國 立 交 通 大 學

應用數學系

碩　士　論　文

模型式與黎曼函數值

On Modular Forms and zeta values

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摘　　　　要

Apéry 在 1980 年證明了$\zeta(3)=\sum_{n=1}^{\infty}n^{-3}$為無理數，他所使用的方式包含一個有理數數列明確的建構方式並使其收斂至$\zeta(3)$。其後 Beukers 發現可以將 Apéry 的數列以模型式來表示。在此論文中我們將利用 Beukers 的觀點與模型式的理論去創造一個有理數數列並使其收斂至$\zeta(5)$。

On modular forms and zeta values

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Abstract

The irrationality of $\zeta(3)=\sum_{n=1}^{\infty} n^{-3}$ was established by Apéry in the late nineteen eighties. His method involved an explicit construction of a sequence of rational numbers converging to $\zeta(3)$. Later, Beukers found that Apéry's sequence has a modular-form interpretation. In this thesis we shall follow Beuker's approach to construct a sequence of rational numbers that converges to $\zeta(5)$ exponentially.

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目錄

1. **Introduction**

The Riemann zeta function was first introduced by Euler and is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

The series is convergent when s is a complex number with Re s >1. Some special values of $\zeta(s)$ are well known. For example $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$. In general, when $s=2n$ is a positive even integer we have $\zeta(s)=\pi^{2n}r$ for some rational numbers r. In fact, the number r can be expressed in terms of the Bernoulli numbers. For odd integer $2n+1\geq 3$, however, not much about $\zeta(2n+1)$ is known. It is not even known whether $\zeta(2n+1)$ are rational, except for the case $2n+1=3$, which was established relatively recently.

In 1970's, R. Apéry [3] proved that $\zeta(3)=\sum_{n=1}^{\infty}n^{-3}$ is an irrational number by constructing two sequences

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

$$b_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\left\{\sum_{m=1}^{n}\frac{1}{m^3}+\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}\right\},$$

and then showing that b_n/a_n converges to $\zeta(3)$ fast enough to ensure irrationality of $\zeta(3)$. Another remarkable discovery of Apéry is that a_n and b_n satisfy the recursive relation

$$(n+1)^3u_{n+1}=(34n^3+51n^2+27n+5)u_n-n^3u_{n-1}$$

Thus, if we set $A(t)=\sum_{n=0}^{\infty}a_nt^n$ and $B(t)=\sum_{n=0}^{\infty}b_nt^n$, then the functions $A(t)$ and $B(t)$ satisfy the differential equations

$$(1-34t+t^2)D_t^3A+(3t^2-51t)D_t^2A+(3t^2-27t)D_tA+(t^2-5t)A=0$$

and

$$(1-34t+t^2)D_t^3B+(3t^2-51t)D_t^2B+(3t^2-27t)D_tB+(t^2-5t)B=6t,$$

where D_t denotes the differential operator td/dt. It turns out that these differential equations have a modular-function origin. It can be shown that if we choose two linearly independent solutions F_1 and F_2 appropriately, then t is a modular function of $\tau = F_2/F_1$ on $\Gamma_0(6)$, A a modular form of weight 2, and $d^3(B/A)/d\tau^3$ a modular form of weight 4. This connection between $\zeta(3)$ and modular forms was discovered by Beukers [1]. (Note that, in general, if F is a modular form of weight k and t a modular function, then F as a function of t, satisfied an $(k+1)$-st order linear differential equation. See Section 5 below.)

In this thesis we will construct a sequence of rational numbers c_n converging to $\zeta(5)$ using Beukers' idea [1]. Our result is as follows.

Theorem 1. *Let a_n and b_n be two sequences satisfying the recursive relation*

$$n^5 u_n = A_7 u_{n-1} - A_6 u_{n-2} - A_5 u_{n-3} - A_4 u_{n-4} - A_3 u_{n-5} - A_2 u_{n-6} + A_1 u_{n-7} - A_0 u_{n-8},$$

where

$$\begin{aligned}
A_7 &= 24n^5 + 420n^4 + 2960n^3 + 10500n^2 + 18744n + 13468,\\
A_6 &= 92n^5 + 1380n^4 + 8160n^3 + 23760n^2 + 33968n + 18960,\\
A_5 &= 600n^5 + 7500n^4 + 38480n^3 + 101100n^2 + 135704n + 74260,\\
A_4 &= 966n^5 + 9660n^4 + 40800n^3 + 90240n^2 + 103840n + 49472,\\
A_3 &= 600n^5 + 4500n^4 + 14480n^3 + 24660n^2 + 21944n + 8076,\\
A_2 &= 92n^5 + 460n^4 + 800n^3 + 560n^2 + 48n - 80,\\
A_1 &= 24n^5 + 60n^4 + 80n^3 + 60n^2 + 24n + 4,\\
A_0 &= n^5
\end{aligned}$$

with initial values $a_0 = 1$, $a_1 = 4$, $a_2 = 34$, $a_3 = 308$, $a_4 = 3083$, $a_5 = 32696$, $a_6 = 361428$, $a_7 = 4119288$ *and* $b_0 = 0$, $b_1 = 144/25$, $b_2 = 333/10$, $b_3 = 217042/675$, $b_4 = 138004123/43200$, $b_5 = 1144320384083/33750000$, $b_6 = 25297127932859/67500000$, $b_7 = 2422896637170749569/567236250000$. *Then the series* $\{b_n/a_n\}$ *converges to* $\zeta(5)$. *More precisely, we have*

$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{b_n}{a_n} - \zeta(5)\right|} \le (\sqrt{3}-1)^4/4$$

Here we tabulate b_n/a_n for $n = 1 \ldots 10$ below. We remark that the error between b_{10}/a_{10} and $\zeta(5)$ is about 10^{-8}. In order to get the same magnitude of error term using $\sum_{n=1}^{N} n^{-5}$. One would need N to be about 100.

$$\begin{aligned} \zeta(5) &= 1.03692775 \\ \hline b_1/a_1 &= 1.44000000 \\ b_2/a_2 &= 0.97941176 \\ b_3/a_3 &= 1.04397306 \\ b_4/a_4 &= 1.03617900 \\ b_5/a_5 &= 1.03700113 \\ b_6/a_6 &= 1.03692095 \\ b_7/a_7 &= 1.03692836 \\ b_8/a_8 &= 1.03692770 \\ b_9/a_9 &= 1.03692776 \\ b_{10}/a_{10} &= 1.03692775 \end{aligned}$$

The rest of the thesis is organized as follows. We first introduce the basic theory of modula groups, congruence subgroups and modular forms in section 2 and section 3. Then we will describe Beukers' approach to irrationality proof using modular forms in section 4. Next we introduce the result of P. F. Stiller [3] and the method of Y.Yang [5] for determining the differential equation satisfied by modular form in section 5. Finally, we will prove theorem 1 and apply the method of [5] to find the recursive relation given above in the last section.

2. Modular group and congruence subgroup

In this section we briefly recall the definition of modular groups and congruence subgroups. For a ring R with unity 1, we denote by $R^\times$ the group of invertible elements in R. The general linear group $GL_2(R)$ is defined by

$$GL_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \; a,b,c,d \in R \quad and \quad ad-bc \in R^\times \right\}$$

Here we consider the situations when $R = \mathbb{R}$ or $R = \mathbb{Z}$. We set

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \; a,b,c,d \in \mathbb{R} \quad and \quad ad-bc > 0 \right\}$$

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \; a,b,c,d \in \mathbb{R} \quad and \quad ad-bc = 1 \right\}$$

We call $SL_2(\mathbb{Z})$ and its subgroups of finite index *modular groups*. The special linear group $SL_2(\mathbb{Z})$ is also called the *full modular group*. A class of modular groups that are of special interest to number theorists is the *congruence subgroups*. Their definition is given as follows.

Definition 2.1. *For a positive integer N, we define the subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ of $SL_2(\mathbb{Z})$ by*

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \ c \equiv 0 \mod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \ c \equiv 0, a \equiv d \equiv 1 \mod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \ b \equiv c \equiv 0, a \equiv d \equiv 1 \mod N \right\}$$

We note that

$$SL_2(\mathbb{Z}) = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1)$$

and

$$SL_2(\mathbb{Z}) \supset \Gamma_0(1) \supset \Gamma_1(N) \supset \Gamma(N)$$

Further, if $M \mid N$, then

$$\Gamma_0(M) \supset \Gamma_0(N) \quad , \quad \Gamma_1(M) \supset \Gamma_1(N) \quad , \quad \Gamma(M) \supset \Gamma(N).$$

These subgroups are modular groups since $[\Gamma(1) : \Gamma(N)] < \infty$. We call $\Gamma(N)$ a principal congruence modular group, and $\Gamma_0(N)$ and $\Gamma_1(N)$ modular groups of *Hecke type*. We call N the *level* of $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$. A modular group containing a principal congruence modular group is called a congruence modular group.

There is another type of congruence subgroup that is of great interest to number theorists. Let N be a positive integer and n be an integer such that $\gcd(n, N/n) = 1$

$$w_n = \left\{ \frac{1}{\sqrt{n}} \begin{pmatrix} an & b \\ cN & dn \end{pmatrix} : adn^2 - bcN = n \right\}$$

The elements w_n are called *Atkin-Lehner involutions*. The set of all $\Gamma_0^*(N)$ of $\Gamma_0(N)$ union all the Atkin-Lehner involutions lies in the normalizer of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$

and we have $\Gamma_0^*(N)/\Gamma_0(N) \simeq \mathbb{Z}_2^k$, where k is the number of distinct prime divisors of N.

Group action on the upper half-plane

Let $\mathbb{H}$ denote that upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. We define a mapping $SL_2(\mathbb{R}) \times \mathbb{H} \mapsto \mathbb{H}$ by

$$(\alpha, \tau) \mapsto \alpha\tau = \frac{a\tau + b}{c\tau + d},$$

where $\tau \in \mathbb{H}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Then we can check that this mapping is a group action. Moreover, this mapping is called a linear fractional transformation and is also a conformal mapping.

