Multistability
for Delayed Hopfield-type Neural Networks

Student : Kuang-Hui Lin Advisor : Chih-Wen Shih

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan, R.O.C.

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Abstract

The number of stable stationary solutions corresponds to the memory capacity for the neural networks. In this presentation, we investigate existence and stability of multiple stationary solutions and multiple periodic solutions for Hopfield-type neural networks with and without delays. Their associated basins of attraction are also estimated. Such a convergent dynamical behavior is established through formulating parameter conditions based on a suitable geometrical setting. Finally, two examples are given to illustrate our main results.

Contents

1 Introduction

The well-known Hopfield-type neural networks and their various generalizations have attracted much attention of the scientific community, due to their promising potential for tasks of classification, associative memory, and parallel computation and their ability to solve difficult optimization problems. In those applications, the stability of the neural networks is crucial and needs to be prescribed before designing a powerful network model. Especially in associative memories , each particular pattern is stored in the networks as an equilibrium, the stability of associated equilibrium shows that the networks have the ability to retrieve the related pattern. In general, in associative memory neural networks, one expects the networks can store as many patterns as possible. In this sense, information about the basin of attraction of each stable equilibrium helps retrieve exactly the needed memories. It is for this reason that leads to the study of the local stability of each equilibrium and its associated basin of attraction.

The classical Hopfield-type neural networks [24] is described by a system of ordinary differential equations

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(x_j(t)) + I_i, \; i = 1, 2, \cdots, n. \tag{1.1}$$

Here $n \geq 2$ is the number of neurons in the networks. For neuron i, $C_i > 0$ and $R_i > 0$ are the neurons amplifier input capacitance and resistance, respectively, and I_i is the constant input from outside the system. The $n \times n$ matrix $T = (T_{ij})$ represents the connection strengths between neurons, and the function g_j are neuron activation functions.

In hardware implementation, time delays occur due to the finite switching speeds of the amplifiers. The Hopfield-type neural networks with delays [29] is described by a system of functional differential equations

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(x_j(t - \tau_{ij})) + I_i, \; i = 1, 2, \cdots, n \tag{1.2}$$

in which $0 < \tau_{ij} \leq \tau$. Recently, the Hopfield-type neural networks with delays has drawn much attention. Even though the delay does not change the equilibrium points, with the appearance of time delay, the dynamics of the corresponding neural network models can be quite complicated. It is interesting to know under what conditions the delays have no effects to the dynamics. Restated, we hope to obtain the delay-independent stability results which are more applicable in designing a practical network.

In applications to parallel computation and signal processing involving the solution optimization problems, it is required that system (1.1) and (1.2) have a unique equilibrium point that is globally attractive. Thus, the global attractivity of systems is of great importance for both practical and theoretical purposes and has been the major concern of most authors dealing with (1.1) and (1.2). We refer to [9, 20, 41, 44] and [10, 15, 19, 38, 40, 42, 43, 45] for systems with and without delays, respectively. Herein, the constant delays have been studied in [10, 15, 19, 38, 42, 45] and there are some results for the case of variable delays in [40, 43] . In [41], the authors study the estimation of exponential convergence rate and the exponential stability of (1.1). Both local and global exponential convergence are discussed therein. In [44], without assuming the boundedness, monotonicity, and differentiability of the activation functions, by using M-matrix theory, Lyapunov functions are constructed and employed to establish sufficient conditions for global asymptotic stability of (1.1). Both global exponential stability and periodic solutions of Hopfield neural networks are analyzed via the method of constructing suitable Lyapunov functionals, with constant delays and variable delays, in [42] and [43], respectively. In [38], without assuming the monotonicity and differentiability of the activation functions, Lyapunov functionals and Lyapunov-Razumikhin technique are constructed and employed to establish sufficient conditions for global asymptotic stability independent of the delays. In the case of monotone and smooth activation functions, the theory of monotone dynamical systems is applied to obtain criteria for global attractivity, which depends on delays. In [45], without assuming the boundedness, monotonicity and differentiability of the activation functions, the authors present conditions ensuring existence, uniqueness and global asymptotical stability of the equilibrium point of (1.2). In [21], some sufficient conditions for local and global exponential stability of the discrete-time Hopfield neural networks with general activation functions are derived, which generalize those existing results. By means of M-matrix theory and some inequality analysis techniques, the exponential stability is derived and the basin of attraction of the stable equilibrium is estimated.

The existence and stability of equilibria and periodic solutions of cellular neural networks with and without delays have also been extensively studied in [6, 7, 8, 14, 27, 28, 31, 46]. In [27], the authors present two types of matrix stability:

complete stability and strong stability. By using these two properties, they obtain some conditions ensuring uniqueness, exponential stability and global asymptotic stability of the equilibrium point for cellular neural networks. A set of criteria is presented for the global exponential stability and the existence of periodic solutions of delayed cellular neural networks by constructing suitable Lyapunov functionals, introducing many parameters and combining with the elementary inequality technique in [6, 8, 14]. In [28], convergence characteristics of continuous-time and discrete-time cellular neural networks are studied. By using Lyapunov functionals, the authors obtain delay independent sufficient conditions for the networks to converge exponentially toward the equilibria associated with the constant input sources. Halanay-type inequalities are employed to obtain sufficient conditions for the networks to be globally exponentially stable. It is shown that the estimates obtained from the Halanay-type inequalities improve the estimates obtained from the Lyapunov functionals. It is also shown that the convergence characteristics of the continuous-time systems are preserved by the discrete-time analogues without any restriction imposed on the uniform discretization step size. In [7], the authors investigate the absolute exponential stability of a general class of delayed neural networks, which require the activation functions to be partially Lipschitz continuous and monotone nondecreasing only, but not necessarily differentiable or bounded. Three sufficient conditions are derived to ascertain whether or not the equilibrium points of the delayed neural networks with additively diagonally stable interconnection matrices are absolutely exponentially stable by using Halanay-type inequality and Lyapunov functional. The problem of global exponential stability for cellular neural networks with time-varing delays are studied in [46]. The theory for existence of many patterns has been developed for cellular neural networks [13, 26, 36, 37], and there are other interesting studies on delayed neural networks in [2, 16, 17, 39].

What has to be noticed is that the stability in most of these papers is referred to as "monostability". This means that the networks have a unique equilibrium or a unique periodic orbit which is globally attractive. The notion of "multistability" of a neural network is used to describe coexistence of multiple stable patterns such as equilibria or periodic orbits. The purpose of this presentation is to investigate existence and stability of multiple equilibria and multiple periodic solutions, and their associated basins of attraction for the Hopfield-type neural networks with and without delays. In order to illustrate our results, we use Matlab to compute the numerical simulations. The numerical methods and programs adopted can be found in [4, 33, 34].

From a mathematical viewpoint, there are three important methods to treat the stability problem of delayed neural networks: Lyapunov functional, characteristic equation and Halanay-type inequalities. The Lyapunov functional approach can be found in [18, 23], and the characteristic equation approach is used in [3, 5, 35]. Finally, the Halanay-type inequalities approach, we refer to [1, 7, 22]. In this presentation, we use Lyapunov functional method and Halanay-type inequalities to study the stability of Hopfield neural networks with and without delays.

The rest of the paper is organized as follows. In Section 2, we establish conditions for existence of 3^n equilibria for the Hopfield's network. 2^n among them will be shown to be asymptotically stable for the system without delays, through a linearization analysis. In Section 3, we shall verify that under same conditions, 2^n regions, each containing an equilibrium, are positively invariant under the flow generated by the system with or without delays. Subsequently, it is argued that these 2^n equilibria are also exponentially stable, even with presence of delays. In Section 4, under same conditions, we shall confirm that 2^n periodic solutions exist in these 2^n regions, each containing a periodic solution, when the system has a periodic input. Two numerical simulations on the dynamics of two-neuron networks which illustrate the present theory, are given in Section 5.

