Fig. 5(b). Consequently, P_1 is indeed an $[\mathbf{x},\mathbf{z}]$-path of length k and P_2 is indeed an $[\mathbf{z},\mathbf{y}]$-path of length $l-k$.

Finally, we consider the case that $\ell(P_2)=l-k\ge 8$. Since $\mathbf{x},\mathbf{y},\mathbf{z}\in V_0(Q_4)$, we have $(\mathbf{x})^3\notin\{\mathbf{y},\mathbf{z}\}$. By Theorem 1, Q_4^1 has an $[(\mathbf{x})^3,\mathbf{y}]$-path $I_{l-7,k-1}$ of length $l-7$ such that $I_{l-7,k-1}(1)=(\mathbf{x})^3$, $I_{l-7,k-1}(k)=\mathbf{z}$, and $I_{l-7,k-1}(l-6)=\mathbf{y}$. For convenience, we write $I_{l-7,k-1}$ as $\langle(\mathbf{x})^3,I'_{k-1},\mathbf{z},I''_{l-k-6},\mathbf{y}\rangle$, where I'_{k-1} is an $[(\mathbf{x})^3,\mathbf{z}]$-path of length $k-1$ and I''_{l-k-6} is a $[\mathbf{z},\mathbf{y}]$-path of length $l-k-6$. Moreover, we write I''_{l-k-6} as $\langle \mathbf{z},\mathbf{v},I'''_{l-k-7},\mathbf{y}\rangle$, where $\mathbf{v}$ is some vertex adjacent to $\mathbf{z}$ and I'''_{l-k-7} is a $[\mathbf{v},\mathbf{y}]$-path of length $l-k-7$. By Lemma 5, $Q_4^0-\{\mathbf{x}\}$ has a $[(\mathbf{z})^3,(\mathbf{v})^3]$-path R_r of length $r\in\{1,3,5\}$. Then we set $P_1=\langle \mathbf{x},(\mathbf{x})^3,I'_{k-1},\mathbf{z}\rangle$ and $P_2=\langle \mathbf{z},(\mathbf{z})^3,R_5,(\mathbf{v})^3,\mathbf{v},I'''_{l-k-7},\mathbf{y}\rangle$. See Fig. 5(c).

Case 2: Suppose that $\mathbf{z}\in V_1(Q_4)$. Since $\mathbf{x}$ and $\mathbf{z}$ are in different partite sets of Q_4, we have $d_{Q_4}(\mathbf{x},\mathbf{z})\in\{1,3\}$.

Subcase 2.1: Suppose that $d_{Q_4}(\mathbf{x},\mathbf{z})=3$. Since $\mathbf{x},\mathbf{y}\in V_0(Q_4)$, we have $d_{Q_4}(\mathbf{x},\mathbf{y})\ge 2$. Hence, we can find an integer $i\in\{0,1,2,3\}$ such that $(\mathbf{x})_i\ne(\mathbf{y})_i$ and $(\mathbf{x})_i\ne(\mathbf{z})_i$. Since Q_4 is edge-transitive, we assume that $i=3$. Without loss of generality, we assume that $\mathbf{x}$ is on Q_4^0 and both $\mathbf{y}$ and $\mathbf{z}$ are on Q_4^1.

First we consider the case that $\ell(P_1)=k\le 7$ and $\ell(P_2)=l-k\le 7$. Since Q_4^0 is bipanconnected, it has an $[\mathbf{x},(\mathbf{z})^3]$-path S_p of length p for any even integer p from $d_{Q_4}(\mathbf{x},\mathbf{z})-1=2$ to 6. Similarly, Q_4^1 has a $[\mathbf{z},\mathbf{y}]$-path R_q of length q for any odd integer q from $d_{Q_4}(\mathbf{y},\mathbf{z})$ to 7. Then we set $P_1=\langle \mathbf{x},S_{k-1},(\mathbf{z})^3,\mathbf{z}\rangle$ and $P_2=R_{l-k}$. As a result, $P_{l,k}=\langle \mathbf{x},P_1,\mathbf{z},P_2,\mathbf{y}\rangle$ is indeed an $[\mathbf{x},\mathbf{y}]$-path of length l such that $P_{l,k}(k+1)=\mathbf{z}$. The illustration is similar to Fig. 5(a).

Next, we consider the case that $\ell(P_1)=k\ge 9$. Let $\mathbf{b}\in V_1(Q_4^1)$ such that $d_{Q_4}(\mathbf{b},\mathbf{z})=2$ and $(\mathbf{b})^3\ne\mathbf{x}$. Since Q_4^0 is bipanconnected, it has an $[\mathbf{x},(\mathbf{b})^3]$-path H of length six. By Theorem 1, Q_4^1 has a $[\mathbf{b},\mathbf{y}]$-path $I_{l-7,k-7}$ of length $l-7$ such that

Table 3
The required paths for Subcase 2.2.1 of Lemma 6.