Cusps and elliptic points

We now classify the linear fractional transformation defined above. A non-constant element α of $GL_2^+(\mathbb{R})$ is called *elliptic*, *parabolic*, or *hyperbolic*, when it satisfies

$$\operatorname{tr}(\alpha)^2 < 4\det(\alpha) \quad , \quad \operatorname{tr}(\alpha)^2 = 4\det(\alpha), \qquad \textit{or} \qquad \operatorname{tr}(\alpha)^2 > 4\det(\alpha),$$

respectively. When $\tau \in \mathbb{H}^*$ is a fixed point of an elliptic, parabolic or hyperbolic element of Γ, we say that τ is an *elliptic point*, a *parabolic point*, or a *hyperbolic point*, respectively. We also call a parabolic point of Γ a *cusp* of Γ.

Remark. In $SL_2(\mathbb{Z})$ the above classification implies that if $\operatorname{tr}(\alpha) = 0$, then $\alpha^2 = -I$, and if $\operatorname{tr}(\alpha) = \pm 1$, then α is order 3 in $PSL_2(\mathbb{Z})$.

Fundamental domains

Let G denote any subgroup of the modular group $\Gamma(1)$. Two points τ and τ' in the upper half-plane $\mathbb{H}$ are said to be equivalent under G if $\tau' = A\tau$ for some $A \in G$. This is an equivalence relation since G is a group.

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. A fundamental domain for Γ is a connected open subset D of $\mathbb{H}$ such that no two points of D are equivalent under Γ and $\mathbb{H} = \cup\gamma\bar{D}$, where $\bar{D}$ is the closure of D. The standard fundamental domain for $SL_2(\mathbb{Z})$ is shown in Fig 1.

$$\Gamma := \{\tau \in \mathbb{H} \mid \frac{-1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2} \ \textit{and}\ |\tau| \geq 1\}$$

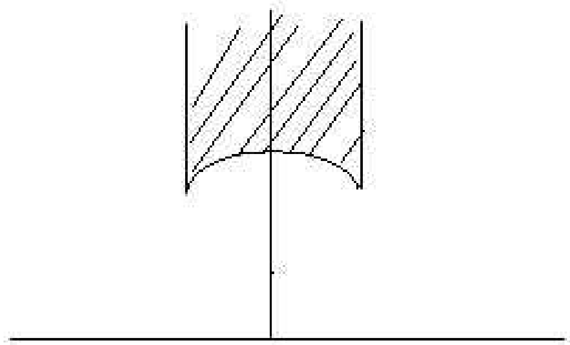

FIGURE 1. Fundamental domain of $\Gamma(1)$

Next we consider the fundamental domain for congruence subgroups. The following fact is well-known.

Proposition 2.1. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$, and let D be a fundamental domain for Γ. Let Γ' be a subgroup of Γ of finite index, and write Γ as a disjoint union of right cosets of Γ':*

$$\Gamma = \Gamma'_{\gamma_1} \cup \cdots \cup \Gamma'_{\gamma_m}$$

Then $D' = \cup \gamma_i D$ is a fundamental domain for Γ'.

Proof. Let $\tau \in \mathbb{H}$. Then $\tau = \gamma\tau'$ for some $\tau' \in \bar{D}$, and $\gamma = \gamma'\gamma_i$ for some $\gamma' \in \Gamma'$ Thus $\tau = \gamma'\gamma_i\tau \in \Gamma'$. If $\gamma D' \cap D' \neq \phi$, then it would contain a transformation of D. But then $\gamma\gamma_i D = \gamma_j D$ for some $i \neq j$, which would imply that $\gamma\gamma_i = \gamma_j$ and this is a contradiction. $\square$

The Riemann surface $\Gamma\backslash\mathbb{H}^*$

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. We now consider the quotient space $\Gamma\backslash\mathbb{H}$. It is in general a non-compact Riemann surface. To compactify it, we begin by adding cusps of Γ to $\mathbb{H}$. Let P_Γ be the set of all cusps of Γ and put $\mathbb{H}^* = \mathbb{H} \cup P_\Gamma$. When Γ has no cusps, $P_\Gamma = \phi$ and $\mathbb{H}^* = \mathbb{H}$. We put

$$U_l = \{\tau \in \mathbb{H} \mid \mathrm{Im}\,(\tau) > l\} \qquad U^* = U_l \cup \{\infty\} \quad l > 0$$

Now we define the topology on $\mathbb{H}^*$ as follows:

(i) for $\tau \in \mathbb{H}$, we take as the fundamental neighborhood system at τ in $\mathbb{H}^*$ that at τ in $\mathbb{H}$.

(ii) for $x \in P_\Gamma$, we take as the fundamental neighborhood system at x the family $\{\sigma^{-1}U_l^* |\, l > 0\}$, where $\sigma \in SL_2(\mathbb{R})$ such that $\sigma x = \infty$.

Then $\mathbb{H}^*$ is also a Hausdorff space under this topology. In fact, put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $x = -d/c$. Then we see that

$$\sigma^{-1}U_l = \{\tau \in \mathbb{H} \,|\, \mathrm{Im}\,(\tau)/|c\tau + d|^2 > l\}$$

and this is the inside of a circle with the radius $(2lc^2)^{-1}$ tangent to the real axis at x. For $x \in P_\Gamma$, we call $\{\sigma^{-1}U_l\}$ a neighborhood of x in H. Since the action of Γ on H is a conformal mapping which maps circles or lines to those, Γ also acts on the topological space $\mathbb{H}^*$. Therefore, the quotient space $\Gamma\backslash\mathbb{H}^*$ can be defined, and we conclude that $\Gamma\backslash\mathbb{H}^*$ is compact.

The genus of $\Gamma\backslash\mathbb{H}^*$

Let $\Re$ be a compact Riemann surface and χ be the Euler-Poincaré characteristic of $\Re$. We define the genus of $\Re$ by

$$\chi = 2 - 2g$$

Then g is a non-negative integer. We now compute the genus of the Riemann surface $\Gamma\backslash\mathbb{H}^*$ introduced above. The main tool we use is the Riemann-Hurwitz formula.

Assuming that $\Re' \longrightarrow \Re$ is a covering of Riemann surfaces, then the Riemann-Hurwitz formula states that

$$2g' - 2 = n(2g - 2) + \sum_{b \in \Re'} (e_b - 1)$$

where g' is the genus of $\Re'$, n is the degree of the covering and e_b is the ramification index at the point b.

Proposition 2.2. *Let Γ be a modular group, and g the genus of $\Gamma\backslash\mathbb{H}^*$ Then*

$$g = 1 + m/12 - v_2/4 - v_3/3 - v_\infty/2$$

where v_2 *is the number of inequivalent elliptic points of order* 2*;* v_3 *is the number of inequivalent elliptic points of order* 3*;* v_∞ *is the number of inequivalent cusps; and* $m = [\bar{\Gamma}(1) : \bar{\Gamma}]$.

Remark. $\bar{\Gamma}(N)$ is denoted the image of $\Gamma(N)$ in $\Gamma(1)\backslash\{\pm I\}$.

Proof. Since $\mathbb{H}^*_\Gamma = \mathbb{H}^*_{\Gamma(1)}$, there exists a natural mapping

$$F : \Re_\Gamma = \Gamma\backslash\mathbb{H}^*_\Gamma \longrightarrow \Re_{\Gamma(1)} = \Gamma(1)\backslash\mathbb{H}^*_{\Gamma(1)}$$

We put $\mathbb{H}^* = \mathbb{H}^*_\Gamma = \mathbb{H}^*_{\Gamma(1)}$, and $\Re = \Re_{\Gamma(1)}$. Let $\pi_\Gamma : \mathbb{H}^* \to \Re_\Gamma$, and $\pi : \mathbb{H}^* \to \Re$ be the natural mappings. For any point b of $\Re_\Gamma$, take a point $\tau \in \mathbb{H}^*$ so that $\pi_\Gamma(\tau) = b$. Hence $\{\Re_\Gamma, F\}$ is a covering of $\Re$ of degree m. Let $e_b = e_{b,F}$ be the ramification index of the covering at b, and put $F(b) = a$. Let a_2,a_3 and a_∞ be the elliptic points of order 2 and 3,and the cusp on $\Re$, respectively. If $a \neq a_2, a_3, a_\infty$, then b is an ordinary point and $e_b = 1$. Suppose $a = a_2$, then $e_b = 1$ or 2. Put

$$t = \#\ \{b \in \Re_\Gamma \mid F(b) = a_2\}.$$

Then $m = v_2 + 2(t - v_2)$. Therefore

$$\sum_2 (e_b - 1) = m - t = (m - v_2)/2, \tag{1}$$

where $\sum_2$ is the summation over the points b such that $F(b) = a_2$. A similar argument implies

$$\sum_3 (e_b - 1) = 2(m - v_3)/3, \tag{2}$$

where $\sum_3$ is the summation over the points b such that $F(b) = a_3$. Next assume $F(b) = a_\infty$. Then b is a cusp in $\Re_\Gamma$, and

$$v_\infty = \#\ \{b \in \Re_\Gamma \mid F(b) = a_\infty\}.$$

Denote by $\sum_\infty$ is the summation over the points b such that $F(b) = a_\infty$. Then $\sum_\infty e_b = m$, so that

$$\sum_\infty (e_b - 1) = m - v_\infty. \tag{3}$$

Consequently, the formula of genus follows from (1), (2), (3) and the Riemann-Hurwitz formula. □

We now restrict our attention to $\Gamma_0(N)\backslash\mathbb{H}^*$. The geometric data can be easily seen to be

$$[\Gamma(1):\Gamma_0(N)] = N\prod_{p|N}(1+\frac{1}{p})$$

$$v_2(\Gamma_0(N)) = \begin{cases} 0, & \text{if } 4\,|\,N. \\ \displaystyle\prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right) & \text{if } 4\nmid N. \end{cases}$$

$$v_3(\Gamma_0(N)) = \begin{cases} 0, & \text{if } 9\,|\,N. \\ \displaystyle\prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right) & \text{if } 9\nmid N. \end{cases}$$

$$v_\infty(\Gamma_0(N)) = \sum_{0<d|N}\phi((d,N/d))$$

where $\phi(n)$ is the Euler function and $(-)$ denote the Legendre symbol. Therefore we can get the genus of $\Gamma_0(N)\backslash\mathbb{H}^*$ by Proposition 2.2 .

Remark. Using the above formula we see that if $\Gamma_0(N)$ is of genus zero, then $N = 1,\ldots,10,12,13,16,18,25$.

