# 國 立 交 通 大 學

## 應 用 數 學 系


## 碩士論文

多頻及多向傳輸模型下的三級式不阻塞克勞斯網路

On Nonblocking Three Stage Clos Networks under the Multirate-Multicast Model


研 究 生: 張惠蘭

指導教授: 黃光明 教授



中 華 民 國 九 十 四 年 六 月

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# 摘要

這篇論文的目的是把討論在多頻及多向傳輸模型下的三級式不阻塞克勞斯 (Clos) 網路上的零碎結果做統整, 並推廣部分結果及補齊一些漏洞。我們也針對目前已有的結果做數值上的比較。期望我們的統整能幫助後來的研究者釐清問題及將此重要的模型應用到通訊及計算機網路上。

中 華 民 國 九 十 四 年 六 月

# On Nonblocking Three Stage Clos Networks under the Multirate-Multicast Model

Student: Huilan Chang Advisor: Frank K. Hwang

Department of Applied Mathematics
National Chiao Tung University

June, 2005

## Abstract

The purpose of this thesis is to survey the scattered piece-meal results on the multirate-multicast model for nonblocking three-stage Clos networks, to fill some gaps and to extend some results. We also do some numerical comparisons among existing results. It is hoped that our survey will facilitate future researchers to identify open problems and to make further inroads into this very important model with applications to communication and computer networks.

# 誌謝

在交大的頭半年過得並不是很順利, 憑著以前念書的習慣反而做不好研究, 直到黃光明老師願意收我做他的研究生後, 這個旁徨無助的時期才露出了曙光; 因為老師的特質讓我們可以學到跟別人不一樣的東西, 看文章的角度、通盤性的思考、做研究的態度及人生哲學, 都是我們親身體驗到的。很感謝黃老師能引我入門, 以及教給我的這一切。

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# Contents

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# 1 Preliminaries

In this section, we introduce some general concepts, terminology and definitions that are used in this article.

## 1.1 Model description

Multistage switching networks are composed of crosspoint switching elements or, more specifically, crosspoints that are usually grouped together into building-block subnetworks called *switch modules*. A *three-stage Clos network*, denoted by $C(n_1, r_1, m, n_2, r_2)$, has $r_1$ switch modules of size $n_1 \times m$ in stage 1, $m$ switch modules of size $r_1 \times r_2$ in stage 2, $r_2$ switch modules of size $m \times n_2$ in stage 3, and exactly one link between every two switch modules in its consecutive stages. A link between two stages is called an *internal link*. Figure 1 shows a $C(2, 4, 3, 3, 2)$ network. In a three-stage network, stage 1 is also referred to as the *input stage*, stage 2 as the *middle stage*, and stage 3 as the *output stage*. For the symmetrical case where $n_1 = n_2 = n$ and $r_1 = r_2 = r$, the three-stage Clos network is denoted as a $C(n, m, r)$ network. If each internal link in a $C(n_1, r_1, m, n_2, r_2)$ is replaced by $d$ links, then the network is denoted by $C(n_1, r_1, m, n_2, r_2, d)$. In general, the set of input ports is denoted as $I = \{1, 2, \cdots, r_1 n_1\}$, the set of output ports is denoted as $\{o_1, o_2, \cdots, o_{r_2 n_2}\}$ and the set of switch modules in the output stage is denoted as $O = \{O_1, O_2, \cdots, O_{r_2}\}$. We also refer to links incident to input and outputs as *external links*.

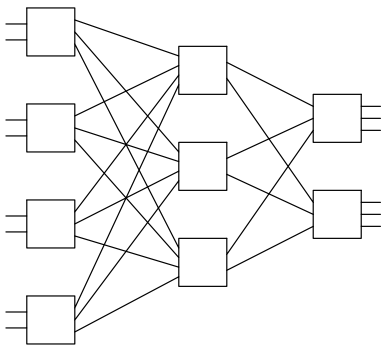

Figure 1: $C(2, 4, 3, 3, 2)$

## 1.2 Multicast environment

With the demand of multicast transmissions such as video-conference, internet games and distance learning, we need a multicast network to accomplish those demands. In the multicast network, a switch is said to have the *fan-out* capability if the switch itself can route multicast traffic without blocking, i.e., any inlet can be connected to any number of idle outlets regardless of other connections. Usually, a switch module is assumed to have the fan-out capability. A multicast request is denoted as $(x, \{o_1, o_2, \cdots, o_s\})$ where $x$ is asked to be connected with output port $o_j$, $j = 1, 2, \cdots, s$. We often simplify its notation as $C = (x, o(x))$.

If the cardinality of $o(x)$, $|o(x)|$, is restricted to at most $f$, the traffic is called an *$f$-cast traffic*. If $o(x)$ is the set of output ports for all $x$, the traffic is called *broadcast*. And we call a request *point-to-point* if $|o(x)| = 1$.

$f$-cast traffics can be divided into two types according to whether additional receivers can be added after a multicast request is already connected. We will use *open-end traffic* (which allows additions) and *closed-end traffic* (which does not allow) to differentiate these two types. Suppose the input port $x$ has generated $r$ requests $(x^i, o(x^i), \omega_i)$, $1 \leq i \leq r$. If a new request from $x$, carrying the same message of $x^i$, to some output ports unconnected yet is allowed, then we call it the *open-end traffic*; if not allowed, then the traffic is closed-end. Note that for either type of traffic, $x$ is always allowed to generate a new request $(x^{r+1}, o(x^{r+1}), \omega_{r+1})$ with a new message as long as $\sum_{i=1}^{r+1} \omega_i$ is under the capacity.

## 1.3 Multirate environment

The need of a multirate network comes from the desire to integrate multimedia transmissions such as audio, data, image and video into one switching network. As different media require a broad range of bandwidths, each request is associated with its required amount of bandwidth, called its *rate*. A link in the switching network has a capacity and can carry as many requests as desired as long as the sum of their rates does not exceed the capacity of the link.

There are two basic multirate models: *discrete* and *continuous*. The discrete model assumes that there is a finite number of distinct rates and the continuous model assumes that all rates are within a given interval. For the continuous model, it is customary to assume that each internal link has capacity 1 and each request has a normalized rate $\omega$, $b \leq \omega \leq B$, where $b$ and $B$, $0 < b \leq B \leq 1$, are bounds of $\omega$. Each external link is assumed to have capacity $\beta$, $B \leq \beta$, i.e., it can generate any number of requests in a frame as long as the sum of rates of these requests does not exceed $\beta$. We call this the $\beta[b, B]$ model (which can also be used for discrete multirate model). We use $\beta(b, B]$, $\beta[b, B)$, $\beta(b, B)$ to exclude $b$ or $B$, and omit $\beta$ if $\beta = 1$, respectively. Note that, [1,1] represents the classical model. In practice, $\beta$ is usually less than 1, representing a 1-level speed-up.

