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碩　士　論　文

細胞類神經網路:缺陷花樣與穩定性

Cellular Neural Networks : Defect Patterns And Stability

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中 華 民 國 九 十 四 年 六 月

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## 摘　　　要

在這篇論文中，探討的是一維細胞類神經網路在其輸出函數是片段線性輸出函數，且此函數在線性區域以外的斜率 $r>0$，我們在最鄰近的細胞之間採用一組對稱的耦合，用兩個參數來描述細胞本身與最鄰近細胞間各自的權數。在這些條件下我們研究存在穩定缺陷平衡的花樣(參閱定義 1.1 和定義 1.2)。特別地，我們給予一個無窮維觀點的 Gerschgorin 定理並且導出一個 $\delta$-extendability 的概念來決定兩個局部花樣是否可以接合在一起。使用這些工具方法，我們給定一個在 $(r,\ a,\ \beta)$ 空間的區域，其相對應的缺陷花樣擁有非零的空間熵而其相關聯的馬賽克花樣的空間熵卻為零。更有甚者，在那些區域所產生的花樣並不是有限型式的子替換。

**Cellular Neural Networks:**
**Defect Patterns And Stability**

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## ABSTRACT

Of concern is one-dimensional Cellular Neural Networks (CNNs) with a piecewise-linear output function for which the slope of the output outside linear zone is $r > 0$. We impose a symmetric coupling between the nearest neighbors. Two parameters $a$ and $\beta$ are used to describe the weights between the cell with itself and its nearest neighbors, respectively. We study patterns that exist as stable defect equilibrium (see Definition 1.1 and 1.2). In particular, we given an infinite-dimensional version of Gerschgorin's Theorem and derive a concept of $\delta$-extendability to determine whether two local-defect patterns can be glued together. Using such tools, we give a region in $(\alpha, a, \beta)$-space for which the corresponding defect patterns have non-zero spatial entropy. Moreover, the patterns generated in those regions are not subshift of finite type.

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# 目 錄

# 1 Introduction

Of concern is one-dimensional Cellular Neural Networks (CNNs) of the form

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}), \quad i \in \mathbb{Z}. \tag{1.1a}$$

Here $x_i$ denotes the state of a cell $C_i$, and $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1\,, \\ x\,, & \text{if } |x| \leq 1, \\ rx - 1 + r, & \text{if } x \leq -1\,, \end{cases} \tag{1.1b}$$

where $r$ is a positive constant. The quantity $z$ is called a source term or a bias term. The numbers $\alpha$, $a$ and $\beta$ are arranged in a vector form $[\alpha,\, a,\, \beta]$, which is called a space-invariant $A$-template

$$A = [\alpha,\ a,\ \beta]. \tag{1.2}$$

$A$ is called symmetric (resp., antisymmetric) if $\alpha = \beta$ (resp., $\alpha = -\beta$). A cell $C_i$ such that $-1 < x_i < 1$ will be called a linear cell. If it does not operate in the linear zone, i.e, $|x_i| > 1$, then it will be called a saturated cell.

CNNs were first proposed by Chua and Yang [1988a, 1988b]. Their main applications are in image processing and pattern recognition [Chua, 1998]. For additional background information, applications, and theory, see [Special Issue, 1995; Thiran, 1997; Chua, 1998] among others.

A basic and important class of solutions of (1.1) is the stable stationary solutions. Specifically, a stationary solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ of (1.1) satisfies the following equation

$$x_i = z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}), \quad i \in \mathbb{Z}. \tag{1.3}$$

Let $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ be a solution of (1.3). The associated output $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} = (f(x_i))_{i\in\mathbb{Z}}$ is called a pattern. The following two types of stationary solutions are of particular interest.

**Definition 1.1.** *A solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ is called a mosaic solution if $|x_i| > 1$ for all $i \in \mathbb{Z}$. Its associated pattern $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} = (f(x_i))_{i\in\mathbb{Z}}$ is called a mosaic pattern. If $|x_i| \neq 1$ for all $i \in \mathbb{Z}$ and there are $i,\ j \in \mathbb{Z}$ such that $|x_i| < 1$ and $|x_j| > 1$, then $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ and $\mathbf{y} = (f(x_i))_{i\in\mathbb{Z}}$ are called, respectively, a defect solution and a defect pattern. If there exists an $i$ such that $|x_i| = 1$, then $\mathbf{x}$ and $\mathbf{y} = (f(x_i))$ are called, respectively, a transition solution and a transition pattern.*

To define the stability of a non-transition stationary solution, we consider the following linearized stability. Let $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^2$, the linearized operator $\mathcal{L}(\mathbf{x})$ of (1.1) at a stationary solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ is given by

$$(\mathcal{L}(\mathbf{x})\xi)_i = -\xi_i + \alpha f'(x_{i-1})\xi_{i-1} + af'(x_i)\xi_i + \beta f'(x_{i+1})\xi_{i+1}. \tag{1.4}$$

**Definition 1.2.** *Let* $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ *be a solution of* (1.3) *with* $|x_i| \neq 1$ *for all* $i \in \mathbb{Z}$*. The stationary solution* $\mathbf{x}$ *is called* (*linearized*) *stable if all eigenvalues of* $\mathcal{L}(\mathbf{x})$ *have negative real parts. The solution is called unstable if there is an eigenvalue* $\lambda$ *of* $\mathcal{L}(\mathbf{x})$ *such that* $\lambda$ *has a positive real part.*

For $r = 0$, the complexity of stable mosaic and defect solutions of (1.1) with respect to all the parameters has been completely characterized when the template $A$ is symmetric or antisymmetric (see [Thiran et. al., 1995; Juang and Lin, 2000]). For $r > 0$, the complexity of stable mosaic solutions with respect to the parameters' regions has also been studied by methods of map approach and geometric approach(see e.g.,[Hsu, 2000; Chang and Juang, 2004; Lie, Liu and Juang, 2005]). In the case that $r = 0$, the explicit formula for the eigenvalues of the linearized operator $\mathcal{L}$ at a feasible stable defect solutions of (1.1a) can be obtained. This, in turn, gives a sharp conditions on the parameters for the stability problem. Due to the non-flatness of the output function at infinity, it is a nontrivial problem to obtain the explicit formulas for the eigenvalues of the linearized operator $\mathcal{L}$. To overcome such difficulty, we give an infinite-dimensional version of Gerschgorin's Theorem. We also derive a concept of $\delta$-extendability to determine whether two local-defect patterns can be glued together. Using such tools, we give a region in ($r$ , $a$ , $\beta$)-space for which the corresponding defect patterns have non-zero spatial entropy, while the associated mosaic patterns have zero spatial entropy. Moreover, the patterns generated in those regions are not subshift of finite type. The (stable) mosaic patterns for spatially discrete Reaction-Diffusion equations have also been extensively studied by many authors (see e.g., [3], [7]).

