國 立 交 通 大 學

應用數學系

碩 士 論 文

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Generalized Hirota method of KdV type bilinear equation

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摘　　要

很多偏微分方程式都可以轉換成$F(D)f\cdot f=0$ 這種雙線性方程式 。這篇論文我們試著得到這雙線性方程式的廣義解，由於這廣義解擁有Fredholm 行列式值的結構，從中我們發展出一個 GLM 積分方程式，相較於逆散射方法所得到的 GLM 積分方程式，我們所得到的積分方程擁有更寬廣的應用空間

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Abstract

$F(D)f \cdot f = 0$ is an important bilinear equation into which many PDEs can be transformed. In this thesis we try to derive the generalized soliton solutions for this bilinear equation. Owing to the structure of Fredholm's determinant of generalized soliton solution we can develop a GLM integral equation whose application is wider than GLM equation produced in inverse scattering method.

誌謝

這真是一個很漫長的過程，不管怎樣，我總算畢業了，感謝邵老師對我的悉心指導，感謝口試委員的不吝指正，以及所有被我麻煩過的人，我想呢，要報答這些人的最好方法就是，在社會上做一個頂天立地，堂堂正正的男子漢

Contents

Chapter 1

Introduction

1.1 History

The story of soliton starts in 1834 from the findings of Russel[1] , but his discovery didn't evoke many ripples around the scientific circle in British and suffered from attacks. It was not until the 1870's that Russel's work was finally vindicated and its scientific importance can be measured by the eminence of the men who did the job[2]. After 25 years Korteweg and de Vries were to derive their famous equation

$$u_t + 6uu_x + u_{xxx} \tag{1.1}$$

,where subscript denote partial derivatives. However , the KdV equation didn't draw much attention until the remarkable discoveries of Zabusky and Kruskal , who were investigating Fermi-Pasta-Ulam problem , in 1965[3]. They observe the particle-like nature of those interacting solitay waves in their numarical experiment. The name "soliton" was born. In a later analysis of the interaction Lax[4] verified their observation rigorously. Afterwards, analytical methods of soliton theory were developed prosperously.

1.2 Three major methods

We introduce briefly below three major methods of soliton theory

Inverse scattering transform

Gardner et al[5] were able to relate equation (1.1) to the eigenvalue problem

$$\varphi_{xx} + u\varphi = \lambda\varphi \tag{1.2}$$

They proved that (1.2) is isospectral in time when u satisfies (1.1) Using inverse scattering they were able to find a GLM equation for the initial value problem of KdV equation and to derive a number of important results , including the explicit solution for the interaction of any number of solitary waves. The success of inverse scattering was explained by a deeper and more general argument by Lax[4], opening the way for more equations to be solved. In 1972, Zakharov and Shabat[6] found an eigenvalue problem with which they were able to solve the nonlieanr Schrodinger equation

$$i\varphi_t + \varphi_{xx} + |\varphi|^2\varphi = 0 \tag{1.3}$$

Also in 1972, Waditi[7,8] applied essentially the same eigenvalue problem to solve the modified KdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0 \tag{1.4}$$

Backlund transformation

Consider a pair of partial differential equations

$$A(u) = 0 \tag{1.5}$$

$$B(\tilde{u}) = 0 \tag{1.6}$$

in which u and $\tilde{u}$ denote the unknown functions, while A and B represent differential operators in m independent variables. Then the set of relations

$$R_j((u), (\tilde{u}), (\xi)) = 0 \quad j = 1, \ldots, n$$

where (u) and ($\tilde{u}$) denote a finite sequence of partial derivatives of u and $\tilde{u}$ respectively.The number of each sequence need not be equal. (ξ) represent parameters. In reality, A and B are not restricted to the average differential operator. Hirota[9] created a new form of Backlund transformations,in which case A and B are specialized differential operator, the combinations of D operator. For example, let f and f' be two solutions of the bilinearized KdV equation , in symbols

$$D_x(D_t + D_x^3)f \cdot f = 0 \tag{1.7}$$

$$D_x(D_t + D_x^3)f' \cdot f' = 0 \tag{1.8}$$

,where D_x and D_t will be explained later.

$$(D_t + 3\lambda D_x + D_x^3)f' \cdot f = 0 \tag{1.9}$$

$$D_x^2 f' \cdot f = \lambda f' \cdot f \tag{1.10}$$

constitute the Backlund transformation of the KdV equation in the bilinear fornalism

Hirota direct method

A perturbational series is a commonly used technique to solve a PDE. The same technique applies to Hirota bilinear equation. For example if we put $\varphi = G/F$ then nonlinear Schrodinger equation $i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0$ can be transformed into

$$(iD_t + D_x^2 - \lambda)G \cdot F = 0 \tag{1.11}$$

$$(D_x^2 - \lambda) = -2G \cdot G^* \tag{1.12}$$

(1.11) and (1.12) are nonlinear Schrodinger equation in the bilinear form. λ is an parameter to be determined. F and G can be set equal to:

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \tag{1.13}$$

$$G = g_0(1 + \varepsilon g_1 + \varepsilon g_2 + \cdots) \tag{1.14}$$

Substituting (1.13) and (1.14) into (1.11) and (1.12) and colleting terms with the same power of ε. In this case, the calculated result indicates that both F and G turn out to have finite terms , i.e , $F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots + \epsilon^m f_m$ and $G = g_0(1 + \varepsilon g_1 + \varepsilon g_2 + \cdots + \varepsilon^n g_n)$(say), provided the solutions to f_1, g_0 and g_1 are properly selected.

Generally speaking , a perturbational series applied to a PDE may not turn out to have finite terms , perhaps not even be convergent. The advantage of Hirota method is able to truncate the series after a number of finite terms.

The marvel of math is that different approaches may lead to the same conclusion. Hirota[9] used the concept of Backlund transformations to show that a new forms of the Backlund transformations lead to the known inverse scattering methods of solutions of the initial-value problem for the respective nonlinear evolution equations.

The drawback of bilinear method is unable to solve a initial value problem for a soliton equation. Oishi studied using bilinear method to solve initial-value problems whose solutions may be expressed as a determinant. The thesis is primarily a review on Oishi's papers[11][12].

Chapter 2

D operator and Bilinearization

2.1 The properties of D operator

The D operator is defined by

$$D_t^m D_x^n a(t,x)\cdot b(t',x') = (\partial_t - \partial_{t'})^m(\partial_x - \partial_{x'})^n a(t,x)b(t',x')|_{t'=t,x'=x} \tag{2.1}$$

For example:

$$D_x a\cdot b = a_x b - a b_x$$

$$D_x^2 a\cdot b = a_{xx}b - 2a_x b_x + a b_{xx}$$

$$D_x^3 a\cdot b = a_{xxx}b - 3a_{xx}b_x + 3a_x b_{xx} - a b_{xxx}$$

From the definition (2.1) we have the following lemma

Lemma 2.1

$$D_x^m a(x)\cdot 1 = \partial_x^m a(x) \tag{2.2}$$

$$D_{x_1}^{m_1} D_{x_2}^{m_2}\dots D_{x_n}^{m_n} a\cdot b = (-1)^{m_1+m_2+\dots+m_n} D_{x_1}^{m_1} D_{x_2}^{m_2}\dots D_{x_n}^{m_n} b\cdot a \tag{2.3}$$

$$D_x^m a\cdot a = 0 \text{ for odd m} \tag{2.4}$$

$$D_x D_t a\cdot 1 = D_x D_t 1\cdot a = \partial_x\partial_t a \tag{2.5}$$

Proof

i(2.3) $D_{x_1}^{m_1} D_{x_2}^{m_2}\dots D_{x_n}^{m_n} a\cdot b$

$$= (\partial_{x_1} - \partial_{x_1'})^{m_1}(\partial_{x_2} - \partial_{x_2'})^{m_2}\cdots(\partial_{x_n} - \partial_{x_n'})^{m_n} a\cdot b|_{x_1'=x_1,x_2'=x_2\dots\dots x_n'=x_n}$$
$$= (-1)^{m_1+m_2+\dots+m_n}(\partial_{x_1'} - \partial_{x_1})^{m_1}(\partial_{x_2'-\partial_{x_2}})^{m_2}\cdots(\partial_{x_n'} - \partial_{x_n})^{m_n} a\cdot b|_{x_1'=x_1,x_2'=x_2\dots\dots x_n'=x_n}$$
$$= (-1)^{m_1+m_2+\dots+m_n} D_{x_1}^{m_1} D_{x_2}^{m_2}\dots D_{x_n}^{m_n} b\cdot a$$

ii(2.4) By (2.3) $D_x^m a \cdot a = (-1)^m D_x^m a \cdot a = -D_x^m a \cdot a$
$\Rightarrow$

$$D_x^m a \cdot a = 0$$

QED

Also using definition (2.1) we have

$$D_{x_1}^{m_1} D_{x_2}^{m_2} \dots D_{x_n}^{m_n} exp(\mathbf{\Phi}_1 \cdot \mathbf{x}) \cdot exp(\mathbf{\Phi}_2 \cdot \mathbf{x})$$

$$= (\phi_1^1 - \phi_1^2)^{m_1} (\phi_2^1 - \phi_2^2)^{m_2} \cdots (\phi_n^1 - \phi_n^2)^{m_n} exp[(\mathbf{\Phi_1} + \mathbf{\Phi}_2) \cdot \mathbf{x}] \tag{2.6}$$

with $\mathbf{\Phi}_i$ and $\mathbf{x}$ being vectors

$$\mathbf{\Phi}_i = (\phi_1^i, \phi_2^i, \dots, \phi_n^i) \ i = 1, 2$$

$$\mathbf{x} = (x_1, x_2, \dots x_n)$$

$$\mathbf{\Phi}_i \cdot \mathbf{x} \equiv \sum_{k=1}^{n} \phi_k^i x_k$$

In particular , we have

$$D_{x_1}^{m_1} D_{x_2}^{m_2} \dots D_{x_n}^{m_n} exp(\mathbf{\Phi} \cdot \mathbf{x}) \cdot exp(\mathbf{\Phi} \cdot \mathbf{x}) = 0 \tag{2.7}$$

Generally speaking , if F is a multipolynomial in $D_{x_1}, D_{x_2} \dots D_{x_m}$, then

$$F(\mathbf{D}) \exp(\mathbf{\Phi}_1 \cdot \mathbf{x}) \cdot \exp(\mathbf{\Phi}_2 \cdot \mathbf{x}) = F(\mathbf{\Phi_1} - \mathbf{\Phi}_2) \exp[(\mathbf{\Phi_1} + \mathbf{\Phi_2}) \cdot \mathbf{x}] \tag{2.8}$$

$$F(\mathbf{D}) f \cdot 1 = F(\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}) f \tag{2.9}$$

The following lemma are essential to bilinearize an evolution equation

Lemma 2.2

$$\exp(\delta D_x) a(x) \cdot b(x) = a(x + \delta) b(x - \delta) \tag{2.10}$$

Proof

$$\begin{aligned}
\exp(\delta D_x) a(x) \cdot b(x) &= \exp[\delta(\partial_x - \partial_{x'})] a(x) b(x')|_{x'=x} \\
&= \exp(\delta \partial_x) a(x) \exp(-\delta \partial_{x\prime}) b(x\prime)|_{x\prime=x} \\
&= [1 + \delta \partial_x + (\delta^2 \partial_x^2)/2! + \cdots] a(x) \times \\
&\quad [1 - \delta \partial_x + (\delta^2 \partial_x^2)/2! - \cdots] b(x\prime)|_{x'=x}
\end{aligned}$$