y	k	$l-k$	Paths between $\mathbf{x}=0000$ and $\mathbf{y}$
1010	9	3	$\langle \mathbf{x}=0000, 0100, 1100, 1101, 0101, 0111, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1011, 1010=\mathbf{y}\rangle$
	9	5	$\langle \mathbf{x}=0000, 0100, 1100, 1101, 0101, 0111, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1011, 1111, 1110, 1010=\mathbf{y}\rangle$
	11	3	$\langle \mathbf{x}=0000, 0100, 1100, 1110, 1111, 1101, 0101, 0111, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1011, 1010=\mathbf{y}\rangle$
	1	9	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001, 1000, 1100, 1110, 1010=\mathbf{y}\rangle$
	3	9	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001, 1000, 1100, 1110, 1010=\mathbf{y}\rangle$
	5	9	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001, 1000, 1100, 1110, 1010=\mathbf{y}\rangle$
	1	11	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0111, 0011, 1011, 1111, 1101, 1001, 1000, 1100, 1110, 1010=\mathbf{y}\rangle$
	3	11	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 0111, 0110, 1110, 1100, 1000, 1001, 1101, 1111, 1011, 1010=\mathbf{y}\rangle$
	1	13	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0100, 0110, 0111, 0011, 1011, 1111, 1101, 1001, 1000, 1100, 1110, 1010=\mathbf{y}\rangle$
1100	9	3	$\langle \mathbf{x}=0000, 0100, 0101, 0111, 1111, 1110, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1101, 1100=\mathbf{y}\rangle$
	9	5	$\langle \mathbf{x}=0000, 0100, 0101, 0111, 1111, 1110, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1011, , 1010, 1000, 1100=\mathbf{y}\rangle$
	11	3	$\langle \mathbf{x}=0000, 0100, 0101, 0111, 1111, 1011, 1010, 1110, 0110, 0010, 0011, 0001=\mathbf{z}, 1001, 1101, 1100=\mathbf{y}\rangle$
	1	9	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 1000, 1100=\mathbf{y}\rangle$
	3	9	$\langle \mathbf{x}=0000, 0010, 0011, 0001=\mathbf{z}, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 1000, 1100=\mathbf{y}\rangle$
	5	9	$\langle \mathbf{x}=0000, 0100, 0110, 0010, 0011, 0001=\mathbf{z}, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 1000, 1100=\mathbf{y}\rangle$
	1	11	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 0111, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 1000, 1100=\mathbf{y}\rangle$
	3	11	$\langle \mathbf{x}=0000, 0010, 0011, 0001=\mathbf{z}, 0101, 0100, 0110, 1110, 1010, 1000, 1001, 1011, 1111, 1101, 1100=\mathbf{y}\rangle$
	1	13	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 0111, 0110, 0100, 0101, 1101, 1111, 1110, 1010, 1011, 1001, 1000, 1100=\mathbf{y}\rangle$
1111	9	3	$\langle \mathbf{x}=0000, 1000, 1010, 0010, 0110, 0100, 0101, 0111, 0011, 0001=\mathbf{z}, 1001, 1011, 1111=\mathbf{y}\rangle$
	9	5	$\langle \mathbf{x}=0000, 1000, 1010, 0010, 0110, 0100, 0101, 0111, 0011, 0001=\mathbf{z}, 1001, 1101, 1100, 1110, 1111=\mathbf{y}\rangle$
	11	3	$\langle \mathbf{x}=0000, 1000, 1100, 1110, 1010, 0010, 0110, 0100, 0101, 0111, 0011, 0001=\mathbf{z}, 1001, 1011, 1111=\mathbf{y}\rangle$
	1	9	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111=\mathbf{y}\rangle$
	3	9	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111=\mathbf{y}\rangle$
	5	9	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111=\mathbf{y}\rangle$
	1	11	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0111, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111=\mathbf{y}\rangle$
	3	11	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 0010, 0110, 1110, 1010, 1011, 1001, 1000, 1100, 1101, 1111=\mathbf{y}\rangle$
	1	13	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0100, 0110, 0111, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111=\mathbf{y}\rangle$

$I_{l-7,k-7}(1)=\mathbf{b}$, $I_{l-7,k-7}(k-6)=\mathbf{z}$, and $I_{l-7,k-7}(l-6)=\mathbf{y}$. For convenience, we write $I_{l-7,k-7}$ as $\langle \mathbf{b}, I'_{k-7}, \mathbf{z}, I''_{l-k}, \mathbf{y}\rangle$, where I'_{k-7} is a $[\mathbf{b},\mathbf{z}]$-path of length $k-7$ and I''_{l-k} is a $[\mathbf{z},\mathbf{y}]$-path of length $l-k$. Then we set $P_1=\langle \mathbf{x}, H, (\mathbf{b})^3, \mathbf{b}, I'_{k-7}, \mathbf{z}\rangle$ and $P_2=I''_{l-k}$. As a result, $P_{l,k}=\langle \mathbf{x}, P_1, \mathbf{z}, P_2, \mathbf{y}\rangle$ is an $[\mathbf{x},\mathbf{y}]$-path of length l such that $P_{l,k}(k+1)=\mathbf{z}$. The illustration is similar to Fig. 5(b).

