國 立 交 通 大 學

應用數學系

碩　士　論　文

二維曲面上的花樣

Patterns Generation in Two-Dimensional Surfaces

研 究 生：胡文貴

指導老師：林松山　教授

中 華 民 國 九 十 四 年 六 月

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摘　　　　要

在這一篇碩士論文中，主要研究的是在二維曲面的花樣，這裡討論的二維曲面包含有圓柱(cylinder)、環面(torus)和球面(sphere)。研究的主要目的是去找這些二維曲面上花樣生成的遞迴公式。

Patterns Generation in Two-Dimensional Surfaces

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ABSTRACT

In this thesis, we study the patterns generation of two-dimensional surfaces. The surfaces include cylinder, torus and sphere. Our purpose is to find the recursive formulas for patterns generation in two-dimensional surfaces.

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目　　　　錄

1 Preliminaries

Since this study bases on some part of [4], we use some definitions and some results in [4]. Therefore we list them in following.

Let $\mathcal{S}$ be a finite set of p elements (symbols, colors or letters of an alphabet). Where $\mathbf{Z}^d$ denotes the integer lattice on $\mathbf{R}^d$, and $d \geq 1$ is a positive integer representing the lattice dimension. Then, function $U : \mathbf{Z}^d \to \mathcal{S}$ is called a global pattern. For each $\alpha \in \mathbf{Z}^d$, we write $U(\alpha)$ as u_α. The set of all patterns $U : \mathbf{Z}^d \to \mathcal{S}$ is denoted by

$$\Sigma_p^d \equiv \mathcal{S}^{\mathbf{Z}^d},$$

i.e., Σ_p^d is the set of all patterns with p different colors in d-dimensional lattice. As for local patterns, i.e., functions defined on (finite) sublattices, for a given d-tuple $N = (N_1, N_2, \cdots, N_d)$ of positive integers, let

$$\mathbf{Z}_N = \{(\alpha_1, \alpha_2, \cdots, \alpha_d) : 1 \leq \alpha_k \leq N_k, 1 \leq k \leq d\}$$

be an $N_1 \times N_2 \times \cdots N_d$ finite rectangular lattice. Denoted by $\widetilde{N} \geq N$ if $\widetilde{N_k} \geq N_k$ for all $1 \leq k \leq d$. The set of all local patterns defined on $\mathbf{Z}_N$ is denoted by

$$\Sigma_N \equiv \Sigma_{N,p} \equiv \{U|_{\mathbf{Z}_N} : U \in \Sigma_p^d\}.$$

Under many circumstances, only a(proper) subset $\mathcal{B}$ of Σ_N is admissible (allowable or feasible). In this case, local patterns in $\mathcal{B}$ are called basic patterns and $\mathcal{B}$ is called the basic set. In a one dimensional case, $\mathcal{S}$ consists of letters of an alphabet, and $\mathcal{B}$ is also called a set of allowable words of length N.

Consider a fixed finite lattice $\mathbf{Z}_N$ and a given basic set $\mathcal{B} \subset \Sigma_N$. For larger finite lattice $\mathbf{Z}_{\widetilde{N}} \supset \mathbf{Z}_N$, the set of all local patterns on $\mathbf{Z}_{\widetilde{N}}$ which can be generated by $\mathcal{B}$ is denoted as $\Sigma_{\widetilde{N}}(\mathcal{B})$. Indeed, $\Sigma_{\widetilde{N}}(\mathcal{B})$ can be characterized by

$$\begin{aligned}\Sigma_{\widetilde{N}}(\mathcal{B}) = \{ \quad & U \in \Sigma_{\widetilde{N}} : U_{\alpha+N} = V_N \; for\ any\ \alpha \in \mathbf{Z}^d \; with\ \mathbf{Z}_{\alpha+N} \subset \mathbf{Z}_{\widetilde{N}} \\ & and\ some\ V_N \in \mathcal{B}\},\end{aligned}$$

where

$$\alpha + N = \{(\alpha_1 + \beta_1, \cdots, \alpha_d + \beta_d) : (\beta_1, \cdots, \beta_d) \in N\},$$

and

$$U_{\alpha+N} = V_N \ means \ u_{\alpha+\beta} = v_\beta \ for \ each \ \beta \in \mathbf{Z}_N .$$

Similarly, the set of all global patterns which can be generated by $\mathcal{B}$ is denoted by

$$\Sigma(\mathcal{B}) = \{U \in \Sigma_p^d : U_{\alpha+N} = V_N \ for \ any \ \alpha \in \mathbf{Z}^d \ with \ some \ V_N \in \mathcal{B}\}.$$

For clarity, we begin by the studying two symbols, i.e., $\mathcal{S} = \{0, 1\}$. On a fixed finite lattice $\mathbf{Z}_{m_1 \times m_2}$, we first give a ordering $\chi = \chi_{m_1 \times m_2}$ on $\mathbf{Z}_{m_1 \times m_2}$ by

$$\chi((\alpha_1, \alpha_2)) = m_2(\alpha_1 - 1) + \alpha_2 \ ,$$

i.e.,

m_2	$2m_2$		$m_1 m_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$m_2 + 1$		$(m_1 - 1)m_2 + 1$

.

The ordering χ of (2.1) on $\mathbf{Z}_{m_1 \times m_2}$ can now be passed to $\Sigma_{m_1 \times m_2}$. Indeed, for each $U = (u_{\alpha_1,\alpha_2}) \in \Sigma_{m_1 \times m_2}$, define

$$\begin{aligned} \chi(U) \ &\equiv \ \chi_{m_1 \times m_2}(U) \\ &= \ 1 + \sum_{\alpha_1=1}^{m_1} \sum_{\alpha_2=1}^{m_2} u_{\alpha_1 \alpha_2} 2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)}. \end{aligned}$$

All patterns in $\Sigma_{2 \times n}$ can be arranged by the ordering matrix

$$\mathbf{X}_n = \left[\ x_{n;i_1 i_2} \ \right],$$

a $2^n \times 2^n$ matrix with entry $x_{n;i_1 i_2} = x_{n;i_1} \oplus x_{n;i_2}$, where $\chi(U_1) = i_1$ and $\chi(U_2) = i_2$, $1 \leq i_1, i_2 \leq 2^n$.

Theorem For any $n \geq 2$, $\Sigma_{2 \times n} = \{y_{j_1 \cdots j_n}\}$, where $y_{j_1 \cdots j_n}$ is given in (2.26). Furthermore, the ordering matrix $\mathbf{X}_n$ can be decomposed by n Z-maps successively as

$$\mathbf{X}_n = \begin{bmatrix} Y_{n;1} & Y_{n;2} \\ Y_{n;3} & Y_{n;4} \\ & \end{bmatrix},$$

$$Y_{n;j_1\cdots j_k} = \begin{bmatrix} Y_{n;j_1\cdots j_k 1} & Y_{n;j_1\cdots j_k 2} \\ Y_{n;j_1\cdots j_k 3} & Y_{n;j_1\cdots j_k 4} \end{bmatrix},$$

for $1 \leq k \leq n-2$, and

$$Y_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} y_{j_1\cdots j_{n-1}1} & y_{j_1\cdots j_{n-1}2} \\ y_{j_1\cdots j_{n-1}3} & y_{j_1\cdots j_{n-1}4} \end{bmatrix}.$$

From the proof of the above Theorem, we have

$$i_{n+1;1} = 2i_{n;1} - 1 + [\frac{j_{n+1}-1}{2}],$$

and

$$i_{n+1;2} = 2i_{n;2} - 1 + \{j_{n+1} - 1 - 2[\frac{j_{n+1}-1}{2}]\}.$$

Define

$$v_{j_1 j_2 \cdots j_n} = v_{j_1 j_2} v_{j_2 j_3} \cdots v_{j_{n-1} j_n},$$

and

$$\mathbf{H}_n = [v_{j_1 j_2 \cdots j_n}],$$

then the transition matrix $\mathbf{H}_n$ for $\mathcal{B}$ defined on $\mathbf{Z}_{2\times n}$ is a $2^n \times 2^n$ matrix with entries $v_{j_1\cdots j_n}$, which are either 1 or 0, by substituting $y_{j_1\cdots j_n}$ by $v_{j_1\cdots j_n}$ in $\mathbf{X}_n$.

For any two matrices $A = (a_{ij})$ and $B = (b_{kl})$, the Kronecker product (tensor product) of $A \otimes B$ is defined by

$$A \otimes B = (a_{ij}B).$$

On the other hand, for any two $n \times n$ matrices

$$C = (c_{ij}) \textit{ and } D = (d_{ij}),$$

where c_{ij} and d_{ij} are numbers or matrices. Then, Hadamard product of $C \circ D$ is defined by

$$C \circ D = (c_{ij} d_{ij}),$$

where the product $c_{ij} \cdot d_{ij}$ of c_{ij} and d_{ij} may be multiplication of numbers, numbers and matrices or matrices whenever it is well-defined. For instance, c_{ij} is number and d_{ij} is matrix.