## 1.4 Multirate-multicast connections

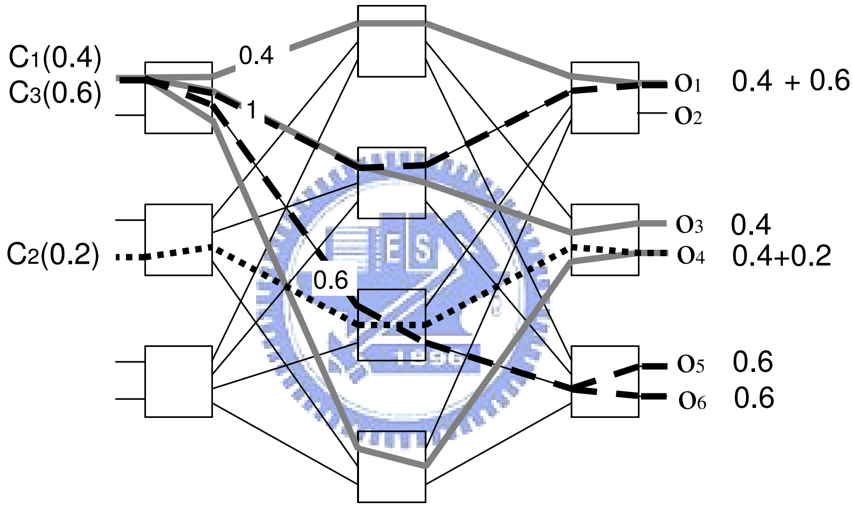


Figure 2: $C(2, 4, 3)$ symmetric Clos Network with connection requests under multirate-multicast model.

In a multirate network, we call an input port or an output port $\omega$-*idle*, $0 \leq \omega \leq 1$, if the fraction of available bandwidth for that input port or output port is at least $\omega$. A multicast connection with weight $\omega$ is a connection from an $\omega$-idle input port to a set of $\omega$-idle output ports using the same fraction of the bandwidth, $\omega$, on every intermediate link of this multicast connection in the network. We will assume that every switch module in our multicast networks has fan-out capability.

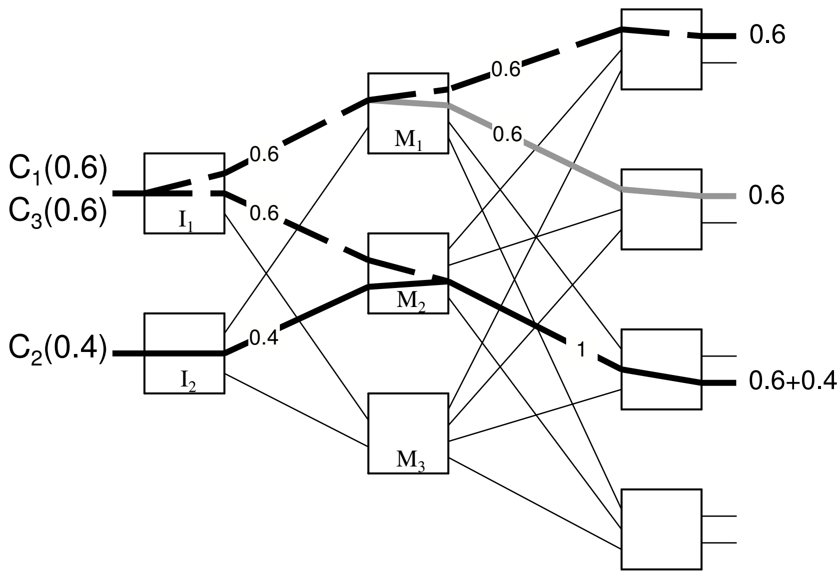


Figure 3: $C(1, 2, 3, 2, 4)$ Clos Network with connection requests under open-end traffic.

In a $C(n_1, r_1, m, n_2, r_2)$ Clos network, a multirate-multicast request is denoted as $\boldsymbol{C} = (x, o(x), \omega)$, where $x \in \{1, \cdots, n_1r_1\}$ is an input port, $o(x)$ is a subset of $\{o_1, o_2, \cdots, o_{r_2n_2}\}$, and $\omega$ is a required weight of the request. Figure 2 shows three multicast connection requests, where $\boldsymbol{C}_1 = (1, \{o_1, o_3, o_4\}, 0.4)$, $\boldsymbol{C}_2 = (4, \{o_4\}, 0.2)$ and $\boldsymbol{C}_3 = (1, \{o_1, o_5, o_6\}, 0.6)$. Again, if $|o(x)|$ is restricted to at most $f$, the traffic is called a multirate $f$-cast traffic. Multirate $f$-cast networks are also divided into open-end traffic and closed-end traffic. Figure 3 shows three request connections on $C(1, 2, 3, 2, 4)$ Clos network for (0,1] open-end 3-cast assignments: $\mathbf{C}_1 = (1, \{o_1, o_6\}, 0.6)$ and $\mathbf{C}_2 = (2, \{o_6\}, 0.4)$. Since $|o(1)| = 2 < 3$, we can add a new receiver $o_2$ to request $\mathbf{C}_1$, that is the request $\mathbf{C}_3$. Notice that, the load of the link between $I_1$ and $M_1$ remains 0.6 after $\mathbf{C}_3$ is routed.

We said a connection request is compatible to the existing configuration if adding this request does not cause capacity overflows for any external links.

Nonblocking switching networks can be categorized into three types:

1) Strictly Nonblocking Switching Network (**SNB**): A connection request compatible to the existing configuration can always be routed.

2) Rearrangeable Nonblocking Switching Network (**RNB**): A connection

request compatible to the existing configuration can always be routed, although it may need to rearrange the existing connections.

3) Wide-Sense Nonblocking Switching Network (**WSNB**): An algorithm exists for setting connections such that a new connection request compatible to the existing configuration can always be routed.

Since a closed-end traffic sequence is also an open-end traffic sequence, while a routing for open-end traffic is also one for closed-end traffic, we have

**Theorem 1.4.1.** *A network is multirate multicast nonblocking under the open-end traffic implies it is so under the closed-end traffic.*

# 2 Strictly and Wide-Sense Nonblocking Network

## 2.1 Strictly nonblocking

There is no literature on SNB Clos networks under the multirate-multicast model. We now give a general result true for all networks.

**Theorem 2.1.1.** *A network is SNB under the open-end traffic for the multirate-multicast model if and only if it is so for the closed-end traffic.*

*Proof.* By Theorem 1.4.1, it suffices to prove the "if" part. Further, it suffices to consider the new request consisting of a single output port since we can decompose a $d$-output-port request into $d$ single output port requests. Consider a request $C = (x, \{o_1\}, \omega)$ which has the same message as the $i$th request of $x$, i.e., $(x^i, o(x^i), \omega)$ already connected to some other outputs. Let $T(x)$ denote the set of existing connections involving the $i$th request of $x$. Since this network is SNB for the closed-end traffic, $C$ can be routed in a path $p$ if $T(X)$ is ignored. Further, when $T(x)$ is put back and intersects $p$ in a link, that link carries both paths with a combined load $\omega$ (not $2\omega$) since they can share the message. When the two paths need to split, then the fan-out capability of a switch is needed. So $C$ is routed in the network in addition to existing connections. □

## 2.2 The no-split rule

The *no-split rule* specifies that output ports in the same output switch in a multicast request must be connected by using the fan-out of that output switch, i.e., using only one path to connect an input port and an output switch. Since a split routing can never help, we assume all WSNB and RNB algorithms use the no-split rule. Under the no-split rule, a request can be represented as $(x, O(x), \omega)$, where $O(x) \subseteq O$. Here, we call a traffic $f$-cast if $|O(x)| \leq f \leq r_2$.