Finally, we consider the case that $\ell(P_2)=l-k\geq 9$. Since $d_{Q_4}(\mathbf{x},\mathbf{z})=3$, we have $(\mathbf{x})^3\neq\mathbf{z}$. Because Q_4^1 is relay-bipanpositionable between $(\mathbf{x})^3$ and $\mathbf{y}$, it has an $[(\mathbf{x})^3,\mathbf{y}]$-path $J_{l-7,k-1}$ of length $l-7$ such that $J_{l-7,k-1}(1)=(\mathbf{x})^3$, $J_{l-7,k-1}(k)=\mathbf{z}$, and $J_{l-7,k-1}(l-6)=\mathbf{y}$. For convenience, we write $J_{l-7,k-1}$ as $\langle (\mathbf{x})^3, J'_{k-1}, \mathbf{z}, J''_{l-k-6}, \mathbf{y}\rangle$, where J'_{k-1} is an $[(\mathbf{x})^3,\mathbf{z}]$-path of length $k-1$ and J''_{l-k-6} is a $[\mathbf{z},\mathbf{y}]$-path of length $l-k-6$. Furthermore, we can write J''_{l-k-6} as $\langle \mathbf{z}, \mathbf{v}, J'''_{l-k-7}, \mathbf{y}\rangle$, where $\mathbf{v}$ is some vertex adjacent to $\mathbf{z}$ and J'''_{l-k-7} is a $[\mathbf{v},\mathbf{y}]$-path of length $l-k-7$. By Lemma 5, $Q_4^0-\{\mathbf{x}\}$ has a $[(\mathbf{z})^3, (\mathbf{v})^3]$-path R_r of length $r\in\{1,3,5\}$. Then we set $P_1=\langle \mathbf{x}, (\mathbf{x})^3, J'_{k-1}, \mathbf{z}\rangle$ and $P_2=\langle \mathbf{z}, (\mathbf{z})^3, R_5, (\mathbf{v})^3, \mathbf{v}, J'''_{l-k-7}, \mathbf{y}\rangle$. The illustration is similar to Fig. 5(c).

Subcase 2.2: Suppose that $d_{Q_4}(\mathbf{x},\mathbf{z})=1$. Since $\mathbf{x}$ and $\mathbf{y}$ are in the same partite set of Q_4, we have $d_{Q_4}(\mathbf{x},\mathbf{y})\in\{2,4\}$. Thus, there exists an integer i of $\{0,1,2,3\}$ such that $(\mathbf{x})_i\neq(\mathbf{y})_i$ and $(\mathbf{x})_i=(\mathbf{z})_i$. Since Q_4 is edge-transitive, we assume that $i=3$. Without loss of generality, we assume that $\mathbf{x},\mathbf{z}\in V(Q_4^0)$ and $\mathbf{y}\in V(Q_4^1)$. Since $\mathbf{y}$ and $\mathbf{z}$ are in different partite sets of Q_4, we have $d_{Q_4}(\mathbf{y},\mathbf{z})\in\{1,3\}$. Therefore, we distinguish the following subcases.

Subcase 2.2.1: Suppose that $d_{Q_4}(\mathbf{y},\mathbf{z})=3$. On the one hand, we concern the case that $\ell(P_1)=k\leq 7$ and $\ell(P_2)=l-k\leq 7$. Since Q_4^0 is bipanconnected, it has an $[\mathbf{x},\mathbf{z}]$-path S_p of length $p\in\{1,3,5,7\}$. Since $d_{Q_4}(\mathbf{y},\mathbf{z})=3$, we have $(\mathbf{z})^3\neq\mathbf{y}$. Similarly, Q_4^1 has a $[(\mathbf{z})^3,\mathbf{y}]$-path R_r of length $r\in\{2,4,6\}$. Then we set $P_1=S_k$ and $P_2=\langle \mathbf{z}, (\mathbf{z})^3, R_{l-k-1}, \mathbf{y}\rangle$. On the other hand, we concern the case that $\ell(P_1)=k\geq 9$ or $\ell(P_2)=l-k\geq 9$. Without loss of generality, we assume that $\mathbf{x}=0000$ and $\mathbf{z}=0001$. Since $\mathbf{y}\in V(Q_4^1)$ and $d_{Q_4}(\mathbf{y},\mathbf{z})=3$, we have $\mathbf{y}\in\{1010, 1100, 1111\}$. Then the required paths obtained by brute force are listed in Table 3.

Subcase 2.2.2: Suppose that $d_{Q_4}(\mathbf{y},\mathbf{z})=1$. Without loss of generality, we assume that $\mathbf{x}=0000$, $\mathbf{y}=1001$, and $\mathbf{z}=0001$. Then we list the required paths obtained by brute force in Table 4. □

We apply Lemma 6 to prove the following theorem.

Theorem 2. *The n-cube Q_n is relay-bipanpositionable between every pair of distinct vertices in the same partite set if $n\geq 4$.*

Proof. We prove this theorem by induction on n. The induction basis follows from Lemma 6. As the inductive hypothesis, we assume that Q_{n-1} is relay-bipanpositionable between every pair of distinct vertices in the same partite set for $n\geq 5$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two distinct vertices in the same partite set of Q_n and let $\mathbf{z}$ be any vertex of $V(Q_n)-\{\mathbf{x},\mathbf{y}\}$. We have to construct an $[\mathbf{x},\mathbf{y}]$-path $P_{l,k}$ of length l such that $P_{l,k}(1)=\mathbf{x}$, $P_{l,k}(k+1)=\mathbf{z}$, and $P_{l,k}(l+1)=\mathbf{y}$ for any even integer l from $d_{Q_n}(\mathbf{x},\mathbf{z})+d_{Q_n}(\mathbf{y},\mathbf{z})$ to 2^n-2 and for any integer k satisfying $d_{Q_n}(\mathbf{x},\mathbf{z})\leq k\leq 2^n-d_{Q_n}(\mathbf{y},\mathbf{z})-2$ and $2|(k-d_{Q_n}(\mathbf{x},\mathbf{z}))$. For convenience, we write $P_{l,k}=\langle \mathbf{x}, P_1, \mathbf{z}, P_2, \mathbf{y}\rangle$ with $\ell(P_1)=k$ and $\ell(P_2)=l-k$. Without loss of generality, we assume that