From Taylor series we know the right hand side

$$= a(x + \delta) b(x' - \delta)|_{x\prime=x} = a(x + \delta) b(x - \delta)$$

QED

2.2 Bilinearization

There are several techniques to transform nonlinear partial differential equation

$$L(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \ldots) = 0$$

into bilinear forms. We introduce some of them here. Let us do some calculations

$$\partial_x[\frac{a(x)}{b(x)}] = \frac{a_x b - b_x a}{b^2} = \frac{(\partial_x - \partial_{x'})a(x)b(x')}{b^2}|_{x'=x} = \frac{D_x a \cdot b}{b^2}$$

$$\begin{aligned}\partial_x^2[\frac{a(x)}{b(x)}] &= \frac{b^2(a_x b - b_x a)_x - 2bb_x(a_x b - b_x a)}{b^4} \\ &= \frac{a_{xx}b - 2a_x b_x + ab_{xx}}{b^2} - \frac{a}{b}\frac{2}{b^2}(b_{xx}b - b_x^2) \\ &= \frac{D_x^2 a \cdot b}{b^2} - \frac{a}{b}\frac{D_x^2 b \cdot b}{b^2}\end{aligned}$$

Therefore a change of dependent variable $u = \frac{a}{b}$ plays a natural role in bilinearization. To avoid tedious calculations in transforming ∂_x^n for $n \geq 3$ we need a lemma

Lemma 2.3

$$\exp(\delta\partial_x)\frac{a}{b} = \frac{\exp(\delta D_x)a \cdot b}{\cosh(\delta D_x)b \cdot b} \tag{2.11}$$

proof

By making use of (2.10) we have

$$\begin{aligned}\cosh(\delta D_x)b \cdot b = \frac{[\exp(\delta D_x) + \exp(-\delta D_x)]}{2} b \cdot b &= \frac{b(x+\delta)b(x-\delta)}{2} \\ + \frac{b(x-\delta)b(x+\delta)}{2} &= b(x+\delta)b(x-\delta)\end{aligned}$$

Therefore

$$\exp(\delta\partial_x)\frac{a}{b} = \frac{a(x+\delta)}{b(x+\delta)} = \frac{a(x+\delta)b(x-\delta)}{b(x+\delta)b(x-\delta)} = \frac{\exp(\delta D_x)a \cdot b}{\cosh(\delta D_x)b \cdot b}$$

QED

Expanding Taylor series on both sides of (2.11) with respect to the parameter δ , we have

$$(1 + \delta\partial_x + \frac{\delta^2}{2}\partial_x^2 + \frac{\delta^3}{6}\partial_x^3 + \cdots)\frac{a}{b} = \frac{(1 + \delta D_x + 1/2\delta^2 D_x^2 + 1/6\delta^3 D_x^3 + \cdots)a \cdot b}{(1 + 1/2\delta^2 D_x^2 + 1/24\delta^4 D_x^4 + \cdots)b \cdot b}$$

$$= (\frac{a}{b} + \delta\frac{D_x a \cdot b}{b^2} + \delta^2/2\frac{D_x^2 a \cdot b}{b^2} + \cdots) \times (1 + \delta^2/2\frac{D_x^2 b \cdot b}{b^2} + \delta^4/24\frac{D_x^4 b \cdot b}{b^2} + \cdots)^{-1}$$

Expanding the denominator using $(1+X)^{-1} = 1 - X + X^2 + \cdots$, and collecting terms in powers of δ , we can obtain formulae which express derivatives of $u = a/b$ in terms of the D-operator. The change of dependent variable $u = 2(\log f)_{xx}$ also plays a natural role in bilinearization. Likewise , we also need a lemma to avoid those messy calculations

Lemma 2.4

$$2\cosh(\delta\partial_x)\log f(x) = \log[\cosh(\delta D_x)f(x)\cdot f(x)] \tag{2.12}$$

proof

$$\begin{aligned}
2\cosh(\delta\partial_x)\log f(x) &= 2[\frac{\exp(\delta\partial_x)+\exp(-\delta\partial_x)}{2}]\log f(x) = \log f(x+\delta) + \log f(x-\delta) \\
&= \log f(x+\delta)f(x-\delta) = log[\frac{f(x+\delta)f(x-\delta)}{2} + \frac{f(x-\delta)f(x+\delta)}{2}] \\
&= \log[\frac{\exp(\delta D_x)f\cdot f}{2} + \frac{\exp(-\delta D_x)f\cdot f}{2}] \\
&= \log[\cosh(\delta D_x)f(x)\cdot f(x)]
\end{aligned}$$

QED

Expanding (2.12) with respect to δ , we have

$$2\partial_x^2 \log f = \frac{D_x^2 f\cdot f}{f^2} \tag{2.13}$$

$$2\partial_x\partial_t \log f = \frac{D_x D_t f\cdot f}{f^2} \tag{2.14}$$

$$2\partial_x^4 \log f = \frac{D_x^4 f\cdot f}{f^2} - 3(\frac{D_x^2 f\cdot f}{f^2})^2 \tag{2.15}$$

$$2\partial_x^6 \log f = \frac{D_x^6 f\cdot f}{f^2} - 15\frac{D_x^4 f\cdot f}{f^2}\frac{D_x^2 f\cdot f}{f^2} + 30(\frac{D_x^2 f\cdot f}{f^2})^3 \tag{2.16}$$

Now we apply these formulae to some equations. Making the dependent variable transformation $u = 2(\log f)_{xx}$ to KdV equation (1.1) we obtain

$$2(\log f)_{xxt} + 6(2\log f)_{xx}(2\log f)_{xxx} + (2\log f)_{xxxxx} = 0$$

Integrate with respect to x once

$$2(\log f)_{xt} + 3[(2\log f)_{xx}]^2 + (2\log f)_{xxxx} = 0$$

By making use of (2.13) , (2.14) , (2.15) we have

$$\frac{D_x D_t f\cdot f}{f^2} + 3(\frac{D_x^2 f\cdot f}{f^2})^2 + \frac{D_x^4 f\cdot f}{f^2} - 3(\frac{D_x^2 f\cdot f}{f^2})^2 = 0$$

Multiplying f^2 on both sides we have

$$D_x(D_t + D_x^3)f \cdot f = 0 \tag{2.17}$$

Now we proceed to apply the same transformation of dependent variable to K-P equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

Set $u = 2(\log f)_{xx}$ and integrate with respect to x twice we obtain

$$-4(2\log f)_{xt} + (2\log f)_{xxxx} + 3[(2\log f)_{xx}]^2 + 3(2\log f)_{yy} = 0$$

Use (2.13) , (2.14) , and (2.15). Then

$$-4\frac{D_xD_tf \cdot f}{f^2} + \frac{D_x^4 f \cdot f}{f^2} - 3(\frac{D_x^2 f \cdot f}{f^2})^2 + 3(\frac{D_x^2 f \cdot f}{f^2})^2 + 3\frac{D_y^2 f \cdot f}{f^2} = 0$$

Multiplying f^2 on both side we obtain the KP equation in the bilinear form

$$(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 0 \tag{2.18}$$

Chapter 3

Generalized Soliton Solution

3.1 Introduction

Many soliton equations can be transformed into bilinear equations such as:

Ex1 The KdV equation : $u_t + 6uu_x + u_{xxx} = 0 \quad u = 2(\log f)_{xx}$
$\Rightarrow Dx(D_t + D_x^3) f \cdot f = 0$

Ex2 The Sawada-Kotera equation:
$u_t + 15(u^3 + uu_{xx}) + u_{xxxxx} = 0 \quad u = 2(\log f)_{xx}$
$\Rightarrow D_x(D_t + D_x^5) f \cdot f = 0$

Ex3 The K-P equation:$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0 \quad u = 2(\log f)_{xx}$
$\Rightarrow (D_x^4 - 4D_xD_t + 3D_y^2) f \cdot f = 0$

These equations , which can be written in the form

$$F(\mathbf{D}) = 0$$

and F is a multipolynomial in $\mathbf{D} = (D_{x_1}, D_{x_2}, \ldots D_{x_m})$ with $x = x_1$, $y = x_2$, $t = x_3$....etc. , have three conditions in common.

$$F(-\mathbf{D}) = F(\mathbf{D}) \tag{3.1}$$

$$F(\mathbf{0}) = 0 \tag{3.2}$$

$$\sum_{\sigma=1,-1} F(\sum_{i=1}^{N} \sigma_i \mathbf{P}_i) \prod_{i<j}^{(N)} F(\sigma_i \mathbf{P}_i - \sigma_j \mathbf{P}_j)\sigma_i\sigma_j = 0 \ , \ for\ N = 1, 2, \ldots \tag{3.3}$$

(3.3) is well known as Hirota condition. In this thesis we try to solve bilinear equation

$$F(\mathbf{D}) f \cdot f = 0$$

under these three conditions. In what follows we tacitly assume these conditions

Lemma 3.1 If F is a multipolynomial in $D_{x_1}, D_{x_2}, \ldots D_{x_m}$ satisfying condition (3.1) , then

$$F(\mathbf{D})f \cdot g = F(\mathbf{D})g \cdot f \tag{3.4}$$

pf : Since $F(\mathbf{D}) = F(D_{x_1}, D_{x_2}, \ldots, D_{x_m})$ is a multipolynomial in $D_{x_1}, D_{x_2}, \ldots D_{x_m}$, then by using (2.3)

$$\begin{aligned} F(\mathbf{D})f \cdot g &= \sum_{0 \le n_1, n_2 \ldots n_m \le N} \alpha_{n_1 n_2 \ldots n_m} D_{x_1}^{n_1} D_{x_2}^{n_2} \ldots D_{x_m}^{n_m} f \cdot g \\ &= \sum_{0 \le n_1, n_2 \ldots n_m \le N} (-1)^{n_1 + n_2 + \ldots + n_m} \alpha_{n_1 n_2 \ldots n_m} D_{x_1}^{n_1} D_{x_2}^{n_2} \ldots D_{x_m}^{n_n} g \cdot f \\ &= F(\mathbf{D})g \cdot f \ , \ if \ n_1 + n_2 + \ldots + n_m \ is \ even \end{aligned}$$

and the evenness of $n_1 + n_2 + \ldots + n_m$ is exactly what (3.1) assures

QED

3.2 The pure N-soliton solutions

Now we proceed to solve

$$F(\mathbf{D})f \cdot f = F(D_{x_1}, D_{x_2}, \ldots, D_{x_m})f \cdot f = 0$$

What we tackle this problem is making use of perturbation method , so we put

$$f = 1 + \varepsilon^1 f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots$$

Substituting this perturbational series of f into the above bilinear equation and collecting like powers of ε , we have

ε^0

$$F(\mathbf{D})1 \cdot 1 = 0 \ , \ \text{this is why (3.2) is required} \tag{3.5}$$

ε^1

$$F(\mathbf{D})(f_1 \cdot 1 + 1 \cdot f_1) = 0 \tag{3.6}$$

ε^2

$$F(\mathbf{D})(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0 \tag{3.7}$$

ε^3

$$F(\mathbf{D})(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0 \tag{3.8}$$

ε^4

$$F(\mathbf{D})(f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) = 0 \tag{3.9}$$

Running through the above procedures with a proper solution to f_1 , for example , $f_1 = \exp(\phi_1 x_1 + \phi_2 x_2 + \cdots + \phi_m x_m) \equiv \exp(\mathbf{\Phi}\cdot\mathbf{x})$, we can derive a truncated series , thus an explicit solution is obtained without resorting to summing the series. Now we set $f_1 = \exp(\mathbf{\Phi}\cdot\mathbf{x})$ then ε^1

$$\begin{aligned} F(\mathbf{D})(f_1\cdot 1 + 1\cdot f_1) &= 2F(\mathbf{D})f_1\cdot 1 = 2F(\partial_{x_1},\partial_{x_2},\ldots\partial_{x_m})\exp(\mathbf{\Phi}\cdot\mathbf{x}) \\ &= 2F(\phi_1,\phi_2\ldots\phi_m)\exp(\mathbf{\Phi}\cdot\mathbf{x}) = 0 \\ &\Rightarrow F(\phi_1,\phi_2\ldots\phi_m) = 0 \end{aligned} \tag{3.10}$$

the above equalities are on account of (2.9) and (3.4). (3.10) is a confinement implying that each $\phi_i, i = 1,2\ldots m$ can be expressed by $(m-1)$ independent variables such as $\phi_i = \phi_i(p,q,\ldots,z)$

