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Long paths in hypercubes with conditional node-faults

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ABSTRACT

Let F be a set of $f \leqslant 2n-5$ faulty nodes in an n-cube Q_n such that every node of Q_n still has at least two fault-free neighbors. Then we show that $Q_n - F$ contains a path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between any two nodes of odd (respectively, even) distance. Since the n-cube is bipartite, the path of length $2^n - 2f - 1$ (or $2^n - 2f - 2$) turns out to be the longest if all faulty nodes belong to the same partite set. As a contribution, our study improves upon the previous result presented by [J.-S. Fu, Longest fault-free paths in hypercubes with vertex faults, Information Sciences 176 (2006) 759–771] where only $n-2$ faulty nodes are considered.

1. Introduction

In many parallel computer systems, processors are connected on the basis of *interconnection networks*, referred to as *networks* henceforth. Among various kinds of networks, hypercube is one of the most attractive topologies discovered for its suitability in both special-purpose and general-purpose tasks [11]. One important issue to address in hypercubes is how to embed other networks into hypercubes. By definition [11], embedding one guest network G into another host network H is a form of injective mapping, η, from the node set of G to the node set of H. A link of G corresponds to a path of H under η. Often embedding takes cycles, paths, or meshes as guest networks [3–5,19,20] because these architectures are extensively applied in parallel systems.

Fault-tolerant embedding in hypercubes has been widely addressed in researches [2,6,7,9,13,15–18]. For example, Latifi et al. [9] proved that an *n-dimensional hypercube* (or *n-cube*), Q_n, is Hamiltonian even if it has $n-2$ faulty links. On the other hand, Tsai et al. [15] showed that Q_n $(n \geqslant 3)$ is both Hamiltonian laceable and strongly Hamiltonian laceable even if it has $n-2$ faulty links. Recently, Tsai and Lai [17] addressed the conditional edge-fault-tolerant edge-bipancyclicity of hypercubes. As Tseng [18] showed, a faulty n-cube, containing $f_e \leqslant n-4$ faulty links and $f_v \leqslant n-1$ faulty nodes with $f_e + f_v \leqslant n-1$, has a fault-free cycle of length at least $2^n - 2f_v$. Furthermore, Fu [6] showed that a fault-free cycle of length at least $2^n - 2f$ can be embedded into an n-cube with $1 \leqslant f \leqslant 2n-4$ faulty nodes. Fu [7] also proved that a fault-free path of length at least $2^n - 2f - 1$ (or $2^n - 2f - 2$) can be embedded to join two arbitrary nodes of odd (or even) distance in an n-cube with $f \leqslant n-2$ faulty nodes.

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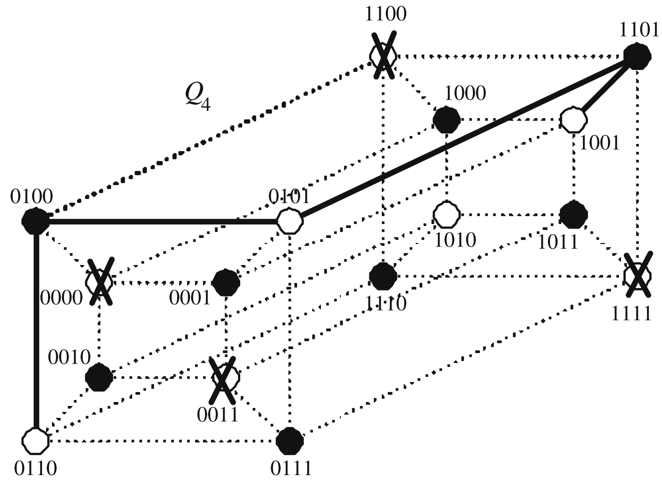

Fig. 1. A conditionally faulty Q_4 with four faulty nodes. Every faulty node is marked by an "X" symbol. The length of the longest path between nodes 0110 and 1001 is 4.

Basically, the components of a network may fail independently. It is unlikely that all failures would be close to each other. Based on this phenomenon, the *conditional node-faults* [10] were defined in such a way that each node of a faulty network still has at least g fault-free neighbors. In this paper, we concern that $g = 2$. More precisely, a network is said to be *conditionally faulty* if and only if every node has at least two fault-free neighbors. Under this premise, we would like to extend Fu's result [7] by showing that a conditionally faulty n-cube with $f \leqslant 2n-5$ faulty nodes still contains a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between any two fault-free nodes of odd (respectively, even) distance. Consider a 4-cube with four faulty nodes, 0000, 0011, 1100, and 1111, as shown in Fig. 1, in which every node has at least two fault-free neighbors. Then the length of the longest path between nodes 0110 and 1001 is $4 < 2^4 - 2 \cdot 4 - 2$. This is why we concentrate only on $f \leqslant 2n-5$ faulty nodes.

It is sufficient to assume that every node should have at least two fault-free neighbors while a long path is constructed between every pair of fault-free nodes. Consider the scenario that u is a fault-free node with only one fault-free neighbor, namely v. Then the longest path between u and v happens to be of length 1. To avoid such a degenerate situation, it is necessary that, for any pair u, v of adjacent nodes, u has some fault-free neighbor other than v, and vice versa. On the other hand, it is also statistically reasonable to require that every node needs to have at least two fault-free neighbors. Suppose, with a random fault model, the probabilities of node failures are identical and independent. Let $Pr(n)$ denote the probability that every node of the n-cube Q_n, containing $2n-5$ faulty nodes, is adjacent to at least two fault-free neighbors. Because Q_n has 2^n nodes, there are $\binom{2^n}{2n-5}$ ways to distribute $2n-5$ faulty nodes. In the random fault model, all these fault distributions have equal probability of occurrence. Clearly, $Pr(3) = 1$ and $Pr(4) = 1 - \frac{2^4 \times \binom{4}{3}}{\binom{2^4}{3}} = \frac{31}{35}$, where $2^4 \times \binom{4}{3}$ is the number of faulty node distributions that there exists some node having three faulty neighbors. When $n \geqslant 5$, the number of faulty node distributions that there exists some node having n faulty neighbors is $2^n \times \binom{2^n - n}{n-5}$. Moreover, the number of faulty node distributions that there exists some node having exactly $n-1$ faulty neighbors is $2^n \times \binom{n}{n-1}\binom{2^n-n}{n-4}$. Since $\binom{2^n-n}{n-4} \geqslant \binom{2^n-n}{n-5}$ for $n \geqslant 5$, we can derive that

$$Pr(n) = 1 - Pr(\text{some node has at least } n-1 \text{ faulty neighbors}) = 1 - \frac{2^n \times \binom{2^n-n}{n-5+2^n} \times \binom{n}{n-1}\binom{2^n-n}{n-4}}{\binom{2^n}{2n-5}}$$

$$\geqslant 1 - \frac{2^n \times (1+n) \times \binom{2^n-n}{n-4}}{\binom{2^n}{2n-5}} = 1 - \frac{2^n \times (1+n) \times (2^n - 2n + 5) \times \prod_{k=n-3}^{2n-5} k}{\prod_{k=2^n-n+1}^{2^n} k}$$

$$= 1 - \frac{(n-3)(n-2)}{2^n-n+1} \times \frac{n-1}{2^n-n} \times \cdots \times \frac{2n-5}{2^n-3} \times \frac{n+1}{2^n-2} \times \frac{2^n-2n+5}{2^n-1} \triangleq L(n).$$

It is not difficult to compute $Pr(n)$ numerically, such as $Pr(5) = \frac{6157}{6293}$, $Pr(6) = \frac{9696527}{9706503}$, etc. Since $\lim_{n\to\infty} L(n) = 1$, $Pr(n)$ approaches to 1 as n increases.

The rest of this paper is organized as follows. In Section 2, basic definitions and notations are introduced. In Section 3, a partition procedure, named *PARTITION*, is proposed to divide a conditionally faulty n-cube into two conditionally faulty subcubes. In Section 4, we show that a conditionally faulty n-cube with $f \leqslant 2n-5$ faulty nodes has a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between any two fault-free nodes of odd (respectively, even) distance. Finally, the conclusion and discussion are presented in Section 5.

2. Preliminaries

Throughout this paper, we concentrate on loopless undirected graphs. For the graph definitions, we follow the ones given by Bondy and Murty [1]. A *graph* G consists of a node set $V(G)$ and a link set $E(G)$ that is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V(G)\}$. It is *bipartite* if its node set can be partitioned into two disjoint partite sets, $V_0(G)$ and $V_1(G)$, such that every link joins a node of $V_0(G)$ and a node of $V_1(G)$.

A *path* P of length k from node x to node y in a graph G is a sequence of distinct nodes $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $x = v_1$, $y = v_{k+1}$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leqslant i \leqslant k$ if $k \geqslant 1$. Moreover, a path of length 0 consisting of a single node x is denoted by $\langle x\rangle$. For convenience, we write P as $\langle v_1, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$, where $Q = \langle v_i, \ldots, v_j\rangle$. The ith node of P is denoted by $P(i)$; i.e., $P(i) = v_i$. A *cycle* is a path with at least three nodes such that the last node is adjacent to the first one. For clarity, a cycle of length k is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A path (or cycle) in a graph G is a *Hamiltonian path* (or *Hamiltonian cycle*) if it spans G. A bipartite graph is *Hamiltonian laceable* [14] if there exists a Hamiltonian path between any two nodes that are in different partite sets. Furthermore, a Hamiltonian laceable graph G is *hyper-Hamiltonian laceable* [12] if, for any node $v \in V_i(G)$ and $i \in \{0,1\}$, there exists a Hamiltonian path of $G - \{v\}$ between any two nodes of $V_{1-i}(G)$. Later Hsieh et al. [8] introduced *strongly Hamiltonian laceability*. A Hamiltonian laceable graph G is strongly Hamiltonian laceable if there exists a path of length $|V(G)| - 2$ between any two nodes in the same partite set.

Let $u = b_n \ldots b_i \ldots b_1$ be an n-bit binary string. For $1 \leqslant i \leqslant n$, we use $(u)^i$ to denote the binary string $b_n \ldots \bar{b}_i \ldots b_1$. Moreover, we use $[u]_i$ to denote the bit b_i of u. The *Hamming weight* of u, denoted by $w_H(u)$, is $|\{1 \leqslant j \leqslant n \mid [u]_j = 1\}|$. The n-cube Q_n consists of 2^n nodes and $n2^{n-1}$ links. Each node corresponds to an n-bit binary string. Two nodes, u and v, are adjacent if and only if $v = (u)^i$ for some i, and we call the link $(u,(u)^i)$ *i-dimensional*. We define $dim((u,v)) = i$ if $v = (u)^i$. The *Hamming distance* between u and v, denoted by $h(u,v)$, is defined to be $|\{1 \leqslant j \leqslant n \mid [u]_j \neq [v]_j\}|$. Hence two nodes, u and v, are adjacent if and only if $h(u,v) = 1$. It is well known that Q_n is a bipartite graph with partite sets $V_0(Q_n) = \{u \in V(Q_n) \mid w_H(u)$ is even$\}$ and $V_1(Q_n) = \{u \in V(Q_n) \mid w_H(u)$ is odd$\}$.

A graph G is *node-transitive* if, for any pair v_1, v_2 of $V(G)$, there exists some automorphism $\mu : V(G) \to V(G)$ such that $\mu(v_1) = v_2$. A graph G is *link-transitive* if, for any two links $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ of G, there exists some automorphism $\psi : V(G) \to V(G)$ such that $\psi(u_1) = u_2$ and $\psi(v_1) = v_2$. As introduced in [11], Q_n is both node-transitive and link-transitive. The following two theorems reveal the link-fault-tolerant Hamiltonian laceability of hypercubes.

Theorem 1 [15]. *Let $n \geqslant 3$. Suppose that $F \subseteq E(Q_n)$ is a set of utmost $n-2$ faulty links. Then $Q_n - F$ is Hamiltonian laceable and strongly Hamiltonian laceable.*

Theorem 2 [15]. *Let $n \geqslant 3$. Suppose that $F \subseteq E(Q_n)$ is a set of utmost $n-3$ faulty links. Then $Q_n - F$ is hyper-Hamiltonian laceable.*

3. Partition of faulty hypercubes

In this section, we show that a conditionally faulty n-cube can be partitioned into two conditionally faulty subcubes if it has $2n-5$ or less faulty nodes. First of all, we introduced some notations to be used later. For $1 \leqslant j \leqslant n$ and $i \in \{0,1\}$, let $Q_n^{j,i}$ be a subgraph of Q_n induced by $\{u \in V(Q_n) \mid [u]_j = i\}$. Obviously, $Q_n^{j,i}$ is isomorphic to Q_{n-1}. Then the node partition of Q_n into subgraphs $Q_n^{j,0}$ and $Q_n^{j,1}$ is called *j-partition*. For convenience, we use $F(G)$ to denote the set of all faulty nodes in graph G. For any node u of G, its *neighborhood* $N_G(u)$ is defined by $N_G(u) = \{v \in V(G) \mid (u,v) \in E(G)\}$. In addition, let $N_G^F(u)$ denote the set $N_G(u) \cap F(G)$.