3. Modular functions and Modular forms

In this section we are concerned with modular functions and modular forms. In particular, we will introduce the classes of modular forms, namely, the Eisenstein series and the Dedekind $\eta-$function.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $f(\tau)$ be a function on $\mathbb{H}^*$ with values in $\mathbb{C}\cup\{\infty\}$. Let k be an integer. The action of γ on f is defined to be

$$f(\tau)\big|_{[\gamma]_k} = (c\tau+d)^{-k}f(\gamma\tau) \quad for \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

More generally, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. We define

$$f(\tau)\big|_{[\gamma]_k} = (\det\gamma)^{k/2}(c\tau+d)^{-k}f(\gamma\tau) \quad for \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$$

We now define modular functions, modular forms and cusp forms for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$. Let N be the level of Γ. We set $q_N = e^{2\pi i\tau/N}$.

Definition 3.1. *Let $f(\tau)$ be a meromorphic function on $\mathbb{H}$, and let Γ be a congruence subgroup of level N, i.e., $\Gamma \supset \Gamma(N)$. Let $k \in \mathbb{Z}$. We call $f(\tau)$ a modular function of weight k for Γ if it satisfies the following two conditions:*

(a) $f\big|_{[\gamma]_k} = f \quad for \quad all \quad \gamma \in \Gamma$ *and,*

(b) *if for any $\gamma_0 \in SL_2(\mathbb{Z})$, $f\big|_{[\gamma_0]_k}$ has the form $\sum a_n q_N^n$ with $a_n = 0$ for $n \ll 0$.*

Here $a_n = 0$ for $n \ll 0$ means that $a_n = 0$, for $n \leq -M$ for some fixed integer M. We call such an $f(\tau)$ a modular form of weight k for Γ if it is holomorphic on $\mathbb{H}$ and if for all $\gamma_0 \in SL_2(\mathbb{Z})$ we have $a_n = 0$ for all $n < 0$ in condition (b). We call a modular form a cusp-form if in addition $a_0 = 0$ in condition (b) for all $\gamma_0 \in SL_2(\mathbb{Z})$. We let $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the set of modular forms of weight k for Γ and the set of cusp-forms of weight k for Γ, respectively. Now we illustrate two examples of modular forms which will appear later.

Eisenstein series

Let k be an even integer greater than 2 and write

$$G_k(\Lambda) = \sum_{\omega \in \Lambda} \omega^{-k} \tag{4}$$

define

$$G_k(\tau) = G_k(\tau\mathbb{Z} + \mathbb{Z})$$

where Λ is denote the lattice spanned by 1 and τ. Because k is at least 4, the double sum (4) is absolutely convergent and uniformly convergent in any compact subset of $\mathbb{H}$. Hence $G_k(\tau)$ is a holomorphic function on $\mathbb{H}$. It is obvious that $G_k(\tau) = G_k(\tau + 1)$, and that the Fourier expansion of $G_k(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n$ has no negative terms. Because $G_k(\tau)$ approaches a finite limit as $\tau \to i\infty$:

$$\lim_{\tau \to i\infty} G_k(\tau) = \sum_{n\in\mathbb{Z}, n\neq 0} n^{-k} = 2\zeta(k)$$

Finally, we easily check that

$$\tau^{-k} G_k(-1/\tau) = G_k(\tau)$$

Thus we have proved that $G_k \in M_k(SL_2(\mathbb{Z}))$.

We now compute the q-expansion coefficients for $G_k(\tau)$. We shall find these coefficients are essentially the arithmetic functions

$$\sigma_k(n) = \sum_{d|n} d^k \tag{5}$$

of n.

Proposition 3.1. *Let k be an even integer greater than 2, and let $\tau \in \mathbb{H}$. Then the modular form $G_k(\tau)$ has q-expansion*

$$G_k(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where $q = e^{2\pi i\tau}$, and the Bernoulli number B_k are defined by setting

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

Proof. The logarithmic derivative of the product formula for sine is

$$\pi\cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty}\left(\frac{1}{a+n} + \frac{1}{a-n}\right), \quad a \in \mathbb{H} \tag{6}$$

If we write the left side as $\pi i(e^{\pi a} + e^{-\pi a})/(e^{\pi a} + e^{-\pi a}) = \pi + 2\pi i/(e^{2\pi i a} - 1)$, multiply both sides by a, replace $2\pi i a$ by x, and expand both series in powers of x, we obtain the well-known formula for $\zeta(k)$:

$$\zeta(k) = -2(\pi i)^k \frac{B_k}{2k!} \quad for\ \ k > 0\ \ even$$

Next, if we successively differentiate both sides of (6) with respect to a and then replace a by $m\tau$, we obtain:

$$\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}e^{2\pi i m n\tau} = -\frac{2k}{B_k}\zeta(k)\sum_{d=1}^{\infty} d^{k-1}q^{dm}$$

Thus

$$G_k(\tau) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^k} = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{m,d=1} d^{k-1}q^{dm}\right)$$

Collecting coefficient of a fixed power q^n in the last double sum, we obtain the sum in (5) as the coefficient of q^n. This completes the proof. □

Because of Proposition 3.1, it is useful to define the normalized Eisenstein series, obtained by dividing $G_k(\tau)$ by the constant $2\zeta(k)$ in:

$$E_k(\tau) = \frac{G_k}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Thus, $E_k(\tau)$ is defined so as to have rational q-expansion coefficients.The first few E_k are:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n;$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n;$$

$$E_8(\tau) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n$$

Next we consider the second example.

Dedekind eta function

Definition 3.2. *Let τ be a complex number with $\operatorname{Im}\tau > 0$. The ordinary Dedekind eta function is defined by*

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i\tau})$$

This function plays an important role in the study of the theory of modular function and its applications to other areas. One of the most important properties of the eta function is the transformation formula, which will be used in this paper.

Proposition 3.2. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the transformation formula for $\eta(\tau)$ is given by, for $c = 0$*

$$\eta(\tau + b) = e^{\pi ib/12}\eta(\tau),$$

and,for $c \neq 0$

$$\eta(\gamma\tau) = \epsilon(a,b,c,d)\sqrt{\frac{c\tau + d}{i}}\eta(\tau)$$

with

$$\epsilon(a,b,c,d) = \begin{cases} \left(\dfrac{d}{c}\right) i^{(1-c)/2} e^{\pi i(bd(1-c^2)+c(a+d))/12}, & \text{if } c \text{ is odd.} \\ \left(\dfrac{c}{d}\right) e^{\pi i(ac(1-d^2)+d(b-c+3))/12}, & \text{if } d \text{ is odd.} \end{cases}$$

where $\left(\frac{d}{c}\right)$ *is the Legendre-Jacobi symbol*

M. Newman [2] gave criteria for a product of $\eta-$function to be modular on $\Gamma_0(N)$

Proposition 3.3. *If* $f(\tau) = \prod_{d|N} \eta(d\tau)^{r_d}$ *satisfies*

(1) $\prod(d^{r_d})$ *is a square,*

(2) $\sum dr_d \equiv 0 mod\ 24$,

(3) $\sum \frac{N}{d} r_d \equiv 0\ mod\ 24$,

then $f(\tau)$ *is a modular function on* $\Gamma_0(N)$ *of weight* $\frac{1}{2}\sum r_d$.

We will use these results to find modular functions suitable for our purpose.

Finally, we discuss the properties of the action of Atkin-Lehner involutions on modular forms.

Proposition 3.4. *Let* n *be a positive integer with* $\gcd(n, N/n) = 1$. *Let* $f(\tau)$ *be a modular form of weight* k *on* $\Gamma_0(N)$. *Then* $f\big|_{[\gamma]}$ *for* $\gamma \in w_n$ *is again a modular form of weight* k *on* $\Gamma_0(N)$, *and that* $f\big|_{[\gamma_1]} = f\big|_{[\gamma_2]}$ *for all* $\gamma_1, \gamma_2 \in w_n$.*Note that* $f(\tau)\big|_{[\gamma]}$ *is defined by* $f(\tau)\big|_{[\gamma]} = (\det\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau)$

Proof. Recall that w_n normalizes $\Gamma_0(N)$. That is, $w_n\Gamma_0(N)w_n^{-1} = \Gamma_0(N)$. We have, for all $\alpha \in \Gamma_0(N)$,

$$f\big|_{[\gamma]}\big|_{[\alpha]} = f\big|_{[\alpha'\gamma]},$$

where α' is an element in $\Gamma_0(N)$. Since f is a modular form on $\Gamma_0(N)$, it follows that

$$f\big|_{[\gamma]}\big|_{[\alpha]} = f\big|_{[\gamma]}.$$

That is, $f\big|_{[\gamma]}$ is modular on $\Gamma_0(N)$. The assertion that $f\big|_{[\gamma_1]} = f\big|_{[\gamma_2]}$ for all $\gamma_1, \gamma_2 \in w_n$ follows from the fact that $\gamma_1^{-1}\gamma_2 \in \Gamma_0(N)$. This proves the proposition. □

4. **Beukers' Proof**

In [1], Beukers gave a modular form interpretation of the Apéry sequence. Now we will describe the Beukers' proof in this section.

Let $t(q) = \sum_{n=0}^{\infty} t_n q^n$ be a power series convergent for all $|q| < 1$ and $W(q)$ be another analytic function on $|q| < 1$. Then consider W as function of t. In general it will be a multivalued function over which we have no control. However, we shall introduce some assumptions. First, $t_0 = 0$, $t_1 \neq 0$. Let $q(t)$ be the local inverse of $t(q)$ with $q(0) = 0$. Choose $W(q(t))$ for the value of w around $t = 0$. Then in order to determine the radius of convergence of power series $W(q(t)) = \sum_{n=0}^{\infty} w_n t^n$ we introduce branching values of t. We say that t branches above t_0, if either t_0 is not in the image of t, or if $t'(q_0) = 0$ for some q_0 with $t(q_0) = t_0$. Now assume that t has a discrete set of branching values $t_1, t_2, \cdots$ where we have exclude zero as a possible value and suppose $|t_1| < |t_2| < \cdots$. It is clear now that the radius of convergence is in general t_1. We shall be interested in cases where the radius of convergence is larger than $|t_1|$. Let γ be a closed contour in the complex t-plane beginning and ending at the origin, not passing through any t_i and which encircles the point t_1 exactly once. Suppose that analytic continuation of $W(q(t))$ along γ again yields the same branch of $W(q(t))$. Then $W(q(t))$ can be continued analytically to the disc $|t| < |t_2|$ with exception of possible isolated singularity t_1. If $W(q(t))$ remains bound around t_1 we can conclude that the radius of convergence is at least $|t_2|$.

