國 立 交 通 大 學

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碩 士 論 文

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摘要

在文獻 [3] 中, 陳尙寬學長、黃光明老師、以及劉昱綺學姊提出了「混合的弦環式網路」的一個新的網路架構。他們推導出「混合的弦環式網路」的直徑的上下界, 發現「混合的弦環式網路」的直徑可達到$\sqrt{2N}$ (N爲網路中的節點數), 相較於使用相同數量硬體的雙環式網路而言, 這是一項很大的改進。在這篇論文中, 我們提出一個只花$O(\log N)$時間的計算「混合的弦環式網路」的直徑的演算法。

關鍵字: 弦環式網路, 雙環式網路, 直徑, 連通度。

中 華 民 國 九 十 四 年 六 月

On the Diameter of a Mixed Chordal Ring Network

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Abstract

Recently, Chen, Hwang and Liu [3] proposed a new network called the mixed chordal ring network which is very comparable to the double-loop network. They proved the surprising result that the mixed chordal ring network can achieve diameter about $\sqrt{2N}$ which is a huge improvement over the double-loop network (here N is the number of nodes in the network). They derived the upper and the lower bounds for the diameter of a mixed chordal ring network. The purpose of this thesis is to propose an $O(\log N)$-time algorithm for deriving the exact value of the diameter of a mixed chordal ring network.

Keywords: Chordal ring network, double-loop network, diameter, connectivity.

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1 Introduction

Wong and Coppersmith [11] introduced the multi-loop networks. The most studied multi-loop network is the double-loop network $DL(N; a, b)$, which can be represented as a digraph with N nodes $0, 1, \cdots, N-1$ and $2N$ links of two types:

$$i \to (i+a) \mod N, \ \forall\, i = 0, 1, \cdots, N-1,$$
$$i \to (i+b) \mod N, \ \forall\, i = 0, 1, \cdots, N-1.$$

It is well-known that the double-loop network has short diameter and hence small transmission delay. The double-loop network can achieve diameter about $\sqrt{3N}$ (see [11]). In [4], Cheng and Hwang proposed an efficient $O(\log N)$-time algorithm for computing the diameter of a double-loop network [4].

In [1], Arden and Lee proposed the undirected chordal ring network. More specifically, an *undirected chordal ring network* $UCR(N; 1, b)$, where N is even and b is odd, has N nodes $0, 1, \cdots, N-1$ and $3N/2$ edges of two types:

$$(i, (i+1) \mod N) \ \ \forall\, i = 0, 1, 2, \cdots, N-1,$$
$$(i, (i+b) \mod N) \ \ \forall\, i = 1, 3, 5, \cdots, N-1.$$

In [8], Hwang and Wright proposed the directed version of the undirected chordal ring. A *directed chordal ring network* $DCR(N; a, b)$, where N is even and both a and b are odd, has N nodes $0, 1, \cdots, N-1$ and $3N/2$ links of two types:

$$i \to (i+a) \mod N, \ \forall\, i = 0, 1, 2, \cdots, N-1,$$
$$i \to (i+b) \mod N, \ \forall\, i = 1, 3, 5, \cdots, N-1.$$

Recently, Chen, Hwang and Liu [3] proposed the mixed chordal ring network. A *mixed chordal ring network* $MCR(N; a, b)$, where N is even and both a and b are odd, has N nodes

$0, 1, ..., N-1$ and $2N$ links of the following types (see Figure 1(a) for an example):

$$\text{ring links}: \ i \to (i+a) \mod N, \ \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$\text{chordal links}: \ i \to (i+b) \mod N, \ \forall\, i = 1, 3, 5, \cdots, N-1,$$

$$\text{chordal links}: \ i \to (i-b) \mod N, \ \forall\, i = 0, 2, 4, \cdots, N-2.$$

Chen, Hwang and Liu [3] proved that $MCR(N; a, b)$ is strongly connected if and only if $\gcd(N, a, b) = 1$. Since we will only talk about strongly connected mixed chordal ring networks, we assume $\gcd(N, a, b) = 1$. If $a = b$ or $a + b = N$, $MCR(N; a, b)$ will contain multiple links between two nodes, which means a waste of the hardware. Thus throughout this thesis, we assume

$$\gcd(N, a, b) = 1, \ a \neq b, \ \text{and } a + b \neq N.$$

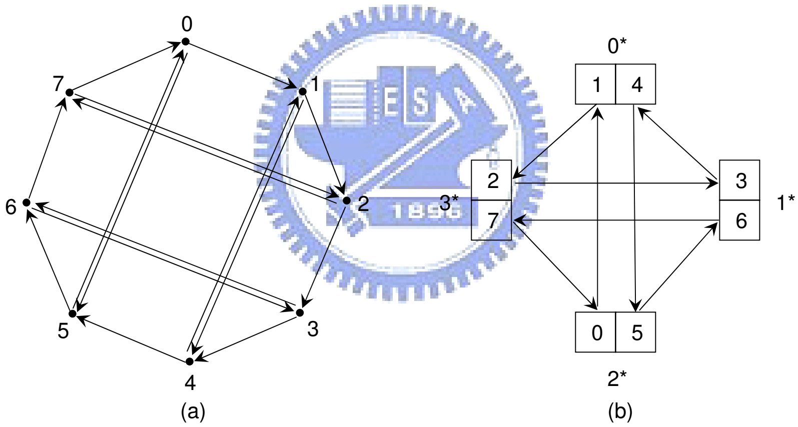


Figure 1: (a) $MCR(8; 1, 3)$. (b) The corresponding double-loop network $DL(4; 2, 3)$.