Suppose Q_n, $n \geqslant 4$, is conditionally faulty with $f \leqslant 2n-5$ faulty nodes. Moreover, suppose u, v, and w are three nodes of this faulty n-cube, and each of them has only two fault-free neighbors. Then we discuss how the faulty nodes will be distributed conditionally. For simplification, let $U = N_{Q_n}^F(u)$, $V = N_{Q_n}^F(v)$, and $W = N_{Q_n}^F(w)$.

If $|V \cap W| = 0$, then we have $f \geqslant |V \cup W| = |V| + |W| = 2n-4$, contradicting the requirement that $f \leqslant 2n-5$. Therefore, $|V \cap W| \geqslant 1$ needs to be satisfied. Similarly, we also have $|U \cap V| \geqslant 1$ and $|U \cap W| \geqslant 1$. Since any two nodes of an n-cube can have utmost two common neighbors, we obtain that $|V \cap W|, |U \cap V|, |U \cap W| \in \{1,2\}$. We first consider the case that at least one of $|V \cap W|$, $|U \cap V|$, and $|U \cap W|$ is equal to 1. Without loss of generality, we suppose $|V \cap W| = 1$.

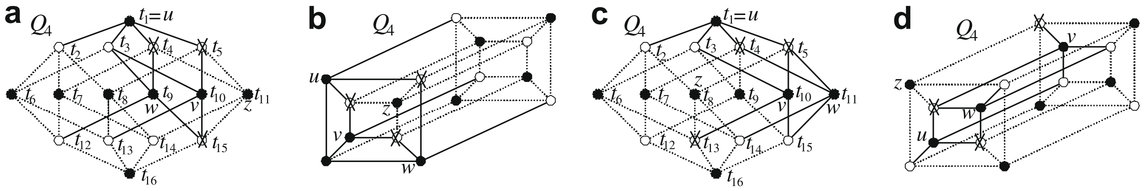

Fig. 2. Every faulty node is marked by an "X" symbol. (a) The Q_4 with $|N^F_{Q_4}(u) \cap N^F_{Q_4}(v)| = |N^F_{Q_4}(v) \cap N^F_{Q_4}(w)| = |N^F_{Q_4}(u) \cap N^F_{Q_4}(w)| = 1$; (b) a layout isomorphic to (a); (c) the Q_4 with $|N^F_{Q_4}(u) \cap N^F_{Q_4}(v)| = |N^F_{Q_4}(v) \cap N^F_{Q_4}(w)| = 1$ and $|N^F_{Q_4}(u) \cap N^F_{Q_4}(w)| = 2$; (d) a layout isomorphic to (c).

I. First, we concern the case that $|V \cap W| = |U \cap V| = |U \cap W| = 1$. If $|U \cap V \cap W| \geqslant 1$, we have $2n-5 \geqslant f \geqslant |U \cup V \cup W| = 3(n-2) - (1+1+1) + 1 = 3n-8$; i.e., $n \leqslant 3$. Since $n \geqslant 4$, we only concern $|U \cap V \cap W| = 0$. Then we have $2n-5 \geqslant f \geqslant |U \cup V \cup W| \geqslant 3(n-2) - (1+1+1) = 3n-9$; i.e., $n \leqslant 4$. Fig. 2a depicts a faulty 4-cube with $|V \cap W| = |U \cap V| = |U \cap W| = 1$ and $|U \cap V \cap W| = 0$. Fig. 2b is a cube-styled layout isomorphic to Fig. 2a. We can examine Fig. 2a in a top-down viewpoint. Since hypercube is node-transitive, we can assume that $u = t_1$. By link-transitivity, we assume that t_4 and t_5 are faulty neighbors of u. Since $|U \cap V| = 1$, we obtain $v \in \{t_7, t_8, t_9, t_{10}\}$. Without loss of generality, we assume that $v = t_{10}$. Since $|U \cap W| = |V \cap W| = 1$ and $|U \cap V \cap W| = 0$, we see that $w = t_9$ and $V \cap W = \{t_{15}\}$. As a consequence, this happens to be the only possibility. However, node t_{11} has only one fault-free neighbor. Thus it is not conditionally faulty.

II. Secondly, we consider the case that $|V \cap W| = |U \cap V| = 1$ and $|U \cap W| = 2$. By the definition of hypercube, we see that $|N_{Q_n}(u) \cap N_{Q_n}(v) \cap N_{Q_n}(w)| \leqslant 1$. Obviously, we have $|U \cap V \cap W| \leqslant |N_{Q_n}(u) \cap N_{Q_n}(v) \cap N_{Q_n}(w)|$. In particular, we claim that $|U \cap V \cap W| = 1$. Suppose, by contradiction, that $|U \cap V \cap W| = 0$. Then we have $U \cap V \cap W = (U \cap V) \cap (U \cap W) = \emptyset$. Since $U \cap V \neq \emptyset$ and $U \cap W \neq \emptyset$, we conclude that $V \cap W = \emptyset$. That is, the assumption of $|U \cap V \cap W| = 0$ leads to a contradiction between $|V \cap W| = 1$ and $V \cap W = \emptyset$. As a result, $|U \cap V \cap W|$ is equal to 1. Accordingly, we have $2n-5 \geqslant f \geqslant |U \cup V \cup W| = 3(n-2) - (1+1+2) + 1 = 3n-9$; i.e., $n \leqslant 4$. See Fig. 2c for illustration. For clarity, Fig. 2d is an isomorphic layout of Fig. 2c. Similarly, we can examine Fig. 2c in a top-down viewpoint. By node-transitivity, we assume that $u = t_1$. By link-transitivity, we assume that t_4 and t_5 are faulty neighbors of u. Since $|U \cap W| = 2$, we have $w = t_{11}$. Since $|V \cap W| = |U \cap V| = 1$ and $|U \cap V \cap W| = 1$, we obtain $v \in \{t_7, t_8, t_9, t_{10}\}$. Without loss of generality, we assume that $v = t_{10}$. Then this turns out to be the only possibility. It is noticed that node t_8 has only two fault-free neighbors.

III. Next, we concern the case that $|V \cap W| = 1$ and $|U \cap V| = |U \cap W| = 2$. Similarly, we have $|U \cap V \cap W| = 1$. Since $(U \cap V) \cup (U \cap W) \subseteq U$, we have $|(U \cap V) \cup (U \cap W)| \leqslant |U|$. However, we have a contradiction that $|(U \cap V) \cup (U \cap W)| = |U \cap V| + |U \cap W| - |U \cap V \cap W| = 2 + 2 - 1 = 3 > n - 2 = |U|$ if $n \leqslant 4$. In what follows, we suppose that $n \geqslant 5$. As a consequence, we have $2n-5 \geqslant f \geqslant |U \cup V \cup W| = 3(n-2) - (1+2+2) + 1 = 3n-10$; i.e., $n = 5$. See Fig. 3a. Again, we examine Fig. 3a in a top-down viewpoint. By node-transitivity, we assume that $u = t_1$. By link-transitivity, we assume that t_4, t_5, and t_6 are faulty neighbors of u. Since $|U \cap V| = |U \cap W| = 2$, we have $\{v, w\} \subset \{t_{14}, t_{15}, t_{16}\}$. Without loss of generality, we assume that $v = t_{14}$ and $w = t_{16}$. Since $|V \cap W| = 1$, we have $t_{26} \notin V \cup W$. Moreover, we have $2n-5 \geqslant f \geqslant |V \cup W| = |V| + |W| - |V \cap W| = (n-2) + (n-2) - 1$

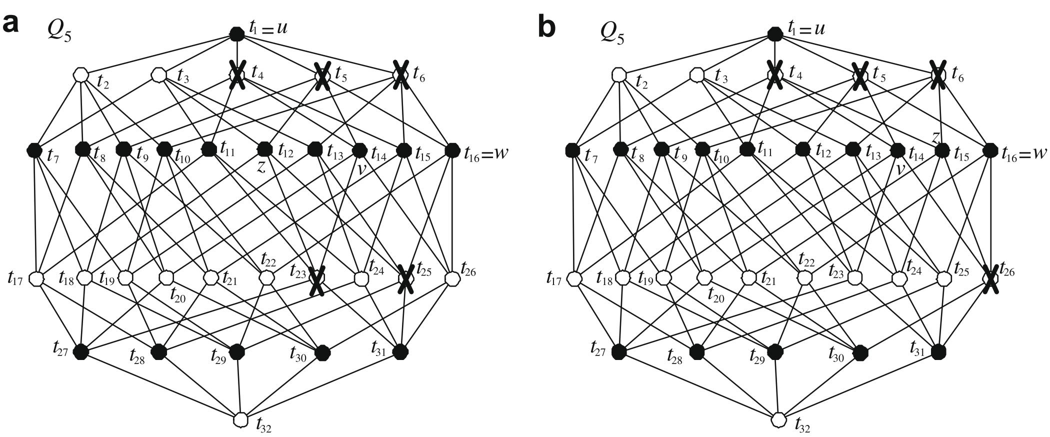

Fig. 3. Every faulty node is marked by an "X" symbol. Each of u, v, w, and z has only two fault-free neighbors. (a) The Q_5 with $|N^F_{Q_5}(v) \cap N^F_{Q_5}(w)| = 1$ and $|N^F_{Q_5}(u) \cap N^F_{Q_5}(v)| = |N^F_{Q_5}(u) \cap N^F_{Q_5}(w)| = 2$; (b) the Q_5 with $|N^F_{Q_5}(u) \cap N^F_{Q_5}(v)| = |N^F_{Q_5}(v) \cap N^F_{Q_5}(w)| = |N^F_{Q_5}(u) \cap N^F_{Q_5}(w)| = 2$.

$= 2n-5$; that is, $f = 2n-5$ and $U \subseteq V \cup W$. Then we have either $t_{20} \in V$ or $t_{23} \in V$. Without loss of generality, we assume that $t_{23} \in V$. Similarly, we can assume that $t_{25} \in W$. As a result, this is the only possibility. It is noted that node $t_{12} = z$ has three faulty neighbors, and $|N^F_{Q_5}(x)| \leqslant 2$ for each $x \in V(Q_5) - \{u, v, w, z\}$.