The construction of the function $t(q)$ and $W(q)$ will proceed using modular forms and functions. The value for which Beukers obtain irrationality results are in fact values at integral points of Dirichlet series associated to modular forms.

We recall two propositions of Beukers. For completeness we also include their proofs.

Proposition 4.1 (Beukers). *Let $f_0(t), f_1(t), \cdots, f_k(t)$ be power series in t. Suppose that for any $n \in \mathbf{N}$, $i = 0, 1 \ldots, k$ the n-th coefficient in the Taylor series of f_i is rational and has denominator dividing $d^n[1, \ldots, n]^r$ where r, d are certain fixed positive integers and $[1, \ldots, n]$ is the lowest common multiple of $1, \ldots, n$. Suppose there exist real numbers $\theta_1, \theta_2, \ldots, \theta_k$ such that $f_0(t) + \theta_1 f_1(t) + \theta_2 f_2(t) + \cdots + \theta_k f_k(t)$ has radius of convergence ρ and infinitely many nonzero Taylor coefficients. If $\rho > de^r$,then at least one of $\theta_1, \theta_2, \cdots \theta_k$ is irrational.*

Proof. Choose $\varepsilon > 0$ such that $\rho > de^{r(1+\varepsilon)}$. Let $f_i(t) = \sum_{n=0}^{\infty} a_{in} t^n$ Since the radius of convergence of $f_0(t) + \theta_1 f_1(t) + \theta_2 f_2(t) + \cdots + \theta_k f_k(t)$ is ρ, we have for sufficiently large n, $|a_{0n} + a_{1n}\theta_1 + \cdots + a_{kn}\theta_k|$. Suppose $\theta_1 \theta_2, \ldots, \theta_k$ are all rational and we have common denominator D. Then $A_n = Dd^n[1, \ldots, n]^r |a_{0n} + a_{1n}\theta_1 + \cdots + a_{kn}\theta_k|$ is an integer smaller than $Dd^n[1, \ldots, n]^r (\rho - \varepsilon)^{-n}$. By the prime number theorem we have $[1,\ldots,n] < e^{(1+\varepsilon)n}$ for sufficiently large n, hence $|A_n| < D(\frac{de^{(1+\varepsilon)r}}{\rho-\varepsilon})^n$. Since $de^{r(1+\varepsilon)}(\rho-\varepsilon)^{-1} < 1$ this implies that $A_n = 0$ for sufficiently large n, in contradiction with our assumption $A_n \neq 0$ for infinitely many n. Thus our proposition follows. $\square$

Proposition 4.2 (Beukers). *Let $F(\tau) = \sum_{n=1}^{\infty} a_n q^n$, $q = e^{2\pi i\tau}$, be a Fourier series convergent for $|q| < 1$,such that for some k,$N \in \mathbf{N}$,*

$$F(-1/N\tau) = \varepsilon(-i\tau\sqrt{N})^k F(\tau)$$

where $\varepsilon = \pm 1$. Let $f(\tau)$ be the Fourier series

$$f(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} q^n$$

Let

$$L(F,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and finally,

$$h(\tau) = f(\tau) - \sum_{0 \le r < \frac{k-2}{2}} \frac{L(F,k-r-1)}{r!}(2\pi i\tau)^r$$

Then

$$h(\tau) - D = (-1)^{k-1}\varepsilon(-i\tau\sqrt{N})^{k-2} h(-1/N\tau)$$

where D=0 if k is odd and $D = L(F,\frac{k}{2})(2\pi i\tau)^{\frac{k}{2}-1}/(\frac{k}{2}-1)!$ if k is even. Moreover, $L(F,\frac{k}{2}) = 0$ if $\varepsilon = -1$

Proof. We apply a lemma of Hecke, see [4] with $G(\tau) = \varepsilon F(\tau)/(i\sqrt{N})^k$
to obtain

$$f(\tau) - \varepsilon(-1)^{k-1}(-i\tau\sqrt{(N)})^{k-2} f(-1/N\tau) = \sum_{r=0}^{k-2} \frac{L(F,k-r-1)}{r!}(2\pi i\tau)^r$$

Split the summation on the right hand side into summation over $r < \frac{k}{2}-1$, $r > \frac{k}{2}-1$ and, possibly ,$r = \frac{k}{2}-1$.For the region $r > \frac{k}{2}-1$ we apply the functional equation

$$\frac{L(F,k-r-1)}{r!} = \varepsilon(-1)^k(-i\sqrt{N})^{k-2}(-1/N)^{k-r-2}(2\pi i)^{k-2r-2}\frac{L(F,r+1)}{(k-r-2)!}$$

and substitute r by $k-2-r$. $\square$

Having introduced these two propositions, we start to describe Beukers' proof. He first defined a modular function t on $\Gamma_0(6)+w_6$, and found the branching values

$$t(i\infty)=0, \qquad t(i/\sqrt{6})=(\sqrt{2}-1)^4, \qquad t(2/5+i/5\sqrt{6})=(\sqrt{2}+1)^4, \qquad t(1/2)=\infty$$

of t. Thus, if one writes a modular form $E(\tau)$ on $\Gamma_0(6)$ as a series of t, the series in general has a radius of convergence $(\sqrt{2}-1)^4$. Then Beukers found a modular form $F(\tau)$ of weight 4 such that the conditions in Proposition 4.2 holds with $F(-1/6\tau) = -36\tau^4F(\tau)$ (that is, $\epsilon=-1$) and $L(F,3)=\zeta(3)$. Thus, choosing $E(\tau)$ to be of weight 2 with $E(-1/6\tau) = -6\tau^2E(\tau)$ and setting $(\frac{d}{d\tau})^3 f(\tau) = (2\pi i)^3F(\tau)$, by Proposition 4.2, we see that

$$E(-1/6\tau)(f(-1/6\tau)-\zeta(3)) = E(\tau)(f(\tau)-\zeta(3))$$

From this Beukers concluded that the radius of convergence of $E(t)(f(t)-\zeta(3))$ equals at least the next branching value. He also checked that the coefficients of $E(t)\in\mathbb{Z}[t]$ and $E(t)f(t)=\sum_{n=1}^{\infty} b_nt^n$,where $b_n\in\mathbb{Z}/[1,2\ldots n]^3$. Finally, he proved that $\zeta(3)$ is irrational by applying Proposition 4.1.

Our construction of sequences converging to $\zeta(5)$ basically follows Beuker's approach. However, our result is not strong enough to conclude that $\zeta(5)$ is irrational. Here we give a weaker version of Proposition 4.1 applicable to our situation.

Proposition 4.3. *Let $f_0(t)=\sum a_nt^n$, $f_1(t)=\sum b_nt^n$ be power series in t. Suppose that θ is a real number such that $f_0(t)-\theta f_1(t)$ has radius of convergence α and*

$$\limsup_{n\to\infty} 1/(\alpha|b_n|^{1/n}) < 1,$$

then a_n/b_n converges to θ.

5. **Differential equations satisfied by modular forms**

In this section, we will give the result of [3] and introduce the the method of [5].

Theorem 2 (Stiller). *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. Suppose that $t = t(q)$ is a non-constant (meromorphic) modular function invariant under Γ, and $F(t) = F(t(q))$ is a (meromorphic) modular form of weight k on the group Γ with respect to a multiplier system χ. Then the functions $F(t)$, $\tau F(t)$, ..., $\tau^k F(t)$ are linearly independent solutions of a $(k+1)$-st order linear differential equation.*

The idea of Theorem 2 is to consider the vector-valued function

$$V(\tau) = (F(\tau), \tau F(\tau), \ldots, \tau^k F(\tau)).$$

This function behaves like a modular function (of weight 0), and so do the derivatives $d^m V/dt^m$. Thus the coefficients of the linear relation among $k+2$ vectors $d^m V/dt^m, m = 0 \ldots k+1$, are Γ-invariant, and thus are algebraic functions of t.

From the general theory of differential equations, we know that the differential equations in Theorem 2 can be expressed in terms of the Wronskians. In practice, we find the following proof of Y. Yang more suitable for the computational purpose.

Theorem 3 (Yang). *Setting*

$$G_1 = \frac{D_q t}{t}, \qquad G_2 = \frac{D_q F}{F}$$

and

$$p_1(t) = \frac{D_q G_1 - 2G_1 G_2/k}{G_1^2}, \qquad p_2(t) = -\frac{D_q G_2 - G_2^2/k}{G_1^2},$$

then the differential equations satisfied by F and t are
for $k = 1$,

$$D_t^2 F + p_1 D_t F + p_2 F = 0,$$

for $k = 2$,

$$D_t^3 F + 3p_1 D_t^2 F + (2p_1^2 + tp_1' + 2p_2) D_t F + (2p_1 p_2 + tp_2')F = 0,$$

and in general $r_m(t)$ are polynomials of t, p_1, p_2, and derivatives of p_1 and p_2.

Remark. The notation in Theorem 3 D_t and D_q are denoted by $D_t = t\frac{d}{dt}$, $D_q = t\frac{d}{dq}$

Y. Yang first prove a lemma showing that the functions p_1 and p_2 in the statement of Theorem 3 are indeed algebraic functions of t.

Lemma 5.1. *Let t, F, G_1 and G_2 be given as in Theorem 3. Then G_1 is a meromorphic modular form of weight 2, while $D_qG_1 - 2G_1G_2/k$ and $D_qG_2 - G_2^2/k$ are meromorphic modular forms of weight 4.*

Proof. Throughout the proof of the lemma we let $\dot{f}(\tau)$ denote the derivative of a function $f(\tau)$ with respect to τ.

The meromorphic property of the functions concerned is clear. We now show that the functions have the claimed modular property. Since, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$t(\gamma\tau) = t(\tau), \qquad F(\gamma\tau) = \chi(\gamma)(c\tau + d)^k F(\tau),$$

taking the logarithmic derivatives of the above equalities with respect to τ, we obtain

$$\frac{\dot{t}}{t}(\gamma\tau) = (c\tau + d)^2 \frac{\dot{t}}{t}(\tau), \qquad \frac{\dot{F}}{F}(\gamma\tau) = kc(c\tau + d) + (c\tau + d)^2 \frac{\dot{F}}{F}(\tau),$$

or equivalently,

$$G_1(\gamma\tau) = (c\tau + d)^2 G_1(\tau), \qquad G_2(\gamma\tau) = \frac{1}{2\pi i} kc(c\tau + d) + (c\tau + d)^2 G_2(\tau). \tag{7}$$

This shows that G_1 is a meromorphic modular form of weight 2.