2 Existence of Multiple Equilibria and their Stability

In this section, we shall formulate sufficient conditions for the existence of multiple stationary solutions for Hopfield neural networks with and without delays. Our approach is based on a geometrical observation. The derived parameter conditions are concrete and can be examined easily. We also establish stability criteria of these equilibria for the system without delays, through estimations on the eigenvalues of the linearized system. Stability for the system with delays will be discussed in the next section. After rearranging the parameters, we consider system (1.1) in the following forms: for the network without delay

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t)) + J_i,\ i = 1, 2, \cdots, n; \tag{2.1}$$

for the network with delays

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \cdots, n. \tag{2.2}$$

Herein, $b_i > 0$, $0 < \tau_{ij} \le \tau := \max_{1\le i,j\le n} \tau_{ij}$. While (2.1) is a system of ordinary differential equations, (2.2) is a system of functional differential equations. The initial condition for (2.2) is

$$x_i(\theta) = \phi_i(\theta), \ \ -\tau \le \theta \le 0, \ i = 1, 2, \cdots, n,$$

and it is usually assumed that $\phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R})$. Let $\ell > 0$. For $\mathbf{x} \in \mathcal{C}([-\tau, \ell], \mathbb{R}^n)$, and $t \in [0, \ell]$, we define

$$\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta), \theta \in [-\tau, 0]. \tag{2.3}$$

Let us denote $\tilde{F} = (\tilde{F}_1, \cdots, \tilde{F}_n)$, where $\tilde{F}_i$ is the right hand side of (2.2),

$$\tilde{F}_i(\mathbf{x}_t) := -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i,$$

where $\mathbf{x} = (x_1, \cdots, x_n)$. A function $\mathbf{x}$ is called a solution of (2.2) on $[-\tau, \ell)$ if $\mathbf{x} \in \mathcal{C}([-\tau, \ell), \mathbb{R}^n)$, $\mathbf{x}_t$ defined as (2.3) lies in the domain of $\tilde{F}$ and satisfies (2.2) for $t \in [0, \ell)$. For a given $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, let us denote by $\mathbf{x}(t; \phi)$ the solution of (2.2) with $\mathbf{x}_0(\theta; \phi) := \mathbf{x}(0 + \theta; \phi) = \phi(\theta)$, for $\theta \in [-\tau, 0]$.

The activation functions g_j usually have sigmoidal configuration or are non-decreasing with saturations. Herein, we consider the typical logistic or Fermi function: for all $j = 1, 2, \cdots, n$,

$$g_j(\xi) = g(\xi) := \frac{1}{1 + e^{-\xi/\varepsilon}}, \quad \varepsilon > 0. \tag{2.4}$$

One may also adopt $g_j(\xi) = 1/(1 + e^{-\xi/\varepsilon_j})$, $\varepsilon_j > 0$. Notably, the stationary equation for systems (2.1) and (2.2) are identical; namely,

$$F_i(\mathbf{x}) := -b_i x_i + \sum_{j=1}^{n} \omega_{ij} g_j(x_j) + J_i = 0, \quad i = 1, 2, \cdots, n, \tag{2.5}$$

where $\mathbf{x} = (x_1, \cdots, x_n)$. For our formulation in the following discussions, we introduce a single neuron analogue (no interaction among neurons)

$$f_i(\xi) := -b_i \xi + \omega_{ii} g(\xi) + J_i, \ \xi \in \mathbb{R}.$$

Let us propose the first parameter condition.

$$(\mathrm{H}_1): 0 < \frac{b_i\varepsilon}{\omega_{ii}} < \frac{1}{4},\ i = 1, 2, \cdots, n.$$

Lemma 2.1: Under condition (H_1), there exist two points p_i and q_i with $p_i < 0 < q_i$, such that $f_i'(p_i) = 0,\ f_i'(q_i) = 0$, for $i = 1, 2, \cdots, n$.

Proof: We compute that

$$g'(\xi) = \frac{1}{\varepsilon}(1 + e^{-\xi/\varepsilon})^{-2} e^{-\xi/\varepsilon}. \tag{2.6}$$

Notably, the graph of function $g'(\xi)$ is concave down and has its maximal value at $\xi = 0$. Notably, g is strictly increasing. We let $y = g(\xi),\ \xi \in \mathbb{R}$. Then $y \in (0, 1)$ and $g(0) = 1/2$. It follows from (2.6) that

$$g'(\xi) = \frac{1}{\varepsilon} y^2 (\frac{1}{y} - 1) = \frac{1}{\varepsilon}(y - y^2).$$

On the other hand, for each i, since $f_i'(\xi) = -b_i + \omega_{ii} g'(\xi)$, we have $f_i'(\xi) = 0$ if and only if $b_i = \omega_{ii} g'(\xi)$; equivalently,

$$\frac{b_i\varepsilon}{\omega_{ii}} = y - y^2.$$

From the configuration in Figure 1, it follows that, for each i, there exist two points $p_i,\ q_i,\ p_i < 0 < q_i$, such that $f_i'(p_i) = f_i'(q_i) = 0$, if the parameter condition $0 < b_i\varepsilon/\omega_{ii} < 1/4$ holds. This completes the proof.

Notably, condition (H_1) implies $\omega_{ii} > 0$ for all $i = 1, 2, \cdots, n$, since each b_i is already assumed a positive constant. We define, for $i = 1, 2, \cdots, n$,

$$\begin{aligned} \hat{f}_i(\xi) &= -b_i\xi + \omega_{ii} g(\xi) + k_i^+ \\ \check{f}_i(\xi) &= -b_i\xi + \omega_{ii} g(\xi) + k_i^-, \end{aligned}$$

where

$$k_i^+ := \sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i,\ k_i^- := -\sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i.$$

It follows that

$$\check{f}_i(x_i) \le F_i(\mathbf{x}) \le \hat{f}_i(x_i), \tag{2.7}$$

for all $\mathbf{x} = (x_1, \cdots, x_n)$ and $i = 1, 2, \cdots, n$, since $0 \le g_j \le 1$ for all j.

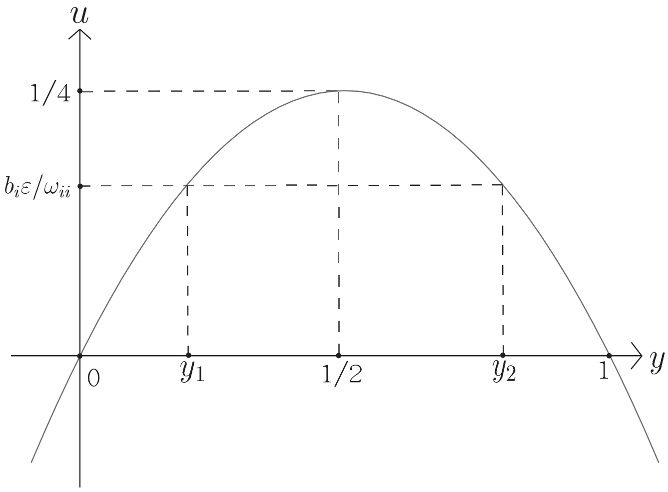

Figure 1: The graph for function $u(y) = y - y^2$ and $y_1 = g(p_i),\ y_2 = g(q_i)$.

We consider the second parameter condition which is concerned with the existence of multiple equilibria for (2.1) and (2.2).

$$(\mathrm{H}_2) : \hat{f}_i(p_i) < 0, \check{f}_i(q_i) > 0, i = 1, 2, \cdots, n.$$

The configuration that motivates (H_2) is depicted in Figure 2. Such a configuration is due to the characteristics of the output function g. Under assumptions (H_1) and (H_2), there exist points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\check{a}_i, \check{b}_i, \check{c}_i$ with $\check{a}_i < \check{b}_i < \check{c}_i$ such that $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$.

Theorem 2.1 : Under (H_1) and (H_2), there exist 3^n equilibria for systems (2.1) and (2.2).

Proof: The equilibria of systems (2.1) and (2.2) are zeros of (2.5). Under condition (H_1) and (H_2), the graphs of $\hat{f}_i$ and $\check{f}_i$ defined above are as depicted as Figure 2. According to the configurations, there are 3^n disjoint closed regions in $\mathbb{R}^n$. Set

$$\begin{aligned}\Omega_i^{\mathrm{l}} &:= \{x \in \mathbb{R} |\ \check{a}_i \le x \le \hat{a}_i\}\\ \Omega_i^{\mathrm{m}} &:= \{x \in \mathbb{R} |\ \hat{b}_i \le x \le \check{b}_i\}\\ \Omega_i^{\mathrm{r}} &:= \{x \in \mathbb{R} |\ \check{c}_i \le x \le \hat{c}_i\},\end{aligned} \tag{2.8}$$

and let $\mathbf{\Omega}^\alpha = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in \Omega_i^{\alpha_i}\}$ with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_i =$"l" or "m" or "r". Herein,"l", "m", "r" mean respectively "left", "middle", "right".