If the no-split rule is the only constraint in routing, we call the routing algorithm the *no-split algorithm.* Note that for point-to-point traffic, nonblocking under the no-split algorithm is equivalent to SNB.

We first give a fundamental relation between the open-end and closed-end type of traffic under the no-split algorithm.

**Theorem 2.2.1.** *A three-stage Clos network is multirate f-cast WSNB under the no-split algorithm for the closed-end traffic, then it is so for the open-end traffic.*

The proof is analogous to the one in Theorem 2.1.1 by replacing $o_1$ by $O_1$.

All WSNB results in this Section are under the no-split algorithm. Svinnset [11] proved

**Theorem 2.2.2.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for $\beta[b, B]$ broadcast assignments if*

$$m \geq \left\lfloor \frac{r_2(n_1\beta - B)}{1 - B + \epsilon} \right\rfloor + \left\lfloor \frac{n_2\beta - B}{1 - B + \epsilon} \right\rfloor + 1.$$

*where $\epsilon$ is a positive number approaching zero.*

With $d$-fold internal links the corresponding result is

$$m \geq \left\lfloor \frac{r_2(n_1\beta - B)}{(1 - B + \epsilon)d} \right\rfloor + \left\lfloor \frac{n_2\beta - B}{(1 - B + \epsilon)d} \right\rfloor + 1.$$

Notice that, if $1-\omega < b$, $1-\omega$ won't be a request weight, so we can modify Svinnset's result in Theorem 2.2.2 to $m \geq \max_{b \leq \omega \leq B} \lfloor \frac{r_2(n_1\beta - \omega)}{M(\omega)} \rfloor + \lfloor \frac{n_2\beta - \omega}{M(\omega)} \rfloor + 1$, where $M(\omega) = \max\{1 - \omega + \epsilon, b\}$.

Following the concept used by Svinnset, we extend Theorem 2.2.2 (the modified version) to $f$-cast model.

**Theorem 2.2.3.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for $\beta[b, B]$ closed-end $f$-cast assignments if*

$$m \geq \max_{b \leq \omega \leq B} \left\lfloor \frac{f(n_1\beta - \omega)}{M(\omega)} \right\rfloor + \left\lfloor \frac{n_2\beta - \omega}{M(\omega)} \right\rfloor + 1$$

*where $M(\omega) = \max\{1-\omega+\epsilon, b\}$ and $\epsilon$ is a positive number approaching zero.*

Based on Theorem 2.2.3, we have the following two results:

**Corollary 2.2.4.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for $\beta[b, B]$ closed-end $f$-cast assignments with $b + B \leq 1$ if*

$$m \geq \left\lceil \frac{f(n_1\beta - B)}{1 - B} \right\rceil + \left\lceil \frac{n_2\beta - B}{1 - B} \right\rceil + 1$$

**Corollary 2.2.5.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for $\beta[b, B]$ closed-end $f$-cast assignments with $b + B > 1$ and $b \leq 1/2$ if*

$$m \geq \left\lfloor \frac{f(n_1\beta - 1 + b)}{b} \right\rfloor + \left\lfloor \frac{n_2\beta - 1 + b}{b} \right\rfloor + 1$$

We tighten the condition in Corollary 2.2.5.

**Theorem 2.2.6.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for $[b, B]$ closed-end $f$-cast assignments with $b + B > 1$ if*

$$m > \lfloor 1/b \rfloor (n_1 - 1)f + \lfloor 1/b \rfloor (n_2 - 1).$$

*Proof.* In the $[b, B]$ with $b + B > 1$ model, once a link carries a request with weight $B$, it can't carry a request anymore. This condition is the same with $[b, 1]$ multirate condition. Let $T(n, \omega, t)$ denote the number of output links with weight greater than $1 - \omega$ in the input switch associated with the new connection request, where $n$ is the number of input ports in the input switch, $\omega$ is the weight of the new connection, and $t$ is the maximum number of fanouts allowed for each connection at the input stage. Yang proved $\max_{b \leq \omega \leq 1} T(n, \omega, t) = \lfloor 1/b \rfloor (n-1)t$ in [13] under $[b, 1]$ condition. This implies $\max_{b \leq \omega \leq B} T(n, \omega, t) = \lfloor 1/b \rfloor (n - 1)t$ for the $[b, B]$ with $b + B > 1$ model. Since the sufficient condition is $m > \max_{b \leq \omega \leq B} T(n_1, \omega, f) + T(n_2, \omega, 1)$, this theorem follows. □

Setting $f = 1$ in Theorem 2.2.3 implies the result in [6] for the symmetric Clos network.

**Corollary 2.2.7.** *A $C(n, m, r)$ network is SNB for $\beta[b, B]$ point-to-point assignments if*

$$m \geq 2 \max_{b \leq \omega \leq B} \left\lfloor \frac{n\beta - \omega}{M(\omega)} \right\rfloor + 1.$$

*where $M(\omega) = \max\{1 - \omega + \epsilon, b\}$ and $\epsilon$ is a positive number approaching zero.*

The corresponding results of Corollary 2.2.4 and 2.2.5 for point-to-point (setting $f = 1$) symmetric model are the following two results in [6].

**Corollary 2.2.8.** *A $C(n, m, r)$ network is SNB for $\beta[b, B]$ point-to-point assignments with $b + B \leq 1$ if*

$$m \geq 2 \left\lceil \frac{n\beta - B}{1 - B} \right\rceil + 1$$

**Corollary 2.2.9.** *A $C(n, m, r)$ network is SNB for $\beta[b, B]$ point-to-point assignments with $b + B > 1$ and $b \leq 1/2$ if*

$$m \geq 2 \left\lfloor \frac{n\beta - 1 + b}{b} \right\rfloor + 1$$

Setting $f = 1$ in Theorem 2.2.6 implies the sufficient side of the following result with $\beta = 1$ in [6]:

**Corollary 2.2.10.** *$C(n, m, r)$ is SNB for the $\beta[b, B]$ point-to-point assignments with $b + B > 1$ if and only if $m \geq 2\lfloor \beta/b \rfloor (n - 1) + 1$.*

Setting $b = 1$ in Theorem 2.2.6, then there is rate 1, i.e., it is the regular phone switching network model. Then Theorem 2.2.6 is comparable to Hwang [6] as follows except the boundary condition:

**Corollary 2.2.11.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB for closed-end $f$-cast assignments if*

$$m \geq \min\{(n_1 - 1)f + n_2, (N_1 - 1)f + 1, N_2\}.$$

## 2.3 $k$-limited algorithm

For the Multicast WSNB condition, Yang and Masson [12] first explicitly suggested an algorithm called *k-limited algorithm*, which is defined by the constraint that any request can use at most $k$ middle switches. Notice that, the $f$-cast results in this subsection are all for the closed-end traffic. They gave the following result:

**Theorem 2.3.1.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a k-limited algorithm for $f$-cast assignments if*

$$m > (n_1 - 1)k + (n_2 - 1)f^{1/k}.$$

Note that, the above theorem shows that the result depends on how we choose $k$. In [13], it was shown that $k = O(\frac{\ln f}{\ln \ln f})$ is an optimal choice for symmetric model.