Table 4
The required paths for Subcase 2.2.2 of Lemma 6.

k	$l-k$	Paths between $\mathbf{x}=0000$ and $\mathbf{y}=1001$
1	1	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
1	3	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1001=\mathbf{y}\rangle$
1	5	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001=\mathbf{y}\rangle$
1	7	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1100, 1101, 1001=\mathbf{y}\rangle$
1	9	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1010, 1000, 1100, 1101, 1001=\mathbf{y}\rangle$
1	11	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0111, 0011, 1011, 1111, 1110, 1010, 1000, 1100, 1101, 1001=\mathbf{y}\rangle$
1	13	$\langle \mathbf{x}=0000, 0001=\mathbf{z}, 0101, 0100, 0110, 0111, 0011, 1011, 1111, 1110, 1010, 1000, 1100, 1101, 1001=\mathbf{y}\rangle$
3	1	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
3	3	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1001=\mathbf{y}\rangle$
3	5	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001=\mathbf{y}\rangle$
3	7	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1100, 1101, 1001=\mathbf{y}\rangle$
3	9	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1010, 1000, 1100, 1101, 1001=\mathbf{y}\rangle$
3	11	$\langle \mathbf{x}=0000, 0100, 0101, 0001=\mathbf{z}, 0011, 0111, 0110, 0010, 1010, 1011, 1111, 1110, 1100, 1101, 1001=\mathbf{y}\rangle$
5	1	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
5	3	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1001=\mathbf{y}\rangle$
5	5	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1101, 1001=\mathbf{y}\rangle$
5	7	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1100, 1101, 1001=\mathbf{y}\rangle$
5	9	$\langle \mathbf{x}=0000, 0100, 0110, 0111, 0101, 0001=\mathbf{z}, 0011, 1011, 1111, 1110, 1010, 1000, 1100, 1101, 1001=\mathbf{y}\rangle$
7	1	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1011, 0011, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
7	3	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1001=\mathbf{y}\rangle$
7	5	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1100, 1000, 1001=\mathbf{y}\rangle$
7	7	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1100, 1110, 1010, 1000, 1001=\mathbf{y}\rangle$
9	1	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1110, 1010, 1011, 0011, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
9	3	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1110, 1010, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1001=\mathbf{y}\rangle$
9	5	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1110, 1010, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1100, 1000, 1001=\mathbf{y}\rangle$
11	1	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1110, 1100, 1000, 1010, 1011, 0011, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$
11	3	$\langle \mathbf{x}=0000, 0010, 0110, 0111, 1111, 1110, 1100, 1000, 1010, 1011, 0011, 0001=\mathbf{z}, 0101, 1101, 1001=\mathbf{y}\rangle$
13	1	$\langle \mathbf{x}=0000, 0010, 0110, 0100, 0101, 0111, 1111, 1110, 1100, 1000, 1010, 1011, 0011, 0001=\mathbf{z}, 1001=\mathbf{y}\rangle$

$\mathbf{x}, \mathbf{y} \in V_0(Q_n)$. Since $\mathbf{x}$ and $\mathbf{y}$ are in the same partite set of Q_n, we have $d_{Q_n}(\mathbf{x},\mathbf{y}) \geq 2$. Therefore, there exists an integer i of $\{0, 1, \ldots, n-1\}$ such that $(\mathbf{x})_i \neq (\mathbf{y})_i$. Since Q_n is edge-transitive, we assume that $i = n-1$. Thus, Q_n can be partitioned into Q_n^0 and Q_n^1 so that $\mathbf{x}$ and $\mathbf{y}$ are on different subcubes. Without loss of generality, we assume that $\mathbf{x} \in V(Q_n^0)$ and $\mathbf{y}, \mathbf{z} \in V(Q_n^1)$. By Theorem 1 and Lemma 2, both Q_n^0 and Q_n^1 are bipanconnected. Then we distinguish the following cases.

Case 1: Suppose that $d_{Q_n}(\mathbf{x},\mathbf{z}) = 1$; i.e., $\mathbf{z} = (\mathbf{x})^{n-1}$. We concern the following subcases.

