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Nonlinear generalized synchronization of chaotic systems by pure error dynamics and elaborate nondiagonal Lyapunov function

Zheng-Ming Ge *, Ching-Ming Chang

Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan, ROC

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Abstract

By applying pure error dynamics and elaborate nondiagonal Lyapunov function, the nonlinear generalized synchronization is studied in this paper. Instead of current mixed error dynamics in which master state variables and slave state variables are presented, the nonlinear generalized synchronization can be obtained by pure error dynamics without auxiliary numerical simulation. The elaborate nondiagonal Lyapunov function is applied rather than current monotonous square sum Lyapunov function deeply weakening the powerfulness of Lyapunov direct method. Both autonomous and nonautonomous double Mathieu systems are used as examples with numerical simulations.

1. Introduction

In secure communication [1,2], biological systems [3,4], and many other fields [5–25], chaos synchronization has been widely used. Generalized synchronization is a complex type of chaos synchronization and gives rise to extensive investigations recently [26–33]. By applying pure error dynamics and elaborate nondiagonal Lyapunov function, the nonlinear generalized synchronization is studied in this paper.

The auxiliary numerical simulation is unavoidable for current mixed error dynamics in which master state variables and slave state variables are presented while their maximum values must be determined by simulation [34–38]. However, the pure error dynamics can be analyzed theoretically without additional numerical simulation. Besides, monotonous and self-limited square sum Lyapunov function, $V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$, is currently used [39–44], but Lyapunov function can be chosen in a variety of elaborate and ingenious forms for different systems. Restricting Lyapunov function to square sum makes a long river brooklike, deeply weakens the powerfulness of Lyapunov direct method. Instead of current plain square sum Lyapunov function, the elaborate nondiagonal Lyapunov function is applied in this paper.

* Corresponding author. Fax: +886 3 5720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

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A systematic method of designing Lyapunov function is proposed based on Lyapunov direct method [45]. Nonlinear generalized synchronization is achieved for both autonomous and nonautonomous double Mathieu systems by applying this technique. This paper is organized as follows. In Section 2, the method of designing Lyapunov function is presented, and nonlinear generalized synchronization is obtained. Section 3 contains the examples of autonomous and nonautonomous double Mathieu systems, and numerical simulations show the feasibility of the proposed method. Finally, the conclusions are drawn.

2. Design of Lyapunov function

Consider the master and slave nonlinear dynamic systems described by

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{2.1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \tag{2.2}$$

where $\mathbf{x}, \mathbf{y} \in R^n$ are master and slave state vectors, $\mathbf{f} : R_+ \times R^n \to R^n$ is a nonlinear vector function, and $\mathbf{u} : R_+ \times R^n \times R^n \to R^n$ is controller vector.

Generalized synchronization means that there is a functional relation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ between master and slave states as time goes to infinity, where $\mathbf{g} : R^n \to R^n$ is a continuously differentiable nonlinear vector function. Define $\mathbf{e} = \mathbf{y} - \mathbf{g}(\mathbf{x})$ as generalized synchronization error vector, and the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{g}}(\mathbf{x}) = \dot{\mathbf{y}} - \frac{d\mathbf{g}(\mathbf{x})}{d\mathbf{x}}\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{y}) - \frac{d\mathbf{g}(\mathbf{x})}{d\mathbf{x}}\mathbf{f}(t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}). \tag{2.3}$$

Eq. (2.3) can be rewritten in the following form:

$$\dot{\mathbf{e}} = \mathbf{p}(t, \mathbf{e}) + \mathbf{q}(t, \mathbf{x}, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \tag{2.4}$$

where $\mathbf{p} : R_+ \times R^n \to R^n$ and $\mathbf{q} : R_+ \times R^n \times R^n \to R^n$ are continuous vector functions represent the error variable terms and the state variable terms in the error dynamics, respectively.

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(t, \mathbf{x}, \mathbf{y}) = -\mathbf{q}(t, \mathbf{x}, \mathbf{y}) + \mathbf{v}(t, \mathbf{e}), \tag{2.5}$$

where $\mathbf{v} : R_+ \times R^n \to R^n$ is a continuous vector function.

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t, \mathbf{e}) + \mathbf{v}(t, \mathbf{e}). \tag{2.6}$$

Based on Lyapunov direct method [45], the scheme of nonlinear generalized synchronization and the procedure of designing elaborate nondiagonal Lyapunov function are described as follows:

Step 1 Construct a Lyapunov function

$$V(t, \mathbf{e}) = \sum_{i=1}^{n} \frac{1}{2}\mathbf{e}_i^{\mathrm{T}}\Lambda_i(t)\mathbf{e}_i = \left[\frac{1}{2}\lambda_{11}(t)e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}(t)e_2^2\right] + \cdots + \left[\frac{1}{2}\lambda_{nn}(t)e_n^2 + \lambda_{n1}e_ne_1 + \frac{1}{2}\lambda_{11}(t)e_1^2\right], \tag{2.7}$$

where $\mathbf{e}_i = \begin{bmatrix} e_i \\ e_{i+1} \end{bmatrix}$ $(i = 1, 2, \ldots, n-1)$, $\mathbf{e}_n = \begin{bmatrix} e_n \\ e_1 \end{bmatrix}$, $\mathbf{\Lambda}_i(t) = \begin{bmatrix} \lambda_{ii}(t) & \lambda_{ii+1} \\ \lambda_{ii+1} & \lambda_{i+1i+1}(t) \end{bmatrix}$ $(i = 1, 2, \ldots, n-1)$, $\mathbf{\Lambda}_n(t) = \begin{bmatrix} \lambda_{nn}(t) & \lambda_{n1} \\ \lambda_{n1} & \lambda_{11}(t) \end{bmatrix}$, and $\mathbf{\Lambda}_i(t) \in R^{2\times 2}$ $(i = 1, 2, \ldots, n)$ are unknown continuously differentiable positive definite matrices to be designed and $\mathbf{\Lambda}_i(t)$, $\mathbf{\Lambda}_n(t)$ are nondiagonal. According to Sylvester's criterion, $\mathbf{\Lambda}_i(t)$ have to be chosen that

$$\begin{aligned} \forall t \geqslant 0, \quad & \lambda_{ii}(t) > 0, \lambda_{ii}(t)\lambda_{i+1i+1}(t) - \lambda_{ii+1}^2 > 0 \quad (i = 1, 2, \ldots, n-1), \\ & \lambda_{nn}(t) > 0, \ \ \lambda_{nn}(t)\lambda_{11}(t) - \lambda_{n1}^2 > 0 \quad (i = n) \end{aligned} \tag{2.8}$$

and

$$\forall t \geqslant 0, \quad 0 < \lambda_{mii} \leqslant \lambda_{ii}(t) \leqslant \lambda_{Mii} \quad (i = 1, 2, \ldots, n), \tag{2.9}$$

where λ_{mii}, λ_{Mii} are positive constants.

Step 2 The derivative of Lyapunov function is

$$\begin{aligned}\dot{V}(t,\mathbf{e}) &= \sum_{i=1}^{n}\left[\dot{\mathbf{e}}_i^{\mathrm{T}}\mathbf{\Lambda}_i(t)\mathbf{e}_i + \frac{1}{2}\mathbf{e}_i^{\mathrm{T}}\dot{\mathbf{\Lambda}}_i(t)\mathbf{e}_i\right] \\ &= \left[\lambda_{11}(t)e_1\dot{e}_1 + \lambda_{12}\dot{e}_1e_2 + \lambda_{12}e_1\dot{e}_2 + \lambda_{22}(t)e_2\dot{e}_2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2\right] + \cdots \\ &\quad + \left[\lambda_{nn}(t)e_n\dot{e}_n + \lambda_{n1}\dot{e}_ne_1 + \lambda_{n1}e_n\dot{e}_1 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{nn}(t)e_n^2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2\right]. \qquad (2.10)\end{aligned}$$

Eq. (2.10) can be rewritten in the following form:

$$\begin{aligned}\dot{V}(t,\mathbf{e}) &= F_1(\dot{\lambda}_{11},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_1^2 + \cdots + F_n(\dot{\lambda}_{nn},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_n^2 \\ &\quad + G_1(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_1e_2 + \cdots + G_m(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_{n-1}e_n \\ &\quad + (2\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{n1}v_n)e_1 + \cdots + (2\lambda_{nn}v_n + \lambda_{n1}v_1 + \lambda_{n-1n}v_{n-1})e_n, \qquad (2.11)\end{aligned}$$

where $F_i(\dot{\lambda}_{ii},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)$ $(i=1,2,\ldots,n)$, $G_j(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)$ $(j=1,2,\ldots,m,\ m=\frac{n(n-1)}{2})$ are continuous differentiable functions, and v_i $(i=1,2,\ldots,n)$ are controllers to be determined.