The $k$-limited algorithm can be extended to the multirate-multicast environment.

Svinnset proved

**Lemma 2.3.2.** *For a request $(x, O(x), \omega)$ in $C(n_1, r_1, m, n_2, r_2)$ with $\beta[b, B]$ broadcast assignments, if there are $\lfloor \frac{n_2\beta - B}{1-B+\epsilon} \rfloor + 1$, where $\epsilon \to 0^+$, middle switches available for $x$ to connect to in the existing configuration, then $x$ can be connected to all output switches in $O(x)$.*

It follows

**Theorem 2.3.3.** *A $C(n_1, r_1, m, n_2, r_2)$ network, $\lfloor \frac{n_2\beta - B}{1-B+\epsilon} \rfloor + 1 < r_2$ , is WSNB under this $k$-limited algorithm for $\beta[b, B]$ broadcast assignments if*

$$m \geq \left\lfloor \frac{k(n_1\beta - B)}{1 - B + \epsilon} \right\rfloor + \left\lfloor \frac{n_2\beta - B}{1 - B + \epsilon} \right\rfloor + 1.$$

*where $k = \lfloor \frac{n_2\beta - B}{1-B+\epsilon} \rfloor + 1$ and $\epsilon$ is a positive number approaching zero.*

We describe the relation between Lemma 2.3.2 and Theorem 2.3.3: Since under a $k$-limited algorithm, fanouts of an input switch module is restricted to at most $k$, $\max_{b \leq \omega B} T(n_1, \omega, k) \leq \left\lfloor \frac{k(n_1\beta - B)}{1-B+\epsilon} \right\rfloor$. Hence, there are at least $\lfloor \frac{n_2\beta - B}{1-B+\epsilon} \rfloor + 1$, where $\epsilon \to 0^+$, middle switches available for this new request to connect to. Thus we can use Lemma 2.3.2.

Yang [13] obtained a better result for $B = 1$.

**Theorem 2.3.4.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a k-limited algorithm for $[b, 1]$ multirate broadcast assignments if*

$$m > \lfloor 1/b \rfloor (n_1 - 1)k + \lfloor 1/b \rfloor (n_2 - 1) r_2^{1/k}.$$

For the $f$-cast model, Yang gave the following result.

**Theorem 2.3.5.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a k-limited algorithm for $[b, 1]$ multirate f-cast assignments if*

$$m > \lfloor 1/b \rfloor (n_1 - 1)k + \lfloor 1/b \rfloor (n_2 - 1) f^{1/k}.$$

Note that, setting $b = 1$ in this theorem yields Theorem 2.3.1.

Kabaciński-Danilewicz [7] gave a result for general $B$. Note that Yang's result (Theorem 2.3.4) under symmetric model is a special case of this result:

Define the following functions:

$$P(i,j) = \begin{cases} \lfloor i/j \rfloor & \text{if } i/j \text{ is not an integer or } \lfloor i/j \rfloor = 0, \\ \lfloor i/j \rfloor - 1 & \text{if } i/j \text{ is an integer and } i/j > 0, \\ 0 & \text{if } j = 0. \end{cases}$$

$$R_1(i,j) = \begin{cases} i - jP(i,j) & \text{for } P(i,j) \neq 0 \text{ and } i - jP(i,j) > b, \\ 0 & \text{for } P(i,j) \neq 0 \text{ and } i - jP(i,j) \leq b, \\ \beta & \text{for P(i,j)=0.} \end{cases}$$

$$R_2(i,j) = \begin{cases} \left\lfloor \frac{j}{R_1(i,j)} \right\rfloor & \text{for } R(i,j) \geq b, \\ 0 & \text{for } R_1(i,j) < b. \end{cases}$$

$$R_3(i,j) = \begin{cases} i/j & \text{for } j \neq 0, \\ 0 & \text{for } j = 0. \end{cases}$$

$$R_5(i,j) = \begin{cases} i & \text{for } i \geq b, \\ 0 & \text{for } i < b. \end{cases}$$

**Theorem 2.3.6.** *A $C(n, m, r, d)$ network is WSNB under a k-limited algorithm for $\beta[b, B]$ broadcast assignments if*

$$m > K(n,d)(k + r^{1/k})$$

*where* $1 \leq k \leq \min(K(n,d), r)$ *and* $K(n,d) = K_{\beta,b,B}(n,d)$

$$= \begin{cases} \left\lfloor \dfrac{(n-1)\lfloor \beta/b \rfloor}{d} \right\rfloor & \text{for } B \in (1-b, \beta], \\ \left\lfloor \dfrac{(n-1)\lfloor \beta/b \rfloor + \lfloor (\beta - B)/b \rfloor}{2d} \right\rfloor & \text{for } B \in (1-2b, \frac{1}{2}] \\ & \text{and } \frac{1}{4} < b < \frac{1}{2}, \\ \left\lfloor \frac{(n-1)P(\beta,1-B)+\lfloor R_3(n-1,a)\rfloor + P(\alpha(B),1-B)}{d} \right\rfloor & \text{for other } B. \end{cases}$$

*in which*
$\alpha(B) = [n-1-a\lfloor R_3(n-1,a)\rfloor]R_1(\beta, 1-B) + R_5(\beta - B)$,
$a = R_2(\beta, 1-B) + R_3(1, R_4(R_1(\beta, 1-B), \gamma))$,
$\gamma = \max\{\lim_{\epsilon \to 0^+}[1 - B + \epsilon - R_2(\beta, 1-B)R_1(\beta, 1-B)], b\}$,
$R_4(R_1(\beta, 1-B), \gamma)$

$$= \begin{cases} P(R_1(\beta,1-B),\gamma) & \text{if } P(\beta,1-B) \neq 0, \text{ or } P(\beta,1-B) = 0 \\ & \text{and } 1-B-R_2(\beta,1-B)R_1(\beta,1-B) \geq b, \\ \left\lfloor \dfrac{R_1(\beta,1-B)}{\gamma} \right\rfloor & \text{otherwise.} \end{cases}$$

They also gave a result under discrete model:

**Theorem 2.3.7.** *A* $C(n,m,r,d)$ *network is WSNB under a k-limited algorithm for discrete broadcast assignments with all rates in* $\{\omega_1, \cdots, \omega_h\}$, *where* $\omega_1 = b$, $\omega_h = B$ *and* $b|\omega_i$ *if*

$$m > \hat{K}(n,d)(k + r^{1/k}).$$

*where* $1 \leq k \leq \min\{\hat{K}(n,d), r\}$ *and*

$$\hat{K}(n,d) = \hat{K}_{\beta,b,B}(n,d) = \frac{(n-1)\lfloor \beta/b \rfloor + \lfloor (\beta - B)/b \rfloor}{d(\lfloor (1-B)/b \rfloor + 1)}.$$

We now extend Theorem 2.3.6 to $f$-cast asymmetric model:

**Theorem 2.3.8.** *A* $C(n_1, r_1, m, n_2, r_2)$ *network is WSNB under a k-limited algorithm for* $\beta[b, B]$ *broadcast assignments if*

$$m > K(n_1, 1)k + K(n_2, 1)f^{1/k}$$

*where* $1 \leq k \leq \min(K(n_2, 1), f)$ *and* $K(*,*)$ *is defined in Theorem 2.3.6.*

*Proof.* Yang [13] gave a result that $m > \max_{b \le \omega \le 1} \{T(n_1, \omega, k) + T(n_2, \omega, 1) f^{1/k}\}$ is a sufficient condition for a $C(n_1, r_1, m, n_2, r_2)$ network to be WSNB under a $k$-limited algorithm for $[b, 1]$ $f$-cast assignments, where $1 \le x \le \min(T(n_2, \omega, 1), f)$ and $T(n, \omega, t)$ was defined in proof of Theorem 2.2.6. We can modify this result to: a $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a $k$-limited algorithm for $\beta[b, B]$ $f$-cast assignments if $m > \max_{b \le \omega \le B} \{T(n_1, \omega, k) + T(n_2, \omega, 1) f^{1/k}\}$ where $1 \le x \le \min(T(n_2, \omega, 1), f)$.

Kabaciński and Danilewicz showed $\max_{1 \le \omega \le B} T(n, \omega, t) = K(n, 1)t$ in the $\beta[b, B]$ model. The theorem follows. □

Note that, setting $\beta = B = 1$, we obtain Theorem 2.3.5.

Similarly, we also extend Theorem 2.3.7 to asymmetric $f$-cast model:

**Theorem 2.3.9.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a k-limited algorithm for $\beta[b, B]$ discrete f-cast assignments with all rates in $\{\omega_1, \cdots, \omega_h\}$, where $\omega_1 = b$, $\omega_h = B$ and $b|\omega_i$ if*

$$m > \hat{K}(n_1, 1)k + \hat{K}(n_2, 1) f^{1/k}.$$

*where $1 \le k \le \min\{\hat{K}(n_2, 1), f\}$ and $\hat{K}(*, *)$ is defined in Theorem 2.3.7.*

Chan-Chan-Yeung [2] proved

**Theorem 2.3.10.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under a k-limited algorithm for $\beta[b, B]$ broadcast assignments if*

$$m > \left\lfloor \frac{k(n_1\beta - B)}{1 - B + \epsilon} \right\rfloor + \left\lfloor \frac{n_2\beta - B}{1 - B + \epsilon} \right\rfloor r_2^{1/k}.$$

*where $1 \le k \le \min(\lfloor \frac{n_2\beta - B}{1 - B + \epsilon} \rfloor, r_2)$ and $\epsilon$ is a positive number approaching zero.*

With $d$-fold internal links the corresponding result is

$$m \ge \left\lfloor \frac{k(n_1\beta - B)}{(1 - B + \epsilon)d} \right\rfloor + \left\lfloor \frac{n_2\beta - B}{(1 - B + \epsilon)d} \right\rfloor r_2^{1/k}.$$

where $1 \le k \le \min(r_2, \left\lfloor \frac{n_2\beta - B}{(1 - B + \epsilon)d} \right\rfloor)$ and $\epsilon$ is a positive number approaching zero.

Kim and Du [8] proved

**Theorem 2.3.11.** *Under a k-limited algorithm, a broadcast request with weight $\omega \leq \frac{1}{p+1}$ cannot be blocked in $C(n,m,r)$ for the $\beta[b,B]$ model if*

$$m > \frac{(\beta n - \omega)(p+1)}{p}(k + r^{1/k})$$

We now extend Theorem 2.3.11 to the $f$-cast asymmetric model and the proof is given in Appendix.

**Theorem 2.3.12.** *Under a k-limited algorithm, a $f$-cast request with weight $\leq \frac{1}{p+1}$ cannot be blocked in $C(n_1, r_1, m, n_2, r_2)$ for the $\beta[b,B]$ model if*

$$m > \frac{(\beta n_1 - b)(p+1)}{p}k + \frac{(\beta n_2 - b)(p+1)}{p}f^{1/k}$$

## 2.4 $R(l)$ algorithm

Kim and Du extended a routing algorithm $R(l)$, first used by Gao and Hwang in multirate point-to-point model (see Corollary 2.4.4), to broadcast traffic. It is the algorithm of reserving $l$ middle switches only for large requests (which can overflow to other middle switches) where a request is called large if its weight $\omega > 1/(q+1)$, $q = \lfloor 1/B \rfloor$. Notice that, the $f$-cast results in this subsection are all for the closed-end traffic.

**Theorem 2.4.1.** *A $C(n,m,r)$ network is WSNB under $R(l)$ and a k-limited algorithm for $\beta[b,B]$ multirate broadcast assignments if*

$$m > \begin{cases} \dfrac{\beta n(q+1)(Bq+B+q-1)}{q^2}(k + r^{1/k}) & \text{for } B < 23/32, \\ (\dfrac{15\beta n}{8} + n - 1)(k + r^{1/k}) & \text{for } B \geq 23/32. \end{cases}$$

*and*

$$l = \lceil (\beta n(Bq + B - 1)(q+1)/q^2)(k + r^{1/k}) \rceil.$$

*where $q = \lfloor \frac{1}{B} \rfloor$.*

We now extend the above theorem to $f$-cast asymmetric model and the proof is given in Appendix.

**Theorem 2.4.2.** *A $C(n_1, r_1, m, n_2, r_2)$ network is WSNB under $R(l)$ and a $k$-limited algorithm for $\beta[b, B]$ $f$-cast broadcast assignments if*

$$m \geq s + l,$$

*where*

$$s > \frac{(\beta n_1 - b)(q+1)}{q} k + \frac{(\beta n_2 - b)(q+1)}{q} f^{1/k},$$

$$l = \left\lceil \frac{[(\beta n_1 - \frac{1}{q+1})k + (\beta n_2 - \frac{1}{q+1}) f^{1/k} - s(1-B)](q+1)}{q} \right\rceil$$

*and* $q = \lfloor \frac{1}{B} \rfloor$.