Subcase 1.1: Suppose that $\ell(P_1) = k = 1$. Obviously, $P_1 = \langle \mathbf{x}, \mathbf{z}\rangle$ is the desired path. On the one hand, we consider the case that $\ell(P_2) = l-k \leq 2^{n-1}-1$. Since Q_n^1 is bipanconnected, it has a $[\mathbf{z},\mathbf{y}]$-path T_t of length t for any odd integer t from $d_{Q_n}(\mathbf{y},\mathbf{z})$ to $2^{n-1}-1$. Then we set $P_2 = T_{l-k}$. See Fig. 6(a). On the other hand, we consider the case that $\ell(P_2) = l-k \geq 2^{n-1}+1$. Since $2^{n-1}-1 \geq 15$ for $n \geq 5$, we can write $T_{2^{n-1}-1}$ as $\langle \mathbf{z}, T', \mathbf{w}, \mathbf{b}, \mathbf{y}\rangle$, where $\mathbf{b}$ is some vertex adjacent to $\mathbf{y}$ and $\mathbf{w}$ is some vertex adjacent to $\mathbf{b}$. By Lemma 4, Q_{n-1} is hyper-bipanconnected. Thus, $Q_n^0 - \{\mathbf{x}\}$ has a $[(\mathbf{w})^{n-1}, (\mathbf{y})^{n-1}]$-path R_r of length r for any even integer r from 2 to $2^{n-1}-2$. Then we can set $P_2 = \langle \mathbf{z}, T', \mathbf{w}, (\mathbf{w})^{n-1}, R_{l-k-2^{n-1}+1}, (\mathbf{y})^{n-1}, \mathbf{y}\rangle$. See Fig. 6(b).

Subcase 1.2: Suppose that $\ell(P_1) = k \geq 3$. Let j be an integer of $\{0, 1, \ldots, n-2\}$ such that $((\mathbf{x})^j)^{n-1} \neq \mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{z}$ are adjacent, $((\mathbf{x})^j)^{n-1}$ and $\mathbf{z}$ are also adjacent. By the inductive hypothesis, Q_n^1 has an $[((\mathbf{x})^j)^{n-1}, \mathbf{y}]$-path $I_{p,q}$ of length p such that $I_{p,q}(1) = ((\mathbf{x})^j)^{n-1}$, $I_{p,q}(q+1) = \mathbf{z}$, and $I_{p,q}(p+1) = \mathbf{y}$ for any even integer p from $d_{Q_n}(((\mathbf{x})^j)^{n-1}, \mathbf{z}) + d_{Q_n}(\mathbf{y},\mathbf{z}) = 1 + d_{Q_n}(\mathbf{y},\mathbf{z})$ to $2^{n-1}-2$ and for any odd integer q from 1 to $2^{n-1} - d_{Q_n}(\mathbf{y},\mathbf{z}) - 2$. For convenience, we write $I_{p,q}$ as $\langle ((\mathbf{x})^j)^{n-1}, I_{p,q}^{(1)}, \mathbf{z}, I_{p,q}^{(2)}, \mathbf{y}\rangle$, where $I_{p,q}^{(1)}$ is an $[((\mathbf{x})^j)^{n-1}, \mathbf{z}]$-path of length q and $I_{p,q}^{(2)}$ is a $[\mathbf{z},\mathbf{y}]$-path of length $p-q$.

First we consider the case that $\ell(P_2) = l-k \leq 2^{n-1}-3$. Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{x})^j]$-path R_r of any odd length r in the range from 1 to $2^{n-1}-1$. Then we set $P_1 = \langle \mathbf{x}, R_r, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, I_{q+l-k,q}^{(1)}, \mathbf{z}\rangle$ with $r+q = k-1$ and $P_2 = I_{q+l-k,q}^{(2)}$. See Fig. 6(c).

Now we consider the case that $\ell(P_2) = l-k \geq 2^{n-1}-1$. Let $m = 2^{n-1}-3$ and $A = \{(I_{1+m,1}^{(2)}(i), I_{1+m,1}^{(2)}(i+1)) \mid 1 \leq i \leq m \text{ and } i \equiv 1 \pmod 2\}$. Obviously, we have $|A| = \lceil m/2 \rceil = 2^{n-2}-1$. Since $|A| \geq 7$ for $n \geq 5$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq m$, such that $\{(I_{1+m,1}^{(2)}(\hat{\imath}))^{n-1}, (I_{1+m,1}^{(2)}(\hat{\imath}+1))^{n-1}\} \cap \{\mathbf{x}, (\mathbf{x})^j\} = \emptyset$. For convenience, let $\mathbf{b} = I_{1+m,1}^{(2)}(\hat{\imath})$ and $\mathbf{w} = I_{1+m,1}^{(2)}(\hat{\imath}+1)$. Accordingly, $I_{1+m,1}^{(2)}$ can be written as $\langle \mathbf{z}, I', \mathbf{b}, \mathbf{w}, I'', \mathbf{y}\rangle$. Let $m' = l-k-m-1 = l-k-2^{n-1}+2$. By Lemma 3, Q_n^0 has two vertex-disjoint paths $S_{k-2+m',k-2}^{(1)}$ and $S_{k-2+m',k-2}^{(2)}$ such that $S_{k-2+m',k-2}^{(1)}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length $k-2$ and $S_{k-2+m',k-2}^{(2)}$ is a $[(\mathbf{b})^{n-1}, (\mathbf{w})^{n-1}]$-path of length m'. Then we can set $P_1 = \langle \mathbf{x}, S_{k-2+m',k-2}^{(1)}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, \mathbf{z}\rangle$ and $P_2 = \langle \mathbf{z}, I', \mathbf{b}, (\mathbf{b})^{n-1}, S_{k-2+m',k-2}^{(2)}, (\mathbf{w})^{n-1}, \mathbf{w}, I'', \mathbf{y}\rangle$. See Fig. 6(d).