Step 3 Appropriately design the controllers v_i such that Eq. (2.11) can be reduced to

$$\begin{aligned}V(t,\mathbf{e}) &= \widehat{F}_1(\dot{\lambda}_{11},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_1^2 + \cdots + \widehat{F}_n(\dot{\lambda}_{nn},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_n^2 \\ &\quad + \hat{G}_1(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_1e_2 + \cdots + \hat{G}_m(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)e_{n-1}e_n, \qquad (2.12)\end{aligned}$$

where $\widehat{F}_i(\dot{\lambda}_{ii},\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)$ $(i=1,2,\ldots,n)$ and $\widehat{G}_j(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t)$ $(j=1,2,\ldots,m,m=\frac{n(n-1)}{2})$ are continuous differentiable functions.

Step 4 Assume

$$\forall j,\quad \widehat{G}_j(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t) = 0, \qquad (2.13)$$

then the relationship between λ_{ij} can be obtained.

Step 5 Use the results of step 4 to check if

$$\forall t \geqslant 0,\quad \widehat{F}_i(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t) < 0 \quad (i=1,2,\ldots,n). \qquad (2.14)$$

Step 6 If Eq. (2.14) can be satisfied, the conditions derived from Eq. (2.14) can be obtained. If Eq. (2.14) cannot be satisfied, i.e.,

$$\begin{aligned}\forall t \geqslant 0,\quad &\widehat{F}_j(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t) \geqslant 0 \\ &\widehat{F}_k(\lambda_{11},\ldots,\lambda_{nn},\lambda_{12},\ldots,\lambda_{n1},t) < 0\end{aligned} \qquad (2.15)$$

return to step 3 and modify the controllers v_j by addition of k_je_j, where k_j are constant gains to be determined. Repeat steps 4 and 5, then the conditions guarantee the validity of Eq. (2.14) can be assured.

Step 7 Appropriately design k_j and $\lambda_{ij}(t)$ such that each condition derived from the above procedure holds. Finally, the elaborate nondiagonal Lyapunov function can be obtained and the generalized synchronization is achieved according to Lyapunov direct method.

3. Nonlinear generalized synchronization of double Mathieu systems

In this section, the nonlinear functional relation between master and slave states is $y_i = g_i(x_i) = \alpha x_i^2 + \beta x_i + \gamma$ $(i=1,2,\ldots,n)$. To demonstrate the use of the proposed method, two examples of autonomous and nonautonomous double Mathieu systems are presented.

3.1. Regular and chaotic dynamics of autonomous and nonautonomous double Mathieu systems

The nonlinear damped Mathieu system is [46,47]

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2.\end{aligned} \qquad (3.1)$$

An autonomous double Mathieu system can be constructed by mutual linear coupling of two Mathieu systems:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+x_4)x_1 - (1+x_4)x_1^3 - ax_2 + bx_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -(1+x_2)x_3 - a(1+x_2)x_3^3 - ax_4 + bx_1.
\end{aligned} \tag{3.2}$$

The parameters in simulation are $a = 0.5$, $b = 1$–1.254, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$. The phase portraits, Poincaré Maps, bifurcation diagram, and Lyapunov exponents are shown in Fig. 1. It can be observed that the motion is period 1 for $b = 1.1$, period 4 for $b = 1.243$, and period 8 for $b = 1.246$. For $b = 1.24$, the motion is chaotic.

A nonautonomous double Mathieu system [48] can also be constructed by mutual linear coupling of two Mathieu systems:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + bx_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -(1+\sin\omega t)x_3 - a(1+\sin\omega t)x_3^3 - ax_4 + bx_1.
\end{aligned} \tag{3.3}$$

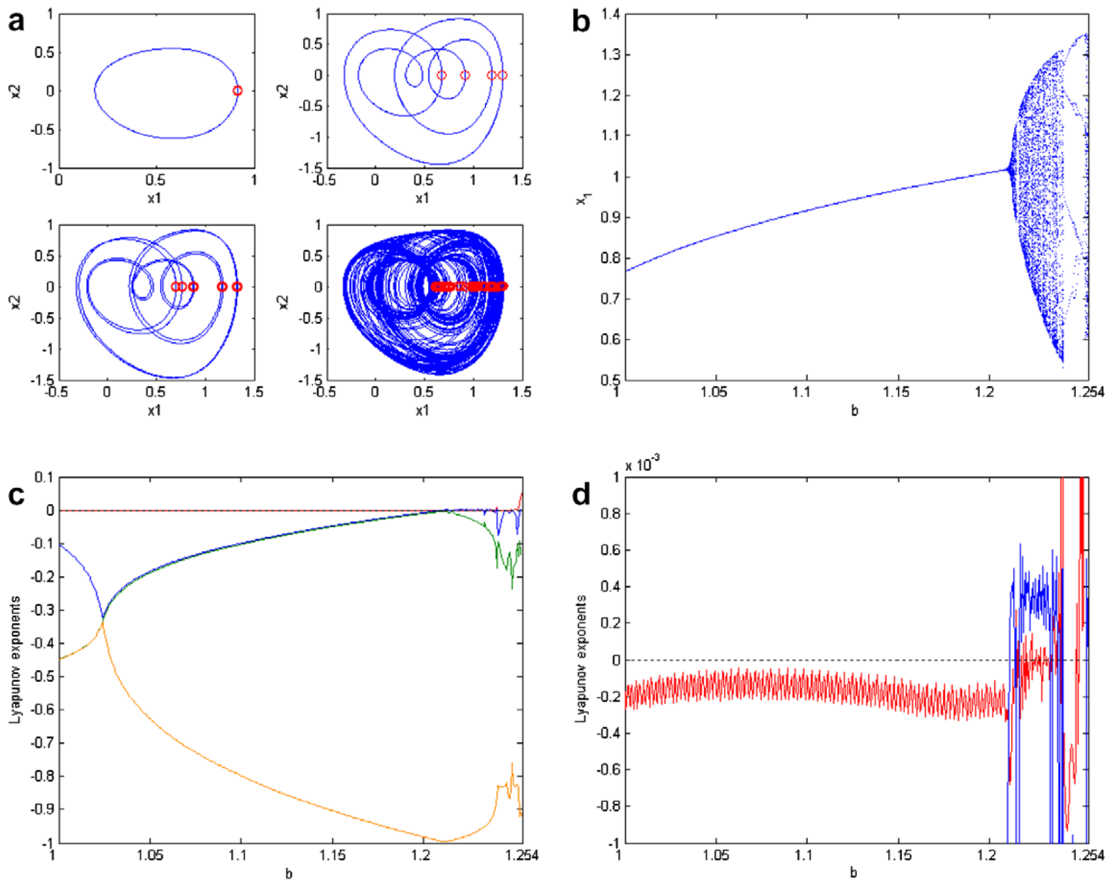

Fig. 1. (a) Phase portraits and Poincaré maps; (b) bifurcation diagram; (c) Lyapunov exponents; (d) local enlargement of Lyapunov exponents for autonomous double Mathieu system.