Our result slightly improves over [8] which replace the two $b$ terms in $s$ and two $\frac{1}{q+1}$ terms in $l$ by 0. To get the point-to-point result, we set $f = k = 1$. For $b \to 0$ and the symmetric case, we obtain

**Corollary 2.4.3.** *A $C(n, m, r)$ network is WSNB under $R(l)$ for $\beta(0, B]$ point-to-point assignments if*

$$m \geq \frac{2\beta n(q+1)(Bq + B + q - 1)}{q^2} - \frac{2}{q}.$$

*and*

$$l = \lceil 2\beta n(Bq + B - 1)(q+1)/q^2 - \frac{2}{q} \rceil.$$

For this model, Gao and Hwang's result [5] proved:

**Corollary 2.4.4.** *A $C(n, m, r)$ network is WSNB under $R(l)$ for $\beta(0, B]$ point-to-point assignments if*

$$m \geq \frac{2\beta n(q+1)(Bq + B + q - 1)}{q^2}.$$

*and*

$$l = \lceil 2\beta n(Bq + B - 1)(q+1)/q^2 \rceil.$$

# 3 Rearrangeable Nonblocking Networks

For the multirate-multicast model, Kim and Du gave a rearrangeable algorithm. The algorithm orders the requests by their weights and routes each of them using at most $k$ middle switches. The requests are routed in the order from heavy to light. To route the next heaviest request, the algorithm would not disturb the heaviest requests which were already routed. It continues to route the other requests until the lightest request is successfully routed.

Let us introduce a multirate model, the recursive half channel model. In this model, there are $h$ rates $\omega_1, \cdots, \omega_h$ with $\omega_1 > \omega_2 > \cdots > \omega_{i-1} > 1/2 \geq \omega_i > \omega_{i+1} > \cdots > \omega_h$ and $\omega_j$ divides $\omega_{j-1}$ for $i+1 \leq j \leq h$.

Kim-Du [8] (1998) proved

**Theorem 3.0.5.** *$C(n, m, r)$ is rearrangeable for the $(0, 1]$ recursive half channel model and broadcast assignments if*

$$m > (n-1) \min_{1 \leq k \leq \min(n-1, r)} (k + r^{1/k}).$$

We extend Theorem 3.0.5 to (i) $f$-cast, (ii) asymmetric Clos network and give a proof in Appendix.

**Theorem 3.0.6.** *$C(n_1, r_1, m, n_2, r_2)$ is rearrangeable for the $(0, 1]$ recursive half channel model and $f$-cast assignments if*

$$m > \min_{1 \leq k \leq \min(n_2-1, f)} \left\{ (n_1 - 1)k + (n_2 - 1) f^{1/k} \right\}.$$

Setting $f = k = 1$ in the above theorem, we get the point-to-point result. And it is the same with the result given by Lin, Du, Hu and Xue [9] under the symmetric model.

**Corollary 3.0.7.** *$C(n, 2n-1, r)$ is rearrangeable for the $(0, 1]$ recursive half channel model and point-to-point assignments.*

Kim and Du observe a case:

**Theorem 3.0.8.** *$C(n, m, r)$ is rearrangeable for broadcast assignments with weights chosen from $\{\omega_1, \cdots, \omega_h\}$, where $1 \geq \omega_1 > \cdots > \omega_h > 0$ and $\omega_j$ divides $\omega_{j-1}$ for $2 \leq j \leq h$ if*

$$m > (n-1) \min_{1 \leq k \leq \min(n-1, r)} (k + r^{1/k}).$$

# 4 Numerical Comparison and Conclusion

In this section, we compare some necessary conditions of $C(n, m, r)$ Clos networks which are WSNB for $[b, B]$ broadcast assignments. Let us denote Kim-Du [8] as KD, Yang [13] as Y, Kabaciński-Danilewicz [7] as KaD and Chan-Chan-Yeung [2] as CY.

| | $n = 10$ | | | | $n = 20$ | | | | $n = 40$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | KD | Y and KaD | | | KD | Y and KaD | | | KD | Y and KaD | | |
| $r$ | $b$ | 0.1 | 0.4 | 0.6 | $b$ | 0.1 | 0.4 | 0.6 | $b$ | 0.1 | 0.4 | 0.6 |
| 4 | 112 | 360 | 72 | 36 | 227 | 760 | 152 | 76 | 457 | 1560 | 312 | 156 |
| 12 | 147 | 477 | 96 | 48 | 299 | 1005 | 201 | 101 | 603 | 2063 | 413 | 207 |
| 20 | 159 | 515 | 103 | 52 | 323 | 1086 | 218 | 109 | 652 | 2229 | 446 | 223 |
| 28 | 168 | 544 | 109 | 55 | 342 | 1147 | 230 | 115 | 689 | 2355 | 471 | 236 |

Table 1: $B = 1$

Note that, for $B = 1$, Yang [13]'s result is the same with Kabaciński-Danilewicz [7]'s. And Kim-Du [8]'s result is better than Yang [13] and Kabaciński-Danilewicz [7]'s result only when $b <= 1/4$ or $1/4 < b \leq 1/3$ and $n > 16$.

The following two tables show some comparisons between the results under $B < 1$ ( except Yang's result which constrains $B = 1$).

| n | 10 | | | 20 | | | 40 | | |
|---|---|---|---|---|---|---|---|---|---|
| | CY | KD | KaD | CY | KD | KaD | CY | KD | KaD |
| $r$ | for all $b$ | | $\leq 0.25$ | for all $b$ | | $\leq 0.25$ | for all $b$ | | $\leq 0.25$ |
| 4 | 146 | 112 | 149 | 306 | 227 | 309 | 626 | 457 | 629 |
| 12 | 193 | 147 | 196 | 404 | 299 | 408 | 828 | 603 | 831 |
| 20 | 208 | 159 | 212 | 437 | 323 | 441 | 894 | 652 | 898 |
| 28 | 220 | 168 | 224 | 461 | 342 | 465 | 944 | 689 | 948 |

Table 2: $B = 0.75$

Finally, we compare the results of Chen-Chen-Yeung and Kim-Du.

| n | 10 | | | | 30 | | | | 100 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | CY | KD | KaD | | CY | KD | KaD | | CY | KD | KaD | |
| $r$ | for all $b$ | | 0.01 | 0.3 | for all $b$ | | 0.01 | 0.3 | for all $b$ | | 0.01 | 0.3 |
| 4 | 62 | 67 | 65 | 57 | 197 | 199 | 197 | 177 | 662 | 661 | 665 | 597 |
| 12 | 82 | 88 | 85 | 75 | 260 | 262 | 260 | 233 | 875 | 873 | 879 | 789 |
| 20 | 88 | 95 | 92 | 81 | 281 | 283 | 281 | 252 | 945 | 943 | 949 | 852 |
| 28 | 93 | 100 | 97 | 85 | 296 | 299 | 296 | 266 | 999 | 997 | 1003 | 900 |

Table 3: $B = 0.4$

Let

$$h_{CY}(n,B) = \frac{n-B}{1-B},$$

$$h_{KD}(n,B) = \begin{cases} \dfrac{n(q+1)(Bq+B+q-1)}{q^2} & \text{for } B < 23/32, \\ (\dfrac{23n}{8} - 1) & \text{for } B \geq 23/32. \end{cases},$$

$$h(B) = \frac{q^2 B}{q^2 - (q+1)(Bq+B+q-1)(1-B)}, \text{where } q = \left\lfloor \frac{1}{B} \right\rfloor.$$

For $B \geq \frac{23}{32}$, $h_{KD}(n,B) < h_{CY}(n,B)$ for $n$ large. Hence Kim-Du's result is better than Chen-Chen-Yeung's for $B \geq \frac{23}{32}$. Table 2 shows this property.