Let $\mathbf{H}_2$ be a transition matrix. Then, for higher order transition matrices $\mathbf{H}_n$, $n \geq 3$, we have the following three equivalent expressions

(I) $\mathbf{H}_n$ can be decomposed into n successive 2×2matrices (or n-successive Z-maps) as follows:

$$\mathbf{H}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix},$$

$$H_{n;j_1 \cdots j_k} = \begin{bmatrix} H_{n;j_1 \cdots j_k 1} & H_{n;j_1 \cdots j_k 2} \\ H_{n;j_1 \cdots j_k 3} & H_{n;j_1 \cdots j_k 4} \end{bmatrix},$$

for $1 \leq k \leq n-2$ and

$$H_{n;j_1 \cdots j_{n-1}} = \begin{bmatrix} v_{j_1 \cdots j_{n-1} 1} & v_{j_1 \cdots j_{n-1} 2} \\ v_{j_1 \cdots j_{n-1} 3} & v_{j_1 \cdots j_{n-1} 4} \end{bmatrix}.$$

Furthermore,

$$H_{n;k} = \begin{bmatrix} v_{k1} H_{n-1;1} & v_{k2} H_{n-1;2} \\ v_{k3} H_{n-1;3} & v_{k4} H_{n-1;4} \end{bmatrix}.$$

(II) Starting from

$$\mathbf{H}_2 = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix},$$

with

$$H_k = \begin{pmatrix} v_{k1} & v_{k2} \\ v_{k3} & v_{k4} \end{pmatrix},$$

$\mathbf{H}_n$ can be obtained from $\mathbf{H}_{n-1}$ by replacing H_k by $H_k \circ \mathbf{H}_2$ according to (3.14).

(III)

$$\mathbf{H}_n = (H_{n-1})_{2^{n-1}\times 2^{n-1}} \circ \left(\begin{array}{c} E_{2^{n-2}} \quad \otimes \quad \left(\begin{array}{cc} H_1 & H_2 \\ H_3 & H_4 \end{array} \right) \\ \\ \end{array} \right),$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries.

2 The cylindrical patterns

In geometry, we can construct a cylinder by pasting one pair of subtenses of a rectangle. So, consider the cylindrical patterns as the finite two-dimensional patterns which have a periodic boundary condition in one direction. For clarity, we study two symbols, i.e., $S = \{0, 1\}$.

According to above, define

$$\sum\nolimits_{m_1\times m_2}^{c} = \left\{ U^c = (u_{\alpha_1\times\alpha_2}) \in \sum\nolimits_{m_1\times(m_2+1)} \mid u_{i,1} = u_{i,(m_2+1)}, 1 \leq i \leq m_1 \right\}$$

to represent the finite two-dimensional patterns which have periodic boundary condition in vertical direction.

(We only study the two-dimensional patterns which have periodic boundary condition in vertical direction, since the two-dimensional patterns which have periodic boundary condition in horizontal direction is similar)

We first define an ordering of patterns for $\sum^{c}_{m_1\times m_2}$as lexicographical ordering in one-dimensional case. On a fixed finite lattice $Z_{m_1\times(m_2+1)}$, we first give a ordering $\chi^c = \chi^c_{m_1\times m_2}$on $Z_{m_1\times(m_2+1)}$ by $\chi^c((\alpha_1, \alpha_2)) = (m_2 + 1)(\alpha_1 - 1) + \alpha_2$, i.e.,

m_2+1	$2(m_2+1)$	$\cdots$	$m_1(m_2+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$(m_2+1)+1$	$\cdots$	$(m_1-1)(m_2+1)+1$

.

The ordering χ^con $Z_{m_1\times(m_2+1)}$can now be passed to $\sum^c_{m_1\times m_2}$. Indeed, for each $U^c=(u_{\alpha_1,\alpha_2})\in\sum^c_{m_1\times m_2}$, define

$$\chi^c(U^c)=\chi^c_{m_1\times m_2}(U^c)=1+\sum_{1\le\alpha_1\le m_1}\sum_{1\le\alpha_2\le m_2}u_{\alpha_1,\alpha_2}2^{m_2(m_1-\alpha_1)+(m_2-\alpha_2)}.$$

Obviously, there is an one-to-one correspondence between local patterns in $\sum^c_{m_1\times m_2}$and positive integers in the set $N_{2^{m_1m_2}}=\{k\in N\mid 1\le k\le 2^{m_1m_2}\}$.

2.1 Ordering matrices

For $1\times(n+1)$ pattern $U^c=(u_k)\,,1\le k\le n+1$ in $\sum^c_{1\times m_2}$,as above, U^c is assigned the number

$$i=\chi^c(U^c)=1+\sum_{1\le k\le n}u_k2^{(n-k)}.$$

As denoted by the $1\times(n+1)$column pattern $x^c_{n;i}$,

$$x^c_{n;i}=\begin{bmatrix}u_{n+1}\\ \vdots\\ u_1\end{bmatrix}\quad or\quad \begin{array}{|c|}\hline u_{n+1}\\ \hline \vdots\\ \hline u_1\\ \hline\end{array}\ ,\text{where } u_1=u_{n+1}\ .$$

In particular, when $n=2$, as denoted by $x^c_i=x^c_{2;i}$, $i=1+2u_1+u_2$ and

$$x^c_i=\begin{bmatrix}u_3\\ u_2\\ u_1\end{bmatrix}\quad or\quad \begin{array}{|c|}\hline u_3\\ \hline u_2\\ \hline u_1\\ \hline\end{array}\ .$$

A 2×3 pattern $U^c=(u_{\alpha_1,\alpha_2})$ can now be obtained by a horizontal direct sum of two 1×3 patterns in $\sum^c_{2\times2}$, i.e.,

$$\begin{aligned}x^c_{i_1,i_2}&\equiv x^c_{i_1}\oplus x^c_{i_2}\\ &\equiv\begin{bmatrix}u_{1,3}&u_{2,3}\\ u_{1,2}&u_{2,2}\\ u_{1,1}&u_{2,1}\end{bmatrix}\quad or\quad \begin{array}{|c|c|}\hline u_{1,3}&u_{2,3}\\ \hline u_{1,2}&u_{2,2}\\ \hline u_{1,1}&u_{2,1}\\ \hline\end{array}\ ,\end{aligned}$$

where

$$i_k=1+2u_{k,1}+u_{k,2},\quad 1\le k\le 2.$$

Therefore, the complete set of all $16(=2^{2\times(3-1)})$ 2×3 patterns in $\sum_{2\times 2}^{c}$ can be listed by a 4×4 matrix $C_2^{*}=[x_{i_1,i_2}^{c}]$ with 2×3 pattern x_{i_1,i_2}^{c} as its entries, i.e.,

$$
\begin{array}{cc}
 & \begin{array}{cccc}
\begin{bmatrix}0\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & \begin{bmatrix}1\\1\\1\end{bmatrix}
\end{array} \\
\begin{array}{c}
\begin{bmatrix}0\\0\\0\end{bmatrix} \\ \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \begin{bmatrix}1\\0\\1\end{bmatrix} \\ \begin{bmatrix}1\\1\\1\end{bmatrix}
\end{array} &
\begin{pmatrix}
\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix} & \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix} & \begin{bmatrix}0&1\\0&0\\0&1\end{bmatrix} & \begin{bmatrix}0&1\\0&1\\0&1\end{bmatrix} \\
\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix} & \begin{bmatrix}0&0\\1&1\\0&0\end{bmatrix} & \begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix} & \begin{bmatrix}0&1\\1&1\\0&1\end{bmatrix} \\
\begin{bmatrix}1&0\\0&0\\1&0\end{bmatrix} & \begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix} & \begin{bmatrix}1&1\\0&0\\1&1\end{bmatrix} & \begin{bmatrix}1&1\\0&1\\1&1\end{bmatrix} \\
\begin{bmatrix}1&0\\1&0\\1&0\end{bmatrix} & \begin{bmatrix}1&0\\1&1\\1&0\end{bmatrix} & \begin{bmatrix}1&1\\1&0\\1&1\end{bmatrix} & \begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix}
\end{pmatrix}
\end{array}
$$

It is easy to verify that $\chi^c\left(x^c_{i_1,i_2}\right) = 4\left(i_1 - 1\right) + i_2$, i.e., we are counting local patterns in $\sum^c_{2\times 2}$ by going through each row successively in above table. Correspondingly, C_2^* can be referred to as an ordering matrix for $\sum^c_{2\times 2}$. Similarly, all patterns in $\sum^c_{2\times n}$ can be arranged by the ordering matrix $C_n^* = \left[x^c_{n;i_1,i_2}\right]$ a $2^n \times 2^n$ matrix with entry $x^c_{n;i_1,i_2} = x^c_{n;i_1} \oplus x^c_{n;i_2}$, where $\chi^c\left(U_1^c\right) = i_1$ and $\chi^c\left(U_2^c\right) = i_2$, $1 \le i_1, i_2 \le 2^n$.

2.2 The relation between C_n^* and X_n

The definitions of X_n and χ are according to the Preliminaries.