The thesis is organized as follows. In Section 2, we study the stability of a defect solution of (1.1a). Section 3 is devoted to the establishment of the $\delta$-extendable local solutions. The main results are recorded in Section 4. In particular we give a region in $(r, a, \beta)$-space for which the corresponding defect patterns have non-zero spatial entropy.

## 2 Stability

In this section, we study the stability of a defect solution of (1.1). To this end, we need to establish an infinite-dimensional version of Gerschgorin's Theorem. Let the operator $\overline{\mathcal{L}}: \ \ell^2 \to \ell^2$ be defined as

$$(\overline{\mathcal{L}}(\xi))_i = \sum_{k=-m}^{m} a_{i-k,i}\xi_{i-k}\,, \quad i \in \mathbb{Z}. \tag{2.1}$$

where $m$ is a fixed positive integer, $\xi = (\xi_i) \in \ell^2$.

**Theorem 2.1.** *Let $\overline{\mathcal{L}}$ be defined as in* (2.1)*, and let $\Lambda_j = \{i \in \mathbb{Z} : a_{i-k,i} = a_{j-k,j}\,,$ for all $k$, $-m \le k \le m\}$. Set*

$$\mathcal{K} = \{j \in \mathbb{Z} : \Lambda_j \textit{ is nonempty}\}, \tag{2.2a}$$

*and assume $\mathcal{K}$ is a minimal set which satisfies the following sense.*

$$if\ j_1 \textit{ and } j_2 \in \mathcal{K}, \textit{ then } \Lambda_{j_1} \textit{ and } \Lambda_{j_2} \textit{ are distinct.} \tag{2.2b}$$

*We further assume that*

$$\text{(i)}\ \ \mathcal{K}\ is\ finite, \tag{2.3a}$$

$$\text{(ii)}\ \ \bigcup_{j\in\mathcal{K}} \Lambda_j = \mathbb{Z}. \tag{2.3b}$$

*Define $\rho_i(\overline{\mathcal{L}}) = \sum_{k=-m}^{m} |a_{i-k,i}|$, for $i \in \mathcal{K}$, and let each of $C_i$ define the circle centered at $a_{i,i}$ with radius $r_i := \rho_i(\overline{\mathcal{L}}) - |a_{i,i}|$. Then each eigenvalue of $\overline{\mathcal{L}}$ lies in some $C_i$.*

*Proof.* Let $\lambda$ lies outside all of $C_i$ . Write

$$(\lambda I - \overline{\mathcal{L}}) = D - (D - (\lambda I - \overline{\mathcal{L}})) := D - K$$

where $D : \ell^2 \to \ell^2$ is the diagonal part of $\overline{\mathcal{L}}$ defined as

$$(D(\xi))_i = (\lambda - a_{i,i})\xi_i.$$

Clearly, $D$ is invertible. Thus, $D - K = D(I - D^{-1}K)$. To complete the proof of the theorem it suffices to prove that $\|D^{-1}K\| < 1$ (see e.g., Theorem 7.3-1 of [12]). To this end, we see that

$$(D^{-1}K(\xi))_i = \frac{1}{\lambda - a_{i,i}} \sum_{\substack{k=-m \\ k\neq 0}}^{m} a_{i-k,i}\xi_{i-k}$$

Clearly,

$$|(D^{-1}K(\xi))_i| \leq \|\xi\|_\infty \max_{j\in\mathcal{K}}(\frac{1}{|\lambda - a_{j,j}|}\sum_{\substack{k=m \\ k\neq 0}}^{m}|a_{j-k,j}|) \leq \|\xi\|_\infty \max_{j\in\mathcal{K}}(\frac{r_j}{|\lambda - a_{j,j}|}).$$

Therefore,

$$\|D^{-1}K\|_\infty \leq \max_{j\in\mathcal{K}} \frac{r_j}{|\lambda - a_{j,j}|} < 1.$$

Note that the assertion in that Theorem is independent of the norm. We, thus complete the proof of the theorem. □

**Corollary 2.1.** *Let* $\mathbf{x} = (x_i)$ *be a defect solution of* (1.1a) *Let* $0 < r < 1$ *and* $a > 0$. *Assume*

$$1 > a + |\alpha| + |\beta|. \tag{2.4}$$

*Then* $\mathbf{x}$ *is stable.*

*Proof.* We first note that the number of the elements in $\mathcal{K}$, as defined in (2.2), with respect to the linearized operator $\mathcal{L}(\mathbf{x})$ in (1.4) is eight. Specifically, if $i \in \mathcal{K}$, then its corresponding interaction three-tuple $(a_{i-1,i}\,, a_{i,i}\,, a_{i+1,i})$ is one of the following eight combinations $(\alpha, -1+a, \beta)$, $(\alpha r, -1+a, \beta r)$, $(\alpha r, -1+a, \beta)$, $(\alpha, -1+a, \beta r)$, $(\alpha r, -1+ar, \beta)$, $(\alpha, -1+ar, \beta r)$, $(\alpha, -1+ar, \beta)$, and $(\alpha r, -1+ar, \beta r)$. It is then easy to see that if the circle $C$, centered at $-1+a$ with radius $|\alpha| + |\beta|$, lies in the open left-half plane, then $\mathbf{x} = (x_i)$ is stable. However, the assumption (2.4) insures the circle $C$ with such property. We just complete the proof of the corollary. □

Unlike the case that $r = 0$, it is a nontrivial problem to obtain the necessary and sufficient condition for a stationary solution being stable with $r > 0$. Moreover, the difficulty in computing the existence of defect solutions is also increasing in the case of $r > 0$. We thus restrict ourselves with the patterns satisfying (i) and (ii) or (i)$'$ and (ii) of the following:

(i) Any linear cell is only adjacent to saturated cell. (2.5a)

(i)$'$ Any linear cell is adjacent to exactly one linear cell. (2.5b)

(ii) Any saturated cell is adjacent to at least one saturated cell of the same sign. In other words, whenever $y_i > 1$, then $y_{i-1} > 1$ or $y_{i+1} > 1$. (2.5c)

**Corollary 2.2.** *Let* $0 < r < 1$ *and* $a > 0$*. Assume*

$$1 > a + (|\alpha| + |\beta|)r, \tag{2.6a}$$

$$1 > ar + |\alpha| + |\beta|r, \tag{2.6b}$$

*and*

$$1 > ar + |\alpha|r + |\beta|. \tag{2.6c}$$

*Then any non-transitional solution of* (1.1a) *satisfying* (2.5a) and (2.5c) *is stable.*