ε^2

$$\begin{aligned} 2F(\mathbf{D})f_2\cdot 1 &= -F(\mathbf{D})f_1\cdot f_1 \\ &= -F(\mathbf{D})[\exp(\mathbf{\Phi}\cdot\mathbf{x})]\cdot[\exp(\mathbf{\Phi}\cdot\mathbf{x})] \\ &= -F(\mathbf{\Phi}-\mathbf{\Phi})\exp[(\mathbf{\Phi}+\mathbf{\Phi})\cdot\mathbf{x}] \\ &= -F(\mathbf{0})\exp(2\mathbf{\Phi}\cdot\mathbf{x}) = 0 \end{aligned}$$

$\Rightarrow f_2$ can be assigned to zero
The above equalities are due to (2.8) and (3.2)

ε^3

$$2F(\mathbf{D})f_3\cdot 1 = -2F(\mathbf{D})f_2\cdot f_1 = 0$$

$\Rightarrow f_3$ can be assigned to zero Going on the calculation for steps $\varepsilon^n, n > 3$ it is readily to have the conclusion that $f_n = 0$ for $n > 3$. Therefore we obtained a truncated series as formerly promised. Collectively , $u = 2(\log f)_{xx}$, where $f = 1 + \exp(\phi_1 x_1 + \phi_2 x_2 + \cdots + \phi_m x_m)$ with condition $F(\phi_1,\phi_2,\ldots\phi_m) = 0$, is a one-soliton solution

If we put $f_1 = \exp(\mathbf{\Phi}_1\cdot\mathbf{x}) + \exp(\mathbf{\Phi}_2\cdot\mathbf{x})$, where $\mathbf{\Phi}_1 = (\phi_1^1,\phi_2^1,\ldots\phi_m^1)$ and $\mathbf{\Phi}_2 = (\phi_1^2,\phi_2^2,\ldots\phi_m^2)$, we have (note that ϕ_i^2 does not mean square)

ε^1

$$\begin{aligned} F(\mathbf{D})(f_1\cdot 1 + 1\cdot f_1) &= 2F(\mathbf{D})f_1\cdot 1 = 2F(\partial_{x_1},\partial_{x_2},\ldots,\partial_{x_m})[\exp(\mathbf{\Phi}_1\cdot\mathbf{x}) \\ &\quad + \exp(\mathbf{\Phi}_2\cdot\mathbf{x})] \\ &= 2[F(\mathbf{\Phi}_1)\exp(\mathbf{\Phi}_1\cdot\mathbf{x}) + F(\mathbf{\Phi}_2)\exp(\mathbf{\Phi}_2\cdot\mathbf{x})] = 0 \\ &\Rightarrow F(\mathbf{\Phi}_1) = F(\mathbf{\Phi}_2) = 0 \end{aligned} \tag{3.11}$$

ε^2

$$
\begin{aligned}
2F(\mathbf{D})f_2 \cdot 1 &= -F(\mathbf{D})f_1 \cdot f_1 \\
&= -2F(\mathbf{D})[\exp(\mathbf{\Phi}_1 \cdot \mathbf{x})] \cdot [\exp(\mathbf{\Phi}_2 \cdot \mathbf{x})] - F(\mathbf{D})[\exp(\mathbf{\Phi}_1 \cdot \mathbf{x})] \cdot [\exp(\mathbf{\Phi}_1 \cdot \mathbf{x})] \\
&- F(\mathbf{D})[\exp(\mathbf{\Phi}_2 \cdot \mathbf{x})] \cdot [\exp(\mathbf{\Phi}_2 \cdot \mathbf{x})] \\
&= -2F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2) exp[(\mathbf{\Phi}_1 + \mathbf{\Phi}_2) \cdot \mathbf{x}]
\end{aligned}
$$

$\Rightarrow$ To meet the above equation we can use (2.9) and put

$$
f_2 = -\frac{F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)} \exp[(\mathbf{\Phi}_1 + \mathbf{\Phi}_2) \cdot \mathbf{x}]
$$

ε^3

$$
\begin{aligned}
2F(\mathbf{D})f_3 \cdot 1 &= -2F(\mathbf{D})f_2 \cdot f_1 \\
&= (const)F(\mathbf{D})\{exp[(\mathbf{\Phi}_1 + \mathbf{\Phi}_2) \cdot \mathbf{x}]\} \cdot \{\exp(\mathbf{\Phi}_1 \cdot \mathbf{x}) + \exp(\mathbf{\Phi}_2 \cdot \mathbf{x})\} \\
&= (const)\{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 - \mathbf{\Phi}_1) exp[(2\mathbf{\Phi}_1 + \mathbf{\Phi}_2) \cdot \mathbf{x}] \\
&+ F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 - \mathbf{\Phi}_2) exp[(\mathbf{\Phi}_1 + 2\mathbf{\Phi}_2) \cdot \mathbf{x}]\} \\
&= 0 \quad by \ (3.11)
\end{aligned}
$$

$\Rightarrow$ f_3 can be assigned zero to meet the above equation

ε^4

$$
2F(\mathbf{D})f_4 \cdot 1 = -2F(\mathbf{D})f_3 \cdot f_1 - F(\mathbf{D})f_2 \cdot f_2 = 0
$$

$\Rightarrow$ f_4 can be assigned zero by $f_3 = 0$, (2.7) and (3.2)

Going on the step for ε^n it is readily to conclude that $f_n = 0 \ \ for \ n > 4$ Collectively , $u = 2(\log f)_{xx}$, where $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 = 1 + \exp(\mathbf{\Phi}_1 \cdot \mathbf{x} + \eta_0) + \exp(\mathbf{\Phi}_2 \cdot \mathbf{x} + \eta_0) + a_{12} \exp[(\mathbf{\Phi}_1 + \mathbf{\Phi}_2) \cdot \mathbf{x} + 2\eta_0]$, $\exp \eta_0 = \varepsilon$, and $a_{12} = -\dfrac{F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)}$, is a two-soliton solution

If we put $f_1 = \sum_{i=1}^{N} \exp \mathbf{\Phi}_i \cdot \mathbf{x}$, then we obtain N-soliton solution? The answer is yes. Here we directly quote N-soliton solution derived by Hirota[10]

$$
f = \sum_{\mu=0,1} exp[\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} A_{ij} \mu_i \mu_j] \tag{3.12}
$$

where $\exp A_{ij} = -\dfrac{F(\mathbf{\Phi}_i - \mathbf{\Phi}_j)}{F(\mathbf{\Phi}_i + \mathbf{\Phi}_j)}$ and $\eta_i = \mathbf{\Phi}_i \cdot \mathbf{x} + (const)$. $\sum_{\mu=0,1}$ means a summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1$, $\sum_{i<j}^{(N)}$ means a summation over all possible pairs (i, j) chosen from the set $\{1, 2, \ldots, N\}$ with the condition that $i < j$

3.3 Construction of generalized soliton solutions

As formerly stated , if we put $f_1 = \sum_{i=1}^{N} \exp(\mathbf{\Phi}_i \cdot \mathbf{x})$, where $\mathbf{\Phi}_i = (\phi_1^i, \phi_2^i, \ldots, \phi_m^i)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, we have a N-soliton solution for arbitrary integer N The starting point of generalized soliton solutions is to express f_1 as a $m-1$ multiple integral , i.e. ,

$$f_1 = \int_{\Gamma_1} \int_{\Gamma_2} \cdots \int_{\Gamma_{m-1}} \exp \mathbf{\Phi} \cdot \mathbf{x} d\tau(p, q \ldots z) \tag{3.13}$$

, which can be abbreviated as $\int_{\Gamma_{(m-1)}} \exp \mathbf{\Phi} \cdot \mathbf{x} d\tau_{m-1}$ with the definitions

$$\mathbf{\Phi} \cdot \mathbf{x} \equiv \sum_{i=1}^{m} \phi_i(p, q, \ldots, z) x_i$$

and

$$\int_{\Gamma_{(m-1)}} d\tau(p, q, \ldots, z) \equiv \int_{\Gamma_1} \int_{\Gamma_2} \cdots \int_{\Gamma_{m-1}} c(p, q, \ldots, z) dp dq \cdots dz$$

The integration represents a multiple complex integral along possibly different path $\Gamma_i, i = 1, 2, \ldots, m-1$ with a properly selected complex function $c(p, q, \ldots, z)$

With f_1 specified by (3.13) now we go through the same perturbational approach as before

ε^1

$$\begin{aligned} F(\mathbf{D}) f_1 \cdot 1 &= F(\mathbf{D})[\int_{\Gamma_{(m-1)}} \exp \mathbf{\Phi} \cdot \mathbf{x} d\tau_{m-1}] \cdot 1 \\ &= \int_{\Gamma_{(m-1)}} F(\mathbf{D})[\exp(\mathbf{\Phi} \cdot \mathbf{x})] \cdot 1 d\tau_{m-1} \\ &= \int_{\Gamma_{(m-1)}} F(\mathbf{\Phi})[\exp(\mathbf{\Phi} \cdot \mathbf{x})] d\tau_{m-1} = 0 \quad \textit{by } (2.9) \end{aligned}$$

To meet the above equation

$$\Rightarrow F(\mathbf{\Phi}) = 0 \tag{3.14}$$

(3.14) explains why each component function $\phi_i = \phi_i(p, q, \ldots z)$ has $m-1$ independable variables. This is also the reason for $m-1$ multiple complex integral.

ε^2

$$
\begin{aligned}
F(\mathbf{D})f_2\cdot 1 &= -\frac{1}{2}F(\mathbf{D})f_1\cdot f_1\\
&= -\frac{1}{2}F(\mathbf{D})[\int_{\Gamma_{(m-1)}}\exp\mathbf{\Phi}\cdot\mathbf{x}d\tau_{m-1}]\cdot[\int_{\Gamma_{(m-1)}}\exp\mathbf{\Phi}\cdot\mathbf{x}d\tau_{m-1}]\\
&= -\frac{1}{2}[\int_{\Gamma^2_{(m-1)}}F(\mathbf{D})[\exp(\mathbf{\Phi}_1\cdot\mathbf{x})]\cdot[\exp(\mathbf{\Phi}_2\cdot\mathbf{x})]d\tau^2_{m-1}\\
&= -\frac{1}{2}\int_{\Gamma^2_{(m-1)}}F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]d\tau^2_{m-1}\quad \textit{by}\ (2.8)
\end{aligned}
$$

where $\mathbf{\Phi}_i$ $(i=1,2)$ means a vector function

$$(\phi_1(p_i,q_i,\ldots z_i),\phi_2(p_i,q_i,\ldots z_i),\ldots,\phi_m(p_i,q_i,\ldots z_i))\ \ \text{for i=1,2}$$

A simple example to clarify the above equation is that

$$[\int_\Gamma f(z)dz]\times[\int_\Gamma f(z)dz]=\int_{\Gamma^2}f(z_1)f(z_2)dz_1dz_2$$

Making the substitution $f_2=c\int_{\Gamma^2_{(m-1)}}\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]d\tau^2_{m-1}$ into the above equation to meet the identity corresponding to ε^2 , we have

$$
\begin{aligned}
&c\int_{\Gamma^2_{(m-1)}}F(\mathbf{D})\{\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]\}\cdot 1d\tau^2_{m-1}\\
&= c\int_{\Gamma^2_{(m-1)}}F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\{\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]\}d\tau^2_{m-1}\quad \textit{by}\ (2.9)\\
&= -\frac{1}{2}\int_{\Gamma^2_{(m-1)}}F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]d\tau^2_{m-1}
\end{aligned}
$$