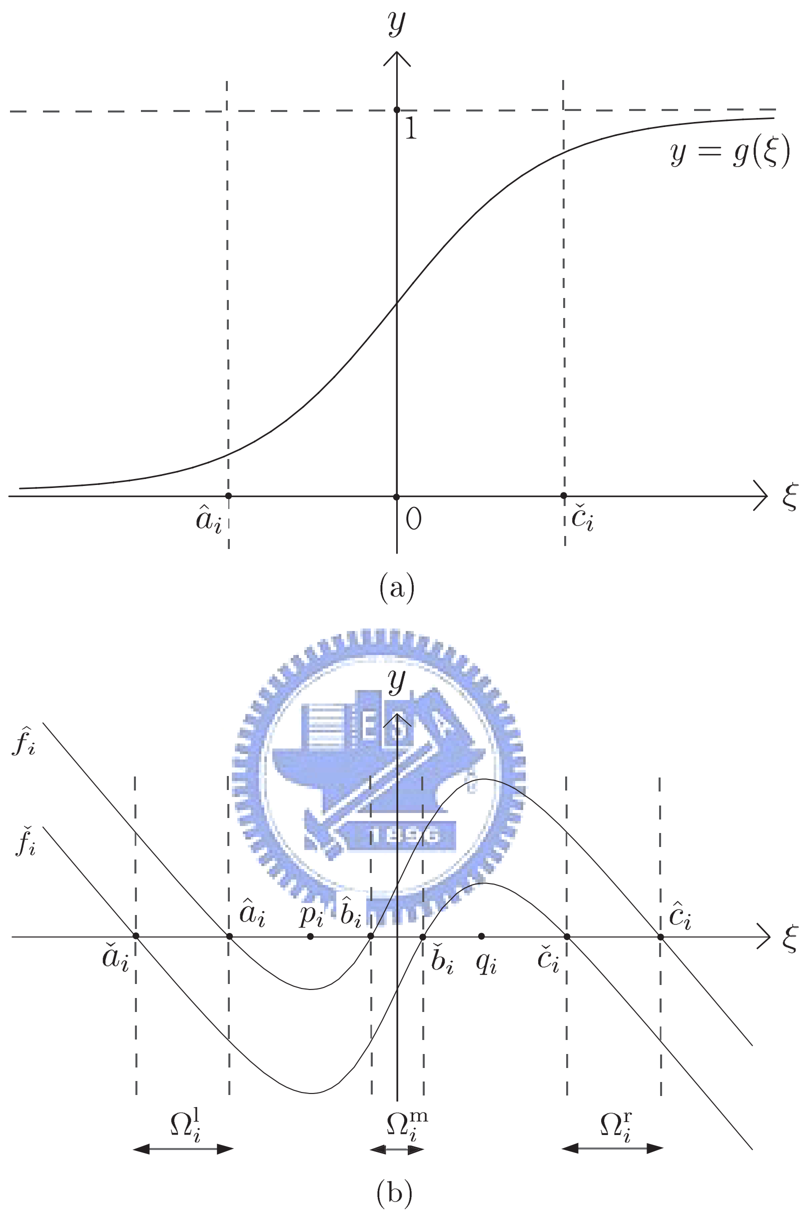

Figure 2: (a) The graph of g with $\varepsilon = 0.5$, (b)Configurations for $\hat{f}_i$ and $\check{f}_i$.

Consider any one of these regions $\mathbf{\Omega}^\alpha$. For a given $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \mathbf{\Omega}^\alpha$, we solve

$$h_i(x_i) := -b_i x_i + \omega_{ii} g(x_i) + \sum_{j=1, j\neq i}^{n} \omega_{ij} g(\tilde{x}_j) + J_i = 0,$$

for $x_i, i = 1, 2, \cdots, n$. According to (2.7), the graph of h_i lies between the graphs of $\hat{f}_i$ and $\check{f}_i$. In fact, the graph of h_i is a vertical shift of the graph for $\hat{f}_i$ or $\check{f}_i$. Thus, one can always find three solutions and each of them lies in one of the regions in (2.8), for each i. Let us pick the one lying in $\Omega_i^{\alpha_i}$ as x_i and define a mapping $\mathbf{H}_\alpha : \mathbf{\Omega}^\alpha \to \mathbf{\Omega}^\alpha$ by $\mathbf{H}_\alpha(\tilde{\mathbf{x}}) = \mathbf{x} = (x_1, x_2, \cdots, x_n)$. Since g is continuous and the graph of h_i is a vertical shift of function $\xi \mapsto -b_i\xi + \omega_{ii} g(\xi)$ by the quantity $\sum_{j=1, j\neq i}^{n} \omega_{ij} g(\tilde{x}_j) + J_i$, the map $\mathbf{H}_\alpha$ is continuous. It follows from the Brouwer's fixed point theorem that there exists one fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ of $\mathbf{H}_\alpha$ in $\mathbf{\Omega}^\alpha$, which is also a zero of the function F, where $F = (F_1, F_2, \cdots, F_n)$. Consequently, there exist 3^n zeros of F, hence 3^n equilibria for system (2.1) and (2.2), and each of them lies in one of the 3^n regions $\mathbf{\Omega}^\alpha$. This completes the proof.

Let

$$g'(\eta) := \max\{g'(\xi) \mid \xi = \check{c}_i, \hat{a}_i, i = 1, 2, \cdots, n\}.$$

We consider the following criterion concerning stability of the equilibria.

$$(\mathrm{H}_3) : b_i > g'(\eta) \sum_{j=1}^{n} |\omega_{ij}|, i = 1, 2, \cdots, n. \tag{2.9}$$

Condition (H_3) implies

$$-b_i + \omega_{ii} g'(x_i) + \sum_{j=1, j\neq i}^{n} |\omega_{ij}| g'(x_j) < 0, \tag{2.10}$$

for $x_i = \check{c}_i, \hat{a}_i, x_j = \check{c}_j, \hat{a}_j, i, j = 1, 2, \cdots, n$, if $\omega_{ii} > 0$ for all i.

Theorem 2.2 : Under conditions (H_1), (H_2) and (H_3), there exist 2^n asymptotically stable equilibria for the Hopfield neural networks without delay (2.1).

Proof: Among the 3^n equilibria in Theorem 2.1, we consider those $\bar{\mathbf{x}} = (\bar{x}_1, \cdots, \bar{x}_n)$ with $\bar{x}_i \in \Omega_i^{\mathrm{l}}$ or Ω_i^{r}, for each i. The linearized system of (2.1) at equilibrium $\bar{\mathbf{x}}$ is

$$\frac{dy_i}{dt} = -b_i y_i + \sum_{j=1}^{n} \omega_{ij} g_j'(\overline{x}_j) y_j, \quad i = 1, 2, \cdots, n.$$

Restated, $\dot{\mathbf{y}} = A\mathbf{y}$ where $DF(\bar{\mathbf{x}}) =: A = [a_{ij}]_{n\times n}$ with

$$[a_{ij}] = \begin{pmatrix} -b_1 + \omega_{11} g'(\bar{x}_1) & \omega_{12} g'(\bar{x}_2) & \cdots & \omega_{1n} g'(\bar{x}_n) \\ \omega_{21} g'(\bar{x}_1) & -b_2 + \omega_{22} g'(\bar{x}_2) & \cdots & \omega_{2n} g'(\bar{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} g'(\bar{x}_1) & \omega_{n2} g'(\bar{x}_2) & \cdots & -b_n + \omega_{nn} g'(\bar{x}_n) \end{pmatrix}.$$

Let

$$r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}| = \sum_{j=1, j\neq i}^{n} |\omega_{ij} g'(\bar{x}_j)| = \sum_{j=1, j\neq i}^{n} |\omega_{ij}| g'(\bar{x}_j), \; i = 1, 2, \cdots, n.$$

According to the Gerschgorin's Theorem,

$$\lambda_k \in \bigcup_{i=1}^{n} B(a_{ii}, r_i),$$

for all $k = 1, 2, \cdots, n$, where λ_k are eigenvalues of A and $B(a_{ii}, r_i) := \{\zeta \in \mathbb{C} \mid |\zeta - a_{ii}| < r_i\}$. Hence, for each k, there exists some $i = i(k)$ such that

$$\mathrm{Re}(\lambda_k) < -b_i + \omega_{ii} g'(\bar{x}_i) + \sum_{j=1, j\neq i}^{n} |\omega_{ij}| g'(\bar{x}_j).$$

Notice that for each j, $g'(\xi) \leq g'(\check{c}_j)$ (resp. $g'(\xi) \leq g'(\hat{a}_j)$), if $\xi \geq \check{c}_j$ (resp. $\xi \leq \hat{a}_j$). Since $\bar{\mathbf{x}}$ is such that $\bar{x}_j \in \Omega_j^{\mathrm{l}}$ or Ω_j^{r}, we have $\bar{x}_j \geq \check{c}_j$ or $\bar{x}_j \leq \hat{a}_j$, for all $j = 1, 2, \cdots, n$. It follows that $\mathrm{Re}(\lambda_k) < 0$, by (2.10). Thus, under (H_3), all the eigenvalues of A have negative real parts. Therefore, there are 2^n asymptotically stable equilibria for system (2.1). The proof is completed.

We certainly can replace condition (H_3) by weaker ones, such as an individual condition for each equilibrium. Let $\bar{\mathbf{x}}$ be an equilibrium lying in $\mathbf{\Omega}^{\alpha}$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\alpha_i = \mathrm{r}$ or $\alpha_i = \mathrm{l}$, that is, $\bar{x}_i \in \Omega_i^{\mathrm{l}}$ or Ω_i^{r}, for each i. For such an equilibrium, we consider, for $i = 1, 2, \cdots, n$,

$$b_i > \omega_{ii} g'(\xi_i) + \sum_{j=1, j\neq i}^{n} |\omega_{ij}| g'(\xi_j), \xi_k = \check{c}_k \text{ if } \alpha_k = \mathrm{r}, \xi_k = \hat{a}_k, \text{ if } \alpha_k = \mathrm{l}, k = 1, 2, \cdots, n.$$

Such conditions are obviously much more tedious than (H_3).