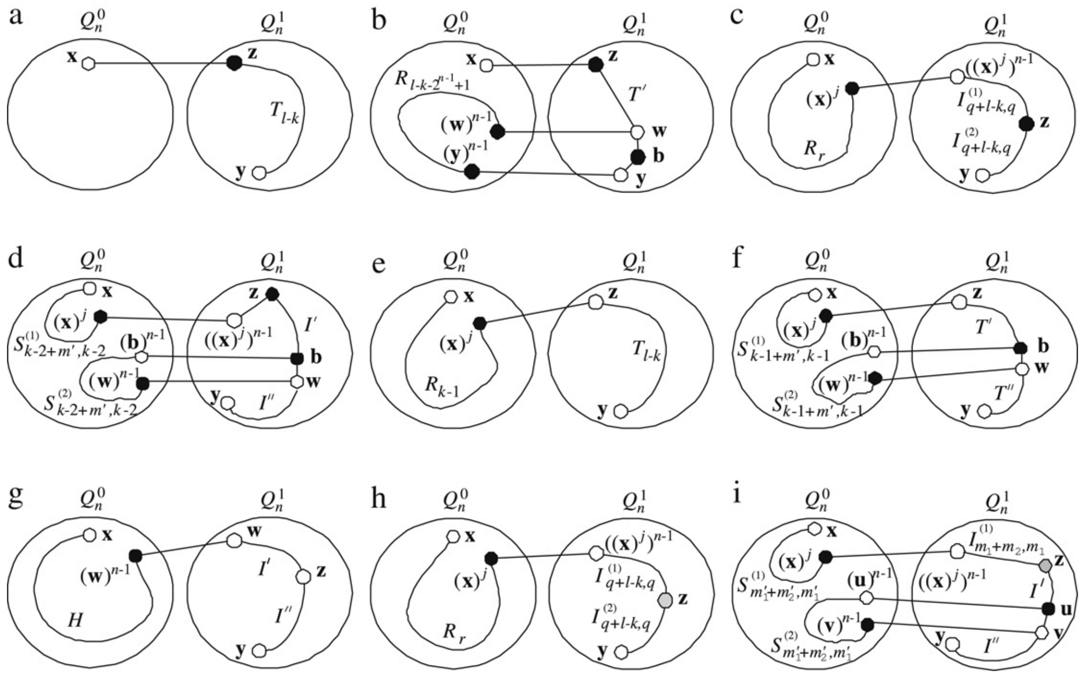

Fig. 6. Illustrations for Theorem 2.

Case 2: Suppose that $d_{Q_n}(\mathbf{x},\mathbf{z}) = 2$. Clearly there exists an integer j of $\{0, 1, \ldots, n-2\}$ such that $(\mathbf{x})_j \neq (\mathbf{z})_j$. Therefore, we have $((\mathbf{x})^j)^{n-1} = \mathbf{z}$. Since Q_n^1 is bipanconnected, it has a $[\mathbf{z},\mathbf{y}]$-path T_t of any even length t from $d_{Q_n}(\mathbf{y},\mathbf{z})$ to $2^{n-1}-2$.

Subcase 2.1: Suppose that $\ell(P_1) = k \leq 2^{n-1}$. Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{x})^j]$-path R_r of any odd length r from 1 to $2^{n-1}-1$. For the case that $\ell(P_2) \leq 2^{n-1}-2$, we can set $P_1 = \langle \mathbf{x}, R_{k-1}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1} = \mathbf{z}\rangle$ and $P_2 = T_{l-k}$. See Fig. 6(e). In what follows, we consider the case that $\ell(P_2) \geq 2^{n-1}$. Let $m = 2^{n-1}-2$ and $A = \{(T_m(i), T_m(i+1)) \mid 2 \leq i \leq m \text{ and } i \equiv 0 \pmod 2\}$. Obviously, we have $|A| = \lceil m/2 \rceil = 2^{n-2}-1$. Since $|A| \geq 7$ for $n \geq 5$, there exists an even integer $\hat{\imath}$, $2 \leq \hat{\imath} \leq m$, such that $\{(T_m(\hat{\imath}))^{n-1}, (T_m(\hat{\imath}+1))^{n-1}\} \cap \{\mathbf{x}, (\mathbf{x})^j\} = \emptyset$. For convenience, let $\mathbf{b} = T_m(\hat{\imath})$ and $\mathbf{w} = T_m(\hat{\imath}+1)$. Accordingly, path T_m can be written as $\langle \mathbf{z}, T', \mathbf{b}, \mathbf{w}, T'', \mathbf{y}\rangle$. Let $m' = l-k-m-1 = l-k-2^{n-1}+1$. By Lemma 3, Q_n^0 has two vertex-disjoint paths $S^{(1)}_{k-1+m',k-1}$ and $S^{(2)}_{k-1+m',k-1}$ such that $S^{(1)}_{k-1+m',k-1}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length $k-1$ and $S^{(2)}_{k-1+m',k-1}$ is a $[(\mathbf{b})^{n-1}, (\mathbf{w})^{n-1}]$-path of length m'. Then we can set $P_1 = \langle \mathbf{x}, S^{(1)}_{k-1+m',k-1}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1} = \mathbf{z}\rangle$ and $P_2 = \langle \mathbf{z}, T', \mathbf{b}, (\mathbf{b})^{n-1}, S^{(2)}_{k-1+m',k-1}, (\mathbf{w})^{n-1}, \mathbf{w}, T'', \mathbf{y}\rangle$. See Fig. 6(f).