The parameters in simulation are $a = 0.5$, $b = 0.9$–1, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$. The phase portraits, Poincaré Maps, bifurcation diagram, and Lyapunov exponents are shown in Fig. 2. It can be observed that the motion is period 1 for $b = 0.9$, period 2 for $b = 0.93$, and period 4 for $b = 0.934$. For $b = 1$, the motion is chaotic.

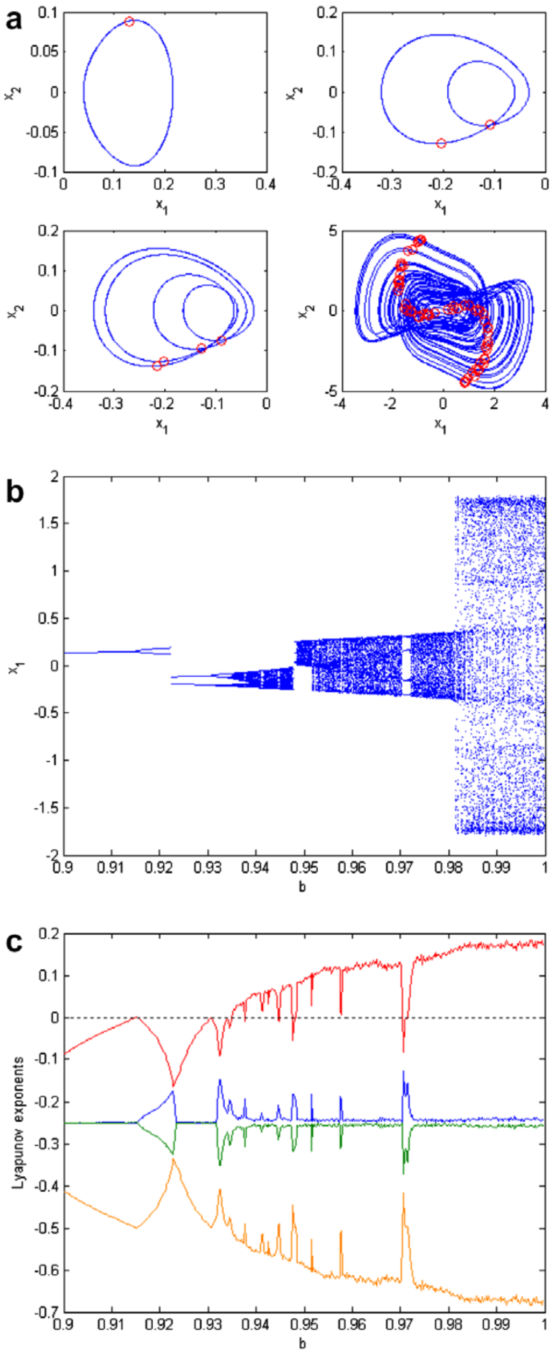

Fig. 2. (a) Phase portraits and Poincaré maps; (b) bifurcation diagram; (c) Lyapunov exponents for nonautonomous double Mathieu system.

3.2. Nonlinear generalized synchronization of autonomous double Mathieu systems

The master and slave autonomous double Mathieu systems can be described by

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a(1+x_4)x_1 - (1+x_4)x_1^3 - ax_2 + bx_3,\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= -(1+x_2)x_3 - a(1+x_2)x_3^3 - ax_4 + bx_1.
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\dot{y}_1 &= y_2 + u_1,\\
\dot{y}_2 &= -a(1+y_4)y_1 - (1+y_4)y_1^3 - ay_2 + by_3 + u_2,\\
\dot{y}_3 &= y_4 + u_3,\\
\dot{y}_4 &= -(1+y_2)y_3 - a(1+y_2)y_3^3 - ay_4 + by_1 + u_4.
\end{aligned} \tag{3.5}$$

The parameters in simulation are $a = 0.5$, $b = 1.24$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $y_1(0) = 0.3$, $y_2(0) = 0.3$, $y_3(0) = 0.4$, $y_4(0) = 0.4$.

Let $e_i = y_i - \alpha x_i^2 - \beta x_i - \gamma$ $(i = 1, \ldots, 4)$ and subtract Eq. (3.4) from Eq. (3.5), then the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(\mathbf{e}) + \mathbf{q}(\mathbf{x},\mathbf{y}) + \mathbf{u}(\mathbf{x},\mathbf{y}), \tag{3.6}$$

where

$$\begin{aligned}
\mathbf{p}(\mathbf{e}) &= [\,p_1(\mathbf{e}) \quad p_2(\mathbf{e}) \quad p_3(\mathbf{e}) \quad p_4(\mathbf{e})\,]^{\mathrm{T}},\\
\mathbf{q}(\mathbf{x},\mathbf{y}) &= [\,q_1(\mathbf{x},\mathbf{y}) \quad q_2(\mathbf{x},\mathbf{y}) \quad q_3(\mathbf{x},\mathbf{y}) \quad q_4(\mathbf{x},\mathbf{y})\,]^{\mathrm{T}},\\
p_1(\mathbf{e}) &= e_2,\\
p_2(\mathbf{e}) &= -ae_1 - ae_2 + be_3,\\
p_3(\mathbf{e}) &= e_4,\\
p_4(\mathbf{e}) &= -e_3 - ae_4 + be_1,\\
q_1(\mathbf{x},\mathbf{y}) &= \alpha x_2^2 - 2\alpha x_1 x_2 + \gamma,\\
q_2(\mathbf{x},\mathbf{y}) &= -\alpha(ax_1^2 - ax_2^2 - bx_3^2) - a(y_4 y_1 - \beta x_4 x_1) + (b-2a)\gamma - [(1+y_4)y_1^3 - \beta(1+x_4)x_1^3]\\
&\quad + 2\alpha x_2[a(1+x_4)x_1 + (1+x_4)x_1^3 - bx_3],\\
q_3(\mathbf{x},\mathbf{y}) &= \alpha x_4^2 - 2\alpha x_3 x_4 + \gamma,\\
q_4(\mathbf{x},\mathbf{y}) &= -\alpha(x_3^2 - ax_4^2 - bx_1^2) - (y_2 y_3 - \beta x_2 x_3) + (b-a-1)\gamma - a[(1+y_2)y_3^3 - \beta(1+x_2)x_3^3]\\
&\quad + 2\alpha x_4[(1+x_2)x_3 + a(1+x_2)x_3^3 - bx_1].
\end{aligned} \tag{3.7}$$

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(\mathbf{x},\mathbf{y}) = -\mathbf{q}(\mathbf{x},\mathbf{y}) + \mathbf{v}(\mathbf{e}). \tag{3.8}$$

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(\mathbf{e}) + \mathbf{v}(\mathbf{e}). \tag{3.9}$$

Step 1 Construct a Lyapunov function

$$V(\mathbf{e}) = \sum_{i=1}^{4} \frac{1}{2}\mathbf{e}_i^{\mathrm{T}}\mathbf{\Lambda}_i\mathbf{e}_i = \left[\frac{1}{2}\lambda_{11}e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}e_2^2\right] + \cdots + \left[\frac{1}{2}\lambda_{44}e_4^2 + \lambda_{41}e_4e_1 + \frac{1}{2}\lambda_{44}e_4^2\right], \tag{3.10}$$

where $\mathbf{\Lambda}_i$ are unknown continuously differentiable positive definite nondiagonal matrices to be designed. According to Sylvester's criterion, $\mathbf{\Lambda}_i$ have to be chosen that

$$\begin{aligned}
&\lambda_{11} > 0, \quad \lambda_{11}\lambda_{22} - \lambda_{12}^2 > 0,\\
&\lambda_{22} > 0, \quad \lambda_{22}\lambda_{33} - \lambda_{23}^2 > 0,\\
&\lambda_{33} > 0, \quad \lambda_{33}\lambda_{44} - \lambda_{34}^2 > 0,\\
&\lambda_{44} > 0, \quad \lambda_{44}\lambda_{11} - \lambda_{41}^2 > 0.
\end{aligned} \tag{3.11}$$