3 Stability of Equilibria and the Basins of Attraction

We plan to investigate the stability of equilibrium for system (2.2), that is, with delays. We shall also explore the basins of attraction for the asymptotically stable equilibria, for both systems (2.1) and (2.2), in this section.

Notably, the function $\xi \mapsto [\omega_{ii} + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|]g'(\xi)$ is continuous, for all $i = 1, 2, \cdots, n$. From (2.10) and $\omega_{ii} > 0$, it follows that there exists a positive constant ϵ_0 such that

$$b_i > \max\{[\omega_{ii} + \sum_{j=1,j\neq i}^{n} |\omega_{ij}|]g'(\xi) : \xi = \hat{a}_i + \epsilon_0, \check{c}_i - \epsilon_0\},\ i = 1, 2, \cdots, n. \tag{3.1}$$

Herein, we choose ϵ_0 such that $\epsilon_0 < \min\{|\hat{a}_i - p_i|,\ |\check{c}_i - q_i|\}$, for all $i = 1, 2, \cdots, n$. For system (2.1), we consider the following 2^n subsets of $\mathbb{R}^n$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i = \mathrm{l}$ or r, and set

$$\tilde{\mathbf{\Omega}}^\alpha = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \tilde{\Omega}_i^{\mathrm{l}} \text{ if } \alpha_i = \mathrm{l}, x_i \in \tilde{\Omega}_i^{\mathrm{r}} \text{ if } \alpha_i = \mathrm{r}\}, \tag{3.2}$$

where

$$\tilde{\Omega}_i^{\mathrm{l}} := \{\xi \in \mathbb{R} \mid \xi \leq \hat{a}_i + \epsilon_0\}, \tilde{\Omega}_i^{\mathrm{r}} := \{\xi \in \mathbb{R} \mid \xi \geq \check{c}_i - \epsilon_0\}.$$

For system (2.2), we consider the following 2^n subsets of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i = \mathrm{l}$ or r, and set

$$\mathbf{\Lambda}^\alpha = \{\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \mid \varphi_i \in \Lambda_i^{\mathrm{l}} \text{ if } \alpha_i = \mathrm{l}, \varphi_i \in \Lambda_i^{\mathrm{r}} \text{ if } \alpha_i = \mathrm{r}\}, \tag{3.3}$$

where

$$\begin{aligned} \Lambda_i^{\mathrm{l}} &:= \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \leq \hat{a}_i + \epsilon_0,\ \text{for all } \theta \in [-\tau, 0]\} \\ \Lambda_i^{\mathrm{r}} &:= \{\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}) \mid \varphi_i(\theta) \geq \check{c}_i - \epsilon_0,\ \text{for all } \theta \in [-\tau, 0]\}. \end{aligned}$$

Theorem 3.1 : Assume that (H_1) and (H_2) hold. Then each $\tilde{\mathbf{\Omega}}^\alpha$ and each $\mathbf{\Lambda}^\alpha$ is positively invariant with respect to the solution flow generated by system (2.1) and system (2.2) respectively.

Proof : We only prove the delay case, i.e., system (2.2). Consider any one of the 2^n sets $\mathbf{\Lambda}^\alpha$. For any initial condition $\phi = (\phi_1, \phi_2, \cdots, \phi_n) \in \mathbf{\Lambda}^\alpha$, we claim that the solution $\mathbf{x}(t;\phi)$ remains in $\mathbf{\Lambda}^\alpha$ for all $t \geq 0$. If this is not true, there exists a component $x_i(t)$ of $\mathbf{x}(t;\phi)$ which is the first one (or one of the first ones) escaping from Λ_i^{l} or Λ_i^{r}. Restated, there exist some i and $t_1 > 0$ such that either $x_i(t_1) = \check{c}_i - \epsilon_0$, $\frac{dx_i}{dt}(t_1) \leq 0$, and $x_i(t) \geq \check{c}_i - \epsilon_0$ for $-\tau \leq t \leq t_1$ or $x_i(t_1) = \hat{a}_i + \epsilon_0$, $\frac{dx_i}{dt}(t_1) \geq 0$ and $x_i(t) \leq \hat{a}_i + \epsilon_0$ for $-\tau \leq t \leq t_1$. For the first case $x_i(t_1) = \check{c}_i - \epsilon_0$ and $\frac{dx_i}{dt}(t_1) \leq 0$, we derive from (2.2) that

$$\frac{dx_i}{dt}(t_1) = -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(x_i(t_1 - \tau_{ii})) + \sum_{j=1, j\neq i}^{n} \omega_{ij} g(x_j(t_1 - \tau_{ij})) + J_i \leq 0. \quad (3.4)$$

On the other hand, recalling (H_2) and previous descriptions of $\check{c}_i$ and ϵ_0, we have $\check{f}_i(\check{c}_i - \epsilon_0) > 0$ which gives

$$\begin{aligned} & -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) + k_i^- \\ = \; & -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) - \sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i > 0. \end{aligned} \quad (3.5)$$

Notice that t_1 is the first time for x_i to escape from Λ_i^{r}. We have $g(x_i(t_1 - \tau_{ii})) \geq g(\check{c}_i - \epsilon_0)$, by the monotonicity of function g. In addition, by $\omega_{ii} > 0$ and $|g(\cdot)| \leq 1$, we obtain from (3.5) that

$$\begin{aligned} & -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(x_i(t_1 - \tau_{ii})) + \sum_{j=1, j\neq i}^{n} \omega_{ij} g(x_j(t_1 - \tau_{ij})) + J_i \\ \geq \; & -b_i(\check{c}_i - \epsilon_0) + \omega_{ii} g(\check{c}_i - \epsilon_0) - \sum_{j=1, j\neq i}^{n} |\omega_{ij}| + J_i > 0, \end{aligned}$$

which contradicts (3.4). Hence, $x_i(t) \geq \check{c}_i - \epsilon_0$ for all $t > 0$. Similar arguments can be employed to show that $x_i(t) \leq \hat{a}_i + \epsilon_0$, for all $t > 0$ for the situation that $x_i(t_1) = \hat{a}_i + \epsilon_0$ and $\frac{dx_i}{dt}(t_1) \geq 0$. Therefore, $\mathbf{\Lambda}^\alpha$ is positively invariant under the flow generated by system (2.2). The assertion for system (2.1) can be justified similarly.

Theorem 3.2 : Under conditions (H_1), (H_2) and (H_3), there exist 2^n exponentially stable equilibria for system (2.2).

Proof: Consider an equilibrium $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \mathbf{\Omega}^\alpha$, for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\alpha_i = \mathrm{l}$ or r, obtained in Theorem 2.2. We consider single-variable functions $G_i(\cdot)$, defined by

$$G_i(\zeta) = b_i - \zeta - \sum_{j=1}^{n} |\omega_{ij}| g'(\xi_j) e^{\zeta \tau_{ij}},$$

where $\xi_j = \hat{a}_j + \epsilon_0$ (resp. $\check{c}_j - \epsilon_0$), if $\alpha_j = \text{l}$ (resp. r). Then, $G_i(0) > 0$, from (3.1) or (H_3). Moreover, there exists a constant $\mu > 0$ such that $G_i(\mu) > 0$, for $i = 1, 2, \cdots, n$, due to continuity of G_i. Let $\mathbf{x}(t)$ be a solution to (2.2) with initial condition $\phi \in \mathbf{\Lambda}^\alpha$ defined in (3.3). Under the translation $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, system (2.2) becomes

$$\frac{dy_i(t)}{dt} = -b_i y_i(t) + \sum_{j=1}^{n} \omega_{ij}[g(x_j(t - \tau_{ij})) - g(\bar{x}_j)], \tag{3.6}$$

where $\mathbf{y} = (y_1, \cdots, y_n)$. Now, consider functions $z_i(\cdot)$ defined by

$$z_i(t) = e^{\mu t}|y_i(t)|, \ i = 1, 2, \cdots, n. \tag{3.7}$$

The domain of definition for $z_i(\cdot)$ is identical to the interval of existence for $y_i(\cdot)$. We shall see in the following computations that the domain can be extended to $[-\tau, \infty)$. Let $\delta > 1$ be an arbitrary real number and let

$$K := \max_{1 \le i \le n} \left\{ \sup_{\theta \in [-\tau, 0]} |x_i(\theta) - \bar{x}_i| \right\} > 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that $z_i(t) < K\delta$, for $t \in [-\tau, 0]$ and all $i = 1, 2, \cdots, n$. Next, we claim that

$$z_i(t) < K\delta, \ \text{for all } t > 0, \ i = 1, 2, \cdots, n. \tag{3.9}$$

Suppose this is not the case. Then there are an $i \in \{1, 2, \cdots, n\}$ (say $i = k$) and a $t_1 > 0$ for the first time such that

$$\begin{aligned} z_i(t) &\le K\delta, \ t \in [-\tau, t_1], \ i = 1, 2, \cdots, n, \ i \ne k, \\ z_k(t) &< K\delta, \ t \in [-\tau, t_1), \\ z_k(t_1) &= K\delta, \ \text{with } \frac{d}{dt} z_k(t_1) \ge 0. \end{aligned}$$