*Proof.* If $i \in \mathcal{K}$, then the corresponding interaction three-tuple $(a_{i-1,i}, a_{i,i}, a_{i+1,i})$ is one of the following four combinations : $(\alpha r, -1 + a, \beta r)$, $(\alpha, -1 + ar, \beta r)$, $(\alpha r, -1 + ar, \beta)$, and $(\alpha r, -1 + ar, \beta r)$. It then follows from(2.6) and Theorem 2.1 that the assertion of corollary holds. □

**Corollary 2.3.** *Let* $0 < r < 1$ *and* $a > 0$*. Assume*

$$1 > a + |\alpha|r + |\beta|, \tag{2.7a}$$

$$1 > a + |\alpha| + |\beta|r, \tag{2.7b}$$

*and*

$$1 > ar + |\alpha| + |\beta|r, \tag{2.7c}$$

*Then any non-transitional solution of (1.1a) satisfying (2.5b) and(2.5c) is stable.*

For a defect pattern containing a string of three consecutive linear cells, it is clear, via Theorem 2.1, that (2.4) is needed to get stability of such defect pattern. As one will see, via Proposition3.1, that if (2.4) holds, then no local mosaic patterns of the form $+++$ and $---$ (see Notations 3.1 and 3.2) will exist. However, the existence of such local mosaic patterns is vital to exhibit the complexity of the patterns. Thus, we will not consider a stable pattern containing a string of more than 2 linear cells in the thesis.

# 3 Extendable Local Patterns

To study the stationary solutions of (1.1a), we start out with the so called local solutions. To begin, we define the following definitions and notations.

**Definition 3.1.** *Given any proper subset* $T \subset \mathbb{Z}$, $\mathbf{x}_T$ *is called a local solution if* $\mathbf{x}_T$ *is a restriction of some solution* $\mathbf{x} = (x_i)$ *of* (1.3) *on* $T$. *The corresponding output* $\mathbf{y}_T \equiv (f(x_i))_T$ *is called a local pattern. If in addition,* $|x_i| < 1$ *for some* $i \in T$ *(resp.,* $|x_i| > 1$ *for all* $i \in T$*), then* $\mathbf{x}_T$ *is called a local defect (resp., mosaic) solution, We define local mosaic and defect patterns accordingly.*

**Notation 3.1.** *For any solution pattern* $\{y_i = f(x_i)\}_{i\in\mathbb{Z}}$, *a cell* $y_i$ *is represented by* $+_i$, $\times_i$, *and* $-_i$ *if* $y_i = f(x_i) > 1$, $|y_i| = |f(x_i)| < 1$, *and* $y_i = f(x_i) < -1$, *respectively.*

**Notation 3.2.** *Let* $T = \{i, i+1, \cdots, i+m\}$, *a local pattern* $y_T$ *is denoted by*

$$\bullet_i \bullet_{i+1} \cdots \bullet_{i+m} \tag{3.1}$$

*where* $\bullet \in \{+, -, \times\}$.

Since the template A of (1.1) is space invariant, the stationary solutions of (1.1a) are also spatial invariant. Hence, should no ambiguity arise, the subscripts in (3.1) is to be omitted.

**Definition 3.2.** *Let* $T = \{i, \cdots, i+m\}$. *Suppose* $\mathbf{y}_T$ *is called a* $(+_i, +_{i+m})$ *(resp.,* $(-_i, -_{i+m})$*;* $(+_i, -_{i+m})$*;* $(-_i, +_{i+m})$*) locally* $\delta$*-extendable pattern of degree 2 provided that for any real numbers* $y_i$, $y_{i+m}$ *satisfying* $1 < y_i\,,\, y_{i+m} \leq 1 + r\delta$*( resp.,* $-1 > y_i\,,\, y_{i+m} \geq -1-r\delta$*;* $1 < y_i\,,\, -y_{i+m} \leq 1+r\delta$*;* $-1 > y_i\,,\, -y_{i+m} \geq -1-r\delta$ *),* $\mathbf{y}_T = y_i \bullet_{i+1} \cdots \bullet_{i+m-1} y_{i+m}$ *is a local pattern. If, in addition, we require* $y_i = y_{i+m}$ *(resp.,* $y_i = -y_{i+m}$ *), then the corresponding* $\mathbf{y}_T$ *is called a* $(+_i\,,\, +_{i+m})$ *or* $(-_i\,,\, -_{i+m})$ *(resp.,*$(+_i\,,\, -_{i+m})$ *or* $(-_i\,,\, +_{i+m})$*) locally* $\delta$*-extendable pattern of degree 1.*

**Remark 3.1.**

(1) *When we say that* $+ + +$ *is a locally* $\delta$*-extendable pattern, it is clear from the context that we meant it is a* $(+_i\,,\, +_{i+2})$ *locally* $\delta$*-extendable pattern. Thus, from here on, when using* (3.1) *to denote* $\mathbf{y}_T$ *being* $(\bullet_i\,,\, \bullet_{i+m})$ *locally* $\delta$*-extendable, we will drop* $(\bullet_i\,,\, \bullet_{i+m})$ *altogether. Here* $\bullet \in \{+\,,\, -\,,\, \times\}$

(2) *The definition* 3.2 *is related to a string of finite cells with Dirichlet boundary conditions. Indeed, let* $x_{i+1}\,,\, x_{i+2}\,,\, \cdots\,,\, x_{i+m-1}$ *satisfy* (1.3) *with boundary conditions* $f(x_i)$ *and* $f(x_{i+m})$ *being arbitrarily prescribed as real numbers in* $(1\,,\,1+r\delta)$ *or* $(-1-r\delta\,,\,-1)$.

We next give conditions so that $+\,+\,+$ and $-\,-\,-$ are locally $\delta$-extendable patterns of degree 2.