Step 2 The derivative of Lyapunov function is

$$\dot{V}(\mathbf{e}) = \sum_{i=1}^{4} \dot{\mathbf{e}}_i^{\mathrm{T}} \mathbf{\Lambda}_i \mathbf{e}_i = [\lambda_{11} e_1 \dot{e}_1 + \lambda_{12} \dot{e}_1 e_2 + \lambda_{12} e_1 \dot{e}_2 + \lambda_{22} e_2 \dot{e}_2] + \cdots + [\lambda_{44} e_4 \dot{e}_4 + \lambda_{41} \dot{e}_4 e_1 + \lambda_{41} e_4 \dot{e}_1 + \lambda_{11} e_1 \dot{e}_1]. \tag{3.12}$$

Eq. (3.12) can be rewritten in the following form:

$$\begin{aligned}\dot{V}(\mathbf{e}) = {} & F_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1^2 + F_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2^2 + F_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_3^2 \\ & + F_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_4^2 + G_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_2 + G_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_3 \\ & + G_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_4 + G_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2e_3 + G_5(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2e_4 \\ & + G_6(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_3e_4 + (2\lambda_{11}v_1 + \lambda_{12}v_2 + \lambda_{41}v_4)e_1 + (2\lambda_{22}v_2 + \lambda_{23}v_3 + \lambda_{12}v_1)e_2 \\ & + (2\lambda_{33}v_3 + \lambda_{34}v_4 + \lambda_{23}v_2)e_3 + (2\lambda_{44}v_4 + \lambda_{41}v_1 + \lambda_{34}v_3)e_4,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &F_1(\lambda_{11},\ldots,\lambda_{41}) = -a\lambda_{12} + b\lambda_{41}, \\ &F_2(\lambda_{11},\ldots,\lambda_{41}) = \lambda_{12} - 2a\lambda_{22}, \\ &F_3(\lambda_{11},\ldots,\lambda_{41}) = b\lambda_{23} - \lambda_{34}, \\ &F_4(\lambda_{11},\ldots,\lambda_{41}) = \lambda_{34} - 2a\lambda_{44}, \\ &G_1(\lambda_{11},\ldots,\lambda_{41}) = 2\lambda_{11} - a\lambda_{12} - 2a\lambda_{22}, \\ &G_2(\lambda_{11},\ldots,\lambda_{41}) = b\lambda_{12} - a\lambda_{23} + b\lambda_{34} - \lambda_{41}, \\ &G_3(\lambda_{11},\ldots,\lambda_{41}) = 2b\lambda_{44} - a\lambda_{41}, \\ &G_4(\lambda_{11},\ldots,\lambda_{41}) = 2b\lambda_{22} - a\lambda_{23}, \\ &G_5(\lambda_{11},\ldots,\lambda_{41}) = \lambda_{23} + \lambda_{41}, \\ &G_6(\lambda_{11},\ldots,\lambda_{41}) = 2\lambda_{33} - a\lambda_{34} - 2\lambda_{44}. \end{aligned} \tag{3.14}$$

Step 3 Design the controllers

$$\begin{aligned} &v_1 = -e_2, \\ &v_2 = ae_1, \\ &v_3 = -e_4, \\ &v_4 = e_3, \end{aligned} \tag{3.15}$$

such that Eq. (3.13) can be reduced to

$$\begin{aligned}\dot{V}(\mathbf{e}) = {} & \widehat{F}_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1^2 + \widehat{F}_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2^2 + \widehat{F}_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_3^2 \\ & + \widehat{F}_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_4^2 + \widehat{G}_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_2 + \widehat{G}_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_3 \\ & + \widehat{G}_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_1e_4 + \widehat{G}_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2e_3 + \widehat{G}_5(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_2e_4 \\ & + \widehat{G}_6(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41})e_3e_4,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &\widehat{F}_1(\lambda_{11},\ldots,\lambda_{41}) = b\lambda_{41}, \\ &\widehat{F}_2(\lambda_{11},\ldots,\lambda_{41}) = -2a\lambda_{22}, \\ &\widehat{F}_3(\lambda_{11},\ldots,\lambda_{41}) = b\lambda_{23}, \\ &\widehat{F}_4(\lambda_{11},\ldots,\lambda_{41}) = -2a\lambda_{44}, \\ &\widehat{G}_1(\lambda_{11},\ldots,\lambda_{41}) = -a\lambda_{12}, \\ &\widehat{G}_2(\lambda_{11},\ldots,\lambda_{41}) = b\lambda_{12} + b\lambda_{34}, \\ &\widehat{G}_3(\lambda_{11},\ldots,\lambda_{41}) = 2b\lambda_{44} - a\lambda_{41}, \\ &\widehat{G}_4(\lambda_{11},\ldots,\lambda_{41}) = 2b\lambda_{22} - a\lambda_{23}, \\ &\widehat{G}_5(\lambda_{11},\ldots,\lambda_{41}) = 0, \\ &\widehat{G}_6(\lambda_{11},\ldots,\lambda_{41}) = -a\lambda_{34}. \end{aligned} \tag{3.17}$$

Step 4 Assume

$$\forall j, \quad \widehat{G}_j(\lambda_{11}, \ldots, \lambda_{41}) = 0, \tag{3.18}$$

then the relationship between λ_{ij} can be obtained:

$$\lambda_{12} = 0, \quad \lambda_{23} = \frac{b}{2a}\lambda_{22}, \quad \lambda_{34} = 0, \quad \lambda_{41} = \frac{b}{2a}\lambda_{44}. \tag{3.19}$$

Step 5 Use the results of step 4 to check if

$$\widehat{F}_i(\lambda_{11}, \ldots, \lambda_{41}) < 0 \quad (i = 1, \ldots, 4). \tag{3.20}$$

It can be obtained that

$$\begin{aligned} &\widehat{F}_1(\lambda_{11}, \ldots, \lambda_{41}) = b\lambda_{41} > 0, \\ &\widehat{F}_2(\lambda_{11}, \ldots, \lambda_{41}) = -2a\lambda_{22} < 0, \\ &\widehat{F}_3(\lambda_{11}, \ldots, \lambda_{41}) = b\lambda_{23} > 0, \\ &\widehat{F}_4(\lambda_{11}, \ldots, \lambda_{41}) = -2a\lambda_{44} < 0. \end{aligned} \tag{3.21}$$

Step 6 Since Eq. (3.20) is not satisfied, i.e.,

$$\begin{aligned} &\widehat{F}_j(\lambda_{11}, \ldots, \lambda_{41}) \geqslant 0 \quad (j = 1, 3), \\ &\widehat{F}_k(\lambda_{11}, \ldots, \lambda_{41}) < 0 \quad (k = 2, 4), \end{aligned} \tag{3.22}$$

return to step 3 and modify the controllers v_1 and v_3 by addition of k_1e_1 and k_3e_3, respectively, where k_1 and k_3 are constant gains to be determined. Because $\dot{V}$ has been modified, Eq. (3.17) becomes

$$\begin{aligned} &\widehat{F}_1(\lambda_{11}, \ldots, \lambda_{41}) = b\lambda_{41} + 2k_1\lambda_{11}, \\ &\widehat{F}_2(\lambda_{11}, \ldots, \lambda_{41}) = -2a\lambda_{22}, \\ &\widehat{F}_3(\lambda_{11}, \ldots, \lambda_{41}) = b\lambda_{23} + 2k_3\lambda_{33}, \\ &\widehat{F}_4(\lambda_{11}, \ldots, \lambda_{41}) = -2a\lambda_{44}, \\ &\widehat{G}_1(\lambda_{11}, \ldots, \lambda_{41}) = (k_1 - a)\lambda_{12}, \\ &\widehat{G}_2(\lambda_{11}, \ldots, \lambda_{41}) = b\lambda_{12} + b\lambda_{34}, \\ &\widehat{G}_3(\lambda_{11}, \ldots, \lambda_{41}) = 2b\lambda_{44} + (k_1 - a)\lambda_{41}, \\ &\widehat{G}_4(\lambda_{11}, \ldots, \lambda_{41}) = 2b\lambda_{22} + (k_3 - a)\lambda_{23}, \\ &\widehat{G}_5(\lambda_{11}, \ldots, \lambda_{41}) = 0, \\ &\widehat{G}_6(\lambda_{11}, \ldots, \lambda_{41}) = (k_3 - a)\lambda_{34}. \end{aligned} \tag{3.23}$$