For $B < \frac{23}{32}$, $h_{CY}(n,B) < h_{KD}(n,B)$ for all $n < h(B)$ and $h_{KD}(n,B) < h_{CY}(n,B)$ for $n$ large. Hence Kim-Du's result is better than Chen-Chen-Yeung's for $B < \frac{23}{32}$ and $n$ large. Table 3 shows this property, where $h(0.4) = 40$. But notice that, for some $B$, $h(B)$ is very large.

# Appendix

Let $\boldsymbol{M}_j$ denote the vector $(\boldsymbol{M}_j(1), \cdots, \boldsymbol{M}_j(r_2))$, where $\boldsymbol{M}_j(k)$ is the sum of weights loaded on link between middle switch $j$ and output switch $k$. In Figure 2, for example, $\boldsymbol{M}_1 = (0.4, 0, 0)$, $\boldsymbol{M}_2 = (0.6, 0.4, 0)$, $\boldsymbol{M}_3 = (0, 0.2, 0.6)$ and $\boldsymbol{M}_4 = (0, 0.4, 0)$ .

***Proof of Theorem 2.3.12.*** Let $\boldsymbol{C} = (x, O(x), \omega)$ be a new $f$-cast request. We have $|O(x)| \leq f$ and $\omega \leq \frac{1}{p+1}$. Under the $k$-limited algorithm let $m'$ be the number of middle switches blocking the new request from the $x$'s input switch $\boldsymbol{B}$. Then,

$$m'(1-\omega) \leq (\beta n_1 - \omega)k$$

implies

$$m' \leq \frac{(\beta n_1 - \omega)(p+1)}{p}k, \tag{1}$$

since $1-\omega \geq \frac{p}{p+1}$.

Consider the $m'' \times |O(x)|$ destination matrix $\boldsymbol{M}$ by discarding at most $m'$ rows whose corresponding middle switches are blocking the new request from the input switch $\boldsymbol{B}$ where those $|O(x)|$ columns are corresponding to output switches in $O(x)$. Suppose that any $k$ middle switches cannot satisfy this new multicast request. Let $t_1(j)$ be the number of elements in the $j$-th row whose values are greater than $1-\omega$ and $t_1 = \min_{1\leq j\leq m''} t_1(j)$. We obtain,

$$m''t_1\frac{p}{p+1} \leq m''t_1(1-\omega) \leq \sum_{j=1}^{m''} t_1(j)(1-\omega) \leq (\beta n_2 - \omega)|O(x)| \leq (\beta n_2 - \omega)f$$

implying

$$m'' \leq \frac{(\beta n_2 - \omega)(p+1)}{p}\frac{f}{t_1}, \tag{2}$$

since $t_1 \neq 0$.

Assume the $s$-th row has the minimum, i.e., $t_1 = t_1(s)$. We can route a part of the request to $f - t_1$ output switches by using the middle switch corresponding to the $s$-th row and delete those $f - t_1$ columns from $\boldsymbol{M}$ for finding the next middle switch to route the remaining destinations. Generally, assume there are only $t_{i-1}$ output switches which are needed to be routed by using $m'' \times t_{i-1}$ destination matrix $\boldsymbol{M}^{(i-1)}$ for $i < k$. Let $t_i(j)$ be the number of elements in the $j$-th row whose values are greater than $1-\omega$ and $t_i$ be the minimum of $t_i(j)$ for all $j$. Then,

$$m''t_i\frac{p}{p+1} \leq m''t_i(1-\omega) \leq \sum_{j=1}^{m''} t_i(j)(1-\omega) \leq (\beta n_2 - \omega)t_{i-1}$$

implies

$$m'' \leq \frac{(\beta n_2 - \omega)(p+1)}{p}\frac{t_{i-1}}{t_i}. \tag{3}$$

where $t_i \neq 0$ for $i < k$. Otherwise, it is a contradiction to the assumption that any $k$ middle switches can not satisfy the new multicast request. When $i = k$, each row vector has at least one element whose value is greater than $1 - \omega$. Therefore,

$$m'' \frac{p}{p+1} \leq m''(1-\omega) \leq \sum_{j=1}^{m''} (1-\omega) \leq (\beta n_2 - \omega) t_{k-1},$$

implies

$$m'' \leq \frac{(\beta n_2 - \omega)(p+1)}{p} t_{k-1}. \tag{4}$$

Since a geometric mean is not less than the minimum of a sequence, the minimum $m''$ can be obtained from (2), (3) and (4) as

$$m'' \leq \frac{(\beta n_2 - \omega)(p+1)}{p} f^{1/k}. \tag{5}$$

$\dfrac{(\beta n_1 - \omega)(p+1)}{p} k + \dfrac{(\beta n_2 - \omega)(p+1)}{p} f^{1/k}$ reaches its maximum at $\omega = b$. Hence, if $m > \dfrac{(\beta n_1 - b)(p+1)}{p} k + \dfrac{(\beta n_2 - b)(p+1)}{p} f^{1/k}$, then the new request can be routed through this network. $\square$

***Proof of Theorem 2.4.2.*** Under the algorithm $R(l)$, assume we partition middle switches $M$ into $M_S$ and $M_L$ whose sizes are $m_S$ and $m_L$, respectively. The algorithm forces a small call ($\omega \leq \frac{1}{q+1}$) to use only $M_S$ but allows a large call ($\omega < \frac{1}{q+1}$) to use not only $M_L$ but also $M_S$. Let $\boldsymbol{C} = (x, O(x), \omega)$ be a request compatible to the existing configuration. First assume $\omega \leq \frac{1}{q+1}$. From Theorem 2.3.12, setting $m_S \geq s > \frac{(\beta n_1 - b)(q+1)}{q} k + \frac{(\beta n_2 - b)(q+1)}{q} f^{1/k}$, $\mathbf{C}$ can be routed. Next assume $\omega > \frac{1}{q+1}$. Let $M'_S$ be a subset of $M_S$ blocking $\mathbf{C}$ from $x$'s input switch $\boldsymbol{B}$ and $M'_L$ be a subset of $M_L$ blocking the request from $\boldsymbol{B}$ and their sizes are $m'_S$ and $m'_L$. Since $q = \lfloor \frac{1}{B} \rfloor$, we have each link from $\boldsymbol{B}$ to $M'_L$ carrying exactly $q$ calls. Because of the compatibility, the maximum total weights going to the middle stage out of the input switch is at most $(\beta n_1 - \omega)k$. Therefore,

$$m'_L \frac{q}{q+1} + m'_S(1-B) \leq m'_L \frac{q}{q+1} + m'_S(1-\omega) \leq (\beta n_1 - \omega)k. \tag{6}$$