**Proposition 3.1.** *Let*

$$\delta>0,\ \ 0<r<1,\ \ z=0,\ \ 0<a<1,\ \text{and}\ \alpha=\beta>0. \tag{3.2}$$

*Then* $+\,+\,+$ *and* $-\,-\,-$ *are locally* $\delta$*-extendable patterns of degree 2 for any* $\delta>0$ *provided that*

$$a+2\beta>1. \tag{3.3}$$

*Proof.* We will only illustrate the case for $+\,+\,+$. Let $y_{i-1}=f(x_{i-1})$ and $y_{i+1}=f(x_{i+1})$ be any number between 1 and $1+r\delta$. It follows from (1.3) that

$$x_i - af(x_i) = \beta p,$$

where $2<p\le 2(1+r\delta)$. If $x_i$ is expected to be greater than 1, then $x_i$ must satisfy

$$x_i=\frac{a-ar+\beta p}{1-ar}>1.$$

However, using (3.3), we have that $x_i=\dfrac{a-ar+\beta p}{1-ar}\ge\dfrac{a-ar+2\beta}{1-ar}>1$. Since $x_{i-1}\,x_i\,x_{i+1}$ a string of cells of length 3, is constructed in a way so that for any $1<x_{i-1}\,,\,x_{i+1}\le 1+\delta$, we can find an $x_i>1$ and that $x_i$ satisfies (1.3). We may then construct $x_{i+1}\,x_{i+2}\,x_{i+3}$ similarly. It is then easy to glue two local solutions $x_{i-1}\,x_i\,x_{i+1}$ and $x_{i+1}\,x_{i+2}\,x_{i+3}$ into a local solution $x_{i-1}\,x_i\,x_{i+1}\ x_{i+2}\,x_{i+3}$ of length 5. We can then extend the patterns in both directions one step at a time. We will eventually construct a global solution $(x_i)_{i\in\mathbb{Z}}$ satisfying (1.3). Thus if (3.3) holds, $+\,+\,+$ is a locally $\delta$-extendable pattern of degree 2, for any $\delta>0$. □

**Remark 3.2.**

(i) *It is clear that if* (3.2) *holds, then* (3.3) *is also a sufficient condition for which* $+ + +$ *and* $- - -$ *are locally* $\delta$*-extendable patterns of degree 1.*

(ii) *From the computation in the proof of Proposition 3.1, we see that* $+ + +$ *and* $+ \times +$ *can not coexist as locally* $\delta$*-extendable patterns.*

We next give conditions on the parameters so that $+ + \times - -$ and $- - \times + +$ are locally $\delta$-extendable patterns of degree 1.

**Proposition 3.2.** *Let* (3.2) *holds. Then* $+ + \times - -$ *and* $- - \times + +$ *are locally* $\delta$*-extendable patterns of degree 1, provided that*

$$(1-a)(1-ar) - 2\beta^2 r > 0, \tag{3.4a}$$

$$a + \beta > 1. \tag{3.4b}$$

*Moreover, if* (3.2) *holds, then* (3.4b) *is also a necessary condition for* $+ + \times - -$ *and* $- - \times + +$ *being locally* $\delta$*-extendable patterns of degree one.*

*Proof.* The computation for $+ + \times - -$ and $- - \times + +$ is exactly the same. We will only illustrate the case for $+ + \times - -$. Let $T = \{0\,, 1\,, 2\,, 3\,, 4\}$ and $\mathbf{y}_T = + + \times - -$. If $\mathbf{y}_T$ is a locally $\delta$-extendable pattern of degree 1, then $x_1$, $x_2$, and $x_3$ satisfy the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\begin{aligned} A &= \begin{bmatrix} 1-ar & -\beta & 0 \\ -\beta r & 1-a & -\beta r \\ 0 & -\beta & 1-ar \end{bmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ &\textit{and} \\ \mathbf{b} &= \begin{bmatrix} \beta p + a(1-r) \\ 0 \\ -\beta p - a(1-r) \end{bmatrix}. \end{aligned} \tag{3.5}$$

Here $p = f(x_0) = -f(x_4)$, and $p$ is any real numbers for which $1 < p \leq 1 + r\delta$. Since the determinant of $A$, denoted by $\Delta$, is equal to

$$\Delta = (1-ar)[(1-a)(1-ar) - 2\beta^2 r] > 0.$$

Thus, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Now, let $x_1$ satisfies the following equation

$$(1-ar)x_1 = \beta p + a(1-r) \tag{3.6}$$

Letting $x_2 = 0$ and $x_3 = -x_1$, we see that the vector $[x_1\,,\,0\,,\,-x_1]^T$ satisfies the linear system (3.5). By the uniqueness of the system (3.5), it then suffices to solve (3.6).
Now,

$$x_1 = \frac{\beta p + a(1-r)}{1-ar} > \frac{\beta + a(1-r)}{1-ar}.$$

we then see that $x_1 > 1$ provided that $\dfrac{\beta + a(1-r)}{1-ar} > 1$, which indeed follows from (3.2) and (3.4b). It is also clear that if (3.2) is assumed, then the last assertion of the Proposition holds true. For otherwise, $x_1 < 1$. To complete the proof, we need to show that such $+ + \times - -$ is a local pattern. To see this, we glue two local patterns $+_0\ +_1\ \times_2\ -_3\ -_4$ and $-_4\ -_5\ \times_6\ +_7\ +_8$ together to produce another local pattern $+_0\ +_1\ \times_2\ -_3\ -_4\ -_5\ \times_6\ +_7\ +_8$. Such procedure can be extended on both sides to construct a global patterns. Thus, $+ + \times - -$ is indeed a locally $\delta$-extendable pattern.

□

**Proposition 3.3.** *Let* (3.2) *holds. Then* $+ + \times \times - -$ *and* $- - \times \times + +$ *are locally* $\delta$*-extendable patterns of degree 1 provided that*

$$(1-ar)(1-a-\beta) - \beta^2 r \neq 0, \tag{3.7a}$$

$$(1-ar)(1-a+\beta r) > \beta r[a(1-r) + \beta(2+r\delta)], \tag{3.7b}$$

*and*

$$a + \sqrt{2}\beta > 1. \tag{3.7c}$$

*Moreover, if* (3.2) *holds, then* (3.7c) *is necessary condition for* $+ + \times \times - -$ *and* $- - \times \times + +$ *being locally* $\delta$*-extendable patterns of degree 1.*

*Proof.* We illustrate only the case for $+ + \times \times - -$. Let $T = \{0, 1, 2, 3, 4, 5\}$ and $\mathbf{y}_T = + + \times \times - -$. If $\mathbf{y}_T$ is a locally $\delta$-extendable pattern of degree 1, then $x_1$, $x_2$, $x_3$ and $x_4$ must satisfy the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1-ar & -\beta & 0 & 0 \\ -\beta r & 1-a & -\beta & 0 \\ 0 & -\beta & 1-a & -\beta r \\ 0 & 0 & -\beta & 1-ar \end{bmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

*and* (3.8)

$$\mathbf{b} = \begin{bmatrix} \beta p + a(1-r) \\ \beta(1-r) \\ -\beta(1-r) \\ -\beta p - a(1-r) \end{bmatrix}.$$

Here $p$ is any real number satisfying $1 < p \leq 1 + r\delta$. Now, $\Delta_A$, the determinant of $A$, is

$$[(1-ar)(1-a) - \beta^2 r]^2 - (1-ar)^2\beta^2$$

$$= [(1-ar)(1-a+\beta) - \beta^2 r][(1-ar)(1-a-\beta) - \beta^2 r] =: t_1[(1-ar)(1-a-\beta) - \beta^2 r].$$