Repeat steps 4 and 5, then the relationship between λ_{ij} becomes

$$\lambda_{12} = 0, \quad \lambda_{22} = \frac{a - k_3}{2b}\lambda_{23}, \quad \lambda_{34} = 0, \quad \lambda_{44} = \frac{a - k_1}{2b}\lambda_{41} \tag{3.24}$$

and Eq. (3.20) can be satisfied if

$$\lambda_{41} < \frac{-2k_1}{b}\lambda_{11}, \quad \lambda_{23} < \frac{-2k_3}{b}\lambda_{33}. \tag{3.25}$$

Step 7 The conditions derived from the above procedure can be summed up as follows:

$$\lambda_{12} = 0, \quad \lambda_{34} = 0, \tag{3.26}$$

$$\lambda_{11} > 0, \quad \lambda_{44} > 0, \quad \lambda_{44}\lambda_{11} - \lambda_{41}^2 > 0, \quad \lambda_{44} = \frac{a - k_1}{2b}\lambda_{41}, \quad \lambda_{41} < \frac{-2k_1}{b}\lambda_{11}, \tag{3.27}$$

$$\lambda_{22} > 0, \quad \lambda_{33} > 0, \quad \lambda_{22}\lambda_{33} - \lambda_{23}^2 > 0, \quad \lambda_{22} = \frac{a - k_3}{2b}\lambda_{23}, \quad \lambda_{23} < \frac{-2k_3}{b}\lambda_{33}. \tag{3.28}$$

Design

$$\begin{aligned}
&k_1 = -a, \quad k_3 = -a,\\
&\lambda_{11} = b, \quad \lambda_{22} = \frac{a^2}{2b}, \quad \lambda_{33} = b, \quad \lambda_{44} = \frac{a^2}{2b},\\
&\lambda_{12} = 0, \quad \lambda_{23} = \frac{a}{2}, \quad \lambda_{34} = 0, \quad \lambda_{41} = \frac{a}{2},
\end{aligned} \tag{3.29}$$

such that each condition holds. Then the elaborate nondiagonal Lyapunov function can be obtained

$$V(\mathbf{e}) = \frac{a^2}{2b}e_2^2 + \frac{a}{2}e_2e_3 + be_3^2 + \frac{a^2}{2b}e_4^2 + \frac{a}{2}e_4e_1 + be_1^2 \tag{3.30}$$

and

$$\dot{V}(\mathbf{e}) = -\frac{3ab}{2}e_1^2 - \frac{a^3}{b}e_2^2 - \frac{3ab}{2}e_3^2 - \frac{a^3}{b}e_4^2. \tag{3.31}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha = 1$, $\beta = 2$, $\gamma = 3$ are chosen in simulation, and the results are shown in Fig. 3.

3.3. Nonlinear generalized synchronization of nonautonomous double Mathieu systems

The master and slave nonautonomous double Mathieu systems can be described by

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + bx_3,\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= -(1+\sin\omega t)x_3 - a(1+\sin\omega t)x_3^3 - ax_4 + bx_1.
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\dot{y}_1 &= y_2 + u_1,\\
\dot{y}_2 &= -a(1+\sin\omega t)y_1 - (1+\sin\omega t)y_1^3 - ay_2 + by_3 + u_2,\\
\dot{y}_3 &= y_4 + u_3,\\
\dot{y}_4 &= -(1+\sin\omega t)y_3 - a(1+\sin\omega t)y_3^3 - ay_4 + by_1 + u_4.
\end{aligned} \tag{3.33}$$

The parameters in simulation are $a = 0.5$, $b = 1$, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $x_4(0) = 0.2$, $y_1(0) = 0.3$, $y_2(0) = 0.3$, $y_3(0) = 0.4$, $y_4(0) = 0.4$.

Let $e_i = y_i - \alpha x_i^2 - \beta x_i - \gamma$ $(i = 1, \ldots, 4)$ and subtract Eq. (3.32) from Eq. (3.33), then the error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t,\mathbf{e}) + \mathbf{q}(t,\mathbf{x},\mathbf{y}) + \mathbf{u}(t,\mathbf{x},\mathbf{y}), \tag{3.34}$$

where

$$\begin{aligned}
\mathbf{p}(t,\mathbf{e}) &= [\,p_1(t,\mathbf{e}) \quad p_2(t,\mathbf{e}) \quad p_3(t,\mathbf{e}) \quad p_4(t,\mathbf{e})\,]^{\mathrm{T}},\\
\mathbf{q}(t,\mathbf{x},\mathbf{y}) &= [\,q_1(t,\mathbf{x},\mathbf{y}) \quad q_2(t,\mathbf{x},\mathbf{y}) \quad q_3(t,\mathbf{x},\mathbf{y}) \quad q_4(t,\mathbf{x},\mathbf{y})\,]^{\mathrm{T}},\\
p_1(t,\mathbf{e}) &= e_2,\\
p_2(t,\mathbf{e}) &= -a(1+\sin\omega t)e_1 - ae_2 + be_3,\\
p_3(t,\mathbf{e}) &= e_4,\\
p_4(t,\mathbf{e}) &= -(1+\sin\omega t)e_3 - ae_4 + be_1,\\
q_1(t,\mathbf{x},\mathbf{y}) &= \alpha x_2^2 - 2\alpha x_1x_2 + \gamma,\\
q_2(t,\mathbf{x},\mathbf{y}) &= -\alpha[a(1+\sin\omega t)x_1^2 - ax_2^2 - bx_3^2] + 2\alpha(1+\sin\omega t)x_1^3x_2 + 2\alpha[a(1+\sin\omega t)x_1x_2 - bx_2x_3]\\
&\quad - (1+\sin\omega t)(y_1^3 - \beta x_1^3) - \gamma[a(1+\sin\omega t) + a - b],\\
q_3(t,\mathbf{x},\mathbf{y}) &= \alpha x_4^2 - 2\alpha x_3x_4 + \gamma,\\
q_4(t,\mathbf{x},\mathbf{y}) &= -\alpha[(1+\sin\omega t)x_3^2 - ax_4^2 - bx_1^2] + 2\alpha a(1+\sin\omega t)x_3^3x_4 + 2\alpha[(1+\sin\omega t)x_3x_4 - bx_1x_4]\\
&\quad - a(1+\sin\omega t)(y_3^3 - \beta x_3^3) - \gamma[1+\sin\omega t + a - b].
\end{aligned} \tag{3.35}$$