Differentiating the expressions in (7) with respect to τ again, we obtain

$$\dot{G}_1(\gamma\tau) = 2c(c\tau + d)^3 G_1(\tau) + (c\tau + d)^4 \dot{G}_1(\tau)$$

and

$$\dot{G}_2(\gamma\tau) = \frac{1}{2\pi i} kc^2(c\tau + d)^2 + 2c(c\tau + d)^3 G_2(\tau) + (c\tau + d)^4 \dot{G}_2(\tau).$$

It follows that

$$\dot{G}_1(\gamma\tau) - \frac{4\pi i}{k} G_1(\gamma\tau) G_2(\gamma\tau) = (c\tau + d)^4 \left\{ \dot{G}_1(\tau) - \frac{4\pi i}{k} G_1(\tau) G_2(\tau) \right\}$$

and

$$\dot{G}_2(\gamma\tau) - \frac{2\pi i}{k} G_2(\gamma\tau)^2 = (c\tau + d)^4 \left\{ \dot{G}_2(\tau) - \frac{2\pi i}{k} G_2(\tau)^2 \right\}.$$

This shows that $D_qG_1 - 2G_1G_2/k$ and $D_qG_2 - G_2^2/k$ are meromorphic modular forms of weight 4, and the proof of the lemma is completed. □

We are now ready to prove our Theorem 3.

Proof of Theorem 3. With a slight abuse of notation we will alternate the use of $F(\tau)$, $F(q)$ and $F(t)$ freely. We first show that F satisfies a $(k+1)$-st order differential equation. By the definitions of D_t, D_q, G_1 and G_2 we have

$$D_tF = t\frac{D_qF}{D_qt} = t\frac{FG_2}{tG_1} = F\frac{G_2}{G_1}, \tag{8}$$

and

$$D_t\frac{G_2}{G_1} = \frac{t}{G_1}\frac{D_qG_2}{D_qt} - t\frac{G_2}{G_1^2}\frac{D_qG_1}{D_qt} = \frac{D_qG_2}{G_1^2} - \frac{G_2}{G_1^3}D_qG_1.$$

By Lemma 5.1 the functions $D_qG_1 - 2G_1G_2/k$ and $D_qG_2 - G_2^2/k$ are meromorphic modular forms of weight 4, and so is G_1^2. Therefore we can write $(D_qG_1 - 2G_1G_2/k)/G_1^2$ and $(D_qG_2 - G_2^2/k)/G_1^2$ as algebraic functions of t, say,

$$p_1(t) = \frac{D_qG_1 - 2G_1G_2/k}{G_1^2}, \qquad p_2(t) = -\frac{D_qG_2 - G_2^2/k}{G_1^2}.$$

Thus, we have

$$D_t\frac{G_2}{G_1} = -\frac{G_2^2}{kG_1^2} - p_1\frac{G_2}{G_1} - p_2. \tag{9}$$

Using (8) and (9) we can now compute higher order derivatives of F inductively. We have

$$D_t^2F = D_t\left(F\frac{G_2}{G_1}\right) = F\left\{(1-1/k)\frac{G_2^2}{G_1^2} - p_1\frac{G_2}{G_1} - p_2\right\}$$

and

$$\begin{aligned} D_t^3F &= F\frac{G_2}{G_1}\left\{(1-1/k)\frac{G_2^2}{G_1^2} - p_1\frac{G_2}{G_1} - p_2\right\} \\ &\qquad + F\left\{2(1-1/k)\frac{G_2}{G_1}D_t\frac{G_2}{G_1} - tp_1'\frac{G_2}{G_1} - p_1D_t\frac{G_2}{G_1} - tp_2'\right\} \\ &= F\left\{(1-1/k)(1-2/k)\frac{G_2^3}{G_1^3} + (3/k-3)p_1\frac{G_2^2}{G_1^2}\right. \\ &\qquad \left. + \left((2/k-3)p_2 - tp_1' + p_1^2\right)\frac{G_2}{G_1} + p_1p_2 - tp_2'\right\}. \end{aligned}$$

It follows that, for $k = 1$,

$$D_t^2F = -p_1F\frac{G_2}{G_1} - p_2F = -p_1D_tF - p_2F,$$

and, for $k = 2$,

$$\begin{aligned} D_t^3 F &= -3p_1 D_t^2 F + F\left\{(-2p_1^2 - tp_1' - 2p_2)\frac{G_2}{G_1} - 2p_1p_2 - tp_2'\right\} \\ &= -3p_1 D_t^2 F + (-2p_1^2 - tp_1' - 2p_2)D_t F - (2p_1p_2 + tp_2')F. \end{aligned}$$

In general, the n-th derivative takes the form

$$D_t^n F = F\left\{\frac{G_2^n}{G_1^n}\prod_{j=1}^{n-1}(1 - j/k) + s_{n,n-1}\frac{G_2^{n-1}}{G_1^{n-1}} + s_{n,n-2}\frac{G_2^{n-2}}{G_1^{n-2}} + \cdots\right\},$$

where $s_{n,j}$ are polynomials of t, p_1, p_2 and their derivatives. When $n = k+1$, the term involving G_2^{k+1}/G_1^{k+1} is annihilated, and we see that $D_t^{k+1}F$ is equal to a linear sum of lower order derivatives of F (with algebraic functions of t as coefficients). □

In the final section, we will apply the method of [5] to find the differential equations for $n = 5$. Moreover, we will obtain the recursive relation of sequences a_n and b_n from these differential equations.

6. **Construction of series converging to** $\zeta(5)$

Now we start to construct a sequence $\{c_n\}$ of rational numbers converging to $\zeta(5)$ in this section. According to our idea described in section 4, we have to find three functions $F(\tau)$, $E(\tau)$ and $t(\tau)$ appropriately and consider the power series $E(t)(f(t)-\zeta(5))$. Now we will divide this section into two parts. We will find the sequence c_n and prove it converges to $\zeta(5)$ in first part. Then we will apply the method of [5] to find the recursive relations in the second part.

6.1

To find suitable t, E and F, our first task is to determine the congruence subgroup that they should be modular on.

First of all, we notice that the zeta functions appear naturally in the L-function associated with an Eisenstien series. Namely, we have

$$\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{d|n}d^{k-1}=\sum_{d=1}^{\infty}\frac{1}{d^{s+1-k}}\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta(s+1-k)\zeta(s).$$

From Proposition 4.2 we see that in order to get $\zeta(5)$ we require the weight k to be 6. Furthermore, taking account that the Atkin-Lehner involution w_N identitify cusps by pairs and that we have two conditions $L(F,4)=0$, $L(F,3)=0$ in Proposition 4.2 that must be fulfilled, the congruence subgroup $\Gamma_0(N)$ must have at least 6 cusps. The smallest positive integer N with this property is 12. That is, we shall choose t, E, F to be modular on $\Gamma_0(12)$.

We first consider the E function. To make the computation of the differential equation satisfied by E easier, we choose E to be a product $\prod_{d|12}\eta(d\tau)^{e_d}$ of η-functions, where $\sum e_d=8$. Thus, if E is to satisfy $E(-1/12\tau)=\pm 12^2\tau^4E(\tau)$, then the sign must be positive. In other words, E satisfies

(1) $E(\tau)\in M_4(\Gamma_0(12))$,

(2) $E(-1/12\tau)=12^2\tau^4E(\tau)$.

Accordingly, by Proposition 3.3, the exponents e_d should satisfy

(i) $\sum e_d=8$,

(ii) $e_1=e_{12}$, $e_2=e_6$, $e_3=e_4$,

(iii) $\prod_{d|12}$ is a square,

(iv) $\sum de_d$,

(v) $E(\tau)$ is holomorphic at $\tau=1/2$ and $\tau=1/3$, Then from these conditions we find that we have the following five choices for E-function:

$$(e_1, e_2, e_3) = (-4, -4, 12) \;\; (-5, 2, 7) \;\; (-6, 8.2) \;\; (-7, 14, -3) \;\; (-8, 20, -8).$$

We now consider the possible choices of the modular form F. Let $f(\tau)$ be determined by $d^5 f(\tau)/d\tau^5$ with the constant term in the Fourier expansion being 0. In order for $f(\tau)$ to satisfy

$$E(-1/12\tau)(f(-1/12\tau) - \zeta(5)) = E(\tau)(f(\tau) - \zeta(5)),$$

by Proposition 4.2, the function F should meet the conditions

(i) $F(\tau) \in M_6(\Gamma_0(12))$, $F(i\infty) = 0$,

(ii) $F(-1/12\tau) = 12^3 \tau^6 F(\tau)$,

(iii) $L(F, 3) = L(F, 4) = 0$ and $L(F, 5) = \zeta(5)$,

where L denotes the Dirichlet series associated with $F(\tau)$. The first condition means that the q-expansion of F start from q. That is $F(q) = a_1 q + a_2 q^2 + \cdots$. Meanwhile, to fulfill condition (ii), the function F should be a linear combination of

$$f_1(\tau) = E_6(\tau) + 12^3 E_6(12\tau) \;,\;\; f_2(\tau) = E_6(2\tau) + 3^3 E_6(6\tau) \;,\;\; f_3(\tau) = 3^3 E_6(3\tau) + 2^6 E_6(4\tau)$$

where $f_i(1/12\tau) = 12^3 \tau^6 f_i(\tau)$, for $i = 1, 2, 3$. That is, we have

$$xF(\tau) = A[E_6(\tau) + 12^3 E_6(12\tau)] + B[E_6(2\tau) + 3^3 E_6(6\tau)] + C[3^3 E_6(3\tau) + 2^6 E_6(4\tau)]$$

where A, B, C, x are constants. Then from the conditions (i) and (iii) we have the following equations:

$$\begin{cases} 13A + B + 7C = 0 \\ 145A + 5B + 25C = \dfrac{4}{7}x \\ 1729A + 28B + 91C = 0 \end{cases}$$

$\Longrightarrow \;\; A = A \;,\; B = -104A \;,\; C = 13A \;,\; x = -175A/2$

where A is a constant. Thus the F function can be decided.