We consider $U^c = (u_{\alpha_1,\alpha_2}) \in \sum^c_{m_1\times m_2} \Longrightarrow u_{i,1} = u_{i,m_2+1}, 1 \le i \le m_1$,then we can decompose U^c as $U^1 \overset{\wedge}{\oplus} U^2$ where$U^1 = (u_{\alpha_1,\alpha_2})^{1\le\alpha_1\le m_1}_{1\le\alpha_2\le m_2} \in \sum_{m_1\times m_2}$ *and* $U^2 = (u_{\alpha_1,\alpha_2})^{1\le\alpha_1\le m_1}_{m_2\le\alpha_2\le m_2+1} \in \sum_{m_1\times 2}$, i.e.,

$$\begin{bmatrix} u_{1,1} & u_{2,1} & \cdots & u_{m_1,1} \\ u_{1,m_2} & u_{2,m_2} & \cdots & u_{m_1,m_2} \\ \vdots & \vdots & \cdots & \vdots \\ u_{1,2} & u_{2,2} & \cdots & u_{m_1,2} \\ u_{1,1} & u_{2,1} & \cdots & u_{m_1,1} \end{bmatrix} = \begin{bmatrix} u_{1,m_2} & u_{2,m_2} & \cdots & u_{m_1,m_2} \\ \vdots & \vdots & \cdots & \vdots \\ u_{1,2} & u_{2,2} & \cdots & u_{m_1,2} \\ u_{1,1} & u_{2,1} & \cdots & u_{m_1,1} \end{bmatrix} \hat{\oplus} \begin{bmatrix} u_{1,1} & u_{2,1} & \cdots & u_{m_1,1} \\ u_{1,m_2} & u_{2,m_2} & \cdots & u_{m_1,m_2} \end{bmatrix}$$

Therefore, we get $\chi^c(U^c) = \chi(U^1)$ and is an one-to-one correspondence between U^c and U^1, i.e., we can determine the $\chi^c(U^c)$ by $\chi(U^1)$.

In particular, consider the pattern $U^c_{2\times n} = \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{1,n} & u_{2,n} \\ \vdots & \vdots \\ u_{1,2} & u_{2,2} \\ u_{1,1} & u_{2,1} \end{bmatrix} \in \sum^c_{v,2\times n}$, then decompoes $U^c_{2\times n}$ into $U^1_{2\times n} \hat{\oplus} U^2_{2\times n}$ where $U^1_{2\times n} = \begin{bmatrix} u_{1,n} & u_{2,n} \\ \vdots & \vdots \\ u_{1,2} & u_{2,2} \\ u_{1,1} & u_{2,1} \end{bmatrix} \in \sum_{2\times n}$ *and* $U^2_{2\times n} = \begin{bmatrix} u_{1,1} & u_{2,1} \\ u_{1,n} & u_{2,n} \end{bmatrix} \in \sum_{2\times 2}$.

Suppose $U^c_{2\times n} = U^c_{2\times n;1} \oplus U^c_{2\times n;2}$ and $U^1_{2\times n} = U^1_{2\times n;1} \oplus U^1_{2\times n;2}$ where

$$U^c_{2\times n;1} = \begin{bmatrix} u_{1,1} \\ u_{1,n} \\ \vdots \\ u_{1,2} \\ u_{1,1} \end{bmatrix}, U^c_{2\times n;2} = \begin{bmatrix} u_{2,1} \\ u_{2,n} \\ \vdots \\ u_{2,2} \\ u_{2,1} \end{bmatrix}, U^1_{2\times n;1} = \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{1,2} \\ u_{1,1} \end{bmatrix} \ \textit{and}\ U^1_{2\times n;2} = \begin{bmatrix} u_{2,n} \\ \vdots \\ u_{2,2} \\ u_{2,1} \end{bmatrix}.$$

And suppose $\chi^c(U^c_{2\times n;1}) = i_1$ and $\chi^c(U^c_{2\times n;2}) = i_2$, then $\chi(U^1_{2\times n;1}) = i_1$ and $\chi(U^1_{2\times n;2}) = i_2$ (By the definition of χ and χ^c).Therefore, $U^c_{2\times n} = x^c_{n;i_1,i_2}$, and $U^1_{2\times n} = x_{n;i_1,i_2}$.

According to Preliminaries, we represent $x_{n;i_1,i_2}$ as $y_{j_1,j_2\ldots j_n}$ where $x_{n;i_1,i_2}$ and $y_{j_1,j_2\cdots j_n}$ defined in [4]. Since the top and bottom layers are the same, then

$$x^c_{n;i_1,i_2} = x_{n;i_1,i_2} \overset{\wedge}{\oplus} y_{j_n,j_1} = y_{j_1,j_2\ldots j_n} \overset{\wedge}{\oplus} y_{j_n,j_1} = y_{j_1,j_2\ldots j_n,j_1}.$$

By the Theorem in the Preliminaries and the argument as above, we can get the following Theorem easily.

Theorem 1 *For any $n \geq 2$, $\sum^c_{2\times n} = \{y_{j_1\cdots j_n,j_1}\}$. Furthermore, the ordering matrix C^*_n can be decomposed by n Z-maps successively as*

$$C^*_n = \begin{bmatrix} Y^c_{n;1} & Y^c_{n;2} \\ Y^c_{n;3} & Y^c_{n;4} \end{bmatrix},$$

$$Y^c_{n;j_1\cdots j_k} = \begin{bmatrix} Y^c_{n;j_1\cdots j_k,1} & Y^c_{n;j_1\cdots j_k,2} \\ Y^c_{n;j_1\cdots j_k,3} & Y^c_{n;j_1\cdots j_k,4} \end{bmatrix},$$

for $1 \leq k \leq n-2$, and

$$Y^c_{n;j_1\cdots j_{n-1}} = \begin{bmatrix} y_{j_1\cdots j_{n-1},1,j_1} & y_{j_1\cdots j_{n-1},2,j_1} \\ y_{j_1\cdots j_{n-1},3,j_1} & y_{j_1\cdots j_{n-1},4,j_1} \end{bmatrix}.$$

2.3 The transition matrices

This part derives the transition matrix C_n for a given basic set $\mathcal{B} \subset \sum_{2\times 2}$.

Given a basic set $\mathcal{B} \subset \sum_{2\times 2}$, the transition matrix C_n can be defined by $C_n = [c_{n;i_1,i_2}]_{1\leq i_1,i_2\leq 2^n}$, the $2^n \times 2^n$ matrix with entries either 0 or 1, according to the following rules:

Since $C^*_n = [x^c_{n;i_1,i_2}]$ and

$$x^c_{n;i_1,i_2} = y_{j_1,j_2\ldots j_n,j_1} = y_{j_1,j_2} \overset{\wedge}{\oplus} y_{j_2,j_3} \cdots \overset{\wedge}{\oplus} y_{j_{n-1},j_n} \overset{\wedge}{\oplus} y_{j_n,j_1},$$

where $j_1 \ldots j_n$ are determined uniquely by i_1 and i_2.Then

$$\begin{cases} c_{n;i_1,i_2} & = 1 \quad \text{if all } y_{j_1,j_2}, \cdots y_{j_{n-1},j_n} \text{and } y_{j_n,j_1} \text{belong to } \mathcal{B} \\ & = 0 \quad \text{otherwise.} \end{cases}$$

Therefore, C_n also can be represented by $C_n = [v_{j_1 j_2 \cdots j_n, j_1}]$, and the transition matrix C_n for $\mathcal{B} \subset \sum_{2\times 2}$ defined on $\mathbf{Z}_{2\times(n+1)}$ is a $2^n \times 2^n$ matrix with entries $v_{j_1 \cdots j_n, j_1}$, which are either 1 or 0, by substituting $y_{j_1 \cdots j_n, j_1}$ by $v_{j_1 \cdots j_n, j_1}$ in C_n^*.

Definition 2 $\widetilde{V}_2 = \begin{bmatrix} \widetilde{V}_{2;1} & \widetilde{V}_{2;2} \\ \widetilde{V}_{2;3} & \widetilde{V}_{2;4} \end{bmatrix}$ *where* $\widetilde{V}_{2;i} = \begin{bmatrix} v_{1i} & v_{2i} \\ v_{3i} & v_{4i} \end{bmatrix} 1 \le i \le 4$, *and* $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem 3 $C_n = \mathbf{H}_n \circ \left(\sum_{1\le i\le 4} e_i \otimes E_{2^{n-2}} \otimes \widetilde{V}_{2;i} \right)$

Proof. Since $C_n = [c_{n;i_1,i_2}]_{1\le i_1,i_2\le 2^n} = [v_{j_1 j_2 \cdots j_n, j_1}] = [v_{j_1 j_2 \cdots j_n} v_{j_n j_1}]$.

Observe the Theorem1.1, we can get

$$\begin{aligned} j_1 &= 1, \quad \text{if } 1 \le i_1, i_2 \le 2^{n-1} \\ j_1 &= 2, \quad \text{if } 1 \le i_1 \le 2^{n-1}, 2^{n-1} \le i_2 \le 2^n \\ j_1 &= 3, \quad \text{if } 2^{n-1} \le i_1 \le 2^n, 1 \le i_2 \le 2^{n-1} \\ j_1 &= 4, \quad \text{if } 2^{n-1} \le i_1, i_2 \le 2^n. \end{aligned}$$

and the j_n in the C_n are

$$\begin{bmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \cdots & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ \vdots & \vdots & \vdots \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \cdots & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{bmatrix}_{2^n \times 2^n}$$

Then the result is easily to prove. ■

2.4 Spatial entropy

First, we save the all components of $\sum_{m\times n}^c$ in a $2^n \times 2^n$ matrix $C_{m\times n}^* = \left[c_{m\times n;i,j}^*\right]_{1\le i,j\le 2^n}$, and the entries

$$c_{m\times n;i,j}^* = \left\{ x_{n;i,k_1,k_2\cdots k_{m-2},j}^c \mid 1 \le k_1, k_2 \cdots, k_{m-2} \le 2^n \right\}$$

are subsets of $\sum_{m\times n}^{c}$.We know $\bigcup\limits_{1\leq i,j\leq 2^n} c_{m\times n;i,j}^{*}=\sum_{m\times n}^{c}$ and $c_{m\times n;i,j}^{*}$ are disjoint to each other.