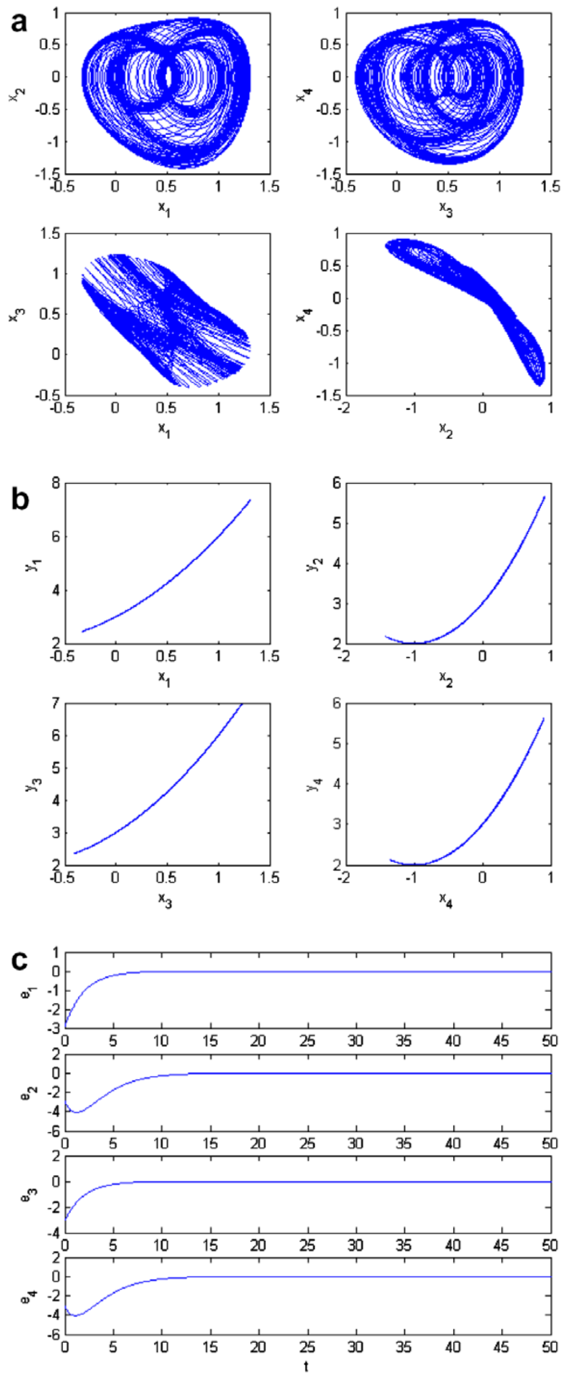

Fig. 3. (a) Phase portraits of master system; (b) phase portraits of x_i to y_i $(i = 1, \ldots, 4)$ when generalized synchronization is obtained; (c) time history of errors.

In order to transform current mixed error dynamics into pure error dynamics, the controller vector is chosen as

$$\mathbf{u}(t, \mathbf{x}, \mathbf{y}) = -\mathbf{q}(t, \mathbf{x}, \mathbf{y}) + \mathbf{v}(\mathbf{e}). \tag{3.36}$$

Now the pure error dynamics can be obtained:

$$\dot{\mathbf{e}} = \mathbf{p}(t, \mathbf{e}) + \mathbf{v}(\mathbf{e}). \tag{3.37}$$

Step 1 Construct a Lyapunov function

$$V(t,\mathbf{e}) = \sum_{i=1}^{4} \frac{1}{2}\mathbf{e}_i^{\mathrm{T}}\mathbf{\Lambda}_i(t)\mathbf{e}_i = \left[\frac{1}{2}\lambda_{11}(t)e_1^2 + \lambda_{12}e_1e_2 + \frac{1}{2}\lambda_{22}(t)e_2^2\right] + \cdots + \left[\frac{1}{2}\lambda_{44}(t)e_4^2 + \lambda_{41}e_4e_1 + \frac{1}{2}\lambda_{44}(t)e_4^2\right], \tag{3.38}$$

where $\mathbf{\Lambda}_i(t)$ are unknown continuously differentiable positive definite nondiagonal matrices to be designed. According to Sylvester's criterion, $\mathbf{\Lambda}_i(t)$ have to be chosen that

$$\begin{aligned}
&\lambda_{11}(t) > 0, \quad \lambda_{11}(t)\lambda_{22}(t) - \lambda_{12}^2 > 0,\\
&\lambda_{22}(t) > 0, \quad \lambda_{22}(t)\lambda_{33}(t) - \lambda_{23}^2 > 0,\\
&\lambda_{33}(t) > 0, \quad \lambda_{33}(t)\lambda_{44}(t) - \lambda_{34}^2 > 0,\\
&\lambda_{44}(t) > 0, \quad \lambda_{44}(t)\lambda_{11}(t) - \lambda_{41}^2 > 0
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
&0 < \lambda_{m11} \leqslant \lambda_{11}(t) \leqslant \lambda_{M11},\\
&0 < \lambda_{m22} \leqslant \lambda_{22}(t) \leqslant \lambda_{M22},\\
&0 < \lambda_{m33} \leqslant \lambda_{33}(t) \leqslant \lambda_{M33},\\
&0 < \lambda_{m44} \leqslant \lambda_{44}(t) \leqslant \lambda_{M44},
\end{aligned} \tag{3.40}$$

where λ_{mii}, λ_{Mii} $(i = 1,\ldots,4)$ are positive constants.

Step 2 The derivative of Lyapunov function is

$$\begin{aligned}
\dot{V}(t,\mathbf{e}) &= \sum_{i=1}^{4} \dot{\mathbf{e}}_i^{\mathrm{T}}\mathbf{\Lambda}_i(t)\mathbf{e}_i\\
&= \left[\lambda_{11}(t)e_1\dot{e}_1 + \lambda_{12}\dot{e}_1e_2 + \lambda_{12}e_1\dot{e}_2 + \lambda_{22}(t)e_2\dot{e}_2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2 + \frac{1}{2}\dot{\lambda}_{22}(t)e_2^2\right] + \cdots\\
&\quad + \left[\lambda_{44}(t)e_4\dot{e}_4 + \lambda_{41}\dot{e}_4e_1 + \lambda_{41}e_4\dot{e}_1 + \lambda_{11}(t)e_1\dot{e}_1 + \frac{1}{2}\dot{\lambda}_{44}(t)e_4^2 + \frac{1}{2}\dot{\lambda}_{11}(t)e_1^2\right].
\end{aligned} \tag{3.41}$$

Eq. (3.41) can be rewritten in the following form:

$$\begin{aligned}
\dot{V}(t,\mathbf{e}) &= F_1(\dot{\lambda}_{11},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1^2 + F_2(\dot{\lambda}_{22},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2^2\\
&\quad + F_3(\dot{\lambda}_{33},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_3^2 + F_4(\dot{\lambda}_{44},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_4^2\\
&\quad + G_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_2 + G_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_3\\
&\quad + G_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_4 + G_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2e_3\\
&\quad + G_5(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2e_4 + G_6(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_3e_4 + (2\lambda_{11}v_1 + \lambda_{12}v_2\\
&\quad + \lambda_{41}v_4)e_1 + (2\lambda_{22}v_2 + \lambda_{23}v_3 + \lambda_{12}v_1)e_2 + (2\lambda_{33}v_3 + \lambda_{34}v_4 + \lambda_{23}v_2)e_3 + (2\lambda_{44}v_4 + \lambda_{41}v_1 + \lambda_{34}v_3)e_4,
\end{aligned} \tag{3.42}$$

where

$$\begin{aligned}
&F_1(\dot{\lambda}_{11},\ldots,t) = \dot{\lambda}_{11} - a(1+\sin\omega t)\lambda_{12} + b\lambda_{41},\\
&F_2(\dot{\lambda}_{22},\ldots,t) = \dot{\lambda}_{22} - 2a\lambda_{22} + \lambda_{12},\\
&F_3(\dot{\lambda}_{33},\ldots,t) = \dot{\lambda}_{33} + b\lambda_{23} - (1+\sin\omega t)\lambda_{34},\\
&F_4(\dot{\lambda}_{44},\ldots,t) = \dot{\lambda}_{44} - 2a\lambda_{44} + \lambda_{34},\\
&G_1(\lambda_{11},\ldots,t) = 2\lambda_{11} - a\lambda_{12} - 2a(1+\sin\omega t)\lambda_{22},\\
&G_2(\lambda_{11},\ldots,t) = b\lambda_{12} - a(1+\sin\omega t)\lambda_{23} + b\lambda_{34} - (1+\sin\omega t)\lambda_{41},\\
&G_3(\lambda_{11},\ldots,t) = 2b\lambda_{44} - a\lambda_{41},\\
&G_4(\lambda_{11},\ldots,t) = 2b\lambda_{22} - a\lambda_{23},\\
&G_5(\lambda_{11},\ldots,t) = \lambda_{23} + \lambda_{41},\\
&G_6(\lambda_{11},\ldots,t) = 2\lambda_{33} - a\lambda_{34} - 2(1+\sin\omega t)\lambda_{44}.
\end{aligned} \tag{3.43}$$