Using (3.7b) and (3.2), we see that

$$(1-ar)(1-a+\beta) > (1-ar)(1-a+\beta r) \geq 2\beta^2 r. \tag{3.9}$$

It then follows from (3.9) that $t_1 > 0$ and so $\Delta_A \neq 0$. Thus, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Consider the following reduced system $A'\mathbf{y} = \mathbf{b}'$, where

$$A' = \begin{bmatrix} 1-ar & -\beta \\ -\beta r & 1-a+\beta \end{bmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

*and* (3.10)

$$\mathbf{b}' = \begin{bmatrix} a - ar + \beta p \\ \beta - \beta r \end{bmatrix}.$$

If $(y_1\,,\,y_2)$ is a solution to the reduced system (3.10), then $(y_1\,,\,y_2\,,\,-y_2\,,\,-y_1)$ is a solution to the original linear system (3.8). By uniqueness of (3.8), it then suffices to study only (3.10). Using Cramer's rule, we obtain that

$$x_1 = \frac{(1-a+\beta)[a(1-r)+\beta p] + \beta^2(1-r)}{(1-ar)(1-a+\beta) - \beta^2 r} =: \frac{\Delta_1(p)}{\Delta_{A'}}$$

and

$$x_2 = \frac{\beta(1-ar)(1-r) + \beta r[a(1-r)+\beta p]}{\Delta_{A'}} = \frac{\Delta_2(p)}{\Delta_{A'}} > 0$$

Using (3.9), we see that $x_1 > 1$ provided that $\Delta_1(1) > \Delta_{A'}$, which is equivalent to $(a + \sqrt{2}\beta - 1)(-a + \sqrt{2}\beta + 1) > 0$. However, $-a + \sqrt{2}\beta + 1 > 0$. Thus $x_1 > 1$ provided that (3.7c) holds. Similarly, $x_2 < 1$ provided that $\Delta_2(1 + r\delta) < \Delta_{A'}$, which is equivalent to (3.7b). Clearly, $+ + \times \times - -$ and $- - \times \times ++$ can be used to construct a global pattern. Hence, $+ + \times \times - -$ and $- - \times \times ++$ are indeed locally $\delta$-extendable patterns. We just complete the proof of the Proposition.

$\square$

# 4 Global Stable Defect Patterns and Entropy

The main results of the thesis is contained in this section. Specifically, we will give conditions on parameters so that globally stable defect patterns can be constructed. Moreover, we will study the complexity of such generated patterns. To begin with, we consider the parameter region in $a-\beta$ plane for which locally $\delta$-extendable patterns $+++$, $---$, $++\times--$ and $--\times++$ can coexist stably.

**Proposition 4.1.** *Assume* (3.2) *is satisfied, and that* $0 < r < \dfrac{1}{2}$. *Let* $\mathbf{\Gamma_1}$ *be the region in* $a-\beta$ *plane such that* (2.6) *and* (3.4b) *are satisfied. Then* $\mathbf{\Gamma_1}$ *is nonempty. Moreover, locally* $\delta$*-extendable patterns*

$$+++,\quad ---,\quad ++\times--,\quad \textit{and}\,--\times++ \tag{4.1}$$

*can coexist stably.*

*Proof.* It is obvious that $\mathbf{\Gamma_1}$ is nonempty, see Figure(4.1). It is also clear that $a+2\beta > a+\beta > 1$. To complete the proof of the Proposition, it remains to show that the stability conditions in (2.6) imply (3.4a). From (2.6), we have $(1-a) > 2\beta r$ and $1-ar > \beta(1+r)$. Hence, $(1-a)(1-ar) > 2\beta^2 r(1+r) > 2\beta^2 r$.

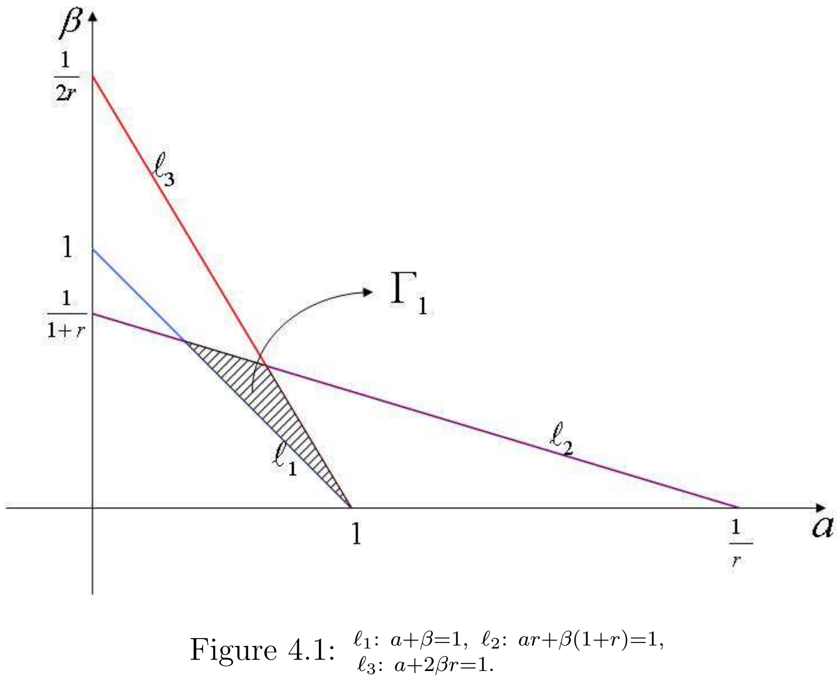


Figure 4.1: $\ell_1$: $a+\beta=1$, $\ell_2$: $ar+\beta(1+r)=1$, $\ell_3$: $a+2\beta r=1$.

$\square$

Write the equality of (3.7b) as

$$ra^2 - ra\beta - r(2+r\delta)\beta^2 - a(r+1) + \beta r + 1 = 0. \tag{4.2}$$

Clearly, (4.2) is a hyperbola, denoted by $\mathbf{H}$. Assume

$$\sqrt{2} - 1 > r. \tag{4.3}$$

We further denote by $\mathbf{H}^+$ the region in $a - \beta$ plane satisfied (3.7b). Let $\mathbf{\Gamma_2}$ be the triangular region, see Figure (4.2), satisfied by (2.7), (3.2), (4.3) and (3.7c).