Given a basic set $\mathcal{B}\subset\sum_{2\times 2}$, the $2^n\times 2^n$ matrix $C_{m\times n}$ can be defined $C_{m\times n}=[c_{m\times n;i,j}]_{1\leq i,j\leq 2^n}$ where $c_{m\times n;i,j}=card\left(c_{m\times n;i,j}^{*}\cap\sum_{m\times n}^{c}(\mathcal{B})\right)$.Then $\sum\limits_{1\leq i,j\leq 2^n} c_{m\times n;i,j}=\Gamma_{m\times n}^{c}(\mathcal{B})$ where $\Gamma_{m\times n}^{c}(\mathcal{B})=card\left(\sum_{m\times n}^{c}(\mathcal{B})\right)$.By the construction of $C_n=[c_{n;i_1,i_2}]_{1\leq i_1,i_2\leq 2^n}$ for the same $\mathcal{B}\subset\sum_{2\times 2}$, we can get

$$c_{m\times n;i,j}=\sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{m-2},j}$$

In particular, when $m=2$, $c_{n;i,j}=c_{2\times n;i,j}$ for $1\leq i,j\leq 2^n$, i.e.,

$$C_n=C_{2\times n}.$$

Theorem 4 $C_{m\times n}=(C_n)^{m-1}$ *for* $m\geq 2$. *Furthermore,*
$$\sum_{1\leq i,j\leq 2^n}\left[(C_n)^{m-1}\right]_{i,j}=\Gamma_{m\times n}^{c}(\mathcal{B}).$$

Proof. We prove this by induction.

(1). When $m=2$, $C_n=C_{2\times n}$, clearly.

(2). Suppose, when $m=l$, $C_{l\times n}=(C_n)^{l-1}$holds, i.e.,

$$c_{l\times n;i,j}=\left[(C_n)^{l-1}\right]_{i,j},1\leq i,j\leq 2^n$$

(3). When $m=l+1$,

$$
\begin{aligned}
c_{(l+1)\times n;i,j} &= \sum_{1\leq k_1,k_2\cdots,k_{l-1}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{l-1},j} \\
&= \sum_{1\leq k_{l-1}\leq 2^n}\left[c_{n;k_{l-1},j}\left(\sum_{1\leq k_1,k_2\cdots,k_{l-2}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{l-2},k_{l-1}}\right)\right] \\
&= \sum_{1\leq k_{l-1}\leq 2^n}\left[c_{n;k_{l-1},j}c_{l\times n;i,k_{l-1}}\right] \\
&= \sum_{1\leq k_{l-1}\leq 2^n}\left[\left[(C_n)^{l-1}\right]_{i,k_{l-1}} c_{n;k_{l-1},j}\right] \\
&= \left[(C_n)^{l}\right]_{i,j},\ for 1\leq i,j\leq 2^n.
\end{aligned}
$$

Since $\sum_{1\leq i,j\leq 2^n} c_{m\times n;i,j} = \Gamma^c_{m\times n}(\mathcal{B})$ and $C_n = C_{2\times n}$, then

$$
\sum_{1\leq i,j\leq 2^n}\left[(C_n)^{m-1}\right]_{i,j} = \Gamma^c_{m\times n}(\mathcal{B}).
$$

The proof is complet. ■

The spatial entropy $h^c(\mathcal{B})$ of $\sum^c(\mathcal{B})$ is defined as follows :
Let $\Gamma^c_{m\times n}(\mathcal{B}) = card(\sum^c_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\sum^c_{m\times n}(\mathcal{B})$. The spatial entropy $h^c(\mathcal{B})$ is defined as

$$
h^c(\mathcal{B}) = \limsup_{m,n\to\infty}\frac{1}{mn}\log\Gamma^c_{m\times n}(\mathcal{B}).
$$

As for spatial entropy $h^c(\mathcal{B})$, we have the following theorem.

Theorem 5 *Given a basic set $\mathcal{B}\subset\sum_{2\times 2}$, let λ^c_n be the largest eigenvalue of the associated transition matrix C_n which is defined above. Then,*

$$
h^c(\mathcal{B}) = \limsup_{n\to\infty}\frac{\log\lambda^c_n}{n}.
$$

Proof. From the construction of $\mathbf{C}_n$, we know that for $m \geq 2$,

$$\begin{aligned}\Gamma^c_{m\times n}(\mathcal{B}) &= \sum_{1\leq i,j\leq 2^n} \left[(C_n)^{m-1}\right]_{i,j} \\ &\equiv \#(C_n^{m-1}).\end{aligned}$$

As in a one dimensional case, we have

$$\lim_{m\to\infty} \frac{\log \#(C_n^{m-1})}{m} = \log \lambda_n^c.$$

Therefore,

$$\begin{aligned}h^c(\mathcal{B}) &= \limsup_{m,n\to\infty} \frac{1}{mn} \log \Gamma^c_{m\times n}(\mathcal{B}) \\ &= \limsup_{n\to\infty} \frac{1}{n} (\lim_{m\to\infty} \frac{\log \Gamma^c_{m\times n}(\mathcal{B})}{m}) \\ &= \limsup_{n\to\infty} \frac{\log \lambda_n^c}{n}.\end{aligned}$$

The proof is complete. ■

Remark 6 *Similarly,the two-dimensional patterns which have periodic boundary condition in horizontal direction have the same arguments.*

3 The toric patterns

In geometry, we can construct a torus by pasting the two pair of subtenses of a rectangle. So, consider the toric patterns as the finite two-dimensional patterns which have double-periodic boundary condition in vertical and horizontal directions. For clarity, we study two symbols, i.e., $S = \{0, 1\}$.

According to above, define

$$\begin{aligned}\sum\nolimits^t_{m_1\times m_2} = \ &\Big\{U^t = (u_{\alpha_1,\alpha_2}) \in \sum\nolimits_{(m_1+1)\times(m_2+1)} \mid u_{i,1} = u_{i,(m_2+1)},\ u_{1,j} = u_{(m_1+1),j}, \\ &for\ \ 1 \leq i \leq m_1+1,\ 1 \leq j \leq m_2+1\}\end{aligned}$$

to represent the finite two-dimensional patterns which have double-periodic boundary condition .

We can save the all components of $\sum_{m\times n}^{t}$ in a $2^n \times 2^n$ matrix $\hat{D}_{m\times n} = \left[\hat{d}_{m\times n;i,j}\right]_{1\leq i,j\leq 2^n,}$ and the entries

$$\hat{d}_{m\times n;i,j} = \left\{x^{c}_{n;i,k_1,k_2\cdots k_{m-2},j,i} \mid 1 \leq k_1, k_2 \cdots , k_{m-2} \leq 2^n\right\}$$

are subsets of $\sum_{m\times n}^{t}$. We know $\bigcup\limits_{1\leq i,j\leq 2^n} \hat{d}_{m\times n;i,j} = \sum_{m\times n}^{t}$ and $\hat{d}_{m\times n;i,j}$ are disjoint to each other.

Given a basic set $\mathcal{B} \subset \sum_{2\times 2}$, the $2^n \times 2^n$ matrix $D_{m\times n}$ can be defined $D_{m\times n} = [d_{m\times n;i,j}]_{1\leq i,j\leq 2^n}$ where $d_{m\times n;i,j} = card\left(\hat{d}_{m\times n;i,j} \cap \sum_{m\times n}^{t}(\mathcal{B})\right)$, then $\sum\limits_{1\leq i,j\leq 2^n} d_{m\times n;i,j} = \Gamma^{t}_{m\times n}(\mathcal{B})$ *where* $\Gamma^{t}_{m\times n}(\mathcal{B}) = card\left(\sum_{m\times n}^{t}(\mathcal{B})\right)$.

By the construction of $C_n = [c_{n;i_1,i_2}]_{1\leq i_1,i_2\leq 2^n}$ for the same $\mathcal{B} \subset \sum_{2\times 2}$ as above, we can get

$$d_{m\times n;i,j} = \sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{m-2},j}c_{n;j,i}.$$

Theorem 7 $D_{m\times n} = (C_n)^{m-1} \circ (C_n)^T$, *where* $(C_n)^T$ *means the transpose of* C_n, *for* $m \geq 2$,*and* $\sum\limits_{1\leq i,j\leq 2^n}\left[(C_n)^{m-1} \circ (C_n)^T\right]_{i,j} = \Gamma^{t}_{m\times n}(\mathcal{B})$.