Deriving c by equating the above equation leads to

$$f_2=\int_{\Gamma^2_{(m-1)}}-\frac{1}{2}\frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)}\exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2)\cdot\mathbf{x}]d\tau^2_{m-1}$$

For convenience , we define $I_3\equiv 2F(\mathbf{D})f_3\cdot 1$, as in (3.8) , we have

ε^3

$$I_3 = -2F(\mathbf{D})f_1 \cdot f_2$$
$$= -2F(\mathbf{D})[\int_{\Gamma_{(m-1)}} \exp(\mathbf{\Phi_1} \cdot \mathbf{x}) d\tau_{m-1}]$$
$$\cdot [\int_{\Gamma^2_{(m-1)}} -\frac{1}{2}\frac{F(\mathbf{\Phi_2}-\mathbf{\Phi_3})}{F(\mathbf{\Phi_2}+\mathbf{\Phi_3})} \exp[(\mathbf{\Phi_2}+\mathbf{\Phi_3})\cdot \mathbf{x}] d\tau^2_{m-1}]$$
$$= \int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)} F(\mathbf{D}) \exp(\mathbf{\Phi}_1 \cdot \mathbf{x}) \cdot \exp[(\mathbf{\Phi}_2+\mathbf{\Phi}_3)\cdot \mathbf{x}] d\tau^3_{m-1}$$
$$= \int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)} F(\mathbf{\Phi}_1-\mathbf{\Phi}_2-\mathbf{\Phi}_3) \exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\cdot \mathbf{x}] d\tau^3_{m-1}$$

Since $\mathbf{\Phi}_i$, $i = 1, 2, 3$ are dummy variables we can write I_3 in such a form

$$I_3 = \frac{1}{3}\int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)} F(\mathbf{\Phi}_3-\mathbf{\Phi}_1-\mathbf{\Phi}_2) \exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\cdot \mathbf{x}] d\tau^3_{m-1}+$$
$$\frac{1}{3}\int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)} F(\mathbf{\Phi}_2-\mathbf{\Phi}_1-\mathbf{\Phi}_3) \exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\cdot \mathbf{x}] d\tau^3_{m-1}+$$
$$\frac{1}{3}\int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)} F(\mathbf{\Phi}_1-\mathbf{\Phi}_2-\mathbf{\Phi}_3) \exp[(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\cdot \mathbf{x}] d\tau^3_{m-1}$$

Using Hirota condition (3.3) we have(see Appendix A.1)

$$I_3 = -\frac{1}{3}\int_{\Gamma^3_{(m-1)}} F(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)}$$
$$\times \exp[\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3] d\tau^3_{m-1}$$

To satisfy the above equation f_3 can be assigned

$$f_3 = -\frac{1}{3!}\int_{\Gamma^3_{(m-1)}} \frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)} \exp[\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3] d\tau^3_{m-1}$$

The mathematical form of f_2 , f_3 leads to a conjecture

$$f_n = \frac{1}{n!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j-\mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j+\mathbf{\Phi}_k)}] \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} \quad n \ge 2$$

To justify this conjecture such f_n must be compatible in the original perturbational series , i.e. ,

$$F(\mathbf{D})(\sum_{l=0}^{n} f_l \cdot f_{n-l}) = 0 \tag{3.15}$$

Is (3.15) an identity? let us calculate the left hand side of (3.15). Firstly we define $F(\mathbf{D})(\sum_{l=0}^{n} f_l \cdot f_{n-l}) \equiv I_n$ Then

$$
\begin{aligned}
I_n &= F(\mathbf{D})[1 \cdot f_n + f_1 \cdot f_{n-1} + \sum_{l=2}^{n-2} f_l \cdot f_{n-l} + f_{n-1} \cdot f_1 + f_n \cdot 1] \\
&= \frac{1}{n!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(\mathbf{D}) 1 \cdot \exp[\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}] d\tau^n_{m-1} + \\
&\frac{1}{(n-1)!}\int_{\Gamma^n_{(m-1)}} \prod_{2\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(\mathbf{D}) \exp(\mathbf{\Phi}_1 \cdot \mathbf{x}) \cdot \exp(\sum_{i=2}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\sum_{l=2}^{n-2} \frac{1}{l!(n-l)!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le l} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] \prod_{l+1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] \times \\
&\qquad F(\mathbf{D}) \exp(\sum_{i=1}^{l} \mathbf{\Phi}_i \cdot \mathbf{x}) \cdot \exp(\sum_{i=l+1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\frac{1}{(n-1)!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n-1} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi_j} + \mathbf{\Phi_k})}] F(\mathbf{D}) \exp(\sum_{i=1}^{n-1} \mathbf{\Phi_i} \cdot \mathbf{x}) \cdot \exp(\mathbf{\Phi_n} \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\frac{1}{n!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\mathbf{\Phi_j} - \mathbf{\Phi_k})}{F(\mathbf{\Phi_j} + \mathbf{\Phi_k})}] F(\mathbf{D}) \exp[\sum_{i=1}^{n} \mathbf{\Phi_i} \cdot \mathbf{x}] \cdot 1 d\tau^n_{m-1}
\end{aligned}
$$

Using (2.8) and we obtain

$$
\begin{aligned}
I_n &= \frac{1}{n!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(-\sum_{i=1}^{n} \mathbf{\Phi}_i) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\frac{1}{(n-1)!}\int_{\Gamma^n_{(m-1)}} \prod_{2\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(\mathbf{\Phi}_1 - \sum_{i=2}^{n} \mathbf{\Phi}_i) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\sum_{l=2}^{n-2} \frac{1}{l!(n-l)!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le l} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] \prod_{l+1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] \times \\
&F(\sum_{i=1}^{l} \mathbf{\Phi}_i - \sum_{i=l+1}^{n} \mathbf{\Phi}_i) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} \quad + \\
&\frac{1}{(n-1)!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n-1} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(\sum_{i=1}^{n-1} \mathbf{\Phi}_i - \mathbf{\Phi}_n) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} + \\
&\frac{1}{n!}\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] F(\sum_{i=1}^{n} \mathbf{\Phi}_i) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{(m-1)} \\
&\equiv I_{n(1)} + I_{n(2)} + I_{n(3)} + I_{n(4)} + I_{n(5)}
\end{aligned}
$$

$I_{n(j)}$ $(j = 1, 2, \ldots 5)$ denotes each integral term respectively

$I_{n(1)}$ can be written as

$$I_{n(1)} = \frac{1}{n!}C_0^n \sum_{\sigma=1,-1} \frac{(1-\sigma_1)(1-\sigma_2)\cdots(1-\sigma_n)}{2^n} \int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] \times$$
$$F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} \quad , \ C_l^n = \frac{n!}{l!(n-l)!} \tag{3.16}$$

$\sum_{\sigma=1,-1}$ means the summation over all possible combinations of $\sigma_1 = 1,-1$ $\sigma_2 = 1,-1$ $\ldots \sigma_n = 1,-1$ Since $\sigma_i = -1$ $i = 1,2,\ldots n$ is the only combination which contributes , (3.16) is valid
Similarly , we have

$$I_{n(2)} = \frac{1}{n!}C_1^n \sum_{\sigma=1,-1} \frac{(1+\sigma_1)(1-\sigma_2)\cdots(1-\sigma_n)}{2^n} \int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] \times$$
$$F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

$$I_{n(3)} = \frac{1}{n!} \sum_{l=2}^{n-2} C_l^n \sum_{\sigma=1,-1} \frac{(1+\sigma_1)\cdots(1+\sigma_l)(1-\sigma_{l+1})\cdots(1-\sigma_n)}{2^n} \times$$
$$\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

$$I_{n(4)} = \frac{1}{n!}C_{n-1}^n \sum_{\sigma=1,-1} \frac{(1+\sigma_1)\cdots(1+\sigma_{n-1})(1-\sigma_n)}{2^n} \int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] \times$$
$$F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

$$I_{n(5)} = \frac{1}{n!}C_n^n \sum_{\sigma=1,-1} \frac{(1+\sigma_1)\cdots(1+\sigma_n)}{2^n} \int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] \times$$
$$F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

$\Longrightarrow$

$$I_n = I_{n(1)} + I_{n(2)} + I_{n(3)} + I_{n(4)} + I_{n(5)}$$
$$= \frac{1}{n!} \sum_{\sigma=1,-1} \sum_{l=0}^{n} C_l^n \frac{(1+\sigma_1)\cdots(1+\sigma_l)(1-\sigma_{l+1})\cdots(1-\sigma_n)}{2^n} \times$$
$$\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j \mathbf{\Phi}_j - \sigma_k \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j \sigma_k] F(\sum_{i=1}^n \sigma_i \mathbf{\Phi}_i) \exp(\sum_{i=1}^n \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

Due to the dummy variables $\mathbf{\Phi}_i$, we can choose $(r_1, r_2, \ldots, r_l)$ from the set $\{1, \ldots, n\}$ with the relation $(r_1 < r_2 < \ldots < r_l)$ and the left was denoted by $(r_{l+1}, r_{l+2}, \ldots, r_n)$ with the relation $(r_{l+1} < r_{l+2} < \ldots < r_n)$ Then

$$I_n = \frac{1}{n!} \sum_{\sigma=1,-1} \sum_{l=0}^{n} C_l^n \frac{(1+\sigma_{r_1})\cdots(1+\sigma_{r_l})(1-\sigma_{r_{l+1}})\cdots(1-\sigma_{r_n})}{2^n} \times$$

$$\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_{r_j}\mathbf{\Phi}_{r_j} - \sigma_{r_k}\mathbf{\Phi}_{r_k})}{F(\mathbf{\Phi}_{r_j} + \mathbf{\Phi}_{r_k})} \sigma_{r_j}\sigma_{r_k}] F(\sum_{i=1}^{n} \sigma_{r_i}\mathbf{\Phi}_{r_i}) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

We denote

$$\int_{\Gamma^n_{(m-1)}} \prod_{1\le j<k\le n} [-\frac{F(\sigma_{r_j}\mathbf{\Phi}_{r_j} - \sigma_{r_k}\mathbf{\Phi}_{r_k})}{F(\mathbf{\Phi}_{r_j} + \mathbf{\Phi}_{r_k})} \sigma_{r_j}\sigma_{r_k}] F(\sum_{i=1}^{n} \sigma_{r_i}\mathbf{\Phi}_{ri}) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1}$$

by C Because C is independent of the selection of $(r_1, \ldots, r_l)$ from the set $\{1, \ldots, N\}$ we have

$$\begin{aligned} I_n &= \frac{1}{n!} \sum_{\sigma=1,-1} \sum_{l=0}^{n} C_l^n \frac{(1+\sigma_{r_1})\cdots(1+\sigma_{r_l})(1-\sigma_{r_{l+1}})\cdots(1-\sigma_{r_n})}{2^n} \times C \\ &= \frac{1}{n!} \sum_{\sigma=1,-1} \sum_{l=0}^{n} \sum_{C_l^n} \frac{(1+\sigma_{r_1})\cdots(1+\sigma_{r_l})(1-\sigma_{r_{l+1}})\cdots(1-\sigma_{r_n})}{2^n} \times C \end{aligned}$$