Note that $z_k(t_1) = K\delta > 0$ implies $y_k(t_1) \ne 0$. Hence $|y_k(t)|$ and $z_k(t)$ are differentiable at $t = t_1$. From (3.6), we derive that

$$\frac{d}{dt}|y_k(t_1)| \le -b_k|y_k(t_1)| + \sum_{j=1}^{n} |\omega_{kj}| g'(\varsigma_j)|y_j(t_1 - \tau_{kj})|, \tag{3.10}$$

for some ς_j between $x_j(t_1-\tau_{kj})$ and $\bar{x}_j$. Hence, from (3.7) and (3.10),

$$\begin{aligned}\frac{dz_k(t_1)}{dt} &\le \mu e^{\mu t_1}|y_k(t_1)| + e^{\mu t_1}[-b_k|y_k(t_1)| + \sum_{j=1}^{n}|\omega_{kj}|g'(\varsigma_j)|y_j(t_1-\tau_{kj})|] \\ &\le \mu z_k(t_1) - b_k z_k(t_1) + \sum_{j=1}^{n}|\omega_{kj}|g'(\varsigma_j)e^{\mu\tau_{kj}}z_j(t_1-\tau_{kj}) \\ &\le -(b_k-\mu)z_k(t_1) + \sum_{j=1}^{n}|\omega_{kj}|g'(\xi_j)e^{\mu\tau_{kj}}[\sup_{\theta\in[t_1-\tau,t_1]} z_j(\theta)], \qquad (3.11)\end{aligned}$$

where $\xi_j = \hat{a}_j + \epsilon_0$ (resp. $\check{c}_j - \epsilon_0$), if $\alpha_j = \mathrm{l}$ (resp. r). Herein, the invariance property of $\mathbf{\Lambda}^\alpha$ in Theorem 3.1 has been applied. Due to $G_i(\mu) > 0$, we obtain

$$\begin{aligned}0 &\le \frac{dz_k(t_1)}{dt} \le -(b_k-\mu)z_k(t_1) + \sum_{j=1}^{n}|\omega_{kj}|g'(\xi_j)e^{\mu\tau_{kj}}[\sup_{\theta\in[t_1-\tau,t_1]} z_j(\theta)] \\ &< -\{b_i - \mu - \sum_{j=1}^{n}|\omega_{ij}|g'(\xi_j)e^{\mu\tau_{kj}}\}K\delta \\ &< 0, \qquad (3.12)\end{aligned}$$

which is a contradiction. Hence the claim (3.9) holds. Since $\delta > 1$ is arbitrary, by allowing $\delta \to 1^+$, we have $z_i(t) \le K$ for all $t > 0$, $i = 1, 2, \cdots, n$. We then use (3.7) and (3.8) to obtain

$$|x_i(t) - \bar{x}_i| \le e^{-\mu t}\max_{1\le j\le n}(\sup_{\theta\in[-\tau,0]}|x_j(\theta)-\bar{x}_j|),$$

for $t > 0$ and all $i = 1, 2, \cdots, n$. Therefore, $\mathbf{x}(t)$ is exponentially convergent to $\bar{\mathbf{x}}$. This completes the proof.

In the following, we employ the theory of local Lyapunov functional [23] and the Halanay-type inequality to establish other sufficient conditions for asymptotic stability and exponential stability for equilibrium of system (2.2).

Lemma 3.3 [7, 22]: Let $v(t)$ be a nonnegative continuous function on $[t_0-\tau, t_0]$, where τ is a positive constant. Suppose

$$\frac{dv(t)}{dt} \le -\alpha v(t) + \beta[\sup_{s\in[t-\tau,t]} v(s)],$$

for $t \ge t_o$. If $\alpha > \beta > 0$, then as $t \ge t_0$, there exist constants $\gamma > 0$ and $k > 0$ such that

$$v(t) \le k e^{-\gamma(t-t_0)},$$

where

$$k = \sup_{s \in [t_0 - \tau, t_0]} v(s)$$

and γ is the unique positive solution of equation

$$\gamma = \alpha - \beta e^{\gamma \tau}.$$

Theorem 3.4 : There exist 2^n asymptotically stable equilibria for system (2.2) under conditions (H_1), (H_2) and one of the following conditions :

$$(H_4) \qquad 2b_i > \sum_{j=1}^{n} |\omega_{ij}| + \sum_{j=1}^{n} |\omega_{ij}| [g'(\xi_j)]^2, \text{ for all } i = 1, 2, \cdots, n,$$

$$(H_5) \qquad 2b_i > \sum_{j=1}^{n} |\omega_{ij}| + [g'(\xi_i)]^2 \sum_{j=1}^{n} |\omega_{ji}|, \text{ for all } i = 1, 2, \cdots, n,$$

$$(H_6) \qquad \min_{1 \le i \le n} [2b_i - \sum_{j=1}^{n} |\omega_{ij}| g'(\xi_j)] > \max_{1 \le i \le n} [\sum_{j=1}^{n} |\omega_{ji}| g'(\xi_i)],$$

where $\xi_k = \hat{a}_k,\ \check{c}_k, k = 1, 2, \cdots, n$.

Proof : The following computations are reserved for solutions in each of the 2^n invariant regions $\mathbf{\Lambda}^\alpha$. (i) As in the proof of Theorem 3.2, there exists a positive constant μ such that

$$2b_i - \mu - \sum_{j=1}^{n} |\omega_{ij}| - \sum_{j=1}^{n} |\omega_{ij}| [g'(\xi_j)]^2 e^{\mu \tau_{ij}} > 0, \tag{3.13}$$

for all $i = 1, 2, \cdots, n$. Define $z_i(t) = e^{\mu t} y_i^2(t)$, where $y_i(t)$ is as in the proof of Theorem 3.2. Recalling (3.6), we derive that

$$\begin{aligned}
\frac{dz_k(t_1)}{dt} &= \mu e^{\mu t_1} [y_k(t_1)]^2 + 2e^{\mu t_1} y_k(t_1)) \dot{y}_k(t_1) \\
&= \mu e^{\mu t_1} [y_k(t_1)]^2 - 2b_k e^{\mu t_1} [y_k(t_1)]^2 + 2 \sum_{j=1}^{n} \omega_{kj} e^{\mu t_1} y_k(t_1) [g(x_j(t_1 - \tau_{kj})) - g(\overline{x}_j)] \\
&\le -(2b_k - \mu) z_k(t_1) + \sum_{j=1}^{n} |\omega_{kj}| e^{\mu t_1} \{[y_k(t_1)]^2 + [g(x_j(t_1 - \tau_{kj})) - g(\overline{x}_j)]^2\} \\
&\le -(2b_k - \mu) z_k(t_1) \\
&\quad + \sum_{j=1}^{n} |\omega_{kj}| e^{\mu t_1} [y_k(t_1)]^2 + \sum_{j=1}^{n} |\omega_{kj}| e^{\mu t_1} [g'(\varsigma_j)]^2 [y_j(t_1 - \tau_{kj})]^2 \\
&\le -[2b_k - \mu - \sum_{j=1}^{n} |\omega_{kj}|] z_k(t_1) + \sum_{j=1}^{n} |\omega_{kj}| [g'(\xi_j)]^2 e^{\mu \tau_{kj}} [\sup_{s \in [t_1 - \tau, t_1]} z_j(s)].
\end{aligned}$$

The assertion under condition (H_4) can be justified by similar arguments as the proof of Theorem 3.2.

(ii) Recall (3.6), and let

$$V(\mathbf{y})(t)=\sum_{i=1}^{n} y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|\int_{t-\tau_{ij}}^{t}[g(x_j(s))-g(\overline{x}_j)]^2 ds.$$

By (H_5), we derive

$$\begin{aligned}
\frac{dV(\mathbf{y})(t)}{dt} &= 2\sum_{i=1}^{n} y_i(t)\{-b_i y_i(t)+\sum_{j=1}^{n}\omega_{ij}[g(x_j(t-\tau_{ij}))-g(\overline{x}_j)]\} \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g(x_j(t))-g(\overline{x}_j)]^2-\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g(x_j(t-\tau_{ij}))-g(\overline{x}_j)]^2 \\
&\le -2\sum_{i=1}^{n} b_i y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|\{y_i^2(t)+[g(x_j(t-\tau_{ij}))-g(\overline{x}_j)]^2\} \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g(x_j(t))-g(\overline{x}_j)]^2-\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g(x_j(t-\tau_{ij}))-g(\overline{x}_j)]^2 \\
&= -2\sum_{i=1}^{n} b_i y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g(x_j(t))-g(\overline{x}_j)]^2 \\
&\le -2\sum_{i=1}^{n} b_i y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|[g'(\xi_j)]^2 y_j^2(t) \\
&= \sum_{i=1}^{n}\{-2b_i+\sum_{j=1}^{n}|\omega_{ij}|+[g'(\xi_i)]^2\sum_{j=1}^{n}|\omega_{ji}|\}y_i^2(t) \\
&< 0.
\end{aligned}$$

We conclude the asymptotical stability for equilibrium $\bar{\mathbf{x}}$, via applying the theory of local Lyapunov functional, cf. [23].