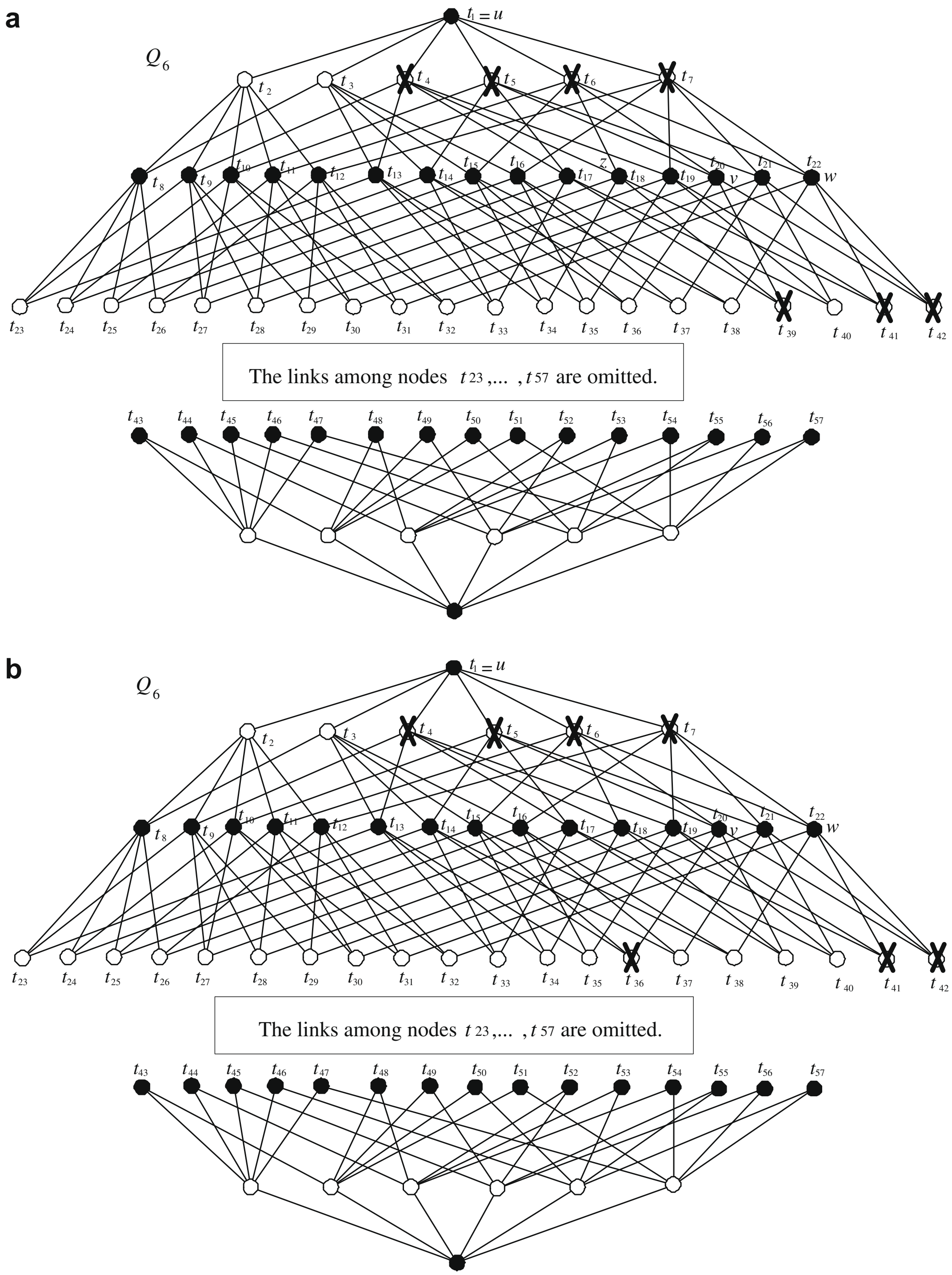

Fig. 4. Every faulty node is marked by an "X" symbol. The Q_6 with $|N^F_{Q_6}(u) \cap N^F_{Q_6}(v)| = |N^F_{Q_6}(v) \cap N^F_{Q_6}(w)| = |N^F_{Q_6}(u) \cap N^F_{Q_6}(w)| = 2$. (a) $|N^F_{Q_6}(u)| = |N^F_{Q_6}(v)| = |N^F_{Q_6}(w)| = |N^F_{Q_6}(z)| = 4$ and $|N^F_{Q_6}(x)| \leqslant 3$ for $x \in V(Q_6) - \{u, v, w, z\}$; (b) $|N^F_{Q_6}(u)| = |N^F_{Q_6}(v)| = |N^F_{Q_6}(w)| = 4$ and $|N^F_{Q_6}(x)| \leqslant 3$ for $x \in V(Q_6) - \{u, v, w\}$.

Now we consider the case that $|V \cap W| = |U \cap V| = |U \cap W| = 2$. Again, we have $|U \cap V \cap W| = 1$. Since $|(U \cap V) \cup (U \cap W)| \leqslant |U|$, we still have a contradiction that $|(U \cap V) \cup (U \cap W)| = |U \cap V| + |U \cap W| - |U \cap V \cap W| = 2 + 2 - 1 = 3 > n - 2 = |U|$ if $n \leqslant 4$. In what follows, we suppose $n \geqslant 5$. Then we have $2n - 5 \geqslant f \geqslant |U \cup V \cup W| = 3(n-2) - (2+2+2) + 1 = 3n - 11$; i.e., $n \in \{5, 6\}$. Note that $|U \cup V \cup W| = 4$ if $n = 5$ and $|U \cup V \cup W| = 7$ if $n = 6$. See Fig. 3b and Fig. 4a and b. In Fig. 3b, it is not difficult to see that $|N^F_{Q_5}(x)| \leqslant 2$ for each $x \in V(Q_5) - \{u, v, w, z\}$. We explain Fig. 4 as follows. By node-transitivity, we assume that $u = t_1$. By link-transitivity, we assume that t_4, t_5, t_6, and t_7 are faulty neighbors of u. Since $|U \cap V| = |U \cap W| = 2$, we deduce that $\{v, w\} \subset \{t_i \mid 17 \leqslant i \leqslant 22\}$. Since $|U \cap V \cap W| = 1$, we can assume that $v = t_{20}$ and $w = t_{22}$. Then we have $|V \cap \{t_{30}, t_{36}, t_{39}, t_{42}\}| = 2$ and $|W \cap \{t_{32}, t_{38}, t_{41}, t_{42}\}| = 2$. Since $|V \cap W| = 2$, we have $V \cap W = \{t_6, t_{42}\}$. If $t_{39} \in V$ and $t_{41} \in W$, then node t_{18} happens to have only two fault-free neighbors (see Fig 4a); otherwise, we have $|N^F_{Q_6}(x)| \leqslant 3$ for each $x \in V(Q_6) - \{u, v, w\}$ (see Fig. 4b, in which nodes t_{36} and t_{41}, for example, are faulty). Hence these figures cover all possibilities.

According to the analysis presented earlier, a conditionally faulty n-cube with $f \leqslant 2n - 5$ faulty nodes is likely to contain three or four nodes such that each of them has only two fault-free neighbors. Since $2n - 5 \leqslant n - 2$ for $n \leqslant 3$, we concentrate only on the case that $n \geqslant 4$. To summarize, we have the following two lemmas.

Lemma 1. *Suppose that an n-cube Q_n $(n \geqslant 4)$ is conditionally faulty with $f \leqslant 2n - 5$ faulty nodes. Let $u, v, w, z \in V(Q_n)$ such that $|N^F_{Q_n}(u)| = |N^F_{Q_n}(v)| = |N^F_{Q_n}(w)| = |N^F_{Q_n}(z)| = n - 2$ and $|N^F_{Q_n}(x)| \leqslant n - 3$ for every $x \in V(Q_n) - \{u, v, w, z\}$. Then the faulty nodes are distributed as illustrated in Figs. 2c, 3a and b, and 4a. In Figs. 2c and 3a, no dimensions can be used to partition Q_n in such a way that both resulting subcubes are conditionally faulty. In Fig. 3b and Fig. 4a, there exists some dimension j of $\{1, 2, \ldots, n\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n - 7$ or less faulty nodes.*

Proof. In Figs. 2c and 3a, we check, by brute force, that either $Q^{k,0}_n$ or $Q^{k,1}_n$ contains a node with only one fault-free neighbor for each $k \in \{1, 2, \ldots, n\}$; that is, there does not exist any dimension to partition Q_n such that both $(n-1)$-cubes are conditionally faulty. In Figs. 3b and 4a, let j be any integer of $\{1, 2, \ldots, n\}$ such that $(u)^j$ is faulty. Then both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n - 7$ or less faulty nodes. □

Lemma 2. *Suppose that an n-cube Q_n $(n \geqslant 4)$ is conditionally faulty with $f \leqslant 2n - 5$ faulty nodes. Let $u, v, w \in V(Q_n)$ such that $|N^F_{Q_n}(u)| = |N^F_{Q_n}(v)| = |N^F_{Q_n}(w)| = n - 2$ and $|N^F_{Q_n}(x)| \leqslant n - 3$ for every $x \in V(Q_n) - \{u, v, w\}$. Then the faulty nodes are distributed as illustrated in Fig. 4b. Moreover, there exists some dimension j of $\{1, 2, \ldots, n\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n - 7$ or less faulty nodes.*

Proof. Let $j \in \{1, 2, \ldots, n\}$ such that $(u)^j \in N^F_{Q_n}(u) \cap N^F_{Q_n}(v) \cap N^F_{Q_n}(w)$. Then both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty with $2n - 7$ or less faulty nodes. □

Lemma 3. *Suppose that an n-cube Q_n $(n \geqslant 4)$ is conditionally faulty with $f \leqslant 2n - 5$ faulty nodes. Let u and v be two nodes of Q_n such that $|N^F_{Q_n}(u)| = |N^F_{Q_n}(v)| = n - 2$ and $|N^F_{Q_n}(x)| \leqslant n - 3$ for every $x \in V(Q_n) - \{u, v\}$. Then there exists some dimension k of $\{1, 2, \ldots, n\}$ such that both $Q^{k,0}_n$ and $Q^{k,1}_n$ are conditionally faulty. When $n \geqslant 5$, both $Q^{k,0}_n$ and $Q^{k,1}_n$ contain $2n - 7$ or less faulty nodes.*

Proof. Since $|N^F_{Q_n}(u)| = |N^F_{Q_n}(v)| = n - 2$ and $f \leqslant 2n - 5$, we have $|N^F_{Q_n}(u) \cap N^F_{Q_n}(v)| \geqslant 1$. Since any two nodes of Q_n can have utmost two common neighbors, we consider the following two cases.

Case 1: Suppose that $|N^F_{Q_n}(u) \cap N^F_{Q_n}(v)| = 2$. Let i and j be two integers such that $\{(u)^i, (u)^j\} = N^F_{Q_n}(u) \cap N^F_{Q_n}(v)$. Obviously, we have $(u)^i = (v)^j$ and $(u)^j = (v)^i$. Then we can partition Q_n along dimension $k \in \{i, j\}$. As a result, both $Q^{k,0}_n$ and $Q^{k,1}_n$ contain at least $n - 3$ faulty nodes. See Fig. 5a.

Case 2: Suppose that $|N^F_{Q_n}(u) \cap N^F_{Q_n}(v)| = 1$. We claim first that this case holds only for $n \geqslant 5$. By contradiction, we suppose $n = 4$. Let p and q be two integers such that both $(u)^p$ and $(u)^q$ are faulty. Since $|N^F_{Q_n}(u) \cap N^F_{Q_n}(v)| = 1$, we have $v \neq ((u)^p)^q$. Thus node $((u)^p)^q$ happens to have only two fault-free neighbors, which contradicts the assumption that $|N^F_{Q_n}(x)| \leqslant n - 3$ for every $x \in V(Q_n) - \{u, v\}$.

Let i and j be two integers such that $\{(u)^i\} = \{(v)^j\} = N^F_{Q_n}(u) \cap N^F_{Q_n}(v)$. Since $|N^F_{Q_n}(u) - \{(u)^i\}| + |N^F_{Q_n}(v) - \{(v)^j\}| = 2(n-3) > n - 2 = |\{1, \ldots, n\} - \{i, j\}|$ for $n \geqslant 5$, there exists some dimension k of $\{1, \ldots, n\} - \{i, j\}$ such that both $(u)^k$ and $(v)^k$ are faulty. As a result, either $Q^{k,0}_n$ or $Q^{k,1}_n$ contains exactly two faulty nodes. See Fig. 5b.

In either case, both $Q^{k,0}_n$ and $Q^{k,1}_n$ are conditionally faulty. □

Lemma 4. *Suppose that an n-cube Q_n $(n \geqslant 4)$ is conditionally faulty with $f \leqslant 2n - 5$ faulty nodes. Let z be a unique node with exactly $n - 2$ faulty neighbors. Then there exists some dimension j of $\{1, 2, \ldots, n\}$ such that both $Q^{j,0}_n$ and $Q^{j,1}_n$ are conditionally faulty. Except for the case depicted in Fig. 5c, both $Q^{j,0}_n$ and $Q^{j,1}_n$ contain $2n - 7$ or less faulty nodes if $n \geqslant 5$.*

Proof. Since Q_n is node-transitive, we assume $z = 0^n$. Since Q_n is also link-transitive, we assume that $(z)^1$ and $(z)^2$ are fault-free. Because z is a unique node with exactly $n - 2$ faulty neighbors, we have $|N^F_{Q_n}(x)| \leqslant n - 3$ for $x \in V(Q_n) - \{z\}$. For every

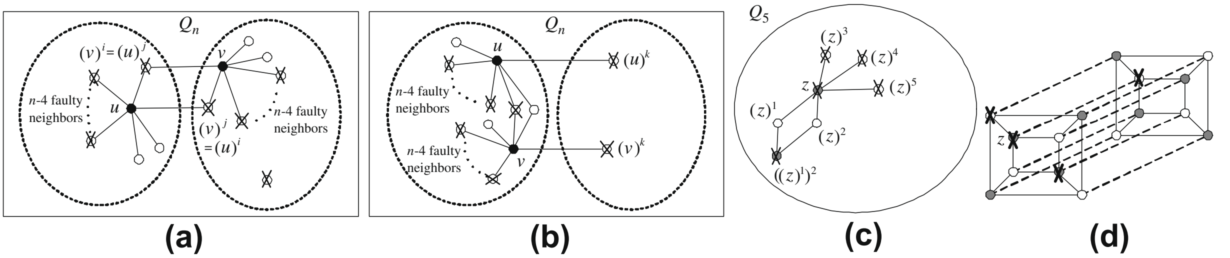

Fig. 5. Every faulty node is marked by an "X" symbol. (a,b) $|N^F_{Q_n}(u)| = |N^F_{Q_n}(v)| = n-2$ and $|N^F_{Q_n}(x)| \leqslant n-3$ for $x \in V(Q_n) - \{u, v\}$; (c) a faulty node distribution on Q_5; (d) a conditionally faulty 4-cube with four faulty nodes.