$where \sum_{C_l^n}$ means the summation over all the selection possibilities

From observation we have

$$\begin{aligned} I_n &= \frac{1}{n!} \sum_{\sigma=1,-1} 2^{-n} \sum_{\epsilon_1\ldots\epsilon_n=1,-1} (1+\epsilon_1\sigma_1)(1+\epsilon_2\sigma_2)\cdots(1+\epsilon_n\sigma_n) \times C \\ &= \frac{1}{n!} \sum_{\sigma=1,-1} 2^{-n} \sum_{\epsilon_2\ldots\epsilon_n=1,-1} [(1+\sigma_1)+(1-\sigma_1)](1+\epsilon_2\sigma_2)\cdots(1+\epsilon_n\sigma_n) \times C \\ &= \frac{1}{n!} \sum_{\sigma=1,-1} 2^{-n+1} \sum_{\epsilon_2\ldots\epsilon_n=1,-1} (1+\epsilon_2\sigma_2)\cdots(1+\epsilon_n\sigma_n) \times C = \cdots = \frac{1}{n!} \sum_{\sigma=1,-1} \times C \\ &= \frac{1}{n!} \int_{\Gamma^n_{(m-1)}} \sum_{\sigma=1,-1} \prod_{1\le j<k\le n} [-\frac{F(\sigma_j\mathbf{\Phi}_j - \sigma_k\mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)} \sigma_j\sigma_k] F(\sum_{i=1}^{n} \sigma_i\mathbf{\Phi}_i) \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} \\ &= 0 \quad \text{by Hirota condition (3.3)} \end{aligned}$$

We summarize the discussion of this section into the following theorem

Theorem 3.1 If $F(\mathbf{D})$, a bilinear operator in a multipolynomial form of $D_{x_1}, \ldots D_{x_m}$, satisfies (3.1)(3.2) and (3.3) , equation $F(\mathbf{D}) f \cdot f = 0$ has a solution

$$f = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n \tag{3.17}$$

where

$$f_1 = \int_{\Gamma_{(m-1)}} \exp(\mathbf{\Phi} \cdot \mathbf{x}) d\tau_{m-1}$$

and

$$f_n = \frac{1}{n!} \int_{\Gamma^n_{(m-1)}} \prod_{1 \le j < k \le n} [-\frac{F(\mathbf{\Phi}_j - \mathbf{\Phi}_k)}{F(\mathbf{\Phi}_j + \mathbf{\Phi}_k)}] \exp(\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{x}) d\tau^n_{m-1} \quad n \ge 2$$

3.4 Two Examples

(i)

KdV equation $u_t + 6uu_x + u_{xxx} = 0$ can be transformed into

$F(D_{x_1}, D_{x_2}) f \cdot f = D_{x_1}(D_{x_2} + D^3_{x_1}) f \cdot f = 0$ with $x_1 = x, x_2 = t, u = 2(\log f)_{xx}$

From (3.13) and (3.14) , in this case $m = 2$, we have

$$f_1 = \int_{\Gamma} \exp[\phi_1(p_1)x_1 + \phi_2(p_1)x_2] d\tau(p_1) \tag{3.18}$$

and

$$F(\phi_1(p_1), \phi_2(p_1)) = \phi_1(\phi_2 + \phi_1^3) = 0 \tag{3.19}$$

From (3.19) we can assign

$$\phi_1(p_1) = p_1 \ , \ \phi_2(p_1) = -p_1^3$$

Using theorem3.1 we have

$$f_1 = \int_{\Gamma} \exp(p_1 x_1 - p_1^3 x_2) d\tau(p_1)$$

$$\begin{aligned} f_n = & \frac{1}{n!} \int_{\Gamma^n} \prod_{1 \le j < k \le n} [-\frac{F(\phi_1(p_j) - \phi_1(p_k), \phi_2(p_j) - \phi_2(p_k))}{F(\phi_1(p_j) + \phi_1(p_k), \phi_2(p_j) + \phi_2(p_k))}] \times \\ & \exp[\sum_{i=1}^{n} \phi_1(p_i)x_1 + \phi_2(p_i)x_2] d\tau^n \\ = & \frac{1}{n!} \int_{\Gamma^n} \prod_{1 \le j < k \le n} \{-\frac{(p_j - p_k)[-p_j^3 + p_k^3 + (p_j - p_k)^3]}{(p_j + p_k)[-p_j^3 - p_k^3 + (p_j + p_k)^3]}\} \times \\ & \exp(\sum_{i=1}^{n} p_i x_1 - \sum_{i=1}^{n} p_i^3 x_2) d\tau^n \\ = & \frac{1}{n!} \int_{\Gamma^n} \prod_{1 \le j < k \le n} (\frac{p_j - p_k}{p_j + p_k})^2 \exp(\sum_{i=1}^{n} p_i x - p_i^3 t) \prod_{i=1}^{n} d\tau(p_i) \end{aligned}$$

for $n \geq 2$, where x_1 and x_2 have been replaced back by x and t respectively. Then

$$f = 1 + \varepsilon \int_{\Gamma} \exp(p_1 x - p_1^3 t) d\tau(p_1) + \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} \int_{\Gamma^n} \prod_{1 \leq j < k \leq n} (\frac{p_j - p_k}{p_j + p_k})^2$$

$$\times \exp(\sum_{i=1}^{n} p_i x - p_i^3 t) \prod_{i=1}^{n} d\tau(p_i) \tag{3.20}$$

Now we set

$$\int_{\Gamma} d\tau(p_i) = \int_{-\infty}^{\infty} \sum_{l=1}^{N} \alpha_l \delta(p_i - p_{0l}) dp_i$$

where α_l and p_{0l} $(l = 1, 2, \ldots, N)$ are real constants with $p_{0j} \neq p_{0k}$ for $j \neq k$ $(1 \leq j, k \leq N)$ and δ is Dirac's delta function. Then from (3.20) f becomes

$$f = 1 + \sum_{l=1}^{N} \varepsilon \int_{-\infty}^{\infty} \exp(p_1 x - p_1^3 t) \alpha_l \delta(p_1 - p_{0l}) dp_1$$

$$+ \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq n} (\frac{p_j - p_k}{p_j + p_k})^2 \exp[\sum_{i=1}^{n} (p_i x - p_i^3 t)] \prod_{i=1}^{n} \sum_{l_i=1}^{N} \alpha_{l_i} \delta(p_i - p_{0l_i}) dp_i$$

$$= 1 + \sum_{l=1}^{N} \varepsilon \alpha_l \exp(p_{0l} x - p_{0l}^3 t) + \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} \sum_{l_1 \cdots l_n = 1}^{N} \alpha_{l_1} \cdots \alpha_{l_n} \prod_{1 \leq j < k \leq n} (\frac{p_{0l_j} - p_{0l_k}}{p_{0l_j} + p_{0l_k}})^2 \times$$

$$\exp[\sum_{i=1}^{n} (p_{0l_i} x - p_{0l_i}^3 t)]$$

We observe that $\prod_{1 \leq j < k \leq n} (\frac{p_{0l_j} - p_{0l_k}}{p_{0l_j} + p_{0l_k}})^2$ makes $l_1 \cdots l_n$ having different values respectively and if $n > N$ it is impossible to make $l_1 \cdots l_n$ having different values respectively thus the terms for $n > N$ contribute nothing. Therefore f can be written as

$$f = 1 + \sum_{l=1}^{N} \varepsilon \alpha_l \exp(p_{0l} x - p_{0l}^3 t) + \sum_{n=2}^{N} \varepsilon^n \sum_{(l_1 \ldots l_n) \in C_n^N} \alpha_{l_1} \cdots \alpha_{l_n} \times$$

$$\prod_{1 \leq j < k \leq n} (\frac{p_{0l_j} - p_{0l_k}}{p_{0l_j} + p_{0l_k}})^2 \exp[\sum_{i=1}^{n} (p_{0l_i} x - p_{0l_i}^3 t)]$$

$$= 1 + \sum_{l=1}^{N} \exp(p_{0l} x - p_{0l}^3 t + c_{0l}) + \sum_{n=2}^{N} \sum_{(l_1 \ldots l_n) \in C_n^N} \prod_{1 \leq j < k \leq ?n} (\frac{p_{0l_j} - p_{0l_k}}{p_{0l_j} + p_{0l_k}})^2$$

$$\exp[\sum_{i=1}^{n} (p_{0l_i} x - p_{0l_i}^3 t + c_{0l_i})]$$

where $c_{0l_i} \equiv \log(\varepsilon\alpha_{l_i})$ and $\sum_{(l_1 \dots l_n)\in C_n^N}$ denotes a summation over $l_1, \dots, l_n$ which are chosen from a set $\{1, \dots, N\}$ as well as $l_1 < l_2 < \cdots < l_n$
Now with

$$p_{0i}x - p_{0i}^3 t + c_{0i} \equiv \eta_i$$

$$(\frac{p_{0i} - p_{0j}}{p_{0i} + p_{0j}})^2 \equiv \exp A_{ij}$$

we have

$$f = 1 + \sum_{i=1}^{N} \exp \eta_i + \sum_{n=2}^{N} \sum_{(l_1,\dots,l_n)\in C_n^N} \exp[\sum_{j=1}^{n} \eta_{l_j} + \sum_{1\le j<k\le n} A_{l_j l_k}] \tag{3.21}$$

Then f expressed in (3.21) equals to f in (3.12) , in other words , we can derive pure N-soliton solution from generalized soliton solution.

(ii)
K-P equation $(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$ can be transformed into

$$F(D_{x_1}, D_{x_2}, D_{x_3}) = (D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2) f \cdot f = 0$$

where $x_1 = x$,$x_2 = y$, $x_3 = t$ and $u = 2(\log f)_{xx}$. From (3.13) and (3.14) , in this case $m = 3$, we have

$$f_1 = \int_{\Gamma}\int_{\Gamma_2} \exp[\phi_1(p,q)x_1 + \phi_2(p,q)x_2 + \phi_3(p,q)x_3] d\tau(p,q) \tag{3.22}$$

and

$$F(\phi_1(p,q), \phi_2(p,q), \phi_3(p,q)) = \phi_1^4 - 4\phi_1\phi_3 + 3\phi_2^2 = 0 \tag{3.23}$$

From (3.23) we can assign

$$\phi_1(p,q) = p - q \ , \ \ \phi_2(p,q) = p^2 - q^2 \text{ and } \phi_3(p,q) = p^3 - q^3 \tag{3.24}$$

Hence from theorem 3.1 and Appendix A.2 we obtain

$$f_n = \frac{1}{n!}\int_{\Gamma_{(2)}^n} \prod_{1\le j<k\le n} [-\frac{F(\phi_1(p_j,q_j) - \phi_1(p_k,q_k), \phi_2(p_j,q_j) - \phi_2(p_k,q_k), \phi_3(p_j,q_j) - \phi_3(p_k,q_k))}{F(\phi_1(p_j,q_j) + \phi_1(p_k,q_k), \phi_2(p_j,q_j) + \phi_2(p_k,q_k), \phi_3(p_j,q_j) + \phi_3(p_k,q_k))}]$$

$$\times \exp[\sum_{i=1}^{n} \phi_1(p_i,q_i)x_1 + \phi_2(p_i,q_i)x_2 + \phi_3(p_i,q_i)x_3] \prod_{i=1}^{n} d\tau(p_i,q_i)$$

$$= \frac{1}{n!}\int_{\Gamma_{(2)}^n} \prod_{1\le j<k\le n} [\frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)}] \exp[\sum_{i=1}^{n} (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 - q_i^3)x_3] \prod_{i=1}^{n} d\tau(p_i,q_i)$$

Now we set

$$\int_{\Gamma_{(2)}} d\tau(p,q) = \int_{\Gamma_1}\int_{\Gamma_2} d\tau(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{l=1}^{N} \alpha_l \delta(p - p_{0l})\delta(q - q_{0l}) dpdq$$