Proof.

$$\begin{aligned}
d_{m\times n;i,j} &= \sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{m-2},j}c_{n;j,i} \\
&= \left(\sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} c_{n;i,k_1}c_{n;k_1,k_2}\ldots c_{n;k_{m-2},j}\right)c_{n;j,i} \\
&= c_{m\times n;i,j}c_{n;j,i} \\
&= \left[(C_{2\times n})^{m-1}\right]_{i,j} c_{n;j,i},\ \ for\ 1 \leq i,j \leq 2^n.
\end{aligned}$$

Then,

$$D_{m\times n} = (C_n)^{m-1} \circ (C_n)^T .$$

Since $\sum\limits_{1\leq i,j\leq 2^n} d_{m\times n;i,j} = \Gamma^t_{m\times n}(\mathcal{B})$,we can get

$$\sum_{1\leq i,j\leq 2^n} \left[(C_n)^{m-1} \circ (C_n)^T\right]_{i,j} = \Gamma^t_{m\times n}(\mathcal{B}) . \ \blacksquare$$

The spatial entropy $h^t(\mathcal{B})$ of $\sum^t(\mathcal{B})$ is defined as follows :
Let $\Gamma^t_{m\times n}(\mathcal{B}) = card(\sum^t_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\sum^t_{m\times n}(\mathcal{B})$. The spatial entropy $h^t(\mathcal{B})$ is defined as

$$h^t(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log \Gamma^t_{m\times n}(\mathcal{B}).$$

Then,

$$h^t(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log(\sum_{1\leq i,j\leq 2^n} \left[(C_n)^{m-1} \circ (C_n)^T\right]_{i,j}).$$

4 The spherical patterns

In geometry, we can suppose that a ball constructed by pasting the four edges of a rectangle into a point. So , consider the spherical patterns as the finite two-dimensional patterns whose boundaries' symbols are all the same. In here, we consider two kinds of the spherical patterns: one kind of the spherical patterns is the finite two-dimensional patterns whose boundaries with four corners(e.x. Fig.1),and the other kind of the spherical patterns is the finite two-dimensional patterns whose boundaries without four corners(e.x. Fig.2).For charity, we study two symbols , i.e, $S = \{0, 1\}$.

⊠	⊠	⊠	⊠	⊠	⊠
⊠					⊠
⊠					⊠
⊠					⊠
⊠	⊠	⊠	⊠	⊠	⊠

Fig.1 Boundary with four corners
(the first kind of the spherical patterns)

	⊠	⊠	⊠	⊠	
⊠					⊠
⊠					⊠
⊠					⊠
	⊠	⊠	⊠	⊠	

Fig.2 Boundary without four corners
(the second kind of the spherical patterns)

4.1 The upper and lower parts of the boundary condition

Before we study the spherical patterns, we consider the top and bottom layers of boundaries firstly.

Definition 8

$$\sum\nolimits^{c_0}_{m_1\times m_2} = \left\{u_{\alpha_1,\alpha_2} \mid u_{\alpha_1,\alpha_2} \in \sum\nolimits_{m_1\times(m_2+2)},\ u_{i,1} = u_{i,m_2+2} = 0,\ 1 \le i \le m_1\right\}$$
$$\sum\nolimits^{c_1}_{m_1\times m_2} = \left\{u_{\alpha_1,\alpha_2} \mid u_{\alpha_1,\alpha_2} \in \sum\nolimits_{m_1\times(m_2+2)},\ u_{i,1} = u_{i,m_2+2} = 1,\ 1 \le i \le m_1\right\}$$

Observe the matrix $C^*_{n+1} = \left[x^c_{n+1;i_1,i_2}\right]_{1\le i_1,i_2\le 2^{n+1}}$. By the Preliminaries and the relation between C^*_{n+1} and X_{n+1}, we can get

$$1 \le i_1, i_2 \le 2^n \quad \Leftrightarrow \quad j_1 = 1 \quad \Longleftrightarrow \quad \text{the top and bottom layers are } \boxed{0}\boxed{0}$$

$$2^n + 1 \le i_1, i_2 \le 2^{n+1} \quad \Leftrightarrow \quad j_4 = 4 \quad \Longleftrightarrow \quad \text{the top and bottom layers are } \boxed{1}\boxed{1}$$

Therefore,

$$\sum\nolimits^{c_0}_{2\times n} = \left\{x^c_{n+1;i_1,i_2} \mid 1 \le i_1, i_2 \le 2^n\right\}$$

and

$$\sum\nolimits^{c_1}_{2\times n} = \left\{x^c_{n+1;i_1,i_2} \mid 2^n + 1 \le i_1, i_2 \le 2^{n+1}\right\}.$$

Furthermore,

$$\sum\nolimits^{c_0}_{m\times n} = \left\{x^c_{n+1;i_1,i_2\cdots i_m} \mid 1 \le i_1, i_2 \cdots i_m \le 2^n\right\}$$

and

$$\sum\nolimits_{m\times n}^{c_1}=\left\{x_{n+1;i_1,i_2\cdots i_m}^{c}\mid 1\leq i_1,i_2\cdots i_m\leq 2^n\right\}$$

We can save $\sum\nolimits_{m\times n}^{c_0}$ in a $2^n\times 2^n$ matrix $\hat{C}_{m\times n}^{0}=\left[\hat{c}_{m\times n;i,j}^{0}\right]_{1\leq i,j\leq 2^n}$, and the entries

$$\hat{c}_{m\times n;i,j}^{0}=\left\{x_{n+1;i,k_1,k_2\cdots k_{m-2},j}^{c}\mid 1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n\right\}$$

are subsets of $\sum\nolimits_{m\times n}^{c_0}$. We know $\underset{1\leq i,j\leq 2^n}{\cup}\hat{c}_{m\times n;i,j}^{0}=\sum\nolimits_{m\times n}^{c_0}$ and $\hat{c}_{m\times n;i,j}^{0}$ are disjoint to each other.

Given a basic set $\mathcal{B}\subset\sum_{2\times 2}$, the $2^n\times 2^n$ matrix $C_{m\times n}^{0}$ can be defined $C_{m\times n}^{0}=\left[c_{0,m\times n;i,j}\right]_{1\leq i,j\leq 2^n}$ where $c_{0,m\times n;i,j}=card\left(\hat{c}_{m\times n;i,j}^{0}\cap\sum\nolimits_{m\times n}^{c_0}(\mathcal{B})\right)$.Then $\sum\limits_{1\leq i,j\leq 2^n}c_{0,m\times n;i,j}=\Gamma_{m\times n}^{c_0}(\mathcal{B})$ *where* $\Gamma_{m\times n}^{c_0}(\mathcal{B})=card\left(\sum\nolimits_{m\times n}^{c_0}(\mathcal{B})\right)$.

By the construction of $C_{n+1}=\left[c_{n+1;i_1,i_2}\right]_{1\leq i_1,i_2\leq 2^{n+1}}$ for the same $\mathcal{B}\subset\sum_{2\times 2}$ and the above arguments, we can get

$$c_{0,m\times n;i,j}=\sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n}c_{n+1;i,k_1}c_{n+1;k_1,k_2}\ldots c_{n+1;k_{m-2},j}$$

Similarly,we can save $\sum\nolimits_{m\times n}^{c_1}$ in a $2^n\times 2^n$ matrix $\hat{C}_{m\times n}^{1}=\left[\hat{c}_{m\times n;i,j}^{1}\right]_{1\leq i,j\leq 2^n}$, and the entries
$\hat{c}_{m\times n;i,j}^{1}=\left\{x_{n+1;i+2^n,k_1,k_2\cdots k_{m-2},j+2^n}^{c}\mid 2^n+1\leq k_1,k_2\cdots,k_{m-2}\leq 2^{n+1}\right\}$ are subsets of $\sum\nolimits_{m\times n}^{c_1}$. We know $\underset{1\leq i,j\leq 2^n}{\cup}\hat{c}_{m\times n;i,j}^{1}=\sum\nolimits_{m\times n}^{c_1}$ and $\hat{c}_{m\times n;i,j}^{1}$ are disjoint to each other.

Given a basic set $\mathcal{B}\subset\sum_{2\times 2}$, the $2^n\times 2^n$ matrix $C_{m\times n}^{1}$ can be defined $C_{m\times n}^{1}=\left[c_{1,m\times n;i,j}\right]_{1\leq i,j\leq 2^n}$ where$c_{1,m\times n;i,j}=card\left(\hat{c}_{m\times n;i,j}^{1}\cap\sum\nolimits_{m\times n}^{c_1}(\mathcal{B})\right)$.Then $\sum\limits_{1\leq i,j\leq 2^n}c_{1,m\times n;i,j}=\Gamma_{m\times n}^{c_1}(\mathcal{B})$ where $\Gamma_{,m\times n}^{c_1}(\mathcal{B})=card\left(\sum\nolimits_{m\times n}^{c_1}(\mathcal{B})\right)$.

By the construction of $C_{n+1} = [c_{n+1;i_1,i_2}]_{1\le i_1,i_2\le 2^{n+1}}$ for the same $\mathcal{B} \subset \sum_{2\times 2}$ and the above arguments, we can get

$$c_{1,m\times n;i,j} = \sum_{2^n+1\le k_1,k_2\cdots,k_{m-2}\le 2^{n+1}} c_{n+1;i+2^n,k_1} c_{n+1;k_1,k_2} \cdots c_{n+1;k_{m-2},j+2^n}$$

In particular $m = 2$, $c_{0,2\times n;i,j} = c_{n+1;i,j}$ and $c_{1,2\times n;i,j} = c_{n+1;i+2^n,j+2^n}$, $1 \le i, j \le 2^n$.