(iii) Recall (3.6), and let

$$W(\mathbf{y})(t)=\frac{1}{2}\sum_{i=1}^{n} y_i^2(t). \tag{3.14}$$

Then,

$$\begin{aligned}
\frac{dW(\mathbf{y})(t)}{dt} &= \sum_{i=1}^{n} y_i(t)\{-b_i y_i(t) + \sum_{j=1}^{n} \omega_{ij}[g(x_j(t-\tau_{ij})) - g(\overline{x}_j)]\} \\
&\leq \sum_{i=1}^{n}\{-b_i y_i^2(t) + \sum_{j=1}^{n} |\omega_{ij}||y_i(t)||y_j(t-\tau_{ij})|g'(\varsigma_j)\} \\
&\leq \sum_{i=1}^{n}\{-b_i y_i^2(t) + \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}|g'(\varsigma_j)[y_i^2(t) + y_j^2(t-\tau_{ij})]\} \\
&= -\sum_{i=1}^{n}[b_i - \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}|g'(\varsigma_j)]y_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}[\sum_{j=1}^{n} |\omega_{ij}|g'(\varsigma_j)y_j^2(t-\tau_{ij})] \\
&= -\sum_{i=1}^{n}[b_i - \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}|g'(\varsigma_j)]y_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}[\sum_{j=1}^{n} |\omega_{ji}|g'(\varsigma_i)y_i^2(t-\tau_{ji})] \\
&\leq -\sum_{i=1}^{n}[b_i - \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}|g'(\varsigma_j)]y_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}[\sum_{j=1}^{n} |\omega_{ji}|g'(\varsigma_i) \sup_{t-\tau\leq s\leq t} y_i^2(s)] \\
&\leq -\sum_{i=1}^{n}[b_i - \frac{1}{2}\sum_{j=1}^{n} |\omega_{ij}|g'(\xi_j)]y_i^2(t) + \frac{1}{2}[\max_{1\leq i\leq n}\sum_{j=1}^{n} |\omega_{ji}|g'(\xi_i)] \sum_{i=1}^{n} \sup_{t-\tau\leq s\leq t} y_i^2(s) \\
&\leq -\alpha W(\mathbf{y})(t) + \beta \sup_{t-\tau\leq s\leq t} W(\mathbf{y})(s),
\end{aligned}$$

where

$$\alpha = \min_{1\leq i\leq n}\left(2b_i - \sum_{j=1}^{n} |\omega_{ij}|g'(\xi_j)\right), \beta = \max_{1\leq i\leq n}\sum_{j=1}^{n} |\omega_{ji}|g'(\xi_i).$$

By (H_6), we have $\alpha > \beta > 0$ and by using Lemma 3.3, we obtain that

$$W(\mathbf{y})(t) \leq \left(\sup_{-\tau\leq s\leq 0} W(\mathbf{y})(s)\right) e^{-\gamma t}, \tag{3.15}$$

for all $t \geq 0$, where γ is the unique solution of $\gamma = \alpha - \beta e^{\gamma\tau}$. It follows that

$$\frac{1}{2}\sum_{i=1}^{n} y_i^2(t) \leq \left[\sup_{-\tau\leq s\leq 0}\left(\frac{1}{2}\sum_{i=1}^{n} y_i^2(s)\right)\right] e^{-\gamma t}. \tag{3.16}$$

Hence, the equilibrium $\bar{\mathbf{x}}$ is asymptotically stable.

Corollary 3.5 : Under conditions (H_1), (H_2), and (H_4) or (H_6), there exist 2^n exponentially stable equilibria for system (2.2).

We observe from equations (2.1) and (2.2) that for every i,

$F_i(\mathbf{x}),\ \tilde{F}_i(\mathbf{x}_t) < 0$ whenever $x_i > 0$ is sufficiently large,

$F_i(\mathbf{x}),\ \tilde{F}_i(\mathbf{x}_t) > 0$ whenever $x_i < 0$ with $|x_i|$ sufficiently large, since $b_i > 0$ and $\sum_{j=1}^{n} \omega_{ij} g_j(x_j(t)) + J_i$, and $\sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i$ are bounded, for any $\mathbf{x}$ and $\mathbf{x}_t$. Therefore, it can be concluded that every solution of (2.1) and (2.2) is bounded in forward time.

4 Periodic Orbits for System with Periodic Inputs

In this section, we study the periodic solutions of the Hopfield-type neural networks with delays and periodic inputs

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j(x_j(t-\tau_{ij})) + J_i(t), \; i = 1, 2, \cdots, n, \tag{4.1}$$

where $J_i : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $i = 1, 2, \cdots, n$, are continuously periodic functions with period T_ω, i.e, $J_i(t + T_\omega) = J_i(t)$.

Theorem 4.1: Under conditions (H_1), (H_2) and (H_3), there exist 2^n exponentially stable T_ω-period solutions for system (4.1).

Proof: We define the norm

$$\parallel \phi \parallel = \max_{1 \le i \le n} \left(\sup_{s \in [-\tau, 0]} |\phi_i(s)| \right).$$

Consider $\varphi, \psi \in \mathbf{\Lambda}^\alpha$, for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\alpha_i =$ l or r, obtained in (3.3). We denote as

$$\mathbf{x}(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \cdots, x_n(t, \varphi))^T,$$

$$\mathbf{x}(t, \psi) = (x_1(t, \psi), x_2(t, \psi), \cdots, x_n(t, \psi))^T,$$

the solutions of (4.1) through $(0, \varphi)$ and $(0, \psi)$, respectively.

Define

$$\mathbf{x}_t(\varphi) = \mathbf{x}(t + \theta, \varphi), \; \theta \in [-\tau, 0], \; t \ge 0,$$

then $\mathbf{x}_t(\varphi) \in \mathbf{\Lambda}^\alpha$ for all $t \ge 0$. From (4.1) we have

$$\begin{aligned} \frac{d}{dt}[x_i(t, \varphi) - x_i(t, \psi)] &= -b_i(x_i(t, \varphi) - x_i(t, \psi)) \\ &\quad + \sum_{j=1}^{n} \omega_{ij} \left[g_j(x_j(t - \tau_{ij}, \varphi)) - g_j(x_j(t - \tau_{ij}, \psi)) \right], \end{aligned}$$

where $t \geq 0,\ i = 1, 2, \cdots, n$. Similar to the proof of Theorem 3.2, we can get

$$|x_i(t,\varphi) - x_i(t,\psi)| \leq e^{-\mu t} \max_{1\leq\ j\leq\ n} \left(\sup_{s\in[-\tau,\ 0]} |x_j(s,\varphi) - x_j(s,\psi)| \right),$$

where $\mu > 0$ is a small constant. Therefore, we have

$$\|\mathbf{x}_t(\varphi) - \mathbf{x}_t(\psi)\| \leq e^{-\mu t}\|\varphi - \psi\|,\ t \geq 0.$$

One can easily obtain from the formula above that

$$\|\mathbf{x}_t(\varphi) - \mathbf{x}_t(\psi)\| \leq e^{-\mu(t-\tau)}\|\varphi - \psi\|,\ t \geq 0. \tag{4.2}$$

We can choose a positive integer m such that

$$e^{-\mu(mT_\omega - \tau)} = K < 1.$$

Define a Poincare mapping $P : \mathbf{\Lambda}^\alpha \to \mathbf{\Lambda}^\alpha$ by

$$P\varphi = \mathbf{x}_{T_\omega}(\varphi).$$

Then we can derive from (4.2) that

$$\|P^m\varphi - P^m\psi\| \leq K\|\varphi - \psi\|.$$

This inequality implies that P^m is a contraction mapping, hence there exists a unique fixed point $\overline{\varphi} \in \mathbf{\Lambda}^\alpha$ such that $P^m\overline{\varphi} = \overline{\varphi}$. Note that

$$P^m(P\overline{\varphi}) = P(P^m\overline{\varphi}) = P\overline{\varphi}.$$

Then $P\overline{\varphi} \in \mathbf{\Lambda}^\alpha$ is also a fixed point of P^m, and so $P\overline{\varphi} = \overline{\varphi}$, i.e.

$$\mathbf{x}_{T_\omega}(\overline{\varphi}) = \overline{\varphi}.$$

Let $\mathbf{x}(t,\overline{\varphi})$ be the solution of (4.1) through $(0,\overline{\varphi})$, then $\mathbf{x}(t+T_\omega,\overline{\varphi})$ is also a solution of (4.1), and note that

$$\mathbf{x}_{t+T_\omega}(\overline{\varphi}) = \mathbf{x}_t(x_{T_\omega}(\overline{\varphi})) = \mathbf{x}_t(\overline{\varphi}),\ t \geq 0;$$

therefore

$$\mathbf{x}(t + T_\omega,\overline{\varphi}) = \mathbf{x}(t,\overline{\varphi}),\ t \geq 0.$$

This shows that $\mathbf{x}(t,\overline{\varphi})$ is exactly one T_ω-period solution of (4.1) in $\mathbf{\Lambda}^\alpha$, and it easy to see that all other solutions of (4.1) in $\mathbf{\Lambda}^\alpha$ converge exponentially to it as $t \to +\infty$. Thus, there are 2^n exponentially stable T_ω-period solutions for system (4.1).