$k \in \{3, \ldots, n\}$, we have $N^F_{Q_n^{k,0}}(x) \subseteq N^F_{Q_n}(x)$ and $N^F_{Q_n^{k,1}}(y) \subseteq N^F_{Q_n}(y)$ for $x \in V(Q_n^{k,0}) - \{z\}$ and $y \in V(Q_n^{k,1})$. Thus we obtain $|N^F_{Q_n^{k,0}}(x)| \leqslant |N^F_{Q_n}(x)| \leqslant n-3$ and $|N^F_{Q_n^{k,1}}(y)| \leqslant |N^F_{Q_n}(y)| \leqslant n-3$ for $x \in V(Q_n^{k,0}) - \{z\}$ and $y \in V(Q_n^{k,1})$. In addition, we have $|N^F_{Q_n^{k,0}}(z)| = (n-2) - 1 = n-3$ for every $k \in \{3, \ldots, n\}$. Let j be an integer of $\{3, \ldots, n\}$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty.

Suppose $f \leqslant 2n-6$. We see that, for any $j \in \{3, \ldots, n\}$, both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.

Suppose $f = 2n-5$. We assume, by contraposition, that either $Q_n^{j,0}$ or $Q_n^{j,1}$ contains $2n-6$ faulty nodes for any $j \in \{3, \ldots, n\}$. Then, for any x of $F(Q_n) - \{(z)^k \mid 3 \leqslant k \leqslant n\}$, we have $[x]_j = [z]_j$ for every $j \in \{3, \ldots, n\}$. Hence we have $F(Q_n) - \{(z)^k \mid 3 \leqslant k \leqslant n\} \subseteq \{z, ((z)^1)^2\}$. Since $|F(Q_n) - \{(z)^k \mid 3 \leqslant k \leqslant n\}| = f - (n-2) = n-3 \leqslant 2 = |\{z, ((z)^1)^2\}|$, we derive that $n \leqslant 5$. That is, if $n \geqslant 6$, there exists some dimension j of $\{3, \ldots, n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes. Since $|F(Q_n) - \{(z)^k \mid 3 \leqslant k \leqslant n\}| = 2$ for $n = 5$, nodes z and $((z)^1)^2$ are faulty; that is, $F(Q_5) = \{z, (z)^3, (z)^4, (z)^5, ((z)^1)^2\}$, as shown in Fig. 5c. Therefore, Fig. 5c happens to be the only possibility that either $Q_n^{j,0}$ or $Q_n^{j,1}$ contains $2n-6$ faulty nodes for every $j \in \{3, \ldots, n\}$. □

Lemma 5. *Suppose that an n-cube Q_n ($n \geqslant 4$) contains $f \leqslant 2n-5$ faulty nodes such that every node has at least three fault-free neighbors. Then there exists some dimension j of $\{1, \ldots, n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty. For $n \geqslant 5$, $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.*

Proof. Since every node has at least three fault-free neighbors, every $(n-1)$-dimensional subcube of Q_n is conditionally faulty. First, we consider the case that $f \leqslant 2n-6$. Let u and v be two distinct faulty nodes, and let $j \in \{1, \ldots, n\}$ such that $[u]_j \neq [v]_j$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.

Now we consider the case that $f = 2n-5$. For $n \geqslant 5$, we claim that there exists some dimension j of $\{1, \ldots, n\}$ such that $|F(Q_n^{j,0})| \leqslant 2n-7$ and $|F(Q_n^{j,1})| \leqslant 2n-7$. For $1 \leqslant k \leqslant n$, we define that $q_k = 1$ if $[u]_k = [v]_k$ for every two distinct faulty nodes $u, v \in F(Q_n)$, and $q_k = 0$ otherwise. Let $q = \sum_{k=1}^{n} q_k$. Clearly, all faulty nodes are located in either $Q_n^{k,0}$ or $Q_n^{k,1}$ if $q_k = 1$. For convenience, let $\{1 \leqslant k \leqslant n \mid q_k = 0\} = \{i_1, \ldots, i_{n-q}\}$. Then both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain at least one faulty node for $j \in \{i_1, \ldots, i_{n-q}\}$. Suppose, by contradiction, either $Q_n^{j,0}$ or $Q_n^{j,1}$ contains only one faulty node for every $j \in \{i_1, \ldots, i_{n-q}\}$. For $v \in F(Q_n)$, let $A(v) = \{1 \leqslant k \leqslant n \mid F(Q_n^{k,0}) = \{v\} \text{ or } F(Q_n^{k,1}) = \{v\}\}$. Since Q_n is node-transitive, we assume that $\mathbf{e} = 0^n$ is a faulty node such that $|A(\mathbf{e})|$ achieves the maximum of set $\{|A(v)| \mid v \in F(Q_n)\}$. For convenience, let $p = |A(\mathbf{e})|$. Obviously, we have $1 \leqslant p \leqslant n-q$. Moreover, let $A(\mathbf{e}) = \{i_1, \ldots, i_p\}$. For $v \in F(Q_n) - \{\mathbf{e}\}$, we see that $[v]_k = 1$ for each $k \in \{i_1, \ldots, i_p\}$. Let $B(k) = \{v \in F(Q_n) - \{\mathbf{e}\} \mid [v]_k \neq [\mathbf{e}]_k\}$ for $k \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since we assumed, by contradiction, that either $Q_n^{j,0}$ or $Q_n^{j,1}$ has only one faulty node for each $j \in \{i_1, \ldots, i_{n-q}\}$, we have $|B(j)| = 1$ for each $j \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since Q_n is link-transitive, we assume that $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$ and $\{i_{p+1}, \ldots, i_{n-q}\} = \{p+1, \ldots, n-q\}$. Then we have $(F(Q_n) - \{\mathbf{e}\}) - \bigcup_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} B(k) \subseteq \{0^{n-p}1^p\}$. Accordingly, we derive that $1 = |\{0^{n-p}1^p\}| \geqslant |(F(Q_n) - \{\mathbf{e}\}) - \bigcup_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} B(k)| \geqslant |F(Q_n)| - |\{\mathbf{e}\}| - \sum_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} |B(k)| = (2n-5) - 1 - (n-q-p)$; that is, $p + q \leqslant 7 - n$. Recall that $p \geqslant 1$ and $q \geqslant 0$. Thus, we have $n \in \{5, 6\}$. Now we can identify all faulty nodes according to the values of p, q, and n.

Case 1: Suppose $(n, q, p) = (5, 0, 1)$. Since $p = 1$, we have $[v]_1 = 1$ for each $v \in F(Q_5) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{2, 3, 4, 5\}$. Thus we have $F(Q_5) = \{00000, 00011, 00101, 01001, 10001\}$. Clearly, node 00001 has five faulty neighbors.

Case 2: Suppose $(n, q, p) = (5, 0, 2)$. Similarly, we have $F(Q_5) = \{00000, 00111, 01011, 10011, 00011\}$. Then node 00011 has three faulty neighbors.

Case 3: Suppose $(n, q, p) = (5, 1, 1)$. We have $F(Q_5) = \{00000, 00011, 00101, 01001, 00001\}$. Again, node 00001 has four faulty neighbors.

Case 4: Suppose $(n, q, p) = (6, 0, 1)$. We have $F(Q_6) = \{000000, 000011, 000101, 001001, 010001, 100001, 000001\}$. Thus, node 000001 has six faulty neighbors.

In short, node $0^{n-p}1^p$ has at least $n-2$ faulty neighbors, which contradicts the requirement that every node has at least three fault-free neighbors. Hence there exists some dimension j of $\{1, \ldots, n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes. □

Suppose that Q_n is conditionally faulty with utmost $2n-5$ faulty nodes. Let $F = F(Q_n)$. For $n \geqslant 5$, we propose a procedure *PARTITION*(Q_n, F) to determine j-partition of Q_n according to the following rules:

(1) Suppose that at least three nodes of Q_n have exactly $n-2$ faulty neighbors, respectively. If Q_n has its faulty nodes distributed as shown in Fig. 3a, it will be partitioned along dimension $j = dim((t_1, t_5))$. Then one resulting subcube has its faulty nodes distributed as in Fig. 2b. Otherwise, Lemma 1 and Lemma 2 ensure that Q_n can be partitioned along some dimension j such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.
(2) Suppose that there exist exactly two nodes of Q_n with $n-2$ faulty neighbors, respectively. By Lemma 3, there exists some dimension j of $\{1,\ldots,n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.
(3) Suppose that there is only one node of Q_n with exactly $n-2$ faulty neighbors. Denote it by z. If the faulty nodes are distributed as in Fig. 5c, we partition Q_n along any dimension $j \in \{i \mid (z)^i$ is faulty$\}$. Then one resulting subcube turns out to have $2n-6$ faulty nodes, distributed as in Fig. 5d. Otherwise, we can apply Lemma 4 to choose a dimension j of $\{1,\ldots,n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.
(4) Suppose that every node of Q_n has at least three fault-free neighbors. Obviously, every $(n-1)$-cube is conditionally faulty. By Lemma 5, there exists some dimension j of $\{1,\ldots,n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ contain $2n-7$ or less faulty nodes.

The following corollary summarizes what is obtained by procedure *PARTITION*(Q_n, *F*). Also, it is a summary of Lemmas 1–5.

Corollary 1. *Suppose that an n-cube Q_n $(n \geqslant 5)$ is conditionally faulty with $f \leqslant 2n-5$ faulty nodes. Except for the cases illustrated in Figs. 2c, 3a, and 5c, there exists some dimension j of $\{1,2,\ldots,n\}$ such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty with $2n-7$ or less faulty nodes.*

4. Long paths in faulty hypercubes

The following theorem was proved by Fu [7].

Theorem 3 [7]. *Let u and v denote two arbitrary fault-free nodes of an n-cube with $f \leqslant n-2$ faulty nodes, where $n \geqslant 3$. If $h(u,v)$ is odd (or even), then there exists a fault-free path of length at least $2^n - 2f - 1$ (or $2^n - 2f - 2$) between u and v.*

To improve the above result, we need the following lemma.

Lemma 6. *Let $z \in V(Q_4)$, $\{i,j,p,q\} = \{1,2,3,4\}$, and $F = \{(z)^i, (z)^j, (z)^p\}$. Suppose that s and t are any two nodes of $Q_4 - F$ such that $\{s,t\} \neq \{z,(z)^q\}$. Then $Q_4 - F$ has a path of length at least 9 or 8 between s and t if $h(s,t)$ is odd or even, respectively.*

Proof. By symmetry, let $z = 0000$, $i = 1$, $j = 2$, $p = 3$, and $q = 4$. We partition Q_4 into $Q_4^{4,0}$ and $Q_4^{4,1}$. Then $Q_4^{4,1}$ is fault-free and $z \in V_0(Q_4^{4,0})$.

Case 1: Both s and t are in $Q_4^{4,0} - F$. Since $Q_4^{4,1}$ is fault-free, Theorem 1 ensures that $Q_4^{4,1}$ contains a path P of length 7 (respectively, 6) between $(s)^4$ and $(t)^4$ if $h(s,t)$ is odd (respectively, even). Thus, $\langle s,(s)^4,P,(t)^4,t\rangle$ is a fault-free path of length 9 (respectively, 8) between s and t if $h(s,t)$ is odd (respectively, even).