Subcase 2.2: Suppose that $\ell(P_1) = k \geq 2^{n-1}+2$. Let $\mathbf{w}$ be a vertex of Q_n^1 such that $d_{Q_n}(\mathbf{w},\mathbf{z}) = 2$. Obviously, $(\mathbf{w})^{n-1}$ and $\mathbf{x}$ are in the different partite sets of Q_n^0. Since Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{w})^{n-1}]$-path H of length $2^{n-1}-1$. By the inductive hypothesis, Q_n^1 has a $[\mathbf{w},\mathbf{y}]$-path $I_{l-2^{n-1},k-2^{n-1}}$ of length $l-2^{n-1}$ such that $I_{l-2^{n-1},k-2^{n-1}}(1) = \mathbf{w}$, $I_{l-2^{n-1},k-2^{n-1}}(k-2^{n-1}+1) = \mathbf{z}$, and $I_{l-2^{n-1},k-2^{n-1}}(l-2^{n-1}+1) = \mathbf{y}$. For convenience, we write $I_{l-2^{n-1},k-2^{n-1}}$ as $\langle \mathbf{w}, I', \mathbf{z}, I'', \mathbf{y}\rangle$, where I' is a $[\mathbf{w},\mathbf{z}]$-path of length $k-2^{n-1}$ and I'' is a $[\mathbf{z},\mathbf{y}]$-path of length $l-k$. Then we set $P_1 = \langle \mathbf{x}, H, (\mathbf{w})^{n-1}, \mathbf{w}, I', \mathbf{z}\rangle$ and $P_2 = I''$. See Fig. 6(g).

Case 3: Suppose that $d_{Q_n}(\mathbf{x},\mathbf{z}) > 2$. Hence there exists an integer j of $\{0, 1, \ldots, n-2\}$ such that $(\mathbf{x})_j \neq (\mathbf{z})_j$ and $((\mathbf{x})^j)^{n-1} \neq \mathbf{y}$. By the inductive hypothesis, Q_n^1 has an $[((\mathbf{x})^j)^{n-1}, \mathbf{y}]$-path $I_{p,q}$ of length p such that $I_{p,q}(1) = ((\mathbf{x})^j)^{n-1}$, $I_{p,q}(q+1) = \mathbf{z}$, and $I_{p,q}(p+1) = \mathbf{y}$ for any even integer p from $d_{Q_n}(((\mathbf{x})^j)^{n-1},\mathbf{z}) + d_{Q_n}(\mathbf{y},\mathbf{z}) = d_{Q_n}(\mathbf{x},\mathbf{z}) + d_{Q_n}(\mathbf{y},\mathbf{z}) - 2$ to $2^{n-1}-2$ and for any integer q satisfying both $d_{Q_n}(\mathbf{x},\mathbf{z})-2 \leq q \leq 2^{n-1}-d_{Q_n}(\mathbf{y},\mathbf{z})-2$ and $2 \mid (q-d_{Q_n}(\mathbf{x},\mathbf{z}))$. For convenience, we write $I_{p,q}$ as $\langle ((\mathbf{x})^j)^{n-1}, I^{(1)}_{p,q}, \mathbf{z}, I^{(2)}_{p,q}, \mathbf{y}\rangle$, where $I^{(1)}_{p,q}$ is an $[((\mathbf{x})^j)^{n-1}, \mathbf{z}]$-path of length q and $I^{(2)}_{p,q}$ is a $[\mathbf{z},\mathbf{y}]$-path of length $p-q$.

First we consider the case that $\ell(P_2) = l-k \leq 2^{n-1} - d_{Q_n}(\mathbf{x},\mathbf{z})$. Because Q_n^0 is bipanconnected, it has an $[\mathbf{x}, (\mathbf{x})^j]$-path R_r of odd length r from 1 to $2^{n-1}-1$. Then we set $P_1 = \langle \mathbf{x}, R_r, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, I^{(1)}_{q+l-k,q}, \mathbf{z}\rangle$ with $r+q = k-1$ and set $P_2 = I^{(2)}_{q+l-k,q}$. See Fig. 6(h).