5 Numerical Illustrations

In this section, we present two examples to illustrate our results.

Example 5.1 : Consider the two-dimensional delayed Hopfield neural networks

$$\frac{dx_1(t)}{dt} = -x_1(t) + 18g_1(x_1(t-10)) + 5g_2(x_2(t-10)) - 9$$
$$\frac{dx_2(t)}{dt} = -3x_2(t) + 5g_1(x_1(t-10)) + 30g_2(x_2(t-10)) - 15,$$

where $g_1(x) = g_2(x) = g(x)$ in (2.4) with $\varepsilon = 0.5$. A computation gives

$$\hat{f}_1(x_1) = -x_1 + 18g(x_1) - 4,$$
$$\check{f}_1(x_1) = -x_1 + 18g(x_1) - 14,$$
$$\hat{f}_2(x_2) = -3x_2 + 30g(x_2) - 10,$$
$$\check{f}_2(x_2) = -3x_2 + 30g(x_2) - 20.$$

Herein, the parameters satisfy our conditions in Theorem 3.2:
Condition (H_1) :

$$0 < \tfrac{b_1\varepsilon}{\omega_{11}} = \tfrac{1}{36} < \tfrac{1}{4}, \quad 0 < \tfrac{b_2\varepsilon}{\omega_{22}} = \tfrac{1}{20} < \tfrac{1}{4}.$$

Condition (H_2) :

$$\hat{f}_1(p_1) = -1.722534 < 0, \quad \check{f}_1(q_1) = 1.722534 > 0,$$
$$\hat{f}_2(p_2) = -4.085501 < 0, \quad \check{f}_2(q_2) = 4.085501 > 0.$$

Condition (H_3) :

$$b_1 = 1 > 0.059932 = \omega_{11}g'(\eta) + |\omega_{12}|g'(\eta),$$
$$b_2 = 3 > 0.091201 = |\omega_{21}|g'(\eta) + \omega_{22}g'(\eta),$$

where $\eta = \pm 3.320288$ is defined in (2.9). Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$ are listed in Table 1.

$\hat{a}_1 = -3.993889$	$p_1 = -1.762747$	$\hat{b}_1 = -0.757751$	q_1=1.762747	$\hat{c}_1$=14
$\check{a}_1 = -14$		$\check{b}_1$=0.757751		$\check{c}_1$=3.993889
$\hat{a}_2 = -3.320288$	$p_2 = -1.443635$	$\hat{b}_2 = -0.452309$	q_2=1.443635	$\hat{c}_2$=6.666650
$\check{a}_2 = -6.666650$		$\check{b}_2$=0.452309		$\check{c}_2$=3.320288

Table 1: Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

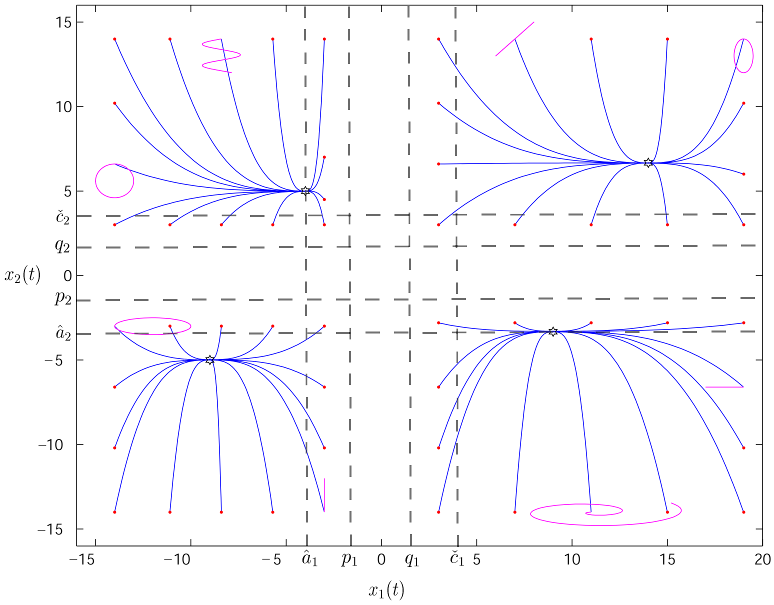

Figure 3: Illustrations for the dynamics in Example 5.1.

The dynamics of this system is illustrated in Figure 3, where evolutions of 56 initial conditions have been tracked. The constant initial conditions are plotted in red colors, and the time-dependent initial conditions are plotted in purple. The evolutions of components $x_1(t)$, $x_2(t)$ are depicted in Figures 4, 5, respectively. There are four exponentially stable equilibria in the system, as confirmed by our theory. The simulations demonstrate the convergence to these four equilibria from initial functions ϕ lying in respective basin of the equilibrium.

Example 5.2 : In this example, we simulate the delayed Hopfield neural networks with continuously periodic inputs.

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 20g_1(x_1(t-10)) + 4g_2(x_2(t-10)) - 10 + 3\mathtt{sin(t)}\\ \frac{dx_2(t)}{dt} &= -3x_2(t) + 4g_1(x_1(t-10)) + 30g_2(x_2(t-10)) - 15 + 3\mathtt{cos(t)}\end{aligned}$$

where $g_1(x) = g_2(x) = g(x)$ in (2.4) with $\varepsilon = 0.5$. A computations gives

$$\begin{aligned}\hat{f}_1(x_1) &= -x_1 + 20g(x_1) - 6 + 3\mathtt{sin(t)},\\ \check{f}_1(x_1) &= -x_1 + 20g(x_1) - 14 + 3\mathtt{sin(t)},\\ \hat{f}_2(x_2) &= -3x_2 + 30g(x_2) - 11 + 3\mathtt{cos(t)},\\ \check{f}_2(x_2) &= -3x_2 + 30g(x_2) - 19 + 3\mathtt{cos(t)}.\end{aligned}$$

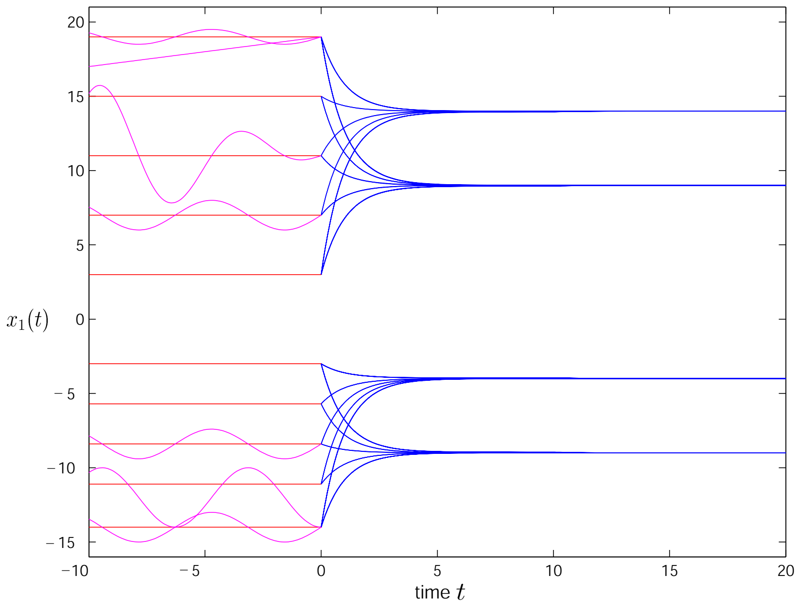

Figure 4: Evolution of state variable $x_1(t)$ in Example 5.1.