Definition 9 $C_n = \begin{bmatrix} C_{n;1} & C_{n;2} \\ C_{n;3} & C_{n;4} \end{bmatrix}$

Therefore, $C^0_{2\times n} = C_{n+1;1}$ and $C^1_{2\times n} = C_{n+1;4}$.

Theorem 10 $C^0_{m\times n} = (C_{n+1;1})^{m-1}$*,and* $C^1_{m\times n} = (C_{n+1;4})^{m-1}$ *for* $m \ge 2$. *Furthermore,* $\sum_{1\le i,j\le 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} = \Gamma^{c_0}_{m\times n}(\mathcal{B})$, *and*

$$\sum_{1\le i,j\le 2^n} \left[(C_{n+1;4})^{m-1}\right]_{i,j} = \Gamma^{c_1}_{m\times n}(\mathcal{B}) \ form \ge 2.$$

Proof. We know

$$\begin{aligned} c_{0,m\times n;i,j} &= \sum_{1\le k_1,k_2\cdots,k_{m-2}\le 2^n} c_{n+1;i,k_1} c_{n+1;k_1,k_2} \cdots c_{n+1;k_{m-2},j} \\ &= \sum_{1\le k_1,k_2\cdots,k_{m-2}\le 2^n} c_{n+1;1;i,k_1} c_{n+1;1;k_1,k_2} \cdots c_{n+1;1;k_{m-2},j} \end{aligned}$$

and

$$\begin{aligned} c_{1,m\times n;i,j} &= \sum_{2^n+1\le k_1,k_2\cdots,k_{m-2}\le 2^{n+1}} c_{n+1;i+2^n,k_1} c_{n+1;k_1,k_2} \cdots c_{n+1;k_{m-2},j+2^n} \\ &= \sum_{1\le k_1,k_2\cdots,k_{m-2}\le 2^n} c_{n+1;4;i,k_1} c_{n+1;4;k_1,k_2} \cdots c_{n+1;4;k_{m-2},j} \end{aligned}$$

We prove the part $C^0_{m\times n} = (C_{n+1;1})^{m-1}$ by induction

(The other part $C^1_{m\times n} = (C_{n+1;4})^{m-1}$ is similar.)
(1). When $m = 2$, $C^0_{2\times n} = C_{n+1;1}$, clearly.

(2). Suppose, when $m = l$, $C^0_{l\times n} = (C_{n+1;1})^{l-1}$holds, i.e.,

$$c_{0,l\times n;i,j} = \left[(C_{n+1;1})^{l-1}\right]_{i,j}, 1 \le i, j \le 2^n$$

(3). When $m = l + 1$,

$$\begin{aligned}
c_{0,(l+1)\times n;i,j} &= \sum_{1\le k_1,k_2\cdots,k_{l-1}\le 2^n} c_{n+1;1;i,k_1} c_{n+1;1;k_1,k_2} \cdots c_{n+1;1;k_{l-1},j} \\
&= \sum_{1\le k_{l-1}\le 2^n} \left[c_{n+1;1;k_{l-1},j} \left(\sum_{1\le k_1,k_2\cdots,k_{l-2}\le 2^n} c_{n+1;1;i,k_1} c_{n+1;1;k_1,k_2} \cdots \right.\right. \\
&\qquad \left.\left. c_{n+1;1;k_{l-2},k_{l-1}} \right)\right] \\
&= \sum_{1\le k_{l-1}\le 2^n} \left[c_{n+1;1;k_{l-1},j} c_{0,l\times n;i,k_{l-1}}\right] \\
&= \sum_{1\le k_{l-1}\le 2^n} \left[\left[(C_{n+1;1})^{l-1}\right]_{i,k_{l-1}} c_{n+1;1;k_{l-1},j}\right] \\
&= \left[(C_{n+1;1})^{l}\right]_{i,j}, \text{ for } 1 \le i, j \le 2^n.
\end{aligned}$$

Since $\sum_{1\le i,j\le 2^n} c_{0,m\times n;i,j} = \Gamma^{c_0}_{m\times n}(\mathcal{B})$ and $c_{0,m\times n;i,j} = \left[(C_{n+1;1})^{m-1}\right]_{i,j}, 1 \le i, j \le 2^n$, then

$$\sum_{1\le i,j\le 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} = \Gamma^{c_0}_{m\times n}(\mathcal{B})$$

The proof is complet. ■

4.2 The boundary with corners

Now, we add the left and right layers of boundaries for the first kind of the spherical patterns(i.e, whose boundaries contain four corners), we do some definition as following:

$$\textstyle\sum^{S_0}_{m_1\times m_2} = \left\{u_{\alpha_1,\alpha_2} \mid u_{\alpha_1,\alpha_2} \in \sum_{(m_1+2)\times(m_2+2)},\ u_{i,1} = u_{i,m_2+2} = u_{1,j} = u_{m_1+2,j} = 0, 1 \le i \le m_1+2, 1 \le j \le m_2+2\right\}$$

and

$$\sum\nolimits_{m_1\times m_2}^{S_1} = \left\{u_{\alpha_1,\alpha_2} \mid u_{\alpha_1,\alpha_2} \in \sum\nolimits_{(m_1+2)\times(m_2+2)},\ u_{i,1} = u_{i,m_2+2} = u_{1,j} = u_{m_1+2,j} = 1, 1 \le i \le m_1+2, 1 \le j \le m_2+2\right\}$$

And then, we can save $\sum_{m\times n}^{S_0}$ in a $2^n \times 2^n$ matrix $\hat{S}^0_{m\times n} = \left[\hat{s}^0_{m\times n;i,j}\right]_{1\le i,j\le 2^n}$, and the entries

$$\hat{s}^0_{m\times n;i,j} = \left\{x^c_{n+1;1,i,k_1,k_2\cdots k_{m-2},j,1} \mid 1 \le k_1, k_2 \cdots, k_{m-2} \le 2^n\right\}$$

are subsets of $\sum_{m\times n}^{S_0}$. We know $\bigcup\limits_{1\le i,j\le 2^n} \hat{s}^0_{m\times n;i,j} = \sum_{m\times n}^{S_0}$ and $\hat{s}^0_{m\times n;i,j}$ are disjoint to each other..

Given a basic set $\mathcal{B} \subset \sum_{2\times 2}$, the $2^n \times 2^n$ matrix $S^0_{m\times n}$ can be defined $S^0_{m\times n} = \left[s^0_{m\times n;i,j}\right]_{1\le i,j\le 2^n}$ where$s^0_{m\times n;i,j} = card\left(\hat{s}^0_{m\times n;i,j} \cap \sum_{m\times n}^{S_0}(\mathcal{B})\right)$.Then $\sum\limits_{1\le i,j\le 2^n} s^0_{m\times n;i,j} = \Gamma^{s_0}_{m\times n}(\mathcal{B})$ where $\Gamma^{s_0}_{m\times n}(\mathcal{B}) = card\left(\sum_{m\times n}^{s_0}(\mathcal{B})\right)$.

By the construction of $C_{n+1;1} = [c_{n+1;1;i_1,i_2}]_{1\le i_1,i_2\le 2^{n+1}}$ for the same $\mathcal{B} \subset \sum_{2\times 2}$ and the above arguments, we can get

$$\begin{aligned} s^0_{m\times n;i,j} &= \sum_{1\le k_1,k_2\cdots,k_{m-2}\le 2^n} c_{n+1;1;1,i}c_{n+1;1;i,k_1}c_{n+1;1;k_1,k_2}\dots c_{n+1;1;k_{m-2},j}c_{n+1;1;j,1} \\ &= \Big(\sum_{1\le k_1,k_2\cdots,k_{m-2}\le 2^n} c_{n+1;1;i,k_1}c_{n+1;1;k_1,k_2}\dots c_{n+1;1;k_{m-2},j}\Big)c_{n+1;1;1,i}c_{n+1;1;j,1} \\ &= c_{0,m\times n;i,j}c_{n+1;1;1,i}c_{n+1;1;j,1} \\ &= \left[(C_{n+1;1})^{m-1}\right]_{i,j} c_{n+1;1;1,i}c_{n+1;1;j,1} \text{ for } 1 \le i,j \le 2^n \end{aligned}$$

i.e.,

$$
\begin{aligned}
S^0_{m\times n} &= (C_{n+1;1})^{m-1} \circ \begin{bmatrix} c_{n+1;1;1,1} & c_{n+1;1;1,1} & \cdots & c_{n+1;1;1,1} \\ c_{n+1;1;1,2} & c_{n+1;1;1,2} & \cdots & c_{n+1;1;1,2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+1;1;1,2^n} & c_{n+1;1;1,2^n} & \cdots & c_{n+1;1;1,2^n} \end{bmatrix} \\
&\quad \circ \begin{bmatrix} c_{n+1;1;1,1} & c_{n+1;1;2,1} & \cdots & c_{n+1;1;2^n,1} \\ c_{n+1;1;1,1} & c_{n+1;1;2,1} & \cdots & c_{n+1;1;2^n,1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+1;1;1,1} & c_{n+1;1;2,1} & \cdots & c_{n+1;1;2^n,1} \end{bmatrix} \\
&= (C_{n+1;1})^{m-1} \circ \left[c_{n+1;1;1,i}c_{n+1;1;j,1}\right]_{1\le i,j\le 2^n}
\end{aligned}
$$