Case 2: Both s and t are in $Q_4^{4,1}$. If $h(s,t)$ is odd, Theorem 1 ensures that $Q_4^{4,1} - \{(1101,1111)\}$ contains a path P of length 7 between s and t. Clearly, path P does not pass through (1101,1111). Since it spans $Q_4^{4,1}$, we have $1111 \in V(P)$. Accordingly, link (1110,1111) or (1011,1111) is on P. Thus P can be written as $\langle s,R_1,1110,1111,R_2,t\rangle$ or $\langle s,T_1,1011,1111,T_2,t\rangle$. As a result, $\langle s,R_1,1110,0110,0111,1111,R_2,t\rangle$ or $\langle s,T_1,1011,0011,0111,1111,T_2,t\rangle$ is a path of length 9 between s and t. On the other hand, if $h(s,t)$ is even, then we consider two cases as follows. Suppose first that $s,t \in V_0(Q_4^{4,1})$. By Theorem 1, $Q_4^{4,1} - \{(1101,1111)\}$ contains a path P of length 6 between s and t. Again, link (1110,1111) or (1011,1111) is on P, and thus the desired path can be constructed as above. Suppose that $s,t \in V_1(Q_4^{4,1})$. By Theorem 2, $Q_4^{4,1} - \{1001\}$ contains a path P of length 6 between s and t. Obviously, link (1110,1111), (1101,1111), or (1011,1111) is on P. Hence the desired path can be constructed similarly.

Case 3: Suppose that s is in $Q_4^{4,0} - F$ and t is in $Q_4^{4,1}$. First, we consider the case that $s \neq z$. If $s \in V_0(Q_4)$, then s is adjacent to node 0111. Clearly, there exists some node v of $\{0110,0101,0011\} - \{s\}$ such that $(v)^4 \neq t$. By Theorem 1, $Q_4^{4,1}$ has a path P of length 6 or 7 between $(v)^4$ and t if $h(s,t)$ is odd or even, respectively. Then $\langle s,0111,v,(v)^4,P,t\rangle$ is a fault-free path of length 9 or 10 if $h(s,t)$ is odd or even, respectively. If $s \in V_1(Q_4)$, then we have $s = 0111$. Obviously, there exists some node u of $\{0110,0101,0011\}$ such that $(u)^4 \neq t$. Similarly, $Q_4^{4,1}$ has a path T of length 7 (respectively, 6) between $(u)^4$ and t if $h(s,t)$ is odd (respectively, even). Then $\langle s,u,(u)^4,T,t\rangle$ is a fault-free path of length 9 (respectively, 8) if $h(s,t)$ is odd (respectively, even).

Next, we consider the case that $s = z$. If $h(s,t)$ is even, it follows from Theorem 1 that $Q_4^{4,1}$ has a path H of length 7 between $(s)^4 = (z)^4$ and t. Then $\langle s = z,(z)^4,H,t\rangle$ is a fault-free path of length 8. If $h(s,t)$ is odd, Theorem 2 ensures that $Q_4^{4,1} - \{1100\}$ has a path R of length 6 between $(z)^4$ and t. Clearly, node 1111 is on R. Accordingly, link (1111,1110), (1111,1101), or (1111,1011) is on R. For example, path R can be written as $\langle (z)^4,R_1,1111,1110,R_2,t\rangle$ if $(1111,1110) \in E(R)$. Then $\langle s = z,(z)^4,R_1,1111,0111,0110,1110,R_2,t\rangle$ is a fault-free path of length 9 between s and t. □

For the sake of readability, the proof of the following theorem will be described in Appendix A.

Theorem 4. *Let F be a set of $f \leqslant 3$ faulty nodes in Q_4 such that every node of Q_4 has at least two fault-free neighbors. Suppose that s and t are two arbitrary nodes of $Q_4 - F$. Then $Q_4 - F$ contains a path of length at least $15 - 2f$ (respectively, $14 - 2f$) between s and t if $h(s,t)$ is odd (respectively, even).*

With Theorem 4 and Lemma 6, we will be able to prove the next theorem.

Theorem 5. *Let F be a set of f faulty nodes in Q_n ($n \geqslant 1$) such that every node of Q_n has at least two fault-free neighbors. Suppose $f = 0$ if $n \in \{1,2\}$, and $f \leqslant 2n-5$ if $n \geqslant 3$. Let s and t be two arbitrary nodes of $Q_n - F$. Then $Q_n - F$ contains a path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between s and t if $h(s,t)$ is odd (respectively, even).*

Proof. The result is trivial for $n \in \{1,2\}$. When $n \in \{3,4\}$, the result follows from Theorem 3 or Theorem 4, respectively. In what follows we consider the case that $n \geqslant 5$. Except for the faulty node distribution illustrated in Fig. 3a, procedure *PARTITION*(Q_n, F) returns j-partition of Q_n such that both $Q_n^{j,0}$ and $Q_n^{j,1}$ are conditionally faulty. If Q_5 has its faulty nodes distributed as in Fig. 3a, then *PARTITION*(Q_5, F) returns j-partition of Q_5 such that one subcube has its faulty nodes distributed as in Fig. 2b. Accordingly, the proof can be justified by the induction on n. Our inductive hypothesis is that the result holds for Q_{n-1}. For convenience, let $F_0 = F(Q_n^{j,0})$ and $F_1 = F(Q_n^{j,1})$. Moreover, let $f_0 = |F_0|$ and $f_1 = |F_1|$. Without loss of generality, we assume that $s \in V_0(Q_n - F)$.

Case 1: Suppose $f_0 \leqslant 2n-7$ and $f_1 \leqslant 2n-7$. Without loss of generality, we assume that $f_0 \leqslant f_1$. In particular, for the case illustrated in Fig. 3a, $Q_5^{j,0}$ is conditionally faulty with $f_0 = 2$ faulty nodes, and $Q_5^{j,1}$ is not conditionally faulty with $f_1 = 3$ faulty nodes distributed as in Fig. 2b.

Subcase 1.1. Both s and t are in $Q_n^{j,0}$. By inductive hypothesis, $Q_n^{j,0} - F_0$ contains a path H_0 of length L at least $2^{n-1} - 2f_0 - 1$ (respectively, $2^{n-1} - 2f_0 - 2$) between s and t if $h(s,t)$ is odd (respectively, even). Clearly, we have $|\{v \in V(Q_n^{j,1}) \,\|\, N^F_{Q_n^{j,1}}(v)| \geqslant n-2\}| \leqslant 1$. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leqslant i \leqslant L \text{ and } i \equiv 1 (\text{mod } 2)\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{L}{2} \rceil > f_1 + 1 \geqslant |F_1 \cup \{v \in V(Q_n^{j,1}) \,\|\, N^F_{Q_n^{j,1}}(v)| \geqslant n-2\}|$ for $n \geqslant 5$, there exists an odd integer $\hat{\imath}$, $1 \leqslant \hat{\imath} \leqslant L$, such that $|F_1 \cap \{(H_0(\hat{\imath}))^j, (H_0(\hat{\imath}+1))^j\}| = 0$, $|N^F_{Q_n^{j,1}}((H_0(\hat{\imath}))^j)| \leqslant n-3$, and $|N^F_{Q_n^{j,1}}((H_0(\hat{\imath}+1))^j)| \leqslant n-3$ are satisfied. Let $x = H_0(\hat{\imath})$ and $y = H_0(\hat{\imath}+1)$. Hence path H_0 can be written as $\langle s, H_0', x, y, H_0'', t \rangle$.

If $Q_n^{j,1}$ is conditionally faulty, our inductive hypothesis asserts that $Q_n^{j,1} - F_1$ has a path H_1 of length at least $2^{n-1} - 2f_1 - 1$ between $(x)^j$ and $(y)^j$. Otherwise, the faulty nodes of $Q_n^{j,1}$ are distributed as in Fig. 2b. Since both $(x)^j$ and $(y)^j$ have two or more fault-free neighbors in $Q_n^{j,1}$, Lemma 6 ensures that $Q_n^{j,1}$ has a fault-free path H_1 of length at least $2^{n-1} - 2f_1 - 1$ between $(x)^j$ and $(y)^j$. Then $\langle s, H_0', x, (x)^j, H_1, (y)^j, y, H_0'', t \rangle$ is a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between s and t if $h(s,t)$ is odd (respectively, even). See Fig. 6a.

Subcase 1.2. Both s and t are in $Q_n^{j,1}$. We consider first that the faulty nodes of $Q_5^{j,1}$ are distributed as depicted in Fig. 2b. Let z denote the node with only one fault-free neighbor r in $Q_5^{j,1}$. Note that $f_0 = 2$ and $f_1 = 3$.

Suppose $\{s,t\} = \{z,r\}$. Then a long path between s and t is constructed as follows. On the one hand, we assume that $s = z$ and $t = r$. Since $|V_0(Q_5^{j,0}) - F_0| \geqslant |V_0(Q_5^{j,0})| - |F_0| = 2^4 - 2 > 4 = |F_1 \cup \{t\}|$, there exists some fault-free node x of $V_0(Q_5^{j,0})$ such that $(x)^j \notin F_1 \cup \{t\}$. By inductive hypothesis, $Q_5^{j,0} - F_0$ has a path H_0 of length at least $2^4 - 2f_0 - 1$ between

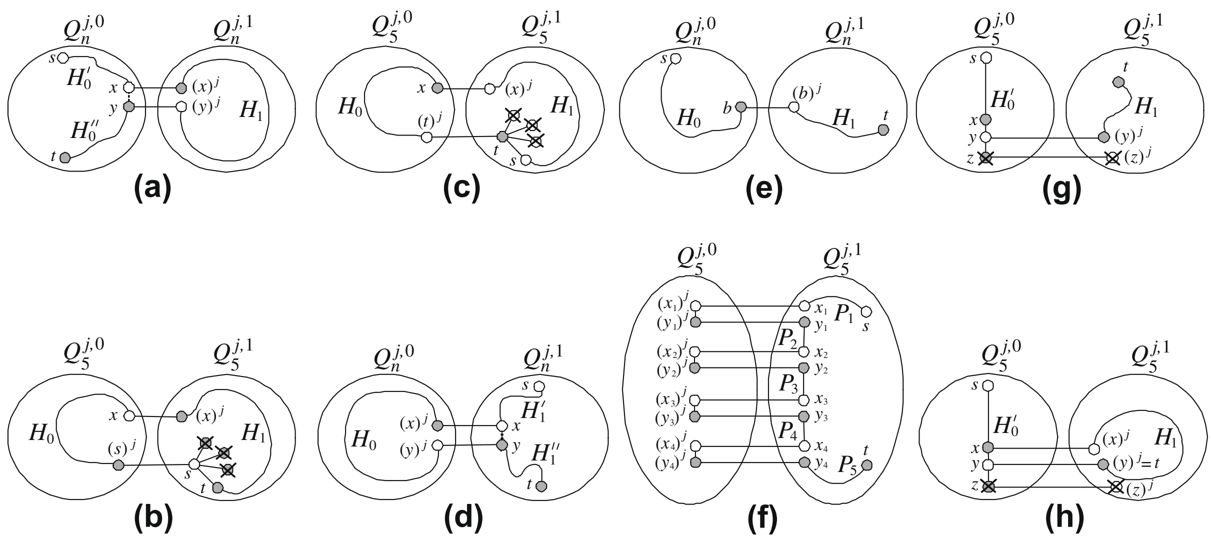

Fig. 6. Illustration for Theorem 5.

$(s)^j$ and x. By Lemma 6, $Q_5^{j,1} - F_1$ has a path H_1 of length at least $2^4 - 2f_1 - 2$ between $(x)^j$ and t. As a result, $\langle s, (s)^j, H_0, x, (x)^j, H_1, t\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ (see Fig. 6b). On the other hand, we assume that $t = z$ and $s = r$. Since $| V_1(Q_5^{j,0}) - F_0 | \geqslant | V_1(Q_5^{j,0}) | - | F_0 | = 2^4 - 2 > 4 = | F_1 \cup \{s\} |$, there exists some fault-free node x of $V_1(Q_5^{j,0})$ such that $(x)^j \notin F_1 \cup \{s\}$. Again, the inductive hypothesis asserts that $Q_5^{j,0}$ has a fault-free path H_0 of length at least $2^4 - 2f_0 - 1$ between x and $(t)^j$; Lemma 6 asserts that $Q^{j,1}$ has a fault-free path H_1 of length at least $2^4 - 2f_1 - 2$ between s and $(x)^j$. Then $\langle s, H_1, (x)^j, x, H_0, (t)^j, t\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ (see Fig. 6c).