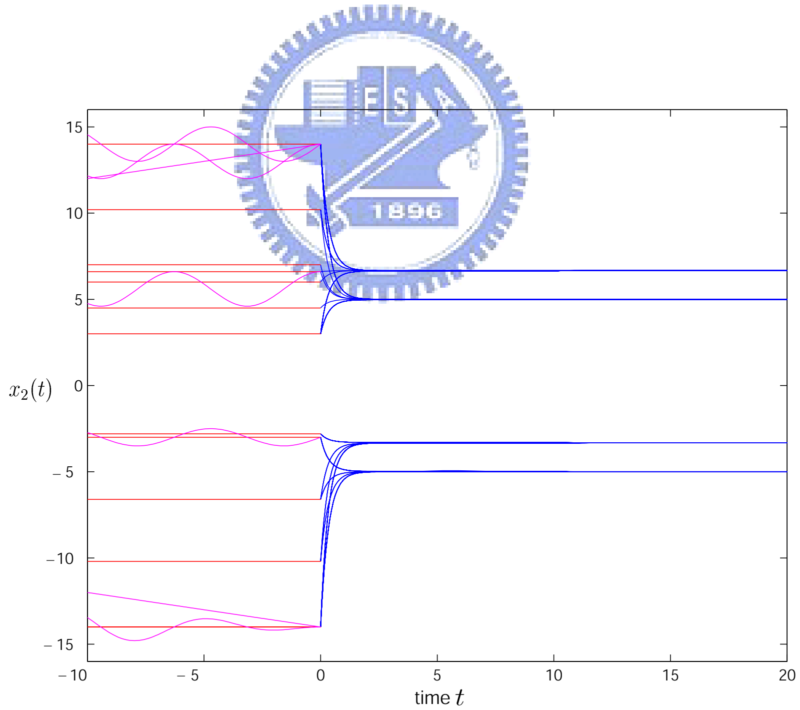

Figure 5: Evolution of state variable $x_2(t)$ in Example 5.1.

Herein,the parameters satisfying our conditions in Theorem 4.1 :

Condition (H_1) :

$$0 < \frac{b_1 \varepsilon}{\omega_{11}} = \frac{1}{40} < \frac{1}{4}, \quad 0 < \frac{b_2 \varepsilon}{\omega_{22}} = \frac{1}{20} < \frac{1}{4}.$$

Condition (H_2) :

$$\begin{aligned} \hat{f}_1(p_1) &= -3.668387 + 3\mathtt{sin(t)} < 0, \quad \check{f}_1(q_1) = 3.668387 + 3\mathtt{sin(t)} > 0, \\ \hat{f}_2(p_2) &= -5.085501 + 3\mathtt{cos(t)} < 0, \quad \check{f}_2(q_2) = 5.085501 + 3\mathtt{cos(t)} > 0. \end{aligned}$$

Condition (H_3) :

$$\begin{aligned} b_1 &= 1 > 0.255133 = \omega_{11} g'(\eta) + |\omega_{12}| g'(\eta), \\ b_2 &= 3 > 0.361438 = |\omega_{21}| g'(\eta) + \omega_{22} g'(\eta), \end{aligned}$$

where $\eta = \pm 2.613229$ is defined in (2.9).

Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$ are listed in Table 2.

p_1=-1.818446	q_1=1.818446	
$\hat{a}_1$=-9.000000∼ -2.944290	$\hat{b}_1$=-1.138254∼ -0.111624	$\hat{c}_1$=11.00000∼ 17.00000
$\check{a}_1$=-17.00000∼ -11.00000	$\check{b}_1$=0.111624∼ 1.138254	$\check{c}_1$=2.944290∼ 9.000000
p_2=-1.443635	q_2=1.443635	
$\hat{a}_2$=-4.665781∼ -2.613229	$\hat{b}_2$=-0.705313∼ -0.083576	$\hat{c}_2$=5.333100∼ 7.333329
$\check{a}_2$=-7.333329∼ -5.333100	$\check{b}_2$=0.083576∼ 0.705313	$\check{c}_2$=2.613229∼ 4.665781

Table 2: Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

The dynamics of this system is illustrated in Figure 6. The evolutions of component $x_1(t), x_2(t)$ are depicted in Figures 7, 8, respectively. There are four periodic solutions in the system , as confirmed by our theory. The simulations demonstrate the convergence to these four periodic solutions from initial functions ϕ lying in respective basin of the periodic solutions.

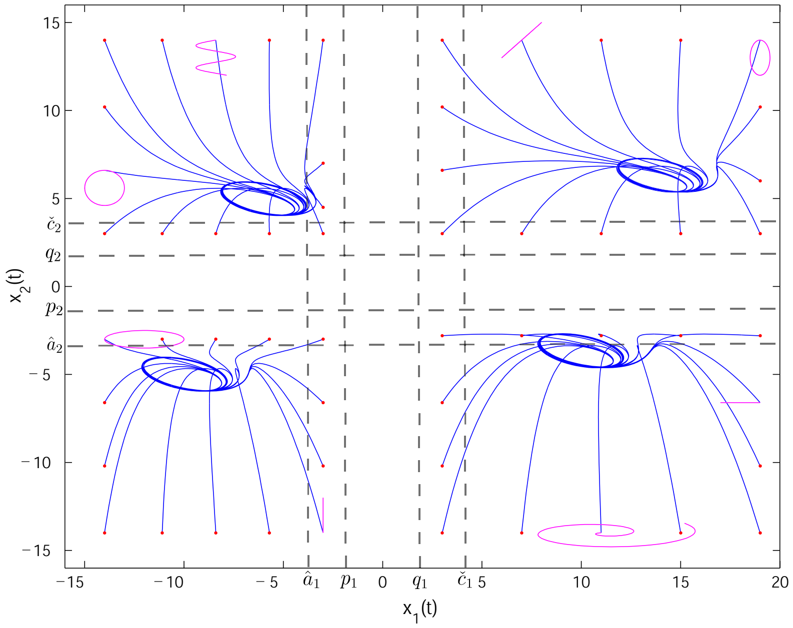

Figure 6: Illustrations for the dynamics in Example 5.2.

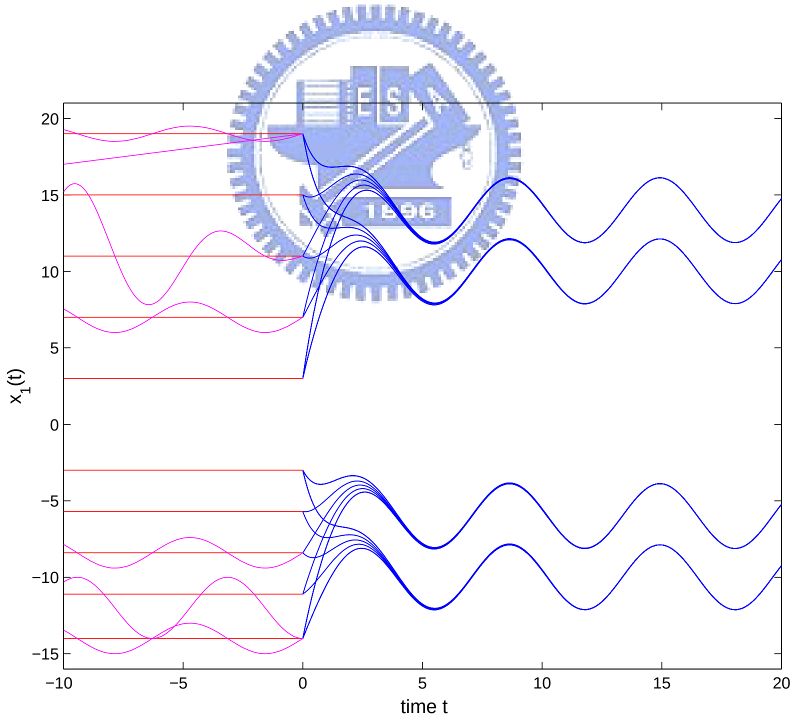

Figure 7: Evolution of state variable $x_1(t)$ in Example 5.2.

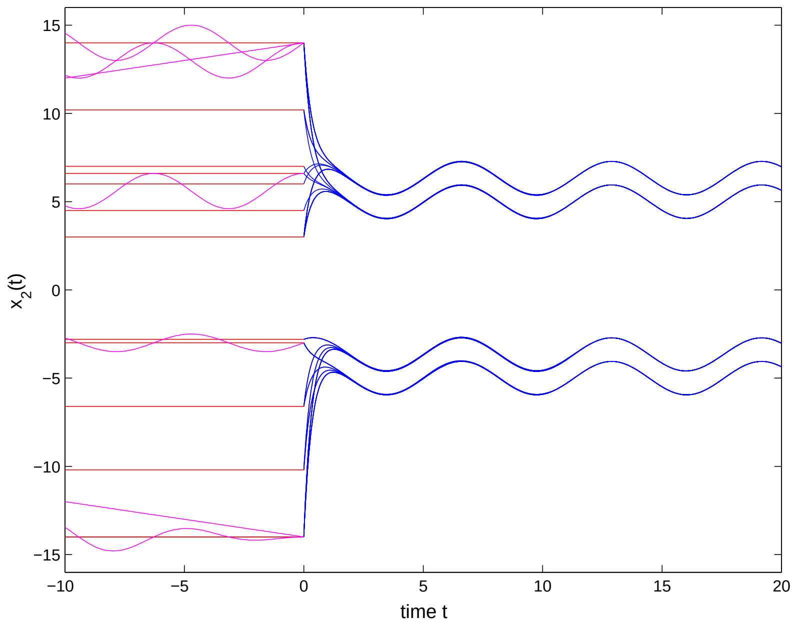

Figure 8: Evolution of state variable $x_2(t)$ in Example 5.2.

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