Then

$$f = 1 + \varepsilon \sum_{l=1}^{N} \alpha_l \exp[(p_{0l} - q_{0l})x_1 + (p_{0l}^2 - q_{0l}^2)x_2 + (p_{0l}^3 - q_{0l}^3)x_3]$$
$$+ \frac{\varepsilon^n}{n!} \sum_{l_1,\ldots,l_n=1}^{N} \alpha_{l_1} \ldots \alpha_{l_n} \prod_{1\le j<k\le n} \frac{(p_{0l_j} - p_{0l_k})(q_{0l_j} - q_{0l_k})}{(p_{0l_j} - q_{0l_k})(q_{0l_j} - p_{0l_k})} \times$$
$$\exp[\sum_{i=1}^{n} (p_{0l_i} - q_{0l_i})x_1 + (p_{0l_i}^2 - q_{0l_i}^2)x_2 + (p_{0l_i}^3 - q_{0l_i}^3)x_3]$$

Now with

$$\varepsilon \alpha_i \exp[(p_{0l_i} - q_{0l_i})x_1 + (p_{0l_i}^2 - q_{0l_i}^2)x_2 + (p_{0l_i}^3 - q_{0l_i}^3)x_3] \equiv \exp \eta_{l_i}$$

$$\frac{(p_{0j} - p_{0k})(q_{0j} - q_{0k})}{(p_{0j} - q_{0k})(q_{0j} - p_{0k})} \equiv \exp A_{jk}$$

The same way as in case **(i)** such f is equivalent to the one in (3.12)

Chapter 4

Relation with GLM equation

In this chapter , we relate the generalized soliton solution for KdV equation to the GLM integral equation. To begin , we need the following lemma

Lemma 4.1 $\prod_{1\leq i<j\leq n}(\frac{p_i-p_j}{p_i+p_j})^2 = det\triangle_n$, where $\triangle_n$ is an n×n matrix with i-j element $\frac{2p_i}{p_i+p_j}$
pf see Appendix A.3

Using Lemma4.1 and letting $\varepsilon = 1$, (3.20) becomes

$$f = 1+\int_\Gamma \exp(px-p^3t)d\tau(p)+\sum_{n=2}^{\infty}\frac{1}{n!}\int_{\Gamma^n} det(\triangle_n)\exp[\sum_{i=1}^{n}(p_ix-p_i^3t)]\prod_{i=1}^{n}d\tau(p_i)$$

Using the definition of determinant

$$A = (a_{ij})\ ,\ \det A = \sum_\sigma sgn(\sigma)a_{1\sigma_{(1)}}a_{2\sigma_{(2)}}\cdots a_{n\sigma(n)}$$

, a summation over all permutations , f becomes

$$f = 1+\int_\Gamma \exp(px-p^3t)d\tau(p)+\sum_{n=2}^{\infty}\frac{1}{n!}\sum_\sigma sgn(\sigma)\int_{\Gamma^n}\frac{2p_1}{p_1+p_{\sigma(1)}}\cdots\frac{2p_n}{p_n+p_{\sigma(n)}}\times$$
$$exp[\sum_{i=1}^{n}(p_ix-p_i^3t)]\prod_{i=1}^{n}d\tau(p_i)$$

Meanwhile

$$\sum_\sigma sgn(\sigma)\int_{\Gamma^n}\frac{2p_1}{p_1+p_{\sigma(1)}}\cdots\frac{2p_n}{p_n+p_{\sigma(n)}}exp[\sum_{i=1}^n(p_ix-p_i^3t)]\prod_{i=1}^n d\tau(p_i)$$

$$=\sum_\sigma sgn(\sigma)\int_{\Gamma^n}p_1\cdots p_n\exp[\sum_{i=1}^n(-p_i^3t)]\frac{2}{p_1+p_{\sigma(1)}}\cdots\frac{2}{p_n+p_{\sigma(n)}}\exp[\sum_{i=1}^n\frac{p_i+p_{\sigma(i)}}{2}x]\prod_{i=1}^n d\tau(p_i)$$

$$=\sum_\sigma sgn(\sigma)(-1)^n\int_{\Gamma^n}p_1\cdots p_n\exp[\sum_{i=1}^n(-p_i^3t)]\int_x^\infty\cdots\int_x^\infty\exp[\sum_{i=1}^n\frac{p_i+p_{\sigma(i)}}{2}s_i]\prod_{i=1}^n ds_i\prod_{i=1}^n d\tau(p_i)$$

$$=\sum_\sigma sgn(\sigma)(-1)^n\int_x^\infty\cdots\int_x^\infty\int_{\Gamma^n}p_1\cdots p_n\exp\sum_{i=1}^n[\frac{(p_i+p_{\sigma(i)})}{2}s_i-p_i^3t]\prod_{i=1}^n d\tau(p_i)\prod_{i=1}^n ds_i$$

$$=\sum_\sigma sgn(\sigma)(-1)^n\int_x^\infty\cdots\int_x^\infty\int_{\Gamma^n}p_1\cdots p_n\exp[\sum_{i=1}^n(\frac{p_is_i}{2}+\frac{p_is_{\sigma^{-1}(i)}}{2}-p_i^3t)]\prod_{i=1}^n d\tau(p_i)\prod_{i=1}^n ds_i$$

rename $\sigma^{-1}=\rho$. The above equation becomes

$$=\int_x^\infty\cdots\int_x^\infty\sum_{\rho^{-1}}sgn(\rho^{-1})(-1)^n\int_{\Gamma^n}p_1\cdots p_n\exp\sum_{i=1}^n[\frac{(s_i+s_{\rho(i)})}{2}p_i-p_i^3t]\prod_{i=1}^n d\tau(p_i)\prod_{i=1}^n ds_i$$

It is noteworthy that we can restrict the path Γ on the left complex plane to avoid the explosion of improper integral at infinity. Now let $F(t,s)=-\int_\Gamma p\exp(\frac{sp}{2}-p^3t)d\tau(p)$ and we know that $\sum_\sigma=\sum_{\sigma^{-1}}=\sum_\rho$ as well as $sgn(\rho)=sgn(\rho^{-1})$

Then the above equation becomes

$$=\int_x^\infty\cdots\int_x^\infty\sum_\rho sgn(\rho)F(t,s_1+s_{\rho(1)})F(t,s_2+s_{\rho(2)})\cdots F(t,s_n+s_{\rho(n)})\prod_{i=1}^n ds_i$$

$$=\int_x^\infty\cdots\int_x^\infty det\Psi_n\prod_{i=1}^n ds_i$$

where Ψ_n is a matrix with $i-j$ element $F(t,s_i+s_j)$
And

$$\int_\Gamma\exp(px-p^3t)d\tau(p)=-\int_\Gamma\int_x^\infty p\exp(ps-p^3t)dsd\tau(p)=-\int_x^\infty\int_\Gamma p\exp(p\frac{s_1+s_1}{2}-p^3t)d\tau(p)ds_1$$

$$=\int_x^\infty F(t,s_1+s_1)ds_1$$

which can be written as $\int_x^\infty det\Psi_1 ds_1$ with $F(t,s_1+s_1)\equiv det\Psi_1$

Therefore f becomes

$$f=1+\sum_{n=1}^\infty\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty det\Psi_n(t;s_1\dots s_n)\prod_{i=1} ds_i \tag{4.1}$$

(4.1) has a form of Fredholm's determinant of a certain integral equation. Inspired by Fredholm's trick[13] we introduce Fredholm's first minor

$$det\Omega_n(t,x,z;s_1\dots s_n) \equiv \begin{vmatrix} F(t,x+z) & F(t,x+s_1) & F(t,x+s_2) & \cdots & F(t,x+s_n) \\ F(t,s_1+z) & F(t,s_1+s_1) & F(t,s_1+s_2) & \cdots & F(t,s_1+s_n) \\ F(t,s_2+z) & F(t,s_2+s_1) & F(t,s_2+s_2) & \cdots & F(t,s_2+s_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F(t,s_n+z) & F(t,s_n+s_1) & F(t,s_n+s_2) & \cdots & F(t,s_n+s_n) \end{vmatrix} \tag{4.2}$$

And particularly

$$det\Omega_0(t,x,z) \equiv F(t,x+z) \tag{4.3}$$

Take cofactor expansion along the first row of (4.2)

$$det\Omega_n = F(t,x+z)det\Psi_n + \sum_{j=1}^{n}(-1)^j F(t,s_j+z)\times$$

$$\begin{vmatrix} F(t,x+s_1) & F(t,x+s_2) & \cdots & F(t,x+s_n) \\ F(t,s_1+s_1) & F(t,s_1+s_2) & \cdots & F(t,s_1+s_n) \\ \vdots & \vdots & \ddots & \vdots \\ F(t,s_{j-1}+s_1) & F(t,s_{j-1}+s_2) & \cdots & F(t,s_{j-1}+s_n) \\ F(t,s_{j+1}+s_1) & F(t,s_{j+1}+s_2) & \cdots & F(t,s_{j+1}+s_n) \\ \vdots & \vdots & \ddots & \vdots \\ F(t,s_n+s_1) & F(t,s_n+s_2) & \cdots & F(t,s_n+s_n) \end{vmatrix}$$

Rearrange the j-th column to the first one

$det\Omega_n =$

$$F(t,x+z)det\Psi_n + \sum_{j=1}^{n}(-1)^j F(t,s_j+z)(-1)^{j-1}\times$$

$$\begin{vmatrix} F(t,x+s_j) & F(t,x+s_1) & \cdots & F(t,x+s_{j-1}) & F(t,x+s_{j+1}) & \cdots & F(t,x+s_n) \\ F(t,s_1+s_j) & F(t,s_1+s_1) & \cdots & F(t,s_1+s_{j-1}) & F(t,s_1+s_{j+1}) & \cdots & F(t,s_1+s_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F(t,s_{j-1}+s_j) & F(t,s_{j-1}+s_1) & \cdots & F(t,s_{j-1}+s_{j-1}) & F(t,s_{j-1}+s_{j+1}) & \cdots & F(t,s_{j-1}+s_n) \\ F(t,s_{j+1}+s_j) & F(t,s_{j+1}+s_1) & \cdots & F(t,s_{j+1}+s_{j-1}) & F(t,s_{j+1}+s_{j+1}) & \cdots & F(t,s_{j+1}+s_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F(t,s_n+s_j) & F(t,s_n+s_1) & \cdots & F(t,s_n+s_{j-1}) & F(t,s_n+s_{j+1}) & \cdots & F(t,s_n+s_n) \end{vmatrix}$$

$$= F(t,x+z)det\Psi_n(t;s_1\dots s_n) - \sum_{j=1}^{n} F(t,s_j+z)det\Omega_{n-1}(t,x,s_j;s_1,\dots,s_{j-1}s_{j+1},\dots s_n)$$

Imposing $\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty\prod_{i=1}^n ds_i$ on both sides of the above equation

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty det\Omega_n(t,x,z;s_1\ldots s_n)\prod_{i=1}^n ds_i$$
$$=F(t,x+z)\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty det\Psi_n(t;s_1,\ldots,s_n)\prod_{i=1}^n ds_i$$
$$-\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty\sum_{j=1}^{n}F(t,s_j+z)det\Omega_{n-1}(t,x,s_j;s_1,\ldots,s_{j-1}s_{j+1},\ldots,s_n)\prod_{i=1}^n ds_i$$

Renaming $s_j=s, s_{j+1}=s_j, s_{j+2}=s_{j+1},\ldots,s_n=s_{n-1}$ for the second summation integral and using (4.1) , the above equation becomes

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty det\Omega_n(t,x,z;s_1\ldots s_n)\prod_{i=1}^n ds_i$$
$$=F(t,x+z)(f-1)-\int_x^\infty F(t,s+z)[\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int_x^\infty\cdots\int_x^\infty \det\Omega_{n-1}(t,x,s;s_1\ldots s_{n-1})\prod_{i=1}^{n-1}ds_i]ds$$