Finally, we consider the t function. Referring to Beukers' proof, now we want to define a t-function which is modular with respect to $\Gamma_0(12) + w_{12}$. The choice of

t-function is similarly with E-function. We also construct the t-function by Proposition 3.3 and Proposition 3.4 as follows:

$$t(\tau) = \prod_{d|12} \eta(d\tau)^{e_d}$$

where d and e_d satisfy the following conditions:

(i) $\sum e_d = o$,

(ii) $e_1 = e_{12}$, $e_2 = e_6$, $e_3 = e_4$,

(iii) $\sum de_d = 24(\mod 24)$,

(iv) t has only one simple zero at $\tau = i\infty$,

From these above conditions we choose the t-function to be

$$t = \left(\frac{\eta(\tau)\eta(12\tau)}{\eta(3\tau)\eta(4\tau)} \right)^4$$

We now determine the branching values of t, which occurs at either the elliptic points or the cusps. In other words, we need to evaluate the values of t at $\tau = 1/2, 1/3, i/\sqrt{12}$, and $(2+\sqrt{-12})/5$. We first note that t and E are modular on $\Gamma_0(12)$, which has six inequivalent cusps 1, 1/2, 1/3, 1/4 and 1/6, 1/12. Furthermore, the function field of modular functions is generated by

$$g = \left(\frac{\eta(\tau)\eta(12\tau)^3}{\eta(4\tau)\eta(3\tau)^3} \right)$$

and the value of g at cusps are given by

$$g(0) = 1/4 \quad g(1/2) = -1/2 \quad g(1/3) = \infty$$

$$g(1/4) = 1 \quad g(1/6) = 1/2 \quad g(1/12) = 0$$

The function $y(-1/12\tau)$ is again invariant on $\Gamma_0(12)$ and one easily checks that

$$y(-1/12\tau) = \frac{y(\tau) - 1/4}{y(\tau) - 1} \tag{10}$$

Moreover, the modular function t and g have the following relation

$$t(\tau) = g(\tau)\frac{1 - 4g(\tau)}{1 - g(\tau)} \tag{11}$$

That $t(i\infty) = 0$, $t(1/3) = \infty$ can been seen from the values $g(i\infty) = 0$, $g(1/3) = \infty$. From (10) it follows that for $\tau = i/\sqrt{12}$ and $g_0 = g(i/\sqrt{12})$ we have $g_0 = (g_0 - 1/4)/(g_0 - 1)$, hence $g_0 = 1 \pm \sqrt{5}/2$ and correspondingly , $t(i/\sqrt{12}) = (\sqrt{3} \pm 1)^4/4$.

On other hand, the next branching value is $t(1/6) = -1$. Therefore, we have the branching values of t as follows:

$$t(i\infty) = 0, \quad t(i/\sqrt{12}) = \frac{(\sqrt{3}-1)^4}{4}, \quad t(1/6) = -1, t((2+\sqrt{-12})/5) = \frac{(\sqrt{3}+1)^4}{4}$$

Now we are ready to give the proof of Theorem 1

Proof of Theorem 1. Let

$$\tfrac{-175}{2}F(\tau) = \left(E_6(\tau) + 12^3E_6(12\tau)\right) - 104\left(E_6(2\tau) + 27E_6(6\tau)\right) + 13\left(27E_6(3\tau) + 2^6E_6(4\tau)\right)$$

$$E(\tau) = \frac{(\eta(3\tau)\eta(4\tau))^{12}}{(\eta(2\tau)\eta(6\tau)\eta(\tau)\eta(12\tau))^4}$$

Notice that $F(\tau) \in M_6(\Gamma_0(12))$ and $F(-1/12\tau) = 12^3\tau^6F(\tau), F(i\infty) = 0$ and $E(\tau) \in M_4(\Gamma_0(12))$, $E(-1/12\tau) = 12^2\tau^4E(\tau)$. Then the Dirichlet series corresponding to $F(\tau)$ reads:

$$L(F,s) = \sum_{n=1}^{\infty}\Bigg\{\left(\frac{-504\sigma_5(n)}{n^s} + 12^3\frac{-504\sigma_5(n)}{(12n)^s}\right) - 104\left(\frac{-504\sigma_5(n)}{(2n)^s} + \frac{-504\sigma_5(n)}{(6n)^s}\right)$$
$$+13\left(27\frac{-504\sigma_5(n)}{(3n)^s} + 2^6\frac{-504\sigma_5(n)}{(4n)^s}\right)\Bigg\}$$

$$= \frac{1008}{175}\left(1 + 12^{3-s} - 104(2^{-s} + 3^36^{-s}) + 13(3^{3-s} + 2^{6-2s})\right)\zeta(s)\zeta(s-5)$$

Define $f(\tau)$ by $(\frac{d}{d\tau})^5f(\tau) = (2\pi i)^5F(\tau)$, $f(i\infty) = 0$. From Proposition 4.2 and the fact that $F(-1/12\tau) = 12^3\tau^6F(\tau)$ follows

$$12^2\tau^4[f(-1/12\tau) - L(F,5)] = [f(\tau) - L(F,5)]$$

and since $L(F,5) = \zeta(5)$, $E(-1/12\tau) = 12^2\tau^4E(\tau)$, we have

$$12^2\tau^4[f(-1/12\tau) - \zeta(5)] = [f(\tau) - \zeta(5)]$$

Multiplication with $E(-1/12\tau) = 12^2\tau^4E(\tau)$ yields

$$E(-1/12\tau)[f(-1/12\tau) - \zeta(5)] = E(\tau)[f(\tau) - \zeta(5)] \tag{12}$$

The function $E(\tau)[f(\tau) - \zeta(5)]$ can be considered as a multivalued function of $t(\tau)$. We choose it at $t = 0$ as follows. From the expansion $t = q - 4q^2 + 2q^3 + \cdots$ one infers the inverse expansion $q = t + 4t^2 + 30t^3 + \cdots$. Then, from $E(\tau) =$

$1 + 4q + 18q^2 + \cdots$ one finds $E(t) = 1 + 4t + 34t^2 + 308t^3 + \cdots$ and similarly , $E(t)f(t) = 144/25t + 1665/50t^2 + \cdots$.

Since the inverse function $t \mapsto \tau$ branches at $t = \frac{(\sqrt{3}-1)^4}{4}$ one expects the radius of convergence of $E(t)[f(t) - \zeta(5)]$ to be $\frac{(\sqrt{3}-1)^4}{4}$. However, by (12), the function $t \mapsto E(t)[f(t)-\zeta(5)]$ has no branch point at $t = \frac{(\sqrt{3}-1)^4}{4}$, and its radius of convergence equals at least the next branching value, which is -1 . Then we can conclude that

$$\limsup_{n\to\infty} \sqrt[n]{|\, b_n - a_n\zeta(5)|} \ll \frac{1}{\beta} \quad where \quad \beta = 1$$

where b_n and a_n are the coefficients of $E(t)f(t)$ and $E(t)$. On other hand, we know the coefficients a_n of $E(t)$ from the branching values. That is

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \frac{1}{\alpha} \quad where \quad \alpha = (\sqrt{3}-1)^4/4 \quad as \quad n \longrightarrow \infty$$

Thus,

$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{b_n}{a_n} - \zeta(5)\right|} \le \frac{\alpha}{\beta}$$

since $\alpha/\beta = (\sqrt{3}-1)^4/4 < 1$. Then $(\alpha/\beta)^n$ tends to 0 as $n \longrightarrow \infty$. Thus b_n/a_n converges to $\zeta(5)$. This completes the proof of Theorem 1. $\square$

6.2

In this subsection, we will find the sequences a_n and b_n by applying the method of Y.Yang. The main purpose of this part is to prove the following proposition.

Proposition 6.1. *Let*

$$t = \left(\frac{\eta(\tau)\eta(12\tau)}{\eta(3\tau)\eta(4\tau)}\right)^4 \qquad E(\tau) = \frac{(\eta(3\tau)\eta(4\tau))^{12}}{(\eta(2\tau)\eta(6\tau)\eta(\tau)\eta(12\tau))^4}$$

Then we have

$$D_t^5E + r_4(t)D_t^4E + r_3(t)D_t^3E + r_2(t)D_t^2E + r_1(t)D_tE + r_0(t)E = 0, \tag{13}$$

$$r_4(t) = 20\frac{(t^2 - 10t - 3)t}{(1+t)(t^2 - 14t + 1)}$$

$$r_3(t) = 80\frac{(2t^5 - 41t^4 + 182t^3 + 158t^2 + 12t - 1)t}{(t+1)^2(t^2 - 14t + 1)^2}$$

$$r_2(t) = 20\frac{(32t^6 - 557t^5 + 1745t^4 + 3310t^3 + 1202t^2 + 31t - 3)t}{(t+1)^3(t^2 - 14t + 1)^2}$$

$$r_1(t) = 8\frac{(160t^7 - 2343t^6 + 4246t^5 + 16963t^4 + 12980t^3 + 2743t^2 + 6t - 3)t}{(t+1)^4(t^2 - 14t + 1)^2}$$

$$r_0(t) = 4\frac{(256t^7 - 3367t^6 + 4740t^5 + 18565t^4 + 12368t^3 + 2019t^2 - 20t - 1)t}{(t+1)^4(t^2 - 14t + 1)^2}$$

If we set $E(t) = \sum_{n=0}^{\infty} a_n t^n$ which satisfy the differential equation (13), then we can easily obtain the recursive relation with the initial conditions:

$$n^5 a_n = A_7 a_{n-1} - A_6 a_{n-2} - A_5 a_{n-3} - A_4 a_{n-4} - A_3 a_{n-5} - A_2 a_{n-6} + A_1 a_{n-7} - A_0 a_{n-8}$$

$a_0 = 1$, $a_1 = 4$, $a_2 = 34$, $a_3 = 308$, $a_4 = 3083$, $a_5 = 32696$, $a_6 = 361428$ $a_7 = 4119288$. Thus we get the sequence a_n from the recursive relation. On other hand, we introduce the following proposition to find the sequence b_n.