Let $M''_S \subseteq M_S \setminus M'_S$ and $M''_L \subseteq M_L \setminus M'_L$ be the subset of $M_S$ and $M_L$ which are available for $\mathbf{C}$, respectively. Their sizes are denoted as $m''_S$ and $m''_L$. To find out the maximum number of blocking links to output switches in $O(x)$, let us consider $(m''_S + m''_L) \times |O(x)|$ destination matrix $\boldsymbol{M}$. Suppose that any $k$ middle switches from $M''_S \cup M''_L$ can not satisfy this new multicast request. We will use the same notation for $t_i(j)$ and $t_i$ as Theorem 2.3.12 but $t_i = \min_{j \in M''_S \cup M''_L} t_i(j)$.

$$\begin{aligned} m''_L t_1 + m''_S t_1(1-B) &\leq \sum_{j \in M''_L} t_1(j) \frac{p}{p+1} + \sum_{j \in M''_S} t_1(j)(1-B) \\ &\leq (\beta n_2 - \omega)|O(x)| \leq (\beta n_2 - \omega) f \end{aligned}$$

implies

$$m''_L \frac{q}{q+1} + m''_S(1_B) \leq (\beta n_2 - \omega) \frac{f}{t_1}, \tag{7}$$

since $t_1 \neq 0$.

We apply a method similar to Theorem 2.3.12 to contruct the destination matrix and obtain the minimum number of middle switches as

$$m''_L \frac{q}{q+1} + m''_S(1-B) \leq (\beta n_2 - \omega) \frac{t_{i-1}}{t_i} \text{for } i < k, \tag{8}$$

$$m''_L \frac{q}{q+1} + m''_S(1-B) \leq (\beta n_2 - \omega) t_{k-1} \text{for } i = k. \tag{9}$$

From (7), (8) and (9), we get

$$m''_L \frac{q}{q+1} + m''_S(1+B) \leq (\beta n_2 - \omega) f^{1/k} \tag{10}$$

Set $m^*_L = m'_L + m''_L \leq m_L$ and $m^*_S = m'_S + m''_S \leq m_S$, we obtain the following by summing up (6) and (10):

$$m^*_L \leq \frac{[(\beta n_1 - \omega)k + (\beta n_2 - \omega) f^{1/k} - m^*_S(1-B)](q+1)}{q}$$

Setting $m^*_S = s > \dfrac{(\beta n_1 - b)(q+1)}{q} k + \dfrac{(\beta n_2 - b)(q+1)}{q} f^{1/k}$ suffices to route all small requests. For this $m^*_S$, if

$$m^*_L > l = \left\lceil \frac{[(\beta n_1 - \frac{1}{q+1})k + (\beta n_2 - \frac{1}{q+1}) f^{1/k} - s(1-B)](q+1)}{q} \right\rceil,$$

the request can be routed.

Therefore, $m = m_L + m_S \geq m_S^* + m_L^* > s + l$ middle switch modules suffice. $\square$

***Proof of Theorem 3.0.6***. Assume $\omega_1, \cdots, \omega_h$ are those $h$ rates with $\omega_1 > \omega_2 > \cdots > \omega_{i-1} > 1/2 \geq \omega_i > \omega_{i+1} > \cdots > \omega_h$ and $\omega_j$ divides $\omega_{j-1}$ for $i+1 \leq j \leq h$. We will prove this theorem by induction on $h$. For $h = 1$, each link can carry no more than one call due to $\omega_1 > 1/2$ so this three-stage Clos network is nonblocking and rearrangeable if $m > \min_{1 \leq k \leq \min(n_2-1,f)} (n_1 - 1)k + (n_2 - 1)f^{1/k}$ (see Corollary 2.3.1). Assume that this Clos network is rearrangeable for $h = h' - 1$. Consider $i$ integers $u_1, u_2, \cdots, u_{i-1}$ and $v$ such that $\omega_j + u_j\omega_{h'} \leq 1 < \omega_j + (u_j+1)\omega_{h'}$ for $j \leq i-1$ and $v\omega_{h'} \leq 1 < (v+1)\omega_{h'}$. If a link blocks a new connected request $\mathbf{C} = (x, O(x), \omega_{h'})$, then the blocking link is carrying either one $\omega_j$-call for some $j \leq i - 1$ and some $\omega_k$-calls, $h \geq k \geq i$, with total weight $u_j\omega_{h'}$ ($U_j$-blocking), or no such $\omega_j$-call but all $\omega_k$-calls with total weight $v\omega_{h'}$ ($V$-blocking). Let us assume that $m'$ middle switches are blocking this new $\omega_{h'}$-call from input stage with $m' > (n_1 - 1)k$. Because all connected requests were able to duplicate their messages at most $k$ times at the input switch, at least $n_1$ input ports should have carry full weights that are $\omega_j + u_j\omega_{h'}$ or $v\omega_{h'}$. This is a contradiction to our assumption for the compatible new $\omega_{h'}$-call. Hence, $m' \leq (n_1 - 1)k$.

Consider $m''$ middle switches by discarding those $m'$ middle switches blocking this new connection request from input stage. Suppose that no $k$ middle switches among $m''$ can route the new $\omega_{h'}$-call. Because each output switch in $O(x)$ has at most $(n_2-1)$ output ports which are either $U_j$-blocking for some $j \leq i-1$ or $V$-blocking for the new call, the total number of blocking links between middle stage and output switches in $O(x)$ is no more than $(n_2-1)|O(x)| \leq (n_2-1)f$. By using the similar approach as Theorem 2.3.12, we can obtain,

$$m'' \leq (n_2 - 1)f/t_1, \tag{11}$$

$$m'' \leq (n_2 - 1)t_{i-1}/t_i \text{ for } i < k, \tag{12}$$

$$m'' \leq (n_2 - 1)t_{k-1} \text{ for } i = k. \tag{13}$$

A minimum of a sequence is not larger than its geometric mean so that, from

(11), (12) and (13), we can obtain,

$$m'' \le \left[(n_2-1)\frac{f}{t_1}\cdot(n_2-1)\frac{t_1}{t_2}\cdot\cdots\cdot(n_2-1)\frac{t_{k-2}}{t_{k-1}}\cdot(n_2-1)t_{k-1}\right]^{1/k} = (n_2-1)f^{1/k}.$$

We showed that the nonblocking multicast Clos network for the switching network is also rearrangeable for multirate-multicast communications for the recursive half channel model. □

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