Now define $E(t,x,z)\equiv\sum_{n=1}^{\infty}\frac{1}{n!}\int_x^\infty\cdots\int_x^\infty det\Omega_n(t,x,z;s_1\ldots s_n)\prod_{i=1}^n ds_i$ and recall (4.3). Then from the above we obtain

$$E(t,x,z)=F(t,x+z)(f-1)-\int_x^\infty F(t,s+z)[\Omega_0(t,x,s)+E(t,x,s)]ds$$
$$=F(t,x+z)(f-1)-\int_x^\infty F(t,s+z)[F(t,x+s)+E(t,x,s)]ds$$
$$\Rightarrow$$
$$E(t,x,z)+F(t,x+z)=F(t,x+z)f(t,x)-\int_x^\infty F(t,s+z)[F(t,x+s)+E(t,x,s)]$$

It is natural to define

$$D(t,x,z)=-E(t,x,z)-F(t,x+z)\qquad(4.4)$$

Then the above equation becomes

$$D(t,x,z)+F(t,x+z)f(t,x)+\int_x^\infty F(t,s+z)D(t,x,s)ds$$

Divide f on both side(of course we assume f is not identical to zero) and put $K(t,x,z)=\frac{D(t,x,z)}{f(t,x)}$ Then we obtain the GLM equation

$$K(t,x,z)+F(t,x+z)+\int_x^\infty K(t,x,s)F(t,s+z)ds\qquad(4.5)$$

Recall $F(t,s) = -\int_\Gamma p\exp(\frac{sp}{2} - p^3t)d\tau(p)$ and $\int_\Gamma d\tau(p) \equiv \int_\Gamma c(p)dp$ with an properly selected $c(p)$ as well as suitable path Γ. Now we set Γ lie in the real-axis and imaginary-axis such that

$$F(t,s) = -\int_{-\infty}^{\infty} p[\exp(\frac{sp}{2} - p^3t)][-\frac{1}{p}\sum_{n=1}^{N} c_n^2(0)\delta(p+2p_n)]dp$$
$$-\int_{-\infty}^{\infty} 2ip[\exp(isp + 8ip^3t)][\frac{ir(p,0)}{4\pi p}]dp$$

where i is imaginary number p_n are different real values and the first integral of right hand side involves the path on real axis while the second integral on the imaginary. Then

$$F(t,s) = \sum_{n=1}^{N} c_n^2(0)\exp(8p_n^3t)exp(-p_ns) + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(p,0)\exp(8ip^3t)exp(ips)dp$$

which can be abbreviated as

$$F(t,s) = \sum_{n=1}^{N} c_n^2(t)\exp(-p_ns) + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(p,t)\exp(ips)dp$$

with $c_n(t) = c_n(0)\exp(4p_n^3t)$ and $r(p,t) = r(p,0)\exp(8ip^3t)$

GLM equation (4.5) with F defined as above entirely coincides with GLM equation described in [14]. Those familiar with[5] would aware that $c_n(t)$ and $r(p,t)$ are nothing but scattering data. [15] also has a result that

$$u = 2\partial_x[K(t,x,x)]$$

Now we prove the same result in our approach. Firstly , we need the following lemma

Lemma 4.2

$$f_x(t,x) = D(t,x,x)$$

pf

$$\partial_x \int_x^\infty \cdots \int_x^\infty det\Psi_n(t;s_1,\ldots,s_n)\prod_{i=1}^{n} ds_i$$
$$= -\sum_{i=1}^{n}\int_x^\infty \cdots \int_x^\infty det\Psi_n(t;s_1,\ldots,s_{i-1},x,s_{i+1},\ldots,s_n)ds_1\ldots ds_{i-1}ds_{i+1}\ldots ds_n \quad (4.6)$$

From the structure of matrix Ψ_n we know we can construct a matrix

$$\Psi_n(s_1,\ldots,s_{i-1},s_{i+1},s_i,s_{i+2},\ldots,s_n)$$

by interchanging i-th and (i+1)-th columns as well as i-th and (i+1)-th rows of $\Psi_n(s_1,\ldots,s_n)$ respectively. Hence we have

$$det\Psi_n(s_1,\ldots,s_{i-1},s_{i+1},s_i,s_{i+2},\ldots,s_n)=(-1)^2 det\Psi_n(s_1,\ldots,s_n)$$

Therefore the r.h.s of (4.6) becomes

$$=-\sum_{i=1}^{n}\int_x^\infty\cdots\int_x^\infty(-1)^{2(i-1)}det\Psi_n(t;x,s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)ds_1\cdots ds_{i-1}ds_{i+1}\cdots ds_n$$

Rename $s_{i+1}=s_i, s_{i+2}=s_{i+1},\ldots,s_n=s_{n-1}$ the above equation becomes

$$=-n\int_x^\infty\cdots\int_x^\infty det\Psi_n(t;x,s_1,\ldots,s_{n-1})\prod_{i=1}^{n-1}ds_i$$

From the structure of Ω_n it is readily to derive

$$det\Psi_n(t;x,s_1,\ldots,s_{n-1})=det\Omega_{n-1}(t,x,x;s_1,\ldots,s_{n-1})$$

Now imposing ∂_x on both sides of (4.1)

$$\begin{aligned}\partial_x f&=\sum_{n=1}^{\infty}\frac{1}{n!}\partial_x\int_x^\infty\cdots\int_x^\infty det\Psi_n(t;s_1,\ldots,s_n)\prod_{i=1}^{n}ds_i\\&=-\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int_x^\infty\cdots\int_x^\infty det\Omega_{n-1}(t,x,x;s_1,\ldots,s_{n-1})\prod_{i=1}^{n-1}ds_i\\&=D(t,x,x)\quad by (4.3)\ and\ (4.4)\end{aligned}$$

QED

And we know $u=2(\log f)_{xx}=2(\frac{f_x}{f})_x$ Then by lemma 4.2

$$u(t,x)=2(\frac{D(t,x,x)}{f(t,x)})_x=2\partial_x[K(t,x,x)]$$

Chapter 5

Perspective

In chapter 4 we derive GLM equation from generalized soliton solutions

$$K(t,x,z) + F(t,x+z) + \int_x^\infty K(t,x,s)F(t,s+z)ds \tag{5.1}$$

with

$$F(t,s) = -\int_\Gamma p\exp(\frac{sp}{2} - p^3 t)d\tau(p) \tag{5.2}$$

As previously stated if the integral path Γ is restricted along the real and imaginary axis , $F(t,s)$ will be transformed into

$$F(t,s) = \sum_{n=1}^{N} c_n^2(t)\exp(-p_n s) + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(p,t)\exp(ips)dp \tag{5.3}$$

The first term of r.h.s of (5.3) represents solitons and the second one represents an oscillatory wave train. Segur[15] has gained the result that an initial disturbance , in general , evolves into solitons as well as an oscillatory wave train and there appears to be no permanent effect on the solitons from the interaction with the oscillatory wave train. Apparently $F(t,s)$ in (5.2) owns more flexibility for us to investigate the interaction between solitons and oscillatory wave train than $F(t,s)$ in (5.3). This is the value of (5.2).

Appendix A

A.1

Here we shall prove

$$
\begin{aligned}
&\frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)}F(\mathbf{\Phi}_3-\mathbf{\Phi}_1-\mathbf{\Phi}_2)+\frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)}F(\mathbf{\Phi}_2-\mathbf{\Phi}_1-\mathbf{\Phi}_3)\\
&+\frac{F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)}F(\mathbf{\Phi}_1-\mathbf{\Phi}_2-\mathbf{\Phi}_3)\\
&=-F(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)\frac{F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(\mathbf{\Phi}_2-\mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(\mathbf{\Phi}_2+\mathbf{\Phi}_3)}
\end{aligned}
$$

this relation in p16 We recall Hirota condition

$$
\sum_{\sigma=1,-1}F(\sum_{i=1}^{N}\sigma_i\mathbf{P}_i)\prod_{i<j}^{(N)}F(\sigma_i\mathbf{P}_i-\sigma_j\mathbf{P}_j)\sigma_i\sigma_j=0\ ,\ for\ N=1,2,\ldots
$$

For $N=3$ the summation has eight terms. They are

$$F(\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(\mathbf{\Phi}_2-\mathbf{\Phi}_3) \tag{A.1}$$

$$F(\mathbf{\Phi}_1+\mathbf{\Phi}_2-\mathbf{\Phi}_3)F(\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(\mathbf{\Phi}_2+\mathbf{\Phi}_3) \tag{A.2}$$

$$F(\mathbf{\Phi}_1-\mathbf{\Phi}_2+\mathbf{\Phi}_3)F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(-\mathbf{\Phi}_2-\mathbf{\Phi}_3) \tag{A.3}$$

$$F(\mathbf{\Phi}_1-\mathbf{\Phi}_2-\mathbf{\Phi}_3)F(\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(-\mathbf{\Phi}_2+\mathbf{\Phi}_3) \tag{A.4}$$

$$F(-\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3)F(-\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(-\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(\mathbf{\Phi}_2-\mathbf{\Phi}_3) \tag{A.5}$$

$$F(-\mathbf{\Phi}_1+\mathbf{\Phi}_2-\mathbf{\Phi}_3)F(-\mathbf{\Phi}_1-\mathbf{\Phi}_2)F(-\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(\mathbf{\Phi}_2+\mathbf{\Phi}_3) \tag{A.6}$$

$$F(-\mathbf{\Phi}_1-\mathbf{\Phi}_2+\mathbf{\Phi}_3)F(-\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(-\mathbf{\Phi}_1-\mathbf{\Phi}_3)F(-\mathbf{\Phi}_2-\mathbf{\Phi}_3) \tag{A.7}$$

$$F(-\mathbf{\Phi}_1-\mathbf{\Phi}_2-\mathbf{\Phi}_3)F(-\mathbf{\Phi}_1+\mathbf{\Phi}_2)F(-\mathbf{\Phi}_1+\mathbf{\Phi}_3)F(-\mathbf{\Phi}_2+\mathbf{\Phi}_3) \tag{A.8}$$

Because $F(-\mathbf{\Phi}) = F(\mathbf{\Phi})$ then $(A.1) = (A.8), (A.2) = (A.7), (A.3) = (A.6), (A.4) = (A.5)$ Therefore Hirota condition for $N = 3$ reduces to $(A.5) + ((A.6) + ((A.7) + ((A.8) = 0$ i.e.

$$\begin{aligned}
&F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_2 + \mathbf{\Phi}_3)F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_2)F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_3)F(\mathbf{\Phi}_2 - \mathbf{\Phi}_3) \\
&+ F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_2 - \mathbf{\Phi}_3)F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_2)F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_3)F(\mathbf{\Phi}_2 + \mathbf{\Phi}_3) \\
&+ F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_2 + \mathbf{\Phi}_3)F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_2)F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_3)F(-\mathbf{\Phi}_2 - \mathbf{\Phi}_3) \\
&+ F(-\mathbf{\Phi}_1 - \mathbf{\Phi}_2 - \mathbf{\Phi}_3)F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_2)F(-\mathbf{\Phi}_1 + \mathbf{\Phi}_3)F(-\mathbf{\Phi}_2 + \mathbf{\Phi}_3) = 0
\end{aligned}$$