Step 3 Design the controllers

$$\begin{aligned} v_1 &= ae_1, \\ v_2 &= -be_3 - ae_1, \\ v_3 &= ae_3, \\ v_4 &= -be_1 - e_3, \end{aligned} \tag{3.44}$$

such that Eq. (3.42) can be reduced to

$$\begin{aligned} \dot{V}(t,\mathbf{e}) = {} & \widehat{F}_1(\dot{\lambda}_{11},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1^2 + \widehat{F}_2(\dot{\lambda}_{22},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2^2 \\ & + \widehat{F}_3(\dot{\lambda}_{33},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_3^2 + \widehat{F}_4(\dot{\lambda}_{44},\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_4^2 \\ & + \widehat{G}_1(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_2 + \widehat{G}_2(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_3 \\ & + \widehat{G}_3(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_1e_4 + \widehat{G}_4(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2e_3 \\ & + \widehat{G}_5(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_2e_4 + \widehat{G}_6(\lambda_{11},\ldots,\lambda_{44},\lambda_{12},\ldots,\lambda_{41},t)e_3e_4, \end{aligned} \tag{3.45}$$

where

$$\begin{aligned} & \widehat{F}_1(\dot{\lambda}_{11},\ldots,t) = \dot{\lambda}_{11} + 2a\lambda_{11} - a(2+\sin\omega t)\lambda_{12}, \\ & \widehat{F}_2(\dot{\lambda}_{22},\ldots,t) = \dot{\lambda}_{22} - 2a\lambda_{22} + \lambda_{12}, \\ & \widehat{F}_3(\dot{\lambda}_{33},\ldots,t) = \dot{\lambda}_{33} + 2a\lambda_{33} - (2+\sin\omega t)\lambda_{34}, \\ & \widehat{F}_4(\dot{\lambda}_{44},\ldots,t) = \dot{\lambda}_{44} - 2a\lambda_{44} + \lambda_{34}, \\ & \widehat{G}_1(\lambda_{11},\ldots,t) = 2\lambda_{11} - 2a(1+\sin\omega t)\lambda_{22}, \\ & \widehat{G}_2(\lambda_{11},\ldots,t) = -a(2+\sin\omega t)\lambda_{23} - (2+\sin\omega t)\lambda_{41}, \\ & \widehat{G}_3(\lambda_{11},\ldots,t) = 0, \\ & \widehat{G}_4(\lambda_{11},\ldots,t) = 0, \\ & \widehat{G}_5(\lambda_{11},\ldots,t) = \lambda_{23} + \lambda_{41}, \\ & \widehat{G}_6(\lambda_{11},\ldots,t) = 2\lambda_{33} - 2(2+\sin\omega t)\lambda_{44}. \end{aligned} \tag{3.46}$$

Step 4 Assume

$$\forall j, \quad \widehat{G}_j(\lambda_{11},\ldots,t) = 0, \tag{3.47}$$

then the relationship between λ_{ij} can be obtained:

$$\lambda_{11} = a(2+\sin\omega t)\lambda_{22}, \quad \lambda_{23} = 0, \quad \lambda_{33} = (2+\sin\omega t)\lambda_{44}, \quad \lambda_{41} = 0. \tag{3.48}$$

Step 5 Use the results of step 4 to check if

$$\forall t \geqslant 0, \quad \widehat{F}_i(\dot{\lambda}_{ii},\ldots,t) < 0 \quad (i = 1,\ldots,4). \tag{3.49}$$

Assume

$$\lambda_{11} = c_1, \quad \lambda_{22} = \frac{c_1}{a(2+\sin\omega t)}, \quad \lambda_{33} = c_2, \quad \lambda_{44} = \frac{c_2}{2+\sin\omega t}, \tag{3.50}$$

where c_1 and c_2 are positive constants to be designed. Eq. (3.49) can be satisfied if the following conditions hold:

$$\begin{aligned} & \widehat{F}_1(\dot{\lambda}_{11},\ldots,t) = 2ac_1 - a(2+\sin\omega t)\lambda_{12} < 0, \\ & \widehat{F}_2(\dot{\lambda}_{22},\ldots,t) = \frac{-2c_1}{2+\sin\omega t} - \frac{c_1\omega\cos\omega t}{a(2+\sin\omega t)^2} + \lambda_{12} < 0, \end{aligned} \tag{3.51}$$

$$\begin{aligned} & \widehat{F}_3(\dot{\lambda}_{33},\ldots,t) = 2ac_2 - (2+\sin\omega t)\lambda_{34} < 0, \\ & \widehat{F}_4(\dot{\lambda}_{44},\ldots,t) = \frac{-2ac_2}{2+\sin\omega t} - \frac{c_2\omega\cos\omega t}{(2+\sin\omega t)^2} + \lambda_{34} < 0. \end{aligned} \tag{3.52}$$

However, both results of Eq. (3.51) and Eq. (3.52) show the contradiction: $\widehat{F}_1 < 0$ and $\widehat{F}_2 < 0$ can not hold in the same time, neither can $\widehat{F}_3 < 0$ and $\widehat{F}_4 < 0$. To simplify the following work, assume only $\widehat{F}_2 < 0$ and $\widehat{F}_4 < 0$ can hold.

Step 6 Since Eq. (3.49) is not satisfied, i.e.,

$$\begin{aligned}\forall t \geqslant 0, \quad & \widehat{F}_j(\lambda_{11}, \ldots, \lambda_{41}) \geqslant 0 \quad (j = 1, 3), \\ & \widehat{F}_k(\lambda_{11}, \ldots, \lambda_{41}) < 0 \quad (k = 2, 4),\end{aligned} \tag{3.53}$$

return to step 3 and modify the controllers v_1 and v_3 by addition of $k_1 e_1$ and $k_3 e_3$, respectively, where k_1 and k_3 are constant gains to be determined. Because $\dot{V}$ has been modified, Eq. (3.46) becomes

$$\begin{aligned}
&\widehat{F}_1(\dot{\lambda}_{11}, \ldots, t) = \dot{\lambda}_{11} + 2(a + k_1)\lambda_{11} - a(2 + \sin \omega t)\lambda_{12}, \\
&\widehat{F}_2(\dot{\lambda}_{22}, \ldots, t) = \dot{\lambda}_{22} - 2a\lambda_{22} + \lambda_{12}, \\
&\widehat{F}_3(\dot{\lambda}_{33}, \ldots, t) = \dot{\lambda}_{33} + 2(a + k_3)\lambda_{33} - (2 + \sin \omega t)\lambda_{34}, \\
&\widehat{F}_4(\dot{\lambda}_{44}, \ldots, t) = \dot{\lambda}_{44} - 2a\lambda_{44} + \lambda_{34}, \\
&\widehat{G}_1(\lambda_{11}, \ldots, t) = 2\lambda_{11} + k_1\lambda_{12} - 2a(1 + \sin \omega t)\lambda_{22}, \\
&\widehat{G}_2(\lambda_{11}, \ldots, t) = -a(2 + \sin \omega t)\lambda_{23} - (2 + \sin \omega t)\lambda_{41}, \\
&\widehat{G}_3(\lambda_{11}, \ldots, t) = k_1\lambda_{41}, \\
&\widehat{G}_4(\lambda_{11}, \ldots, t) = k_3\lambda_{23}, \\
&\widehat{G}_5(\lambda_{11}, \ldots, t) = \lambda_{23} + \lambda_{41} \\
&\widehat{G}_6(\lambda_{11}, \ldots, t) = 2\lambda_{33} + k_3\lambda_{34} - 2(2 + \sin \omega t)\lambda_{44}.
\end{aligned} \tag{3.54}$$