Chen, Hwang and Liu [3] proved the surprising result that the mixed chordal ring network can achieve diameter about $\sqrt{2N}$ which is a huge improvement over the double-loop network (here N is the number of nodes in the network). They derived the upper and the lower bounds for the diameter of a mixed chordal ring network. The purpose of this thesis is to propose an $O(\log N)$-time algorithm for deriving the exact value of the diameter of a mixed chordal ring network. We now summarize current results below.

	double-loop network	mixed chordal ring network
lower bound for the diameter	$\left\lceil \sqrt{3N} \right\rceil - 2$ [11]	$\left\lceil \sqrt{2N} - 3/2 \right\rceil$ [this thesis]
upper bound for the diameter	$\sqrt{3N} + (3N)^{\frac{1}{4}} + \frac{5}{2}$ [9, 10]	$\sqrt{2N} + 3$ [3]
computing the diameter	$O(\log N)$ time [4]	$O(\log N)$ time [this thesis]

Chen, Hwang and Liu [3] also proposed the necessary and sufficient conditions for a mixed chordal ring network to be strongly connected or strongly 2-connected. Since the proof for the strongly 2-connected case in [3] has a flaw, we also correct it in this thesis. Let $D(N)$ denote the smallest diameter of a mixed chordal ring network with N nodes. Obviously it is desirable to find a $MCR(N; a, b)$ which can achieve $D(N)$. In this thesis, we show the computer output of the choices of a, b that can achieve $D(N)$ for $N \leq 5000$. We obtain the surprising result that about 98.88% of these N's, $D(N)$ can be achieved by $a = 1$; moreover, if $N = 2 \times (2k-1) \times (2k+1)$, then $D(N)$ can be achieved by setting $a = 2k - 1$ and $b = 2k + 1$.

This thesis is organized as follows: Section 2 describes previous results of double-loop networks. Section 3 gives some combinatorial results of $MCR(N; a, b)$ and derives the diameter of $MCR(N; a, \frac{N}{2})$. Section 4 derived the minimum distance diagram of $MCR(N; a, b)$. Section 5 provides an $O(\log N)$-time algorithm for computing the diameter of $MCR(N; a, b)$. Section 6 lists some experimental results. Section 7 is the concluding remarks.

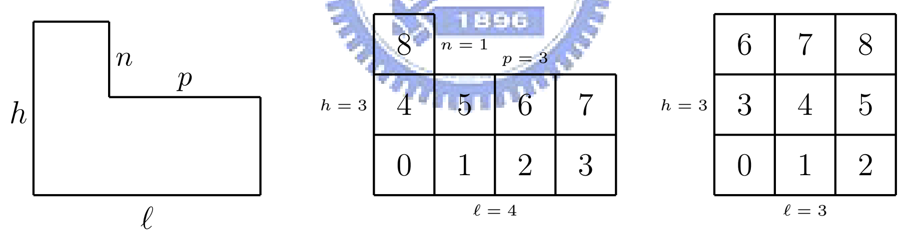
2 Previous results of double-loop networks

In this section, we will briefly review previous results of double-loop networks; see [7] for a recent survey. It is well-known that a double-loop network $DL(N; a, b)$ is strongly connected if and only if $\gcd(N, a, b) = 1$. When $DL(N; a, b)$ is strongly connected, then we can talk about a minimum distance diagram (MDD) which is a diagram with node 0 in cell $(0, 0)$, and node v in cell (i, j) if and only if $ia + jb \equiv v \pmod{N}$ and $i + j$ is the minimum among all (i', j') satisfying the congruence. Namely, a shortest path from 0 to v is through taking i a-links and j b-links (in any order). Note that in a cell (i, j), i is the column index and j is the row index. An MDD includes every node exactly once (in case of two shortest paths, the convention is to choose the cell with the smaller row index, i.e., the smaller j). Since $DL(N; a, b)$ is clearly node-symmetric, there is no loss of generality in assuming: node 0 is the origin of a path.

Wong and Coppersmith [11] proved that the MDD of $DL(N; a, b)$ (their proof for $DL(N;$ 1, $b)$ is easily extended to the general case) is always an L-shape which can be characterized by four parameters ℓ, h, p, n (see Fig. 2 (a)). These four parameters are the lengths of four of the six segments on the boundary of the L-shape. Chen and Hwang [2] showed that necessarily

$$\ell > n \text{ and } h \geq p. \tag{2.1}$$

Fig. 2 (b) illustrates an MDD with a regular L-shape. Fig. 2 (c) illustrates one with an L-shape degenerate into a rectangle.



(a) The four parameters (b) $a = 1$, $b = 4$ (c) $a = 1$, $b = 3$

Figure 2: Minimum distance diagrams and L-shapes.

Fiol, Valero, Yebra, Alegre, and Lang [5], and also Fiol, Yebra, Alegre and Valero [6], showed that an L-shape, degenerate or not, always tessellates the plane (see Figure 3). By considering the relative positions of lattice points occupied by node 0 (see Figure 2), they derived the following congruence:

$$\begin{aligned} \ell a - nb &\equiv 0 \pmod{N} \\ -pa + hb &\equiv 0 \pmod{N}. \end{aligned} \tag{2.2}$$