Now divide $F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)F(\mathbf{\Phi}_1 + \mathbf{\Phi}_3)F(\mathbf{\Phi}_2 + \mathbf{\Phi}_3))$ on both sides. Then

$$\begin{aligned}
&\frac{F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2)}{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)}F(\mathbf{\Phi}_3 - \mathbf{\Phi}_1 - \mathbf{\Phi}_2) + \frac{F(\mathbf{\Phi}_1 - \mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_3)}F(\mathbf{\Phi}_2 - \mathbf{\Phi}_1 - \mathbf{\Phi}_3) \\
&+ \frac{F(\mathbf{\Phi}_2 - \mathbf{\Phi}_3)}{F(\mathbf{\Phi}_2 + \mathbf{\Phi}_3)}F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2 - \mathbf{\Phi}_3) \\
&= -F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 + \mathbf{\Phi}_3)\frac{F(\mathbf{\Phi}_1 - \mathbf{\Phi}_2)F(\mathbf{\Phi}_1 - \mathbf{\Phi}_3)F(\mathbf{\Phi}_2 - \mathbf{\Phi}_3)}{F(\mathbf{\Phi}_1 + \mathbf{\Phi}_2)F(\mathbf{\Phi}_1 + \mathbf{\Phi}_3)F(\mathbf{\Phi}_2 + \mathbf{\Phi}_3)}
\end{aligned}$$

A.2

This is a proof for p22

$$\phi_1(p,q) = p - q \ , \ \phi_2(p,q) = p^2 - q^2 \ , \ \phi_3(p,q) = p^3 - q^3$$

$$F(a,b,c) = a^4 - 4ac + 3b^2$$

$F(\phi_1(p_j,q_j) - \phi_1(p_k,q_k), \phi_2(p_j,q_j) - \phi_2(p_k,q_k), \phi_3(p_j,q_j) - \phi_3(p_k,q_k))$
$= [p_j - q_j - (p_k - q_k)]^4 - 4[p_j - q_j - (p_k - q_k)][p_j^3 - q_j^3 - (p_k^3 - q_k^3)]$
$+ 3[p_j^2 - q_j^2 - (p_k^2 - q_k^2)]^2$

$[p_j - q_j - (p_k - q_k)]^4$
$= p_j^4 + q_j^4 + p_k^4 + q_k^4 - 4p_j^3 q_j - 4p_j q_j^3 - 4p_j^3 p_k - 4p_j p_k^3 + 4p_j^3 q_k + 4p_j q_k^3 + 4q_j^3 p_k + q_j p_k^3 - 4q_j^3 q_k - 4q_j q_k^3 - 4p_k^3 q_k - 4p_k q_k^3 + 6p_j^2 q_j^2 + 6p_j^2 p_k^2 + 6p_j^2 q_k^2 + 6q_j^2 p_k^2 + 6q_j^2 q_k^2 + 6p_k^2 q_k + 12p_j^2 q_j p_k - 12p_j^2 q_j q_k - 12p_j^2 p_k - 12q_j^2 p_j p_k + 12q_j^2 p_j q_k - 12q_j^2 p_k q_k - 12p_k^2 p_j q_j + 12p_k^2 p_j q_k - 12p_k^2 q_j q_k - 12q_k^2 p_j p_j - 12q_k^2 p_j p_k + 12q_k^2 q_j p_k + 24p_j q_j p_k q_k$

$-4[p_j - q_j - (p_k - q_k)][p_j^3 - q_j^3 - (p_k^3 - q_k^3)]$
$= -4(p_j^4 - p_j q_j^3 - p_j p_k^3 + p_j q_k^3 - q_j p_j^3 + q_j^4 + q_j p_k^3 - q_j q_k^3 - p_k p_j^3 + p_k q_j^3 + p_k^4 - p_k q_k^3 + q_k p_j^3 - q_k q_j^3 - q_k p_k^3 + q_k^4)$

$3[p_j^2 - q_j^2 - (p_k^2 - q_k^2)]^2$
$= 3(p_j^4 + q_j^4 + p_k^4 + q_k^4 - 2p_j^2 q_j^2 - 2p_j^2 p_k^2 + 2p_j^2 q_k^2 + 2q_j^2 p_k^2 - 2q_j^2 q_k^2 - 2p_k^2 q_k^2)$

Adding the three terms we obtain
$F(\phi_1(p_j,q_j) - \phi_1(p_k,q_k), \phi_2(p_j,q_j) - \phi_2(p_k,q_k), \phi_3(p_j,q_j) - \phi_3(p_k,q_k)) = 12(p_j -$

$p_k)(p_jq_k^2 + p_jq_jp_k - p_jq_jq_k - p_jp_kq_k - q_j^2p_k + q_j^2q_k - q_jq_k^2 + q_jp_kq_k)$
$= 12(p_j - p_k)(q_j - q_k)(q_jq_k - p_jq_k + p_jp_k - q_jp_k)$

Likewise
$F(\phi_1(p_j, q_j) + \phi_1(p_k, q_k), \phi_2(p_j, q_j) + \phi_2(p_k, q_k), \phi_3(p_j, q_j) + \phi_3(p_k, q_k))$
$= [p_j - q_j + (p_k - q_k)]^4 - 4[p_j - q_j + (p_k - q_k)][p_j^3 - q_j^3 + (p_k^3 - q_k^3)]$
$+ 3[p_j^2 - q_j^2 + (p_k^2 - q_k^2)]^2$
$= [p_j - q_j - (q_k - p_k)]^4 - 4[p_j - q_j - (q_k - p_k)][p_j^3 - q_j^3 - (q_k^3 - p_k^3)] + 3[p_j^2 - q_j^2 - (q_k^2 - p_k^2)]^2$
$= 12(p_j - q_k)(q_j - p_k)(q_jp_k - p_jp_k + p_jq_k - q_jq_k)$
$= -12(p_j - q_k)(q_j - p_k)(q_jq_k - p_jq_k + p_jp_k - q_jp_k)$

Therefore
$-\dfrac{F(\phi_1(p_j, q_j) - \phi_1(p_k, q_k), \phi_2(p_j, q_j) - \phi_2(p_k, q_k), \phi_3(p_j, q_j) - \phi_3(p_k, q_k))}{F(\phi_1(p_j, q_j) + \phi_1(p_k, q_k), \phi_2(p_j, q_j) + \phi_2(p_k, q_k), \phi_3(p_j, q_j) + \phi_3(p_k, q_k))}$
$= \dfrac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)}$

A.3

This is a proof for lemma 4.1 The following proof can be readily extended to $n \times n$ matrix. Now we consider a 3×3 matrix

$$D_3(x_1, x_2, x_3; \alpha_1, \alpha_2, \alpha_3) = \begin{vmatrix} \frac{1}{x_1 - \alpha_1} & \frac{1}{x_1 - \alpha_2} & \frac{1}{x_1 - \alpha_3} \\ \frac{1}{x_2 - \alpha_1} & \frac{1}{x_2 - \alpha_2} & \frac{1}{x_2 - \alpha_3} \\ \frac{1}{x_3 - \alpha_1} & \frac{1}{x_3 - \alpha_2} & \frac{1}{x_3 - \alpha_3} \end{vmatrix} \tag{A.9}$$

If we set $x_i = x_j$ or $\alpha_i = \alpha_j$ for $i \neq j$, then $D_3 = 0$. Hence D_3 is divisible by $\zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3)$ where

$$\zeta^{\frac{1}{2}}(x_1, x_2, x_3) \equiv \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \prod_{1 \le j < k \le 3} (x_k - x_j) = (-1)^{\frac{3(3-1)}{2}} \prod_{1 \le j < k \le 3} (x_j - x_k)$$

Now multiplying the i-th row of D_3 with $u_i = (x_i - \alpha_1)(x_i - \alpha_2)(x_i - \alpha_3)$ we obtain

$$u_1u_2u_3D_3 = \begin{vmatrix} (x_1 - \alpha_2)(x_1 - \alpha_3) & (x_1 - \alpha_1)(x_1 - \alpha_3) & (x_1 - \alpha_1)(x_1 - \alpha_2) \\ (x_2 - \alpha_2)(x_2 - \alpha_3) & (x_2 - \alpha_1)(x_2 - \alpha_3) & (x_2 - \alpha_1)(x_2 - \alpha_2) \\ (x_3 - \alpha_2)(x_3 - \alpha_3) & (x_3 - \alpha_1)(x_3 - \alpha_3) & (x_3 - \alpha_1)(x_3 - \alpha_2) \end{vmatrix}$$

Owing to D_3 being divisible by $\zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3)$ and $u_1u_2u_3D_3$ having the same polynomial degree on $x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3$ as $\zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3)$ we have

$$u_1u_2u_3D_3 = (const)\zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3)$$

Now put $\alpha_i = x_i$ for $i = 1, 2, 3$ then the above equation becomes

$$\begin{vmatrix} (x_1 - x_2)(x_1 - x_3) & 0 & 0 \\ 0 & (x_2 - x_1)(x_2 - x_3) & 0 \\ 0 & 0 & (x_3 - x_1)(x_3 - x_2) \end{vmatrix} = (const)\zeta(x_1, x_2, x_3)$$

$$\Rightarrow (-1)^{\frac{3(3-1)}{2}} \prod_{1 \le j < k \le 3} (x_j - x_k)^2 = (const) \prod_{1 \le j < k \le 3} (x_j - x_k)^2$$

$$(const) = (-1)^{\frac{3(3-1)}{2}}$$

Therefore

$$D_3 = \frac{(-1)^{\frac{3(3-1)}{2}} \zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3)}{u_1 u_2 u_3}$$

Now replace $\alpha_i = -x_i$ for $i = 1, 2, 3$ then

$$\begin{vmatrix} \frac{1}{x_1 + x_1} & \frac{1}{x_1 + x_2} & \frac{1}{x_1 + x_3} \\ \frac{1}{x_2 + x_1} & \frac{1}{x_2 + x_2} & \frac{1}{x_2 + x_3} \\ \frac{1}{x_3 + x_1} & \frac{1}{x_3 + x_2} & \frac{1}{x_3 + x_3} \end{vmatrix} = D_3(x_1, x_2, x_3; -x_1, -x_2, -x_3)$$

$$= \frac{(-1)^{\frac{3(3-1)}{2}} \zeta^{\frac{1}{2}}(x_1, x_2, x_3)\zeta^{\frac{1}{2}}(-x_1, -x_2, -x_3)}{2^3 x_1 x_2 x_3 \prod_{1 \le j < k \le 3}(x_j + x_k)^2}$$

while

$$\zeta^{\frac{1}{2}}(-x_1, -x_2, -x_3) = (-1)^{\frac{3(3-1)}{2}} \prod_{1 \le j < k \le 3} (x_k - x_j) = (-1)^{\frac{3(3-1)}{2}} (-1)^{\frac{3(3-1)}{2}} \prod_{1 \le j < k \le 3} (x_j - x_k)$$

$$= \prod_{1 \le j < k \le 3} (x_j - x_k)$$

$$\Rightarrow \begin{vmatrix} \frac{1}{x_1 + x_1} & \frac{1}{x_1 + x_2} & \frac{1}{x_1 + x_3} \\ \frac{1}{x_2 + x_1} & \frac{1}{x_2 + x_2} & \frac{1}{x_2 + x_3} \\ \frac{1}{x_3 + x_1} & \frac{1}{x_3 + x_2} & \frac{1}{x_3 + x_3} \end{vmatrix} = \frac{1}{2^3 x_1 x_2 x_3} \prod_{1 \le j < k \le 3} (\frac{x_j - x_k}{x_j + x_k})^2$$

$$\Rightarrow \begin{vmatrix} \frac{2x_1}{x_1 + x_1} & \frac{2x_1}{x_1 + x_2} & \frac{2x_1}{x_1 + x_3} \\ \frac{2x_2}{x_2 + x_1} & \frac{2x_2}{x_2 + x_2} & \frac{2x_2}{x_2 + x_3} \\ \frac{2x_3}{x_3 + x_1} & \frac{2x_3}{x_3 + x_2} & \frac{2x_3}{x_3 + x_3} \end{vmatrix} = \prod_{1 \le j < k \le 3} (\frac{x_j - x_k}{x_j + x_k})^2$$

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