Repeat steps 4 and 5, then the relationship between λ_{ij} becomes

$$\begin{aligned}
&\lambda_{11} = a(2 + \sin \omega t)\lambda_{22} - \frac{k_1}{2}\lambda_{12}, \quad \lambda_{23} = 0, \\
&\lambda_{33} = (2 + \sin \omega t)\lambda_{44} - \frac{k_3}{2}\lambda_{34}, \quad \lambda_{41} = 0.
\end{aligned} \tag{3.55}$$

Assume

$$\begin{aligned}
&\lambda_{11} = c_1 - \frac{k_1}{2}c_3, \quad \lambda_{22} = \frac{c_1}{a(2 + \sin \omega t)}, \\
&\lambda_{33} = c_2 - \frac{k_3}{2}c_4, \quad \lambda_{44} = \frac{c_2}{2 + \sin \omega t}, \\
&\lambda_{12} = c_3, \quad \lambda_{34} = c_4,
\end{aligned} \tag{3.56}$$

where c_1, c_2, c_3, c_4 are constants to be designed, and c_1, c_2 are positive numbers. Eq. (3.49) can be satisfied if

$$\begin{aligned}
&2(a + k_1)c_1 < (k_1^2 + ak_1 + 2a + a\sin \omega t)c_3, \\
&(4a + 2a\sin \omega t + \omega \cos \omega t)c_1 > a(2 + \sin \omega t)^2 c_3, \\
&2(a + k_3)c_2 < (k_3^2 + ak_3 + 2 + \sin \omega t)c_4, \\
&(4a + 2a\sin \omega t + \omega \cos \omega t)c_2 > (2 + \sin \omega t)^2 c_4.
\end{aligned} \tag{3.57}$$

Step 7 The conditions derived from the above procedure can be summed up as follows:

$$\lambda_{23} = 0, \quad \lambda_{41} = 0, \tag{3.58}$$

$$\begin{aligned}
&c_1 > 0, \quad c_1 > \frac{k_1}{2}c_3, \\
&2(a + k_1)c_1 < (k_1^2 + ak_1 + 2a + a\sin \omega t)c_3, \\
&(4a + 2a\sin \omega t + \omega \cos \omega t)c_1 > a(2 + \sin \omega t)^2 c_3, \\
&(2c_1 - k_1c_3)c_1 > c_3^2(4a + 2a\sin \omega t).
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
&c_2 > 0, \quad c_2 > \frac{k_3}{2}c_4, \\
&2(a + k_3)c_2 < (k_3^2 + ak_3 + 2 + \sin \omega t)c_4, \\
&(4a + 2a\sin \omega t + \omega \cos \omega t)c_2 > (2 + \sin \omega t)^2 c_4, \\
&(2c_2 - k_3c_4)c_2 > c_4^2(4 + 2\sin \omega t).
\end{aligned} \tag{3.60}$$

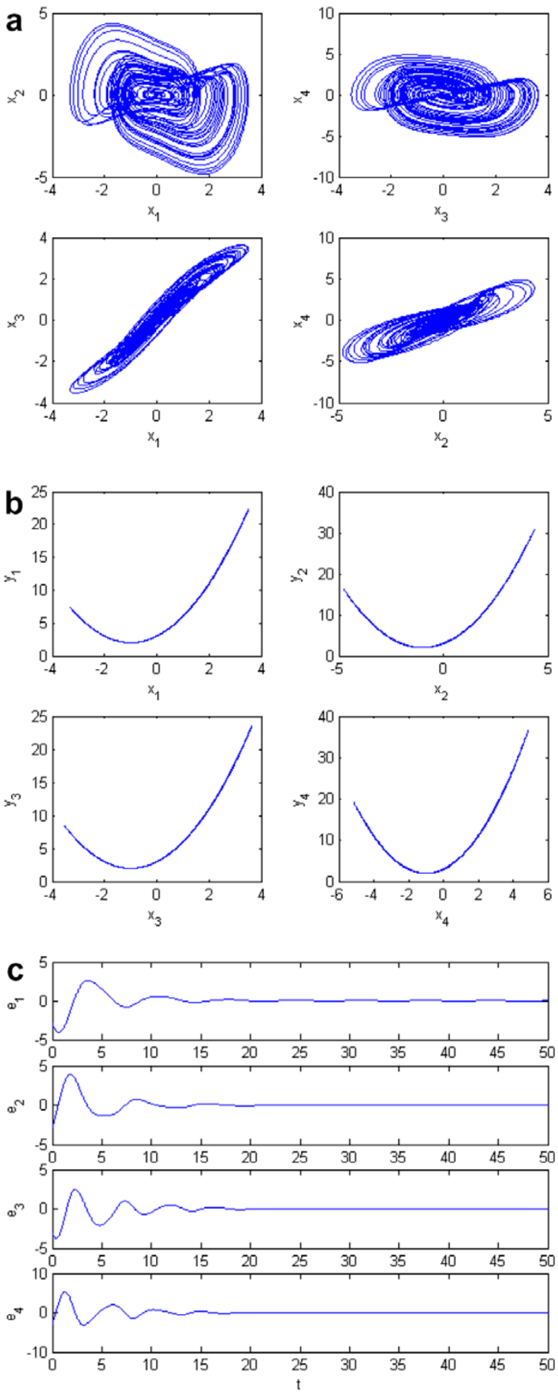

Fig. 4. (a) Phase portraits of master system; (b) phase portraits of x_i to y_i $(i = 1, \ldots, 4)$ when generalized synchronization is obtained; (c) time history of errors.

Design

$$\begin{aligned} &k_1 = -0.4, \quad k_3 = -0.4, \quad \lambda_{23} = 0, \quad \lambda_{41} = 0, \\ &c_1 = 20, \quad c_3 = 9 \quad \Rightarrow \lambda_{12} = 9, \quad \lambda_{11} = 21.8, \quad \lambda_{22} = \frac{20}{a(2 + \sin \omega t)}, \\ &c_2 = 50, \quad c_4 = 11 \Rightarrow \lambda_{34} = 11, \quad \lambda_{33} = 52.2, \quad \lambda_{44} = \frac{50}{2 + \sin \omega t}, \end{aligned} \tag{3.61}$$

such that each condition can be satisfied. Then the elaborate nondiagonal Lyapunov function can be obtained

$$V(t,\mathbf{e}) = 21.8e_1^2 + 9e_1e_2 + \frac{20}{a(2+\sin\omega t)}e_2^2 + 52.2e_3^2 + 11e_3e_4 + \frac{50}{2+\sin\omega t}e_4^2 \tag{3.62}$$

and

$$\begin{aligned}\dot{V}(t,\mathbf{e}) = & -(4.64+4.5\sin t)e_1^2 - \frac{44+4\sin t+40\cos t-9\sin^2 t}{(2+\sin t)^2}e_2^2 \\ & -(11.56+11\sin t)e_3^2 - \frac{56+6\sin t+50\cos t-11\sin^2 t}{(2+\sin t)^2}e_4^2.\end{aligned} \tag{3.63}$$

Since Lyapunov global asymptotical stability theorem is satisfied, the global generalized synchronization is achieved. $\alpha = 1$, $\beta = 2$, $\gamma = 3$ are chosen in simulation, and the results are shown in Fig. 4.

4. Conclusions

The nonlinear generalized synchronization is studied by applying pure error dynamics and elaborate nondiagonal Lyapunov function. This method gives a rigorous theory for generalized synchronization and greatly extends the use of various forms of Lyapunov function, while current method gives semi-simulation theory for generalized synchronization, which must get the maximum values of state variables by simulation, and use monotonous square sum Lyapunov function. By the systematic procedure, the complexity of designing suitable elaborate nondiagonal Lyapunov function is reduced greatly. The proposed method is effectively applied to both autonomous and nonautonomous double Mathieu systems.

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