Suppose $\{s,t\} \neq \{z,r\}$. Then Lemma 6 asserts that $Q_5^{j,1} - F_1$ contains a path H_1 of length L at least $2^4 - 2f_1 - 1$ (respectively, $2^4 - 2f_1 - 2$) between s and t if $h(s,t)$ is odd (respectively, even). Let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leqslant i \leqslant L$ and $i \equiv 1 (\mathrm{mod}\ 2)\}$ be a set of disjoint links. Since $|A| = \lceil \frac{L}{2} \rceil > 2 = f_0$, there exists an odd integer $\hat{\imath}$, $1 \leqslant \hat{\imath} \leqslant L$, such that $F_0 \cap \{(H_1(\hat{\imath}))^j, (H_1(\hat{\imath}+1))^j\} = \emptyset$. Let $x = H_1(\hat{\imath})$ and $y = H_1(\hat{\imath}+1)$. Accordingly, path H_1 can be written as $\langle s, H_1', x, y, H_1'', t\rangle$. Again, the inductive hypothesis asserts that $Q_5^{j,0} - F_0$ has a path H_0 of length at least $2^4 - 2f_0 - 1$ between $(x)^j$ and $(y)^j$. Then $\langle s, H_1', x, (x)^j, H_0, (y)^j, y, H_1'', t\rangle$ is a fault-free path of length at least $2^5 - 2f - 1$ or $2^5 - 2f - 2$ if $h(s,t)$ is odd or even, respectively. See Fig. 6d.

Now we consider the case that faulty nodes of $Q_5^{j,1}$ are not distributed as depicted in Fig. 2b, or $n \geqslant 6$. Then $Q_n^{j,1}$ is conditionally faulty. By inductive hypothesis, $Q_n^{j,1} - F_1$ has a path H_1 of length L at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between s and t if $h(s,t)$ is odd (respectively, even). Similarly, let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leqslant i \leqslant L$ and $i \equiv 1 (\mathrm{mod}\ 2)\}$ be a set of disjoint links. Since $|A| = \lceil \frac{L}{2} \rceil > f_0$ for $n \geqslant 5$, there is a link (x,y) of A such that $F_0 \cap \{(x)^j, (y)^j\} = \emptyset$. Accordingly, path H_1 can be written as $\langle s, H_1', x, y, H_1'', t\rangle$. By inductive hypothesis, $Q_n^{j,0} - F_0$ has a path H_0 of length at least $2^{n-1} - 2f_0 - 1$ between $(x)^j$ and $(y)^j$. Again, $\langle s, H_1', x, (x)^j, H_0, (y)^j, y, H_1'', t\rangle$ is a fault-free path of length at least $2^n - 2f - 1$ or $2^n - 2f - 2$ if $h(s,t)$ is odd or even, respectively. See Fig. 6d.

Subcase 1.3. Suppose that s is in $Q_n^{j,0}$ and t is in $Q_n^{j,1}$. Note that $| \{x \in V(Q_n^{j,1}) \mid | N^F_{Q_n^{j,1}}(x) | \geqslant n-2\} | \leqslant 1$. On the one hand, we consider the case that node t has only one fault-free neighbor, denoted by r, in $Q_n^{j,1}$. On this occasion, n is equal to 5. Since $| V_1(Q_n^{j,0}) - F_0 | \geqslant 2^{n-2} - f_0 > f_1 + 2 = | F_1 \cup \{t, r\} |$ for $n = 5$, there exists a fault-free node b of $V_1(Q_n^{j,0}) - F_0$ such that $(b)^j \notin F_1 \cup \{t, r\}$. On the other hand, we consider the case that node t has at least two fault-free neighbors in $Q_n^{j,1}$. Since $| V_1(Q_n^{j,0}) - F_0 | \geqslant 2^{n-2} - f_0 > f_1 + 2 \geqslant | F_1 | + | \{t\} | + | \{x \in V(Q_n^{j,1}) \mid | N^F_{Q_n^{j,1}}(x) | \geqslant n-2\} | \geqslant | F_1 \cup \{t\} \cup \{x \in V(Q_n^{j,1}) \mid | N^F_{Q_n^{j,1}}(x) | \geqslant n-2\} |$ for $n \geqslant 5$, there exists a fault-free node b of $V_1(Q_n^{j,0}) - F_0$ such that $(b)^j \notin F_1 \cup \{t\} \cup \{x \in V(Q_n^{j,1}) \mid | N^F_{Q_n^{j,1}}(x) | \geqslant n-2\}$.

By inductive hypothesis, $Q_n^{j,0} - F_0$ has a path H_0 of length at least $2^{n-1} - 2f_0 - 1$ between s and b. If the faulty nodes of $Q_n^{j,1}$ are distributed as illustrated in Fig. 2b, Lemma 6 asserts that $Q_n^{j,1} - F_1$ has a path H_1 of length at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between $(b)^j$ and t if $h((b)^j, t)$ is odd (respectively, even); otherwise, the inductive hypothesis asserts that $Q_n^{j,1} - F_1$ has a path H_1 of length at least $2^{n-1} - 2f_1 - 1$ (respectively, $2^{n-1} - 2f_1 - 2$) between $(b)^j$ and t if $h((b)^j, t)$ is odd (respectively, even). Then $\langle s, H_0, b, (b)^j, H_1, t\rangle$ is a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between s and t if $h(s,t)$ is odd (respectively, even). See Fig. 6e.

Case 2: Suppose either $f_0 = 2n - 6$ or $f_1 = 2n - 6$. By Lemmas 1–5, we know that this case may occur while $n = 5$. More precisely, the faulty nodes happen to be distributed as illustrated in Fig. 5c where z is itself a faulty node with three faulty neighbors. Without loss of generality, we assume that $f_0 = 4$; thus, $(z)^j$ is a unique faulty node in $Q_5^{j,1}$.

Subcase 2.1. Both s and t are in $Q_5^{j,0}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{z\})$ contains a path H_0 of length L at least $9 = 2^4 - 2 \cdot 3 - 1$ (respectively, $8 = 2^4 - 2 \cdot 3 - 2$) between s and t if $h(s,t)$ is odd (respectively, even).

First, we consider the case that node z is not on H_0. Let $A = \{(H_0(i), H_0(i+1)) \mid 1 \leqslant i \leqslant L$ and $i \equiv 1 (\mathrm{mod}\ 2)\}$ be a set of disjoint links on H_0. Since $|A| = \lceil \frac{L}{2} \rceil > 1 = f_1$, there exists an odd integer $\hat{\imath}$, $1 \leqslant \hat{\imath} \leqslant L$, such that both $(H_0(\hat{\imath}))^j$ and $(H_0(\hat{\imath}+1))^j$ are fault-free. Let $x = H_0(\hat{\imath})$ and $y = H_0(\hat{\imath}+1)$. Hence path H_0 can be written as $\langle s, H_0', x, y, H_0'', t\rangle$. It follows from inductive hypothesis that $Q_5^{j,1} - \{(z)^j\}$ has a path H_1 of length at least $13 = 2^4 - 2 \cdot 1 - 1$ between $(x)^j$ and $(y)^j$. Then $\langle s, H_0', x, (x)^j, H_1, (y)^j, y, H_0'', t\rangle$ is a fault-free path of length at least $23 > 2^5 - 2 \cdot 5 - 1$ (respectively, $22 > 2^5 - 2 \cdot 5 - 2$) between s and t if $h(s,t)$ is odd (respectively, even).

Now we consider the case that node z is on H_0. Since the length of H_0 is at least 9, we can write H_0 as $\langle s, H_0', x, z, y, H_0'', t\rangle$. Clearly, $(x)^j$ and $(y)^j$ are fault-free nodes in the same partite set of $Q_5^{j,1}$. By Theorem 2, $Q_5^{j,1}$ is hyper-Hamiltonian laceable; thus $Q_5^{j,1} - \{(z)^j\}$ has a path H_1 of length 14 between $(x)^j$ and $(y)^j$. Then $\langle s, H_0', x, (x)^j, H_1, (y)^j, y, H_0'', t\rangle$ is a fault-free path of length at least $23 > 2^5 - 2 \cdot 5 - 1$ (respectively, $22 > 2^5 - 2 \cdot 5 - 2$) between s and t if $h(s,t)$ is odd (respectively, even).

Subcase 2.2. Both s and t are in $Q_5^{j,1}$. For the sake of clarity, we distinguish whether $h(s,t)$ is odd or even.

Suppose that $h(s,t)$ is odd. By inductive hypothesis, $Q_5^{j,1} - \{(z)^j\}$ contains a path H_1 of length L at least 13 between s and t. Obviously, we have $(z)^j \notin V(H_1)$. Consequently, $(v)^j \neq z$ for any $v \in V(H_1)$. Let $A = \{(H_1(i), H_1(i+1)) \mid 1 \leqslant i \leqslant L$ and $i \equiv 1 (\mathrm{mod}\ 2)\}$ be a set of disjoint links on H_1. Since $|A| - |F_0 - \{z\}| = \lceil \frac{L}{2} \rceil - (f_0 - 1) \geqslant 7 - (4 - 1) = 4$, there exist four links of A, namely (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), such that $(x_i)^j$ and $(y_i)^j$ are fault-free for all $i \in \{1, 2, 3, 4\}$. Thus path H_1 can be written as $\langle s, P_1, x_1, y_1, P_2, x_2, y_2, P_3, x_3, y_3, P_4, x_4, y_4, P_5, t\rangle$. Then $\langle s, P_1, x_1, (x_1)^j, (y_1)^j, y_1, P_2, x_2, (x_2)^j, (y_2)^j, y_2, P_3, x_3, (x_3)^j, (y_3)^j, y_3, P_4, x_4, (x_4)^j, (y_4)^j, y_4, P_5, t\rangle$ is a fault-free path of length at least $21 = 2^5 - 2 \cdot 5 - 1$ between s and t. See Fig. 6f.

Suppose that $h(s,t)$ is even. If s and $(z)^j$ belong to the different partite sets of $Q_5^{j,1}$, Theorem 2 asserts that $Q_5^{j,1} - \{(z)^j\}$ has a path H_1 of length 14 between s and t. Similar to the case that $h(s,t)$ is odd, there exist four disjoint links on H_1, namely (x_1, y_1),

(x_2, y_2), (x_3, y_3), and (x_4, y_4), such that $(x_i)^j$ and $(y_i)^j$ are fault-free for all $i \in \{1,2,3,4\}$. Accordingly, we can write $H_1 = \langle s, P_1, x_1, y_1, P_2, x_2, y_2, P_3, x_3, y_3, P_4, x_4, y_4, P_5, t\rangle$. Then $\langle s, P_1, x_1, (x_1)^j, (y_1)^j, y_1, P_2, x_2, (x_2)^j, (y_2)^j, y_2, P_3, x_3, (x_3)^j, (y_3)^j, y_3, P_4, x_4, (x_4)^j, (y_4)^j, y_4, P_5, t\rangle$ is a fault-free path of length at least $22 > 2^5 - 2\cdot 5 - 2$ between s and t. If nodes s and $(z)^j$ belong to the same partite set of $Q_5^{j,1}$, then we construct a fault-free path as follows. Since $Q_5^{j,0}$ is conditionally faulty, we denote by x any fault-free neighbor of z in $Q_5^{j,0}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{z\})$ has a path H_0 of length at least $9 = 2^4 - 2\cdot 3 - 1$ between x and z. We can write path H_0 as $\langle x, H_0', y, z\rangle$, where y is also a fault-free neighbor of z. Without loss of generality, let $j = 5$, $\{x, y\} = \{(z)^1, (z)^2\}$, and $X = \{((z)^j, ((z)^j)^3), ((z)^j, ((z)^j)^4)\}$. Since $|X| = 2$, Theorem 1 ensures that $Q_5^{j,1} - X$ is strongly Hamiltonian laceable; hence it has a path H_1 of length 14 between s and t. Obviously, both $((z)^j, (x)^j)$ and $((z)^j, (y)^j)$ are on H_1, and we can write H_1 as $\langle s, H_1', (x)^j, (z)^j, (y)^j, H_1'', t\rangle$. Then $\langle s, H_1', (x)^j, x, H_0', y, (y)^j, H_1'', t\rangle$ is a fault-free path of length at least $22 > 2^5 - 2\cdot 5 - 2$ between s and t.

Subcase 2.3. Suppose that s is in $Q_5^{j,0}$ and t is in $Q_5^{j,1}$. By inductive hypothesis, $Q_5^{j,0} - (F_0 - \{z\})$ has a path H_0 of length at least 9 (respectively, 8) between s and z if $h(s,z)$ is odd (respectively, even). Accordingly, path H_0 can be written as $\langle s, H_0', x, y, z\rangle$. Since $(z)^j$ is a unique faulty node in $Q_5^{j,1}$, both $(x)^j$ and $(y)^j$ are fault-free.