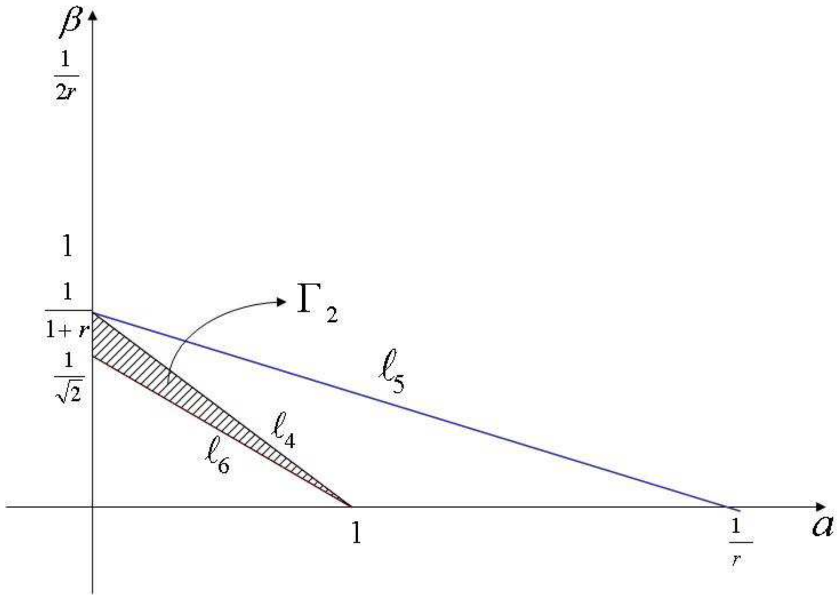


Figure 4.2: $\ell_4$: $a+\beta(1+r)=1$, $\ell_5$: $ar+\beta(1+r)=1$ $\ell_6$: $a+\sqrt{2}\beta=1$.

**Proposition 4.2.** *Let* (3.2) *and* (4.3) *be satisfied. If* $r\delta \geq \sqrt{2}$*, then* $\mathbf{\Gamma_2} \cap \mathbf{H}^+$ *is nonempty. If* $r\delta < \sqrt{2}$ *and* $\delta > 3 + r$*, then* $\mathbf{\Gamma_2} \cap \mathbf{H}^+$ *is also nonempty.*

*Proof.* Let the line $\ell_k$ be defined as

$$a + k\beta = 1, \tag{4.4}$$

where $1 + r < k < \sqrt{2}$.

Substituting (4.4) into (3.7b), we obtain that

$$[r(1+k)a + r(2+r\delta)\beta]\beta < \beta(k+r). \tag{4.5}$$

Multiplying (4.4) by $-\beta r(1+k)$ and adding the resulting equation into (4.5), we get

$$r(2+r\delta-k-k^2)\beta^2 < \beta k(1-r). \tag{4.6}$$

Since the parameter $\beta$ is positive, we have (4.6) reduced to

$$r(2+r\delta-k-k^2)\beta < k(1-r). \tag{4.7}$$

Now, if $r\delta \geq \sqrt{2}$, then for any $1+r<k<\sqrt{2}$, we have

$$\beta < \frac{k(1-r)}{r(2+r\delta-k-k^2)} =: \beta_1 \tag{4.8}$$

where $\beta_1$ is positive. Thus the line segment $\ell_k$, $1+r<k<\sqrt{2}$, defined as

$$\overline{\ell}_k = \{(a\,,\,\beta):\ a+k\beta=1,\ 0<\beta<\beta_1\} \tag{4.9}$$

lie in $\mathbf{\Gamma_2}\cap\mathbf{H}^+$. Suppose $0<r\delta<\sqrt{2}$ and $\delta>3+r$. Letting $k=1+r$, we have

$$(2+r\delta-k-k^2) = 2+r\delta-(1+r)-(1+r)^2 = r(\delta-3-r) > 0.$$

Therefore, there exists a $k^*(r\,,\,\delta)$, such that if $1+r<k<k^*(r\,,\,\delta)$, then $\beta_1$, given as in (4.8) is positive. Consequently, the line segments $\overline{\ell}_k$ in (4.9), where $1+r<k<k^*(r\,,\,\delta)$ lie in $\mathbf{\Gamma_2}\cap\mathbf{H}^+$. We just complete the proof of the Proposition. $\square$

**Proposition 4.3.** *Let* (3.2) *and* (4.2) *be satisfied. We further assume that*

$$r\delta \geq \sqrt{2}, \quad \textit{or} \quad r\delta < \sqrt{2} \ \ \textit{and} \ \ \delta > 3+r. \tag{4.10}$$

*Then locally $\delta$-extendable patterns*

$$\mathbf{+\,+\,+,\ \ -\,-\,-,\ \ +\,+\times\times--}\ \ \textit{and}\ \ \mathbf{-\,-\times\times++} \tag{4.11}$$

*can coexist stably.*

*Proof.* Let $\gamma_\mathbf{1}$ be the curve defined by equality of (3.7a). That is $\gamma_\mathbf{1}=\{(a\,,\,\beta)\ :\ (1-ar)(1-a-\beta)-\beta^2 r=0\}$. Clearly, $\gamma_\mathbf{1}$ is a hyperbola, Since $\mathbf{\Gamma_2}\cap\mathbf{H}^+$ is a region for which its area is positive. Thus $(\mathbf{\Gamma_2}\cap\mathbf{H}^+)-\gamma_\mathbf{1}$ is nonempty. $\square$

We are now ready to state our first main result.

**Theorem 4.1.** *Assume* (3.2) *is satisfied. Let* $0 < r < \frac{1}{2}$ *and* $\delta > 0$. *Suppose* $(a\,,\,\beta) \in \mathbf{\Gamma_1}$. *Then any defect pattern satisfying the following rules is stable.*

(i) *Any linear cell* $y_i$ *is surrounded by two saturated cells of opposite signs.* (4.12a)

(ii) *Two consecutive linear cells are separated by a string of* $k$ *saturated cells of the same sign. Here* $k \geq 3$, *and is odd.*

(4.12b)

*On the other hand, any defect pattern satisfying* (4.12) *can be generated by using locally* $\delta$*-extendable patterns given in* (4.1).