Now we consider the case that $\ell(P_2) = l-k \geq 2^{n-1} - d_{Q_n}(\mathbf{x},\mathbf{z}) + 2$. Let $m_1 = d_{Q_n}(\mathbf{x},\mathbf{z}) - 2$, $m_2 = 2^{n-1} - d_{Q_n}(\mathbf{x},\mathbf{z})$, and $A = \{(I^{(2)}_{m_1+m_2,m_1}(i), I^{(2)}_{m_1+m_2,m_1}(i+1)) \mid 1 \leq i \leq m_2 \text{ and } i \equiv 1 \pmod 2\}$. Obviously, we have $|A| = \lceil m_2/2 \rceil \geq \lceil (2^{n-1}-n)/2 \rceil$. Since $|A| \geq 6$ for $n \geq 5$, there exists an odd integer $\hat{\imath}$, $1 \leq \hat{\imath} \leq m_2$, such that $\{(I^{(2)}_{m_1+m_2,m_1}(\hat{\imath}))^{n-1}, (I^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1))^{n-1}\} \cap \{\mathbf{x}, (\mathbf{x})^j\} = \emptyset$. For convenience, let $\mathbf{u} = I^{(2)}_{m_1+m_2,m_1}(\hat{\imath})$ and $\mathbf{v} = I^{(2)}_{m_1+m_2,m_1}(\hat{\imath}+1)$. Accordingly, $I^{(2)}_{m_1+m_2,m_1}$ can be written as $\langle \mathbf{z}, I', \mathbf{u}, \mathbf{v}, I'', \mathbf{y}\rangle$. For simplicity, let $m'_1 = k-m_1-1$ and $m'_2 = l-k-m_2-1$.

By Lemma 3, Q_n^0 has two vertex-disjoint paths $S^{(1)}_{m'_1+m'_2,m'_1}$ and $S^{(2)}_{m'_1+m'_2,m'_1}$ such that $S^{(1)}_{m'_1+m'_2,m'_1}$ is an $[\mathbf{x}, (\mathbf{x})^j]$-path of length m'_1 and $S^{(2)}_{m'_1+m'_2,m'_1}$ is a $[(\mathbf{u})^{n-1}, (\mathbf{v})^{n-1}]$-path of length m'_2. Then we can set $P_1 = \langle \mathbf{x}, S^{(1)}_{m'_1+m'_2,m'_1}, (\mathbf{x})^j, ((\mathbf{x})^j)^{n-1}, I^{(1)}_{m_1+m_2,m_1}, \mathbf{z}\rangle$ and $P_2 = \langle \mathbf{z}, I', \mathbf{u}, (\mathbf{u})^{n-1}, S^{(2)}_{m'_1+m'_2,m'_1}, (\mathbf{v})^{n-1}, \mathbf{v}, I'', \mathbf{y}\rangle$. See Fig. 6(i).

In summary, P_1 is indeed an $[\mathbf{x}, \mathbf{z}]$-path of length k and P_2 is indeed an $[\mathbf{z}, \mathbf{y}]$-path of length $l - k$. Consequently, the proof is completed. □

According to Theorems 1 and 2, we have the following result.

Theorem 3. *The n-cube Q_n is bipanpositionably bipanconnected if $n \geq 4$.*

By Theorem 1, we have shown that Q_n is not only bipanconnected but also hyper-bipanconnected if $n \geq 2$. Moreover, it is also easy to prove that Q_n is bipancyclic and bipanpositionably Hamiltonian.

Corollary 1 ([15]). *The n-cube Q_n is bipancyclic if $n \geq 2$.*

Corollary 2 ([7]). *The n-cube Q_n is bipanpositionably Hamiltonian if $n \geq 2$.*

4. Conclusion

In this paper, we defined the bipanpositionable bipanconnectedness for bipartite graphs and discussed such property on hypercubes. That is, for any two vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n and for any vertex $\mathbf{z} \in V(Q_n) - \{\mathbf{x}, \mathbf{y}\}$, the Q_n contains an $[\mathbf{x}, \mathbf{y}]$-path $P_{l,k}$ of length l such that $P_{l,k}(1) = \mathbf{x}$, $P_{l,k}(k+1) = \mathbf{z}$, and $P_{l,k}(l+1) = \mathbf{y}$ for any integer l satisfying $h(\mathbf{x}, \mathbf{z}) + h(\mathbf{y}, \mathbf{z}) \leq l \leq 2^n - 1$ and $2|(l - h(\mathbf{x}, \mathbf{z}) - h(\mathbf{y}, \mathbf{z}))$ and for any integer k satisfying both $h(\mathbf{x}, \mathbf{z}) \leq k \leq l - h(\mathbf{y}, \mathbf{z})$ and $2|(k - h(\mathbf{x}, \mathbf{z}))$. In particular, path $P_{l,k}$ turns out to be a Hamiltonian path of Q_n while $l = 2^n - 1$. Recently, Lee et al. [8] presented a method to construct a Hamiltonian path in Q_n with a required vertex in a fixed position. It is noticed that their result is just a special case included in our addressed bipanpositionable bipanconnectedness. Therefore, our study can be thought of a generalization of the previous result. Based on the bipanpositionable bipanconnectedness of hypercubes, many other properties of hypercubes, such as bipancyclicity, bipanconnectedness, bipanpositionable Hamiltonicity, etc., can be easily derived. In other words, our study unifies the related researches in a general sense.

It is straightforward to define the panpositionable panconnectedness for the non-bipartite graphs. That is, a graph G is said to be panpositionably panconnected if, for any two distinct vertices x and y of G and for any vertex $z \in V(G) - \{x, y\}$, it contains a path $P_{l,k}$ of length l such that $P_{l,k}(1) = x$, $P_{l,k}(k+1) = z$, and $P_{l,k}(l+1) = y$ for each integer l from $d_G(x, z) + d_G(y, z)$ to $|V(G)| - 1$ and for each integer k from $d_G(x, z)$ to $l - d_G(y, z)$. Then it is also intriguing to address such issue on various kinds of non-bipartite network topologies.

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