Similarly, we can get $s^1_{m\times n;i,j} = \left[(C_{n+1;4})^{m-1}\right]_{i,j} c_{n+1;4;2^n,i}c_{n+1;4;j,2^n}$ for $1 \le i,j \le 2^n$,i.e.,

$$
\begin{aligned}
S^1_{m\times n} &= (C_{n+1;4})^{m-1} \circ \begin{bmatrix} c_{n+1;4;2^n,1} & c_{n+1;4;2^n,1} & \cdots & c_{n+1;4;2^n,1} \\ c_{n+1;4;2^n,2} & c_{n+1;4;2^n,2} & \cdots & c_{n+1;4;2^n,2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+1;4;2^n,2^n} & c_{n+1;4;2^n,2^n} & \cdots & c_{n+1;4;2^n,2^n} \end{bmatrix} \\
&\quad \circ \begin{bmatrix} c_{n+1;4;1,2^n} & c_{n+1;4;2,2^n} & \cdots & c_{n+1;4;2^n,2^n} \\ c_{n+1;4;1,2^n} & c_{n+1;4;2,2^n} & \cdots & c_{n+1;4;2^n,2^n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+1;4;1,2^n} & c_{n+1;4;2,2^n} & \cdots & c_{n+1;4;2^n,2^n} \end{bmatrix} \\
&= (C_{n+1;4})^{m-1} \circ \left[c_{n+1;4;2^n,i}c_{n+1;4;j,2^n}\right]_{1\le i,j\le 2^n}
\end{aligned}
$$

Since $\sum\limits_{1\le i,j\le 2^n} s^0_{m\times n;i,j} = \Gamma^{s_0}_{m\times n}(\mathcal{B})$ and $\sum\limits_{1\le i,j\le 2^n} s^1_{m\times n;i,j} = \Gamma^{s_1}_{m\times n}(\mathcal{B})$,we can get

$$
\begin{aligned}
\sum_{1\le i,j\le 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} c_{n+1;1;1,i}c_{n+1;1;j,1} &= \Gamma^{s_0}_{m\times n}(\mathcal{B}) \\
\sum_{1\le i,j\le 2^n} \left[(C_{n+1;4})^{m-1}\right]_{i,j} c_{n+1;4;2^n,i}c_{n+1;4;j,2^n} &= \Gamma^{s_1}_{m\times n}(\mathcal{B}).
\end{aligned}
$$

Remark 11 *If we define* $H_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix}$, *then* $C_{n;i} = H_{n;i}\circ\left(E_{2^{n-2}} \otimes \widetilde{V}_{2;i}\right)$, *for* $1 \le i \le 4$.

The spatial entropy $h^{s_0}(\mathcal{B})$ of $\sum^{s_0}(\mathcal{B})$ is defined as follows :
Let $\Gamma^{s_0}_{m\times n}(\mathcal{B}) = card(\sum^{s_0}_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\sum^{s_0}_{m\times n}(\mathcal{B})$.
The spatial entropy $h^{s_0}(\mathcal{B})$ is defined as

$$h^{s_0}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log \Gamma^{s_0}_{m\times n}(\mathcal{B}).$$

Then,

$$h^{s_0}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log(\sum_{1\leq i,j\leq 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} c_{n+1;1;1,i} c_{n+1;1;j,1}).$$

Similarly, the spatial entropy $h^{s_1}(\mathcal{B})$ of $\sum^{s_1}(\mathcal{B})$ is defined as follows :
Let $\Gamma^{s_1}_{m\times n}(\mathcal{B}) = card(\sum^{s_1}_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\sum^{s_1}_{m\times n}(\mathcal{B})$.
The spatial entropy $h^{s_1}(\mathcal{B})$ is defined as

$$h^{s_1}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log \Gamma^{s_1}_{m\times n}(\mathcal{B}).$$

Then,

$$h^{s_1}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log(\left[(C_{n+1;4})^{m-1}\right]_{i,j} c_{n+1;1;2^n,i} c_{n+1;1;j,2^n}).$$

4.3 The boundary without corners

Now, we add the left and right layers of boundaries for the second kind of the spherical patterns (i.e, whose boundaries do NOT contain four corners), we do some definition as following:

Let $U^1 = \left(u^1_{1,\beta_1}\right)_{1\le\beta_1\le n} \in \sum_{1\times n}$, $U^2 = \left(u^2_{\alpha_1,\alpha_2}\right)_{1\le\alpha_1\le m, 1\le\alpha_1\le n+2} \in \sum_{m\times(n+2)}$, and $U^3 = \left(u^3_{1,\beta_2}\right)_{1\le\beta_2\le n} \in \sum_{1\times n}$,define

$$U^1 \circledast U^2 \circledast U^3 = \begin{array}{|c|c|c|c|c|c|}
\cline{2-5} \multicolumn{1}{c|}{} & u^2_{1,n+2} & u^2_{2,n+2} & \cdots & u^2_{m,n+2} & \multicolumn{1}{c}{} \\
\hline u^1_{1,n} & u^2_{1,n+1} & u^2_{2,n+1} & \cdots & u^2_{m,n+1} & u^3_{1,n} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline u^1_{1,1} & u^2_{1,2} & u^2_{2,2} & \cdots & u^2_{m,2} & u^3_{1,1} \\
\hline \multicolumn{1}{c|}{} & u^2_{1,1} & u^2_{2,1} & \cdots & u^2_{m,1} & \multicolumn{1}{c}{} \\
\cline{2-5}
\end{array}.$$

and

$$\overset{\backsim}{\sum}{}^{s_0}_{m_1\times m_2} = \Big\{ u^1_{1,\beta_1} \circledast u^2_{\alpha_1,\alpha_2} \circledast u^3_{1,\beta_2} \mid u^1_{1,\beta_1} \in \sum_{1\times m_2}, u^2_{\alpha_1,\alpha_2} \in \sum_{m_1\times(m_2+2)}, \text{ and } u^3_{1,\beta_2} \in \sum_{1\times m_2}, u^1_{1,i} = u^2_{j,1} = u^2_{j,m_2+2} = u^3_{1,i} = 0, \text{ for } 1\le i\le m_{2,}\ 1\le j\le m_1 \Big\}$$

and

$$\overset{\backsim}{\sum}{}^{s_1}_{m_1\times m_2} = \Big\{ u^1_{1,\beta_1} \circledast u^2_{\alpha_1,\alpha_2} \circledast u^3_{1,\beta_2} \mid u^1_{1,\beta_1} \in \sum_{1\times m_2}, u^2_{\alpha_1,\alpha_2} \in \sum_{m_1\times(m_2+2)}, \text{ and } u^3_{1,\beta_2} \in \sum_{1\times m_2}, u^1_{1,i} = u^2_{j,1} = u^2_{j,m_2+2} = u^3_{1,i} = 1, \text{ for } 1\le i\le m_{2,}\ 1\le j\le m_1 \Big\}$$

And then, we can save $\overset{\backsim}{\sum}{}^{s_0}_{m\times n}$ in a $2^n\times 2^n$ matrix $\tilde{S}^*_{0,m\times n} = \left[\tilde{s}^*_{0,m\times n;i,j}\right]_{1\le i,j\le 2^n}$, and the entries

$$\tilde{s}^*_{0,m\times n;i,j} = \left\{ x_{n;1,i} \overset{\wedge}{\circledast} x^c_{n+1;i,k_1,k_2\cdots k_{m-2},j} \overset{\wedge}{\circledast} x_{n;j,1} \mid 1\le k_1, k_2\cdots, k_{m-2} \le 2^n \right\}$$

are subsets of $\overset{\backsim}{\sum}{}^{s_0}_{m\times n}$,where $\overset{\wedge}{\circledast}$ means "the $\circledast$ need to overlap one layer". Since we know

$$\chi\left(\begin{array}{|c|}\hline u^2_{1,n} \\ \hline \vdots \\ \hline u^2_{1,1} \\ \hline\end{array}\right) = \chi^c\left(\begin{array}{|c|}\hline 0 \\ \hline u^2_{1,n} \\ \hline \vdots \\ \hline u^2_{1,1} \\ \hline 0 \\ \hline\end{array}\right) \text{ and } \chi\left(\begin{array}{|c|}\hline u^2_{m,n} \\ \hline \vdots \\ \hline u^2_{m,1} \\ \hline\end{array}\right) = \chi^c\left(\begin{array}{|c|}\hline 0 \\ \hline u^2_{m,n} \\ \hline \vdots \\ \hline u^2_{m,1} \\ \hline 0 \\ \hline\end{array}\right).$$

We know $\bigcup\limits_{1\le i,j\le 2^n} \tilde{s}^*_{0,m\times n;i,j} = \overset{\backsim}{\sum}{}^{s_0}_{m\times n}$ and $\tilde{s}^*_{0,m\times n;i,j}$ are disjoint to each other.

Given a basic set $\mathcal{B} \subset \sum_{2\times 2}$, the $2^n \times 2^n$ matrix $\tilde{S}^0_{m\times n}$ can be defined $\tilde{S}^0_{m\times n} = \left[\tilde{s}^0_{m\times n;i,j}\right]_{1\leq i,j\leq 2^n}$, where

$$\tilde{s}^0_{m\times n;i,j} = card\left(\tilde{s}^*_{0,m\times n;i,j} \cap \overset{\backsim\ s_0}{\sum}_{m\times n}(\mathcal{B})\right).$$

Then $\sum\limits_{1\leq i,j\leq 2^n} \tilde{s}_{0,m\times n;i,j} = \tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B})$ where $\tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B}) = card\left(\overset{\backsim\ s_0}{\sum}_{m\times n}(\mathcal{B})\right)$.