Proposition 6.2. *Suppose that $t(\tau)$ and $E(\tau)$ are given in Proposition 6.1, and that $F(\tau)$ and $f(\tau)$ be defined as in the proof of Theorem 1. Then $A(\tau) = E(\tau)f(\tau)$ satisfies the inhomogeneous differential equations*

$$D_t^5 A + r_4(t) D_t^4 A + r_3(t) D_t^3 A + r_2(t) D_t^2 A + r_1(t) D_t A + r_0(t) A = H(t) \tag{14}$$

where

$$H(t) = \frac{144}{25}t - \frac{9504}{25}t^2 + \frac{3744}{25}t^4 + \frac{5616}{25}t^5$$

and $G_1 = D_q t/t$. Moreover, if we set $E(t)f(t) = \sum_{n=0}^{\infty} b_n t^n$. Then b_n satisfy the recursive relation given in the statement of Theorem 1 with the initial values $b_0 = 0 \quad b_1 = 144/25 \quad b_2 = 333/10$

$b_3 = 217042/675 \quad b_4 = 138004123/43200 \quad b_5 = 1144320384083/33750000$
$b_6 = 25297127932859/67500000 \quad b_7 = 2422896637170749569/567236250000.$

Before we prove the proposition 6.1 and 6.2, we introduce the formula for the number of zeros of modular form with respect to $\Gamma_0(12)$. This result will be used in the proof of Proposition 6.1.

Lemma 6.1. *Let f be a nonzero modular function of weight k for $\Gamma_0(12)$. For $p \in \mathbb{H}$, let $v_p(f)$ denote the order of zero of $f(\tau)$ at the point p. Let $v_\infty(f)$ denote the index of the first nonvanishing term in the $q-$expansion of $f(\tau)$. Then we have the formula*

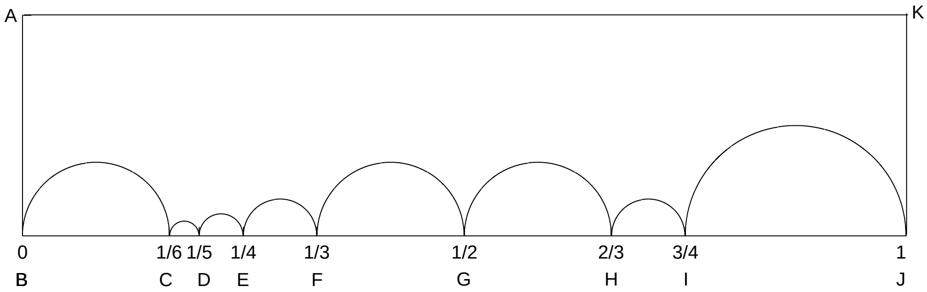

FIGURE 2. Fundamental domain of $\Gamma_0(12)$

$$v_\infty(f) + \sum_{p \in \Gamma_0(12)\backslash\mathbb{H}} v_p(f) = 2k \tag{15}$$

Proof. The idea of the proof is to count the zeros and poles in $\Gamma_0(12)\backslash\mathbb{H}$ by integrating the logarithmic derivative of $f(\tau)$ around the boundary of the fundamental domain F. More precisely, let L be the contour in Figure 2. The top of L is a horizontal line from $K = 1 + iT$ to $A = 0 + iT$, where T is taken larger than the imaginary part of any of the zeros or poles of $f(\tau)$. The rest contour follows around the boundary of F, except that it detours around any zeros or poles on the boundary along circular arcs of small radius ε.

According to the residue theorem, we have

$$\frac{1}{2\pi i}\int_L \frac{f'(\tau)}{f(\tau)}d\tau = \sum_{p \in \Gamma\backslash\mathbb{H}} v_p(f) \tag{16}$$

On the other hand, we evaluate the integral in (16) section by section.

First of all, the integral from A to B cancels the integral from J to K. Next, we evaluate the integral over KA. To do this we make the change of variables $q = e^{2\pi i\tau}$. Let $\tilde{f}(q) = f(\tau) = \sum a_n q^n$ be the q-expansion. Since $f'(\tau) = \frac{d}{dq}\tilde{f}(q)\frac{dq}{d\tau}$, we find that this section of the integral in (15) is equal to the following integral over the circle of radius $e^{-2\pi T}$ centered at zero :

$$\frac{1}{2\pi i}\int \frac{d\tilde{f}/dq}{\tilde{f}(q)}dq.$$

Since the circle is traversed in a clockwise direction as τ goes from K to A, it follows that this integral is minus the order of zero or pole of $\tilde{f}(q)$ at 0, and this is what we mean by $-v_\infty(f)$.

Finally, we consider the integral from B to J. Since $f(\tau)$ is a nonzero modular function of weight k for $\Gamma_0(12)$, we have

$$f(A(\tau)) = (c\tau + d)^k f(\tau),$$

where $A(\tau) = \frac{a\tau+b}{c\tau+d}$. Differentiation of this equation gives us

$$f'(A(\tau))(A'(\tau)) = (c\tau + d)^k f'(\tau) + kc(c\tau + d)^{k-1} f(\tau)$$

From this we find

$$\frac{f'(A(\tau))(A'(\tau))}{f(A(\tau))} = \frac{f'(\tau)}{f(\tau)} + \frac{kc}{c\tau + d}$$

Consequently, for any path γ not through a zero we have

$$\frac{1}{2\pi i}\int_{A(\gamma)} \frac{f'(u)}{f(u)} du = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(\tau)}{f(\tau)} d\tau + \frac{1}{2\pi i}\int_{\gamma} \frac{kc}{c\tau + d} d\tau \tag{17}$$

Therefore the integrals of semicircles along the B to J in Figure do not cancel unless $k = 0$. Now there are eight semicircles between B and J. We note that the transformation $\gamma_1 = \begin{pmatrix} 7 & -1 \\ 36 & -5 \end{pmatrix}$ takes BC to DC, i.e., $\gamma_1\tau$ goes from D to C along the contour as τ goes from B to C along the contour. Similarly, we can find that $\gamma_2 = \begin{pmatrix} -19 & 4 \\ -24 & 5 \end{pmatrix}$ takes DE to JI, $\gamma_3 = \begin{pmatrix} -17 & 5 \\ -24 & 7 \end{pmatrix}$ takes EF to IH and $\gamma_4 = \begin{pmatrix} -7 & 3 \\ -12 & 5 \end{pmatrix}$ takes FG to HG. So the integral from B to J can be evaluated as follows:

$$\frac{1}{2\pi i}\int_{BJ} \frac{f'(\tau)}{f(\tau)} = \frac{1}{2\pi i}\left\{ \int_{BC} + \int_{CD} + \int_{DE} + \int_{DE} + \int_{FG} + \int_{GH} + \int_{HI} + \int_{IJ} \right\}$$

$$= \frac{1}{2\pi i}\left\{ \int_{BC} - \int_{\gamma_1(BC)} + \int_{DE} - \int_{\gamma_2(DE)} + \int_{FG} - \int_{\gamma_3(FG)} + \int_{HI} - \int_{\gamma_4(HI)} \right\}$$

$$= \frac{1}{2\pi i}\left\{ \int_{BC} \frac{36k}{36\tau - 5} d\tau + \int_{DE} \frac{-24k}{-24\tau + 5} d\tau + \int_{FG} \frac{-12k}{-12\tau + 5} d\tau + \int_{EF} \frac{-24k}{-24\tau + 7} d\tau \right\}$$

Therefore, we have the following conclusion:

$$\frac{1}{2\pi i}\int_{BJ} \frac{f'(\tau)}{f(\tau)} = 2k$$

This completes the proof. □

Proof of Proposition 6.1. Referring to the proof of Y.Yang, by using (8) and (9) we can now compute higher order derivatives of F inductively. We have

$$D_t^2 F = D_t\left(F\frac{G_2}{G_1}\right) = F\left\{(1-1/k)\frac{G_2^2}{G_1^2} - p_1\frac{G_2}{G_1} - p_2\right\}$$

and

$$\begin{aligned} D_t^3 F &= F\frac{G_2}{G_1}\left\{(1-1/k)\frac{G_2^2}{G_1^2} - p_1\frac{G_2}{G_1} - p_2\right\} \\ &\quad + F\left\{2(1-1/k)\frac{G_2}{G_1}D_t\frac{G_2}{G_1} - tp_1'\frac{G_2}{G_1} - p_1 D_t\frac{G_2}{G_1} - tp_2'\right\} \\ &= F\Bigg\{(1-1/k)(1-2/k)\frac{G_2^3}{G_1^3} + (3/k-3)p_1\frac{G_2^2}{G_1^2} \\ &\quad + \left((2/k-3)p_2 - tp_1' + p_1^2\right)\frac{G_2}{G_1} + p_1p_2 - tp_2'\Bigg\}. \\ D_t^4 F &= F\Bigg\{(1-1/k)(1-2/k)(1-3/k)\frac{G_2^4}{G_1^4} + 6p_1(1-1/k)(1-2/k)\frac{G_2^3}{G_1^3} + \\ &\quad (7p_1 - 6p_2 - 4p_1't + (-7p_1^2 + 14p_2 + 4p_1't)/k - 8p_2/k^2)\frac{G_2^2}{G_1^2} - \\ &\quad (p_1^3 - 10p_1p_2 + p_1't + p_1^{(2)}t^2 + 4p_2't - 3p_1'tp_1 + (8p_1p_2 - 2p_2't)/k)\frac{G_2}{G_1} - \\ &\quad (p_2't + p_2^{(2)}t^2 + p_1^2p_2 - p_1p_2't - 2p_1'tp_2 - 3p_1^2 + (2p_2^2)/k)\Bigg\}. \end{aligned}$$