If $(y)^j \neq t$, it follows from inductive hypothesis that $Q_5^{j,1} - \{(z)^j\}$ has a path H_1 of length at least 13 (respectively, 12) between $(y)^j$ and t if $h((y)^j, t)$ is odd (respectively, even). Then $\langle s, H_0', x, y, (y)^j, H_1, t\rangle$ is a path of length at least $21 = 2^5 - 2\cdot 5 - 1$ (respectively, $20 = 2^5 - 2\cdot 5 - 2$) between s and t if $h(s,t)$ is odd (respectively, even). See Fig. 6g. Otherwise, if $(y)^j = t$, then our inductive hypothesis asserts that $Q_5^{j,1} - \{(z)^j\}$ has a path H_1 of length at least 13 between $(x)^j$ and $(y)^j$. Then $\langle s, H_0', x, (x)^j, H_1, (y)^j = t\rangle$ is a path of length at least $21 = 2^5 - 2\cdot 5 - 1$ (respectively, $20 = 2^5 - 2\cdot 5 - 2$) between s and t if $h(s,t)$ is odd (respectively, even). See Fig. 6h.

Therefore the proof is completed. □

5. Conclusion

In this paper, we show that a conditionally faulty n-cube with $f \leqslant 2n - 5$ faulty nodes contains a fault-free path of length at least $2^n - 2f - 1$ (respectively, $2^n - 2f - 2$) between any two fault-free nodes of odd (respectively, even) distance. When compared with the previous results presented by Fu [7], our results can tolerate almost double that faulty nodes under an additional condition that every node has two or more fault-free neighbors. It has been well grounded that $2n - 5$ is the maximum number of faulty nodes tolerable in Q_n if $n = 4$. Yet it is not easy to show that a fault-free path of length at least $2^n - 2f - 1$ (or $2^n - 2f - 2$) cannot be embedded to connect any two nodes in a conditionally faulty n-cube with f faulty nodes for $f \geqslant 2n - 4$ and $n \geqslant 5$. In fact, we conjecture that an n-cube may tolerate more than $2n - 5$ faulty nodes with respect to fault-tolerant path embedding. Therefore, we intend to find, in our future work, the tight upper bound to the number of tolerable faulty nodes.

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Appendix A. Proof of Theorem 4

In order to prove Theorem 4, we address the following two lemmas in advance.

Lemma 7. *Suppose that Q_3 is conditionally faulty with $f \leqslant 2$ faulty nodes. Let s and t denote any two fault-free nodes of Q_3. Then Q_3 contains a fault-free path of length at least $7 - 2f$ (respectively, $6 - 2f$) between s and t if $h(s,t)$ is odd (respectively, even).*

Proof. If $f < 2$, this result follows from Theorem 3. Thus we only consider the case that $f = 2$. For convenience, let $F = F(Q_3)$. Since Q_3 is node-transitive, we assume that node 000 is faulty. To require that every node of Q_3 has at least two fault-free neighbors, the other faulty node must be one of $\{001, 010, 100, 111\}$.

Case 1: One of $\{001, 010, 100\}$ is faulty. Obviously, each of $\{001, 010, 100\}$ is adjacent to 000. Since Q_3 is link-transitive, we assume that $001 \in F$; that is, $F = \{000, 001\}$. Then we partition Q_3 into $Q_3^{2,0}$ and $Q_3^{2,1}$. Hence we have $F \subseteq V(Q_3^{2,0})$. See Fig. 7a.

Subcase 1.1. Both s and t are in $Q_3^{2,0} - F$. Without loss of generality, we assume that $s = 101$ and $t = 100$. Obviously, $\langle s = 101, 111, 110, 100 = t\rangle$ is a fault-free path of length $3 = 7 - 2\cdot 2$.

Subcase 1.2. Both s and t are in $Q_3^{2,1}$. If $h(s,t)$ is odd, then $Q_3^{2,1}$ contains a path of length 3 between s and t. Otherwise, $Q_3^{2,1}$ contains a path of length 2 between s and t.

Subcase 1.3. Suppose that s is in $Q_3^{2,0} - F$ and t is in $Q_3^{2,1}$. Without loss of generality, we assume $s = 101$ and list the required path in Table 1.

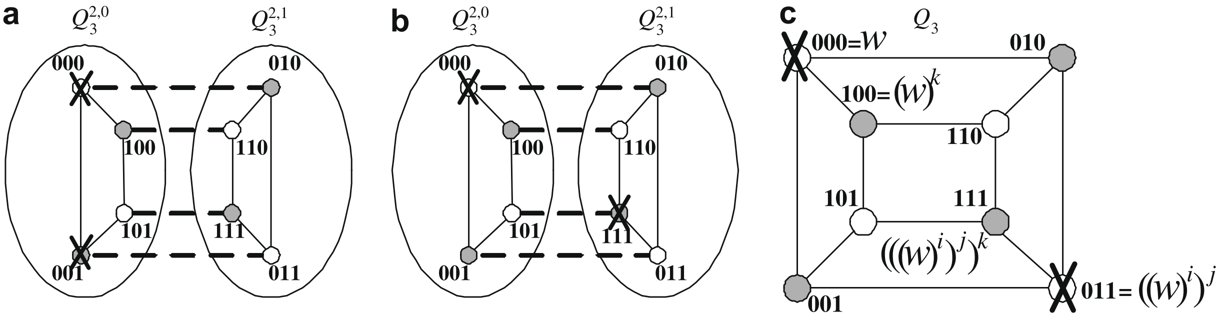

Fig. 7. (a,b) Illustrations for Lemma 7; (c) the distribution of faulty nodes indicated in Lemma 8.

Case 2: Node 111 is faulty. See Fig. 7b for illustration.

Subcase 2.1. Both s and t are in $Q_3^{2,0} - \{000\}$. For every possible combination of s and t, we list the required paths in Table 1.

Subcase 2.2. Both s and t are in $Q_3^{2,1} - \{111\}$. This subcase is symmetric to Subcase 2.1.

Subcase 2.3. Suppose that s is in $Q_3^{2,0} - \{000\}$ and t is in $Q_3^{2,1} - \{111\}$. For every possible combination of s and t, we list the required paths in Table 1.

In summary, $Q_3 - F$ contains a path of length at least $7 - 2f$ (respectively, $6 - 2f$) between s and t if $h(s,t)$ is odd (respectively, even). □

Lemma 8. *Let* $w \in V_0(Q_3)$ *and* $\{i,j,k\} = \{1,2,3\}$. *Suppose that* b_1 *and* b_2 *are two arbitrary nodes of* $V_1(Q_3)$. *Then* $Q_3 - \{w, ((w)^i)^j\}$ *contains a path of length four between* b_1 *and* b_2 *if and only if* $\{b_1, b_2\} \neq \{(w)^k, (((w)^i)^j)^k\}$.

Proof. Since Q_3 is node-transitive and link-transitive, we assume that $w = 000$, $i = 1$, $j = 2$, and $k = 3$. See Fig. 7c. Then we list all the required paths in Table 1. □

Table 1
The required paths for Lemma 7 and Lemma 8.

Subcase 1.3 of Lemma 7		
$s = 101$	$t = 010$	$\langle s = 101, 100, 110, 010 = t\rangle$
	$t = 011$	$\langle s = 101, 111, 011 = t\rangle$
	$t = 110$	$\langle s = 101, 100, 110 = t\rangle$
	$t = 111$	$\langle s = 101, 100, 110, 111 = t\rangle$
Subcase 2.1 of Lemma 7		
$s = 101$	$t = 001$	$\langle s = 101, 100, 110, 010, 011, 001 = t\rangle$
	$t = 100$	$\langle s = 101, 001, 011, 010, 110, 100 = t\rangle$
$s = 001$	$t = 100$	$\langle s = 001, 011, 010, 110, 100 = t\rangle$
Subcase 2.3 of Lemma 7		
$s = 001$	$t = 010$	$\langle s = 001, 011, 010 = t\rangle$
	$t = 011$	$\langle s = 001, 101, 100, 110, 010, 011 = t\rangle$
	$t = 110$	$\langle s = 001, 011, 010, 110 = t\rangle$
$s = 100$	$t = 010$	$\langle s = 100, 110, 010 = t\rangle$
	$t = 011$	$\langle s = 100, 110, 010, 011 = t\rangle$
	$t = 110$	$\langle s = 100, 101, 001, 011, 010, 110 = t\rangle$
$s = 101$	$t = 010$	$\langle s = 101, 100, 110, 010 = t\rangle$
	$t = 011$	$\langle s = 101, 001, 011 = t\rangle$
	$t = 110$	$\langle s = 101, 100, 110 = t\rangle$
Lemma 8		
$b_1 = 001$	$b_2 = 010$	$\langle b_1 = 001, 101, 100, 110, 010 = b_2\rangle$
	$b_2 = 100$	$\langle b_1 = 001, 101, 111, 110, 100 = b_2\rangle$
	$b_2 = 111$	$\langle b_1 = 001, 101, 100, 110, 111 = b_2\rangle$
$b_1 = 010$	$b_2 = 100$	$\langle b_1 = 010, 110, 111, 101, 100 = b_2\rangle$
	$b_2 = 111$	$\langle b_1 = 010, 110, 100, 101, 111 = b_2\rangle$

Theorem 4. *Let F be a set of $f \leqslant 3$ faulty nodes in Q_4 such that every node of Q_4 has at least two fault-free neighbors. Suppose that s and t are two arbitrary nodes of $Q_4 - F$. Then $Q_4 - F$ contains a path of length at least $15 - 2f$ (respectively, $14 - 2f$) between s and t if $h(s,t)$ is odd (respectively, even).*

Proof. If $f < 3$, this result follows from Theorem 3. Thus we concentrate only on the case that $f = 3$. By Lemmas 1–5, Fig. 2c happens to be a unique case that a conditionally faulty Q_4 with three faulty nodes cannot be partitioned along any dimension in such a way that both subcubes are conditionally faulty. On this occasion, we partition Q_4 along an arbitrary dimension j; otherwise, there exists some dimension j such that both $Q_4^{j,0}$ and $Q_4^{j,1}$ are conditionally faulty.

Case 1: Both $Q_4^{j,0}$ and $Q_4^{j,1}$ are conditionally faulty. For convenience, let $F_0 = F(Q_4^{j,0})$ and $F_1 = F(Q_4^{j,1})$. Without loss of generality, we assume that $f_0 = |F_0| = 2$ and $f_1 = |F_1| = 1$. Moreover, we assume $s \in V_0(Q_4 - F)$.

Subcase 1.1. Both s and t are in $Q_4^{j,0}$. By Lemma 7, $Q_4^{j,0} - F_0$ contains a path H_0 of length at least $3 = 7 - 2f_0$ (respectively, $2 = 6 - 2f_0$) between s and t if $h(s,t)$ is odd (respectively, even). Obviously, H_0 can be written as $\langle s = x_0, x_1, x_2, H_0', t\rangle$. If $(x_1)^j$ is faulty, then $(x_0)^j$ and $(x_2)^j$ are fault-free. By Theorem 2, $Q_4^{j,1}$ is hyper-Hamiltonian laceable. Thus $Q_4^{j,1} - \{(x_1)^j\}$ has a Hamiltonian path H_1 between $(x_0)^j$ and $(x_2)^j$. As a result, $\langle s = x_0, (x_0)^j, H_1, (x_2)^j, x_2, H_0', t\rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) when $h(s,t)$ is odd (respectively, even). If $(x_1)^j$ is fault-free, then $(x_0)^j$ or $(x_2)^j$ is fault-free. Suppose, for example, that $(x_0)^j$ is fault-free. By Lemma 7, $Q_4^{j,1} - F_1$ has a fault-free path H_1 of length at least $7 - 2f_1$ between $(x_0)^j$ and $(x_1)^j$. As a result, $\langle s = x_0, (x_0)^j, H_1, (x_1)^j, x_1, x_2, H_0', t\rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) when $h(s,t)$ is odd (respectively, even).

Subcase 1.2. Both s and t are in $Q_4^{j,1}$. First, we consider the case that $h(s,t)$ is odd. By Lemma 7, $Q_4^{j,1} - F_1$ contains a path T_1 of length at least $5 = 7 - 2f_1$ between s and t. Let $A = \{(T_1(i), T_1(i+1)) \mid 1 \leqslant i \leqslant 5 \text{ and } i \equiv 1 (\text{mod} 2)\}$ be a set of disjoint links on T_1. Since $|A| = 3 > f_0$, there exists an odd integer $\hat{\imath}$, $1 \leqslant \hat{\imath} \leqslant 5$, such that both $(T_1(\hat{\imath}))^j$ and $(T_1(\hat{\imath}+1))^j$ are fault-free. Let $w = T_1(\hat{\imath})$ and $b = T_1(\hat{\imath}+1)$. Accordingly, T_1 can be written as $\langle s, T_1', w, b, T_1'', t\rangle$. By Lemma 7, $Q_4^{j,0} - F_0$ has a path T_0 of length at least $7 - 2f_0$ between $(w)^j$ and $(b)^j$. As a result, $\langle s, T_1', w, (w)^j, T_0, (b)^j, b, T_1'', t\rangle$ is a fault-free path of length at least $15 - 2f$ between s and t.