By the construction of $C_{n+1;1} = [c_{n+1;1;i_1,i_2}]_{1\leq i_1,i_2\leq 2^{n+1}}$ *and* $H_n = [h_{n;i_1,i_2}]_{1\leq i_1,i_2\leq 2^{n+1}}$ *for the same* $\mathcal{B} \subset \sum_{2\times 2}$ and the above arguments, we can get

$$\begin{aligned}
\tilde{s}^0_{m\times n;i,j} &= \sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} h_{n;1,i}c_{n+1;1;i,k_1}c_{n+1;1;k_1,k_2}\cdots c_{n+1;1;k_{m-2},j}h_{n;j,1} \\
&= \Big(\sum_{1\leq k_1,k_2\cdots,k_{m-2}\leq 2^n} c_{n+1;1;i,k_1}c_{n+1;1;k_1,k_2}\cdots c_{n+1;1;k_m,j}\Big)h_{n;1,i}h_{n;j,1} \\
&= c_{0,m\times n;i,j}h_{n;1,i}h_{n;j,1} \\
&= \left[(C_{n+1;1})^{m-1}\right]_{i,j} h_{n;1,i}h_{n;j,1} \text{ for } 1\leq i,j\leq 2^n
\end{aligned}$$

i.e.,

$$\begin{aligned}
\tilde{S}_{0,m\times n} &= (C_{n+1;1})^{m-1} \circ \begin{bmatrix} h_{n;1,1} & h_{n;1,1} & \cdots & h_{n;1,1} \\ h_{n;1,2} & h_{n;1,2} & \cdots & h_{n;1,2} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n;1,2^n} & h_{n;1,2^n} & \cdots & h_{n;1,2^n} \end{bmatrix} \\
&\qquad \circ \begin{bmatrix} h_{n;1,1} & h_{n;2,1} & \cdots & h_{n;2^n,1} \\ h_{n;1,1} & h_{n;2,1} & \cdots & h_{n;2^n,1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n;1,1} & h_{n;2,1} & \cdots & h_{n;2^n,1} \end{bmatrix} \\
&= (C_{n+1;1})^{m-1} \circ [h_{n;1,i}h_{n;j,1}]_{1\leq i,j\leq 2^n}
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\tilde{S}_{1,m\times n} &= (C_{n+1;4})^{m-1} \circ \begin{bmatrix} h_{n;2^{n}\cdot,1} & h_{n;2^{n}\cdot,1} & \cdots & h_{n;2^{n}\cdot,1} \\ h_{n;2^n,2} & h_{n;2^n,2} & \cdots & h_{n;2^n,2} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n;2^n,2^n} & h_{n;2^n,2^n} & \cdots & h_{n;2^n,2^n} \end{bmatrix} \\
&\qquad \circ \begin{bmatrix} h_{n;1,2^n} & h_{n;2,2^n} & \cdots & h_{n;2^n,2^n} \\ h_{n;1,2^n} & h_{n;2,2^n} & \cdots & h_{n;2^n,2^n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n;1,2^n} & h_{n;2,2^n} & \cdots & h_{n;2^n,2^n} \end{bmatrix} \\
&= (C_{n+1;4})^{m-1} \circ [h_{n;2^n,i} h_{n;j,2^n}]_{1\le i,j\le 2^n}
\end{aligned}
$$

Since $\sum\limits_{1\le i,j\le 2^n} \tilde{s}^0_{m\times n;i,j} = \tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B})$ and $\sum\limits_{1\le i,j\le 2^n} \tilde{s}^1_{m\times n;i,j} = \tilde{\Gamma}^{s_1}_{m\times n}(\mathcal{B})$,we can get

$$
\begin{aligned}
\sum_{1\le i,j\le 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} [h_{n;1,i} h_{n;j,1}] &= \tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B}) \\
\sum_{1\le i,j\le 2^n} \left[(C_{n+1;4})^{m-1}\right]_{i,j} [h_{n;2^n,i} h_{n;j,2^n}] &= \tilde{\Gamma}^{s_1}_{m\times n}(\mathcal{B}).
\end{aligned}
$$

The spatial entropy $\tilde{h}^{s_0}(\mathcal{B})$ of $\overset{\backsim}{\sum}{}^{s_0}(\mathcal{B})$ is defined as follows :
Let $\tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B}) = card(\overset{\backsim}{\sum}{}^{s_0}_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\overset{\backsim}{\sum}{}^{s_0}_{m\times n}(\mathcal{B})$.
The spatial entropy $\tilde{h}^{s_0}(\mathcal{B})$ is defined as

$$\tilde{h}^{s_0}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log \tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B}).$$

Then,

$$\tilde{h}^{s_0}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log\Big(\sum_{1\le i,j\le 2^n} \left[(C_{v,n+1;1})^{m-1}\right]_{i,j} [h_{n;1,i} h_{n;j,1}]\Big).$$

Similarly, the spatial entropy $\tilde{h}^{s_1}(\mathcal{B})$ of$\overset{\backsim}{\sum}{}^{s_1}(\mathcal{B})$ is defined as follows :
Let $\tilde{\Gamma}^{s_1}_{m\times n}(\mathcal{B}) = card(\overset{\backsim}{\sum}{}^{s_1}_{m\times n}(\mathcal{B}))$,the number of distinct patterns in $\overset{\backsim}{\sum}{}^{s_1}_{m\times n}(\mathcal{B})$.

The spatial entropy $\tilde{h}^{s_1}(\mathcal{B})$ is defined as

$$\tilde{h}^{s_1}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log \tilde{\Gamma}^{s_1}_{m\times n}(\mathcal{B}).$$

Then,

$$\tilde{h}^{s_1}(\mathcal{B}) = \limsup_{m,n\to\infty} \frac{1}{mn} \log(\sum_{1\le i,j\le 2^n} \left[(C_{v,n+1;4})^{m-1} \right]_{i,j} \left[h_{n;2^n,i} h_{n;j,2^n} \right]).$$

5 Conclusion

We get the recusive formulas for patterns generation in two-dimensional surfaces (cylinder, torus and sphere).

1. Cylinder: $C_{m\times n} = (C_n)^{m-1}$ for $m \ge 2$. Furthermore,

$$\sum_{1\le i,j\le 2^n} \left[(C_n)^{m-1} \right]_{i,j} = \Gamma^c_{m\times n}(\mathcal{B}).$$

2. Torus: $D_{m\times n} = (C_n)^{m-1} \circ (C_n)^T$ for $m \ge 2$. Furthermore,

$$\sum_{1\le i,j\le 2^n} \left[(C_n)^{m-1} \circ (C_n)^T \right]_{i,j} = \Gamma^t_{m\times n}(\mathcal{B}).$$

3. Sphere:
 (i) The boundary contains four corners.

$$\begin{aligned} S^0_{m\times n} &= (C_{n+1;1})^{m-1} \circ \left[c_{n+1;1;1,i} c_{n+1;1;j,1} \right] \\ S^1_{m\times n} &= (C_{n+1;4})^{m-1} \circ \left[c_{n+1;4;2^n,i} c_{n+1;4;j,2^n} \right] \end{aligned}$$

 Furthermore,

$$\begin{aligned} \sum_{1\le i,j\le 2^n} \left[(C_{n+1;1})^{m-1} \right]_{i,j} c_{n+1;1;1,i} c_{n+1;1;j,1} &= \Gamma^{s_0}_{m\times n}(\mathcal{B}) \\ \sum_{1\le i,j\le 2^n} \left[(C_{n+1;4})^{m-1} \right]_{i,j} c_{n+1;1;2^n,i} c_{n+1;1;j,2^n} &= \Gamma^{s_1}_{m\times n}(\mathcal{B}). \end{aligned}$$

(ii) The boundary dose NOT contain four corners.

$$\tilde{S}_{0,m\times n} = (C_{n+1;1})^{m-1} \circ [h_{n;1,i}h_{n;j,1}]_{1\leq i,j\leq 2^n}$$
$$\tilde{S}_{1,m\times n} = (C_{n+1;4})^{m-1} \circ [h_{n;2^n,i}h_{n;j,2^n}]_{1\leq i,j\leq 2^n}$$

Furthermpre,

$$\sum_{1\leq i,j\leq 2^n} \left[(C_{n+1;1})^{m-1}\right]_{i,j} [h_{n;1,i}h_{n;j,1}] = \tilde{\Gamma}^{s_0}_{m\times n}(\mathcal{B}).$$
$$\sum_{1\leq i,j\leq 2^n} \left[(C_{n+1;4})^{m-1}\right]_{i,j} [h_{n;2^n,i}h_{n;j,2^n}] = \tilde{\Gamma}^{s_1}_{m\times n}(\mathcal{B}).$$

Future plan: (i) Compare the entropy of the plane with the entropy of these surfaces. If they are not the same, we will find some sufficient conditions to let them equal.
(ii) We will extend the two-dimensional resulte to higher-dimensional cases.And also consider the above problems.

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