*Proof.* If $(a\,,\,\beta)$ is chosen as assumed, then the four local patterns in (4.1) are $\delta$-extendable patterns. Suppose the symbol on the most right of one of the local pattern is the same as that of the most left of the other local pattern. Then those two local patterns can be glued together.
For instance,

$$\begin{array}{ccccccc} + & + & \boxed{+} & & & & \\ & & \boxed{+} & + & \times & - & - \end{array} \quad \Rightarrow \quad + \; + \; \boxed{+} \; + \; \times \; - \; - \tag{4.13a}$$

$$\begin{array}{ccccccccc} - & - & \times & + & \boxed{+} & & & & \\ & & & & \boxed{+} & + & \times & - & - \end{array} \quad \Rightarrow \quad - \; - \; \times \; + \; \boxed{+} \; + \; \times \; - \; - \tag{4.13b}$$

Note that the local pattern in (4.13b) has 3 $+$'s between two linear cells. We also observe that there are even number of $+$'s in the local pattern $+ \; + \; \boxed{+} \; + \; \times \; - \; -$, which is in (4.13a). By adding more $+ \; + \; +$'s to the left of $+ \; + \; \boxed{+} \; + \; \times \; - \; -$, we still get even number of $+$'s in the newly created local patterns. Thus, to get another linear cell in such gluing process, we need to glue $- \; - \; \times \; + \; +$ to the left of newest local patterns. Then the resulting patterns must have odd number of $+$'s between two linear cells. Arguing similarly, we see that if we use the four local patterns in (4.1) to generate the global patterns, then such patterns must satisfy (4.12). The converse is also true. $\square$

**Remark 4.1.** *The parameters' condition for* $+ \; \times \; -$ *is even more friendly than that of* $+ \; + \; \times \; - \; -$. *Considering the following grouping,*

$$\begin{array}{cccc} + & + & \boxed{+} & \\ & & \boxed{+} & \times \; - \end{array} \quad \Rightarrow \quad + \; + \; \times \; - \tag{4.14a}$$

$$- \quad \times \quad \begin{array}{c}\boxed{+}\\ \boxed{+}\end{array} \quad \times \quad - \quad \Rightarrow \quad - \times + \times - \tag{4.14b}$$

*We see in the case of* (4.14a)*, one needs to consider if* $+ + \times$ *is* $\delta$*-extendable. While in the case of* (4.14b)*, one needs to worry about if* $\times + \times$ *is* $\delta$*-extendable. A direct computation would yield that for* $0 < a < 1$*,* $\beta > 0$ *and* $0 < r < 1$*, there is no feasible parameter region to guarantee that* $\times + \times$ *and* $+ + \times$ *are* $\delta$*-extendable. This is the reason why we require two consecutive linear cells are separated by a string of 2 saturated cells.*

**Theorem 4.2.** *Let* (3.2)*,* (4.3)*,and* (4.10) *be satisfied. Suppose* $(a\,,\,\beta) \in (\mathbf{\Gamma_2} \cap \mathbf{H}^+) - \gamma_\mathbf{1}$*. Then any defect pattern satisfying the following rules is stable.*

(i) *Any linear cell belongs to a string of 2 linear cells.* (4.15a)

(ii) *Any string of 2 linear cells is surrounded by two saturated cells of opposite sign.* (4.15b)

(iii) *Two consecutive string of 2 linear cells are separated by a string of* $k$ *saturated cells of the same sign. Here* $k \geq 3$*, and is odd.* (4.15c)

*Conversely, any defect pattern satisfying* (4.15) *can be generated by using locally* $\delta$*-extendable patterns given in* (4.11)*.*

**Remark 4.2.** *Due to the difficulty in obtaining the exact formula for the linearized operator* $\mathcal{L}(\mathbf{x})$*, as given in* (1.4)*, we are unable to rule out the coexistence of stably local defect patterns of size one and two, which is our conjecture.*

Having characterized the set of stable patterns, we now want to measure its complexity. Chow and Mallet-Paret [3] have introduced the notion of spatial entropy to provide a measure of the number of mosaic patterns. Extending it to encompass all combinations of stable defect equilibria, this definition is, in the 1-D case, as follows:

**Definition 4.1.** *Consider the set* $\mathcal{S}$ *of sequences* $\mathbf{x} = (x_i)$*, where* $\mathbf{x}$ *satisfies* (1.3) *as stable equilibria. Let* $\mathcal{S}(M)$ *be the number of different subsequence of* $M$ *cells observed in* $\mathcal{S}$ *through a window of size* $M$ *in the infinite lattice. Then its entropy function is*

$$h(\mathcal{S}) = \lim_{M\to\infty} \frac{1}{M} \log \mathcal{S}(M). \tag{4.16}$$

Note that as the template $A$ in (1.2) is space-invariant, the set $\mathcal{S}$ is translation invariant, and so the position of the window of $M$ cells in the infinite array is not important, only its size $M$ matters [15]. This definition also characterizes the complexity of the patterns, in the sense that the CNNs is said to exhibit spatial chaos if $h > 0$ and pattern formation if $h = 0$ [15].

**Theorem 4.3.** *Let $0 < r < \frac{1}{2}$ and $\delta > 0$. Suppose $(a\,,\,\beta) \in \mathbf{\Gamma_1}$. Then the set $\mathcal{D}_1$ of defect patterns described in* (4.1) *exhibits spatial chaos. Moreover, the spatial entropy $h(\mathcal{D}_1)$ of $\mathcal{D}_1$ is greater or equal than $\frac{\ln 2}{5}$.*

*Proof.* Since $+ + +$ and $- - -$ are $\delta$-extendable, then $+ + + + +$ and $- - - - -$ are $\delta$-extendable. Consider a window of size $5n$ as follows:

$$\underset{B_1}{\boxed{\qquad\qquad}}\;\;\underset{B_2}{\boxed{\qquad\qquad}}\;\cdots\;\underset{B_n}{\boxed{\qquad\qquad}}$$

Here $B_1$ is a sub-window of size 5. Now, using $+ + + + +$, $- - - - -$, $+ + \times - -$ and $- - \times + +$ to fill in $B_k$, $1 \leq k \leq n$, we see that each of $B_i$ has at least 2 choices to make $B_1 B_2 \cdots B_n$ a local pattern. Hence

$$h(\mathcal{D}_1) \geq \lim_{n\to\infty} \frac{\ln 2^n}{5n} = \frac{\ln 2}{5}\,.$$

$\square$

**Remark 4.3.** *Since any two consecutive linear cells are separated by any odd number $k$, $k \geq 3$, of saturated cells of the same sign, $\mathcal{D}_1$ is not a subshift of finite type. It will be interesting to compute the exact entropy of $\mathcal{D}_1$.*

**Theorem 4.4.** *Let $0 < r < \sqrt{2} - 1$ and* (4.10) *be satisfied. Suppose $(a\,,\,\beta) \in (\mathbf{\Gamma_2} \cap \mathbf{H}^+) - \boldsymbol{\gamma_1}$, then the set $\mathcal{D}_2$ of defect patterns given in* (4.15) *exhibits spatial chaos. Moreover, the spatial entropy $h(\mathcal{D}_2)$ of $\mathcal{D}_2$ is greater or equal than $\frac{\ln 2}{6}$.*

*Proof.* We first show that if $(a\,,\,\beta) \in (\mathbf{\Gamma_2} \cap \mathbf{H}^+) - \boldsymbol{\gamma_1}$, then $+ + ++$ and $- - --$ are locally $\delta$-extendable patterns of degree one. We illustrate only the case for $+ + ++$. Use similar approaches as those in proving Propositions (3.2) and (3.3), we consider the following equation.