Next, we consider the case that $h(s,t)$ is even. Hence we have $t \in V_0(Q_4 - F)$. Let u denote the faulty node in $Q_4^{j,1}$. Then we distinguish the following two subcases.

Subcase 1.2.1. Suppose that $u \in V_1(Q_4^{j,1})$. By Theorem 2, $Q_4^{j,1}$ is hyper-Hamiltonian laceable. Thus $Q_4^{j,1} - \{u\}$ has a Hamiltonian path H_1 from s to t. Obviously, the length of H_1 is equal to 6. Let $B = \{(H_1(i), H_1(i+1)) \mid 1 \leqslant i \leqslant 6 \text{ and } i \equiv 1 (\text{mod } 2)\}$ be a set of disjoint links on T_1. Since $|B| = 3 > f_0$, there exists an odd integer $\hat{\imath}$, $1 \leqslant \hat{\imath} \leqslant 6$, such that both $(H_1(\hat{\imath}))^j$ and $(H_1(\hat{\imath}+1))^j$ are fault-free. Let $w = H_1(\hat{\imath})$ and $b = H_1(\hat{\imath}+1)$. Thus H_1 can be written as $\langle s, H_1', w, b, H_1'', t\rangle$. By Lemma 7, $Q_4^{j,0} - F_0$ has a path H_0 of length at least $7 - 2f_0$ between $(w)^j$ and $(b)^j$. As a result, $\langle s, H_1', w, (w)^j, H_0, (b)^j, b, H_1'', t\rangle$ is a fault-free path of length at least $14 - 2f_0 > 14 - 2f$ between s and t.

Subcase 1.2.2. Suppose that $u \in V_0(Q_4^{j,1})$. Since $h(s,t)$ is even, it follows from Lemma 7 that $Q_4^{j,1} - F_1$ has a path T_1 of length at least $6 - 2f_1 = 4$ between s and t. If there exists a link (w,b) on T_1 such that both $(w)^j$ and $(b)^j$ are fault-free, then a path of length at least $14 - 2f$ can be constructed in a way similar to that described in Subcase 1.2.1. Otherwise, we have $F_0 \cap \{(T_1(i))^j, (T_1(i+1))^j\} \neq \emptyset$ for every i. Then we claim that both $(T_1(2))^j$ and $(T_1(4))^j$ are faulty. Since $f_0 = 2$, we see that $|F_0 \cap \{(T_1(1))^j, (T_1(2))^j, (T_1(3))^j\}| = 1$ and $|F_0 \cap \{(T_1(3))^j, (T_1(4))^j, (T_1(5))^j\}| = 1$. Then we have $F_0 \cap \{(T_1(1))^j, (T_1(2))^j, (T_1(3))^j\} = (F_0 \cap \{(T_1(1))^j, (T_1(2))^j\}) \cap (F_0 \cap \{(T_1(2))^j, (T_1(3))^j\}) = \{(T_1(2))^j\}$. Similarly, we have $F_0 \cap \{(T_1(3))^j, (T_1(4))^j, (T_1(5))^j\} = \{(T_1(4))^j\}$. That is, $F_0 = \{(T_1(2))^j, (T_1(4))^j\}$. By Lemma 8, $Q_4^{j,0} - F_0$ contains either a path T_0 of length 4 between $(T_1(1))^j$ and $(T_1(3))^j$ or a path R_0 of length 4 between $(T_1(3))^j$ and $(T_1(5))^j$. As a result,$\langle s = T_1(1), (T_1(1))^j, T_0, (T_1(3))^j, T_1(3), T_1(4), T_1(5) = t\rangle$ or $\langle s = T_1(1), T_1(2), T_1(3), (T_1(3))^j, R_0, (T_1(5))^j, T_1(5) = t\rangle$ is a fault-free path of length $8 = 14 - 2f$.

Subcase 1.3. Suppose that s is in $Q_4^{j,0}$ and t is in $Q_4^{j,1}$. Since $f_0 = 2$, we have $|V_1(Q_4^{j,0}) - F_0| \geqslant 2 = |F_1 \cup \{t\}|$ and $|V(Q_4^{j,0}) - (F_0 \cup \{s\})| = 5 > |F_1 \cup \{t\}|$. If $h(s,t)$ is odd, we choose a node x of $V_1(Q_4^{j,0}) - F_0$ such that $(x)^j$ is fault-free; otherwise, we choose a node x of $V(Q_4^{j,0}) - (F_0 \cup \{s\})$ such that $(x)^j \notin F_1 \cup \{t\}$. By Lemma 7, $Q_4^{j,0} - F_0$ contains a path H_0 of length at least $7 - 2f_0$ (respectively, $6 - 2f_0$) between s and x when $h(s,x)$ is odd (respectively, even). Similarly, $Q_4^{j,1} - F_1$ contains a path H_1 of length at least $7 - 2f_1$ (respectively, $6 - 2f_1$) between $(x)^j$ and t when $h((x)^j, t)$ is odd (respectively, even). As a result, $\langle s, H_0, x, (x)^j, H_1, t\rangle$ is a fault-free path of length at least $15 - 2f$ (respectively, $14 - 2f$) if $h(s,t)$ is odd (respectively, even).

Case 2: Suppose Q_4 has its faulty nodes distributed as in Fig. 2c. To be precise, we assume $F = \{0000, 0011, 1100\}$. Then we partition Q_4 into $Q_4^{4,0}$ and $Q_4^{4,1}$. It is noticed that $Q_4^{4,0}$ is not conditionally faulty.

Subcase 2.1. Both s and t are in $Q_4^{4,0} - \{0000, 0011\}$. By Theorem 3, $Q_4^{4,0} - \{0000\}$ has a path T_0 of length at least 5 (respectively, 4) between s and t if $h(s,t)$ is odd (respectively, even).

Table 2
The required paths in Subcase 2.3 of Theorem 4.

$s = 1101$	$t = 1110$	$\langle s = 1101, 1001, 0001, 0101, 0100, 0110, 0010, 1010, 1110 = t\rangle$
	$t = 1111$	$\langle s = 1101, 1001, 0001, 0101, 0100, 0110, 0010, 1010, 1110, 1111 = t\rangle$
	$t = 1000$	$\langle s = 1101, 0101, 0001, 1001, 1011, 1111, 1110, 1010, 1000 = t\rangle$
	$t = 1001$	$\langle s = 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1010, 1000, 1001 = t\rangle$
	$t = 1010$	$\langle s = 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1001, 1000, 1010 = t\rangle$
	$t = 1011$	$\langle s = 1101, 0101, 0001, 1001, 1000, 1010, 1110, 1111, 1011 = t\rangle$
$s = 1110$	$t = 1111$	$\langle s = 1110, 1010, 1000, 1001, 1101, 0101, 0100, 0110, 0111, 1111 = t\rangle$
	$t = 1000$	$\langle s = 1110, 0110, 0100, 0101, 0001, 1001, 1011, 1010, 1000 = t\rangle$
	$t = 1001$	$\langle s = 1110, 0110, 0100, 0101, 1101, 1111, 1011, 1010, 1000, 1001 = t\rangle$
	$t = 1010$	$\langle s = 1110, 0110, 0100, 0101, 0001, 1001, 1101, 1111, 1011, 1010 = t\rangle$
	$t = 1011$	$\langle s = 1110, 0110, 0100, 0101, 0001, 1001, 1101, 1111, 1011 = t\rangle$
$s = 1111$	$t = 1000$	$\langle s = 1111, 0111, 0110, 0100, 0101, 0001, 1001, 1011, 1010, 1000 = t\rangle$
	$t = 1001$	$\langle s = 1111, 0111, 0101, 0100, 0110, 0010, 1010, 1000, 1001 = t\rangle$
	$t = 1010$	$\langle s = 1111, 0111, 0110, 0100, 0101, 1101, 1001, 1000, 1010 = t\rangle$
	$t = 1011$	$\langle s = 1111, 0111, 0101, 0100, 0110, 0010, 1010, 1000, 1001, 1011 = t\rangle$
$s = 1000$	$t = 1001$	$\langle s = 1000, 1010, 1110, 0110, 0100, 0101, 1101, 1111, 1011, 1001 = t\rangle$
	$t = 1010$	$\langle s = 1000, 1001, 1101, 0101, 0100, 0110, 1110, 1111, 1011, 1010 = t\rangle$
	$t = 1011$	$\langle s = 1000, 1001, 1101, 0101, 0100, 0110, 1110, 1111, 1011 = t\rangle$
$s = 1001$	$t = 1010$	$\langle s = 1001, 1011, 1111, 0111, 0101, 0100, 0110, 1110, 1010 = t\rangle$
	$t = 1011$	$\langle s = 1001, 1000, 1010, 1110, 0110, 0100, 0101, 1101, 1111, 1011 = t\rangle$
$s = 1010$	$t = 1011$	$\langle s = 1010, 1000, 1001, 1101, 0101, 0100, 0110, 0111, 1111, 1011 = t\rangle$

We consider first that $h(s,t)$ is odd. Thus the length of path T_0 is greater than or equal to 5. Then T_0 passes through every node of $V_0(Q_4^{4,0}) - \{0000\}$. In particular, the faulty node 0011 is on T_0. Hence T_0 can be written as $\langle s, T'_0, x, 0011, y, T''_0, t\rangle$. Since $h(0011, 1100) = 4$, both $(x)^4$ and $(y)^4$ are fault-free. Since $h((x)^4, (y)^4)$ is even, Theorem 3 ensures that $Q_4^{4,1} - \{1100\}$ has a path T_1 of length at least 4 between $(x)^4$ and $(y)^4$. As a result, $\langle s, T'_0, x, (x)^4, T_1, (y)^4, y, T''_0, t\rangle$ is a fault-free path of length at least $9 = 15 - 2f$.

Next, we consider the case that $h(s,t)$ is even. We distinguish whether the faulty node 0011 is on T_0. If node 0011 is on T_0, then a path of length at least 8 can be constructed to join s and t in a way similar to that described earlier. Otherwise, there exists a link (w,b) on T_0 such that both $(w)^4$ and $(b)^4$ are fault-free. Hence T_0 can be written as $\langle s, R'_0, w, b, R''_0, t\rangle$. By Theorem 3, $Q_4^{4,1} - \{1100\}$ has a path T_1 of length at least 5 between $(w)^4$ and $(b)^4$. Then $\langle s, R'_0, w, (w)^4, T_1, (b)^4, b, R''_0, t\rangle$ turns out to be a fault-free path of length at least $10 > 14 - 2f$.

Subcase 2.2. Suppose that s is in $Q_4^{4,0} - \{0000, 0011\}$ and t is in $Q_4^{4,1} - \{1100\}$. By Theorem 3, $Q_4^{4,0} - \{0000\}$ has a path T_0 of length at least 5 (respectively, 4) between nodes s and 0011 if $h(s, 0011)$ is odd (respectively, even). Accordingly, we write T_0 as $\langle s, T'_0, x, y, 0011\rangle$. Since $h(0011, 1100) = 4$, both $(x)^4$ and $(y)^4$ is fault-free. On the one hand, we assume $(y)^4 \neq t$. By Theorem 3, $Q_4^{4,1} - \{1100\}$ has a path T_1 of length at least 5 (respectively, 4) between $(y)^4$ and t if $h((y)^4, t)$ is odd (respectively, even). As a result, $\langle s, T'_0, x, y, (y)^4, T_1, t\rangle$ is a fault-free path of length at least $9 = 15 - 2f$ (respectively, $8 = 14 - 2f$) if $h(s,t)$ is odd (respectively, even). On the other hand, if $(y)^4 = t$, then Theorem 3 ensures that $Q_4^{4,1} - \{1100\}$ has a path R_1 of length at least 5 between $(x)^4$ and $(y)^4$. Then $\langle s, T'_0, x, (x)^4, R_1, (y)^4 = t\rangle$ turns out to be a fault-free path of length at least $9 = 15 - 2f$ (respectively, $8 = 14 - 2f$) if $h(s,t)$ is odd (respectively, even).

Subcase 2.3. Both s and t are in $Q_4^{4,1} - \{1100\}$. We list the required paths obtained by brute force in Table 2.

Therefore the proof is completed. □

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