$$
\begin{aligned}
D_t^5F = F\Big\{ & -p_1'tG_2G_1 - 9/2p_2't\frac{G_2}{G_1} - 15/4p_1't - 3p_2^{(2)}t^2 + p_1p_2't + 3p_1'tp_2+ \\
& p_1^{(4)}\frac{G_2}{G_1} - 8p_1p_2^2 - 15/16p_1\frac{G_2^4}{G_1^4} - 15/4p_1't\frac{G2}{G1} - 15/8p_2\frac{G_2^3}{G_1^3} + p_1^3p_2+ \\
& 75/8p_1^2\frac{G_2^3}{G_1^3} - p_2^{(3)}t^3 + 17/2p_2'tp_2 - 45/4p_1^3\frac{G_2^2}{G_1^2} + 39/2p_1^2p_2\frac{G_2}{G_1} - p_1p_2't+ \\
& 17/2p_2^2\frac{G_2}{G_1} - p_2't - p_2't - 3p_1^{(2)}t^2\frac{G_2}{G_1} + 4p_1'tp_1\frac{G_2}{G_1} - 45/8p_2't\frac{G_2^2}{G_1^2}- \\
& 9/2p_2^{(2)}t^2\frac{G_2}{G_1} - 15/4p_1^{(2)}t^2\frac{G_2^2}{G_1^2} + 75/4p_1p_2\frac{G_2^2}{G_1^2} + 16p_1'tp_2\frac{G_2}{G_1} + 25/2p_1p_2't\frac{G_2}{G_1}+ \\
& 75/4p_1'tp_1\frac{G_2^2}{G_1^2} + 3p_1't^2p_2' + p_1p_2^{(2)}t^2 + 3p_1^{(2)}t^2p_2 - p_1(3)t^3\frac{G_2}{G_1} + 4p_1^{(2)}t^2p_1\frac{G_2}{G_1}+ \\
& 3(p_1')^2t^2\frac{G_2}{G_1} - 6p_1'tp_1^2\frac{G_2}{G_1} - 5p_1'tp_1p_2\Big\} \\
= -10p1D_t^4F & - (5p_2 + 35p_1^2 + 10p_1't)D_t^3F - (5p_1't + 50p_1^3 + 15/2p_2't + 5p_1^{(2)}t^2+ \\
& 45p_1'tp_1 + 30p_1p_2)D_t^2F - (p_1't + 4p_2^2 + 7p_1{\prime}^2t^2 + 46p_1'tp_1^2 + 11p_1^{(2)}t^2p_1+ \\
& 24p_1^4 + p_1^{(3)}t^3 + 9/2p_2^{(2)}t^2 + 30p_1p_2't + 3p_1^{(2)}t^2 + 14p_1'tp_2 + 52p_1^2p_2 + 11p_1'tp_1+ \\
& 9/2p_2't)D_tF - (24p_1^3p_2 + 4p_2'tp_2 + p_2^{(3)}t^3 + 7p_1't^2p_2' + 9p_1p_2^{(2)}t^2 + 3p_2^{(2)}t^2+ \\
& 20p_1'tp_1p_2 + 2p_1^{(2)}t^2p_2 + 26p_1^2p_2't + 9p_1p_2't + 2p_1'tp_2 + p_2't + 8p_1p_2^2)
\end{aligned}
$$

Then we find that D_t^5F can be represented as follows:

$$
D_t^5F + r_4(t)D_t^4F + r_3(t)D_t^3F + r_2(t)D_t^2F + r_1(t)D_tF + r_0(t)F = 0, \tag{18}
$$

where $r_m(t)$ are polynomials of t, p_1, p_2 and their derivatives.

Next we want to express the modular functions $p_1 = (D_qG_1 - G_1G_2)/G_1^2$ and $p_2 = (D_qG_2 - G_2^2)/G_1^2$ as a rational functions of t. By Lemam5.1 we know that G_1 is a meromorphic modular form of weight 2, so G_1^2 is a modular form of weight 4 on $\Gamma_0(12)$. On the other hand, by Lemma 6.1 we know that a modular form of weight 4 on $\Gamma_0(12)$ has eight zeros and by (11), the expression for $(D_qG_1 - G_1G_2)/G_1^2$ in terms of t takes the form $(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4)/(b_0 + b_1t + b_2t^2 + b_3t^3 + b_4t^4)$. Comparing the first terms of $(D_qG_1 - G_1G_2)/G_1^2$ with $(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4)/(b_0 + b_1t + b_2t^2 + b_3t^3 + b_4t^4)$, we obtain p_1 as follows:

$$
p_1(t) = \frac{2t^4 - 18t^3 - 26t^2 - 6t}{(t^2 - 14t + 1)(t + 1)^2}
$$

Similarly, we have

$$p_2(t) = \frac{4t^4 - 28t^3 - 36t^2 - 4t}{(t^2 - 14t + 1)(t+1)^2}$$

Finally, from (18) we complete the proof. □

Proof of Proposition 6.2. Recall the the method of variation goes as follows. If u_1, u_2, u_3, u_4, u_5 are solutions of

$$D_t^5 A + r_4(t) D_t^4 A + r_3(t) D_t^3 A + r_2(t) D_t^2 A + r_1(t) D_t A + r_0(t) A = 0$$

then a solution v of

$$D_t^5 A + r_4(t) D_t^4 A + r_3(t) D_t^3 A + r_2(t) D_t^2 A + r_1(t) D_t A + r_0(t) A = H(t)$$

can be solved by assuming $v = p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 + p_5 u_5$ with

$$\begin{cases} (D_t p_1) u_1 + (D_t p_2) u_2 + (D_t p_3) u_3 + (D_t p_4) u_4 + (D_t p_5) u_5 = 0 \\ D_t p_1 D_t u_1 + D_t p_2 D_t u_2 + D_t p_3 D_t u_3 + D_t p_4 D_t u_4 + D_t p_5 D_t u_5 = 0 \\ D_t p_1 D_t^2 u_1 + D_t p_2 D_t^2 u_2 + D_t p_3 D_t^2 u_3 + D_t p_4 D_t^2 u_4 + D_t p_5 D_t^2 u_5 = 0 \\ D_t p_1 D_t^3 u_1 + D_t p_2 D_t^3 u_2 + D_t p_3 D_t^3 u_3 + D_t p_4 D_t^3 u_4 + D_t p_5 D_t^3 u_5 = 0 \\ D_t p_1 D_t^4 u_1 + D_t p_2 D_t^4 u_2 + D_t p_3 D_t^4 u_3 + D_t p_4 D_t^4 u_4 + D_t p_5 D_t^4 u_5 = H(t). \end{cases}$$

and then solving $D_t p_1$, $D_t p_2$, $D_t p_3$, $D_t p_4$, $D_t p_5$. Now, by Stiller's theorem, we have $u_1 = E$, $u_2 = \tau E$, $u_3 = \tau^2 E$, $u_4 = \tau^3 E$, $u_5 = \tau^4 E$. Thus, using the definition of G_1 and G_2, it follows that $D_t u_1 = E G_2 / G_1$, $D_t u_2 = E(1 + \tau G_2)/G_1$, $D_t u_3 = \tau E(2 + \tau G_2)/G_1$, $D_t u_4 = \tau^2 E(3 + \tau G_2)/G_1$,$D_t u_5 = \tau^3 E(4 + \tau G_2)/G_1$, where $G_2 = D_q E/E$. The higher derivatives can be computed analogously. At the end, we find

$$D_t p_1 = \frac{H\tau^4 G_1^4}{24E}, \; D_t p_2 = \frac{-H\tau^3 G_1^4}{6E}, \; D_t p_3 = \frac{H\tau^2 G_1^4}{4E}, \; D_t p_4 = \frac{-H\tau G_1^4}{6E}, \; D_t p_5 = \frac{H G_1^4}{24E}$$

Now let

$$D_t^5 A + r_4(t) D_t^4 A + r_3(t) D_t^3 A + r_2(t) D_t^2 A + r_1(t) D_t A + r_0(t) A = H(t)$$

be a differential equation satisfied by Ef. Reversing the procedure above, we have

$$p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 + p_5 u_5 = Ef$$

Then

$$p_1 + \tau p_2 + \tau^2 p_3 + \tau^3 p_4 + \tau^4 p_5 = f = (2\pi i)^5 \int_\tau \int_\tau \int_\tau \int_\tau \int_\tau F$$

$$D_t p_1 + \tau D_t p_2 + \tau^2 D_t p_3 + \tau^3 D_t p_4 + \tau^4 D_t p_5$$

$$+ p_2/G_1 + 2\tau p_3/G_1 + 3\tau^2 p_4/G_1 + 4\tau^3 p_5/G_1 = -\frac{1}{G_1}\int_\tau\int_\tau\int_\tau\int_\tau F$$

Then

$$\Longrightarrow p_2 + 2\tau p_3 + 3\tau^2 p_4 + 4\tau^3 p_5 = -\int_\tau\int_\tau\int_\tau\int_\tau F$$

$$D_t p_2 + 2\tau D_t p_3 + 3\tau^3 D_t p_4 + 4\tau^3 D_t p_5$$

$$+ 2p_3/G_1 + 6\tau p_4/G_1 + 12\tau^2 p_5/G_1 = \frac{1}{G_1}\int_\tau\int_\tau\int_\tau F$$

$$\Longrightarrow 2p_3 + 6\tau p_4 + 12\tau^2 p_5 = \int_\tau\int_\tau\int_\tau F$$

$$2D_t p_3 + 6\tau D_t p_4 + 12\tau^2 D_t p_5 + 6p_4/G_1 + 24\tau p_5/G_1 = -\frac{1}{G_1}\int_\tau\int_\tau F$$

$$\Longrightarrow 6p_4 + 24\tau p_5 = -\int_\tau\int_\tau F$$

$$6D_t p_4 + 24\tau D_t p_5 + 24p_5/G_1 = \frac{1}{G_1}\int_\tau\int_\tau F$$

$$\Longrightarrow 24p_5 = \int_\tau F$$

$$24D_t p_5 = \frac{HG_1^4}{E} = \frac{-1}{G_1}F$$

Thus

$$H(t) = -EF/G_1^5 = \frac{144}{25}t - \frac{9504}{25}t^2 + \frac{3744}{25}t^4 + \frac{5616}{25}t^5.$$

where the expression of H in terms of t is determined in the same way as in Proposition 6.1. This completes the proof of Proposition 6.2. □

References

[1] F. Beukers. Irrationality proofs using modular forms. *Astérisque*, (147-148):271–283, 345, 1987. Journées arithmétiques de Besançon (Besançon, 1985).

[2] Morris Newman. Construction and application of a class of modular functions. *Proc. London. Math. Soc. (3)*, 7:334–350, 1957.

[3] P. Stiller. Special values of Dirichlet series, monodromy, and the periods of automorphic forms. *Mem. Amer. Math. Soc.*, 49(299):iv+116, 1984.

[4] André Weil. Remarks on Hecke's lemma and its use. In *Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976)*, pages 267–274. Japan Soc. Promotion Sci., Tokyo, 1977.

[5] Yifan Yang. On differential equations satisfied by modular forms. *Math. Z.*, 246(1-2):1–19, 2004.