$$-x + \beta p + a[1 + r(x - 1)] + \beta x = 0.$$

then $$x = \frac{\beta p + (a+\beta)(1-r)}{1 - ar - \beta r} \geq \frac{\beta + (a+\beta)(1-r)}{1 - ar - \beta r}.$$

If $\dfrac{\beta + (a+\beta)(1-r)}{1 - ar - \beta r} > 1$, which is equivalent to $a + 2\beta > 1$, then $x > 1$. However, if $a$ and $\beta$ are chosen as assumed, then $a + 2\beta > 1$. Thus, $+ + ++$ and $- - --$ are locally $\delta$-extendable patterns of degree one. Gluing $+ + ++$ and $+ + +$ together, we have $+ + + + ++$ is $\delta$-extendable of degree one. Similarly, $- - - - --$ is also $\delta$-extendable of degree one. Let $B_i$, $i = 1\,, 2\,, \cdots, n$, be a subwindow of size 6. Considering a window $B_1\, B_2 \cdots B_n$ of size 6, as in the proof of Theorem (4.3), we consider, similarly, that $h(\mathcal{D}_2) \geq \dfrac{\ln 2}{6}$. $\square$

**Remark 4.4.** *We note that the local mosaic patterns of the form* $+ + -$, $+ - +$, $- + +$, $- - +$, $- + -$ *and* $+ - -$ *are locally $\delta$-extendable only if* $a > 1$ (see, [11]). *Thus, if* $(a\,, \beta) \in \mathbf{\Gamma}_1$ *or* $(a\,, \beta) \in (\mathbf{\Gamma}_2 \cap \mathbf{H}^+) - \gamma_1$, *then any global mosaic patterns must either be all* $+'s$ *or all* $-'s$. *Consequently, the associated mosaic patterns have zero spatial entropy.*

Figure 4.3 is a collection of a computer simulation with sets of parameters chosen from the parameters regions in Figure 4.1. and 4.2.. Specifically, we set $\delta = 10$ for all cases. Each collection in Figure 4.3. contains two pairs of two arrays of colors. The first array is the initial outputs. The second array represents the final outputs. If the state $x_j$ of a cell $C_j$ is such that $|x_j| < 1$, then we color it gray. If the state $x_j$ of a cell $C_j$ is less than $-1$ (greater than 1, respectively), then we color it white (black, respectively). Moreover, the final outputs in each of the collection consist of the basic defect patterns allowed in their corresponding parameters region.

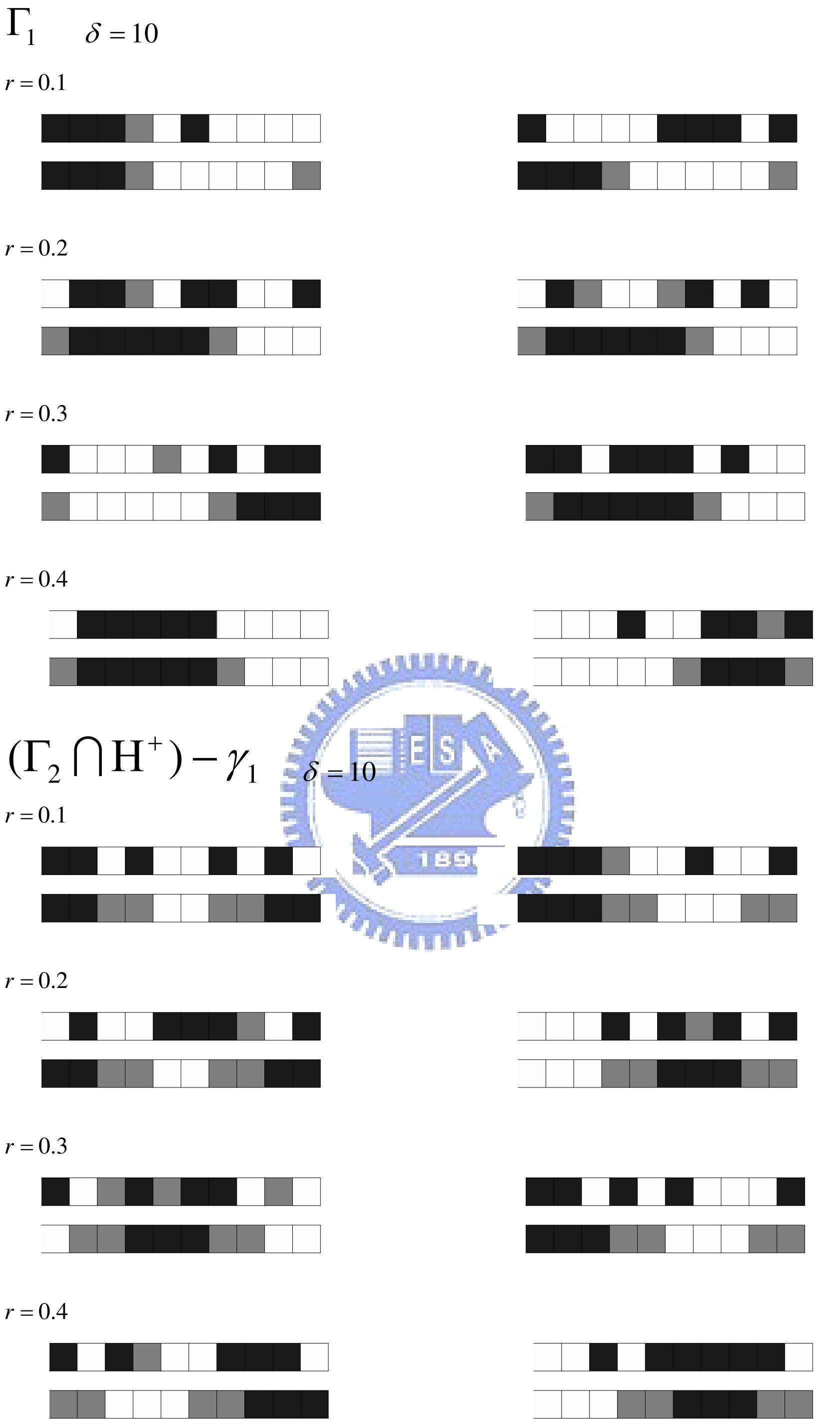
$\Gamma_1$ $\delta = 10$
$r = 0.1$
$r = 0.2$
$r = 0.3$
$r = 0.4$
$(\Gamma_2 \cap H^+) - \gamma_1$ $\delta = 10$
$r = 0.1$
$r = 0.2$
$r = 0.3$
$r = 0.4$

Figure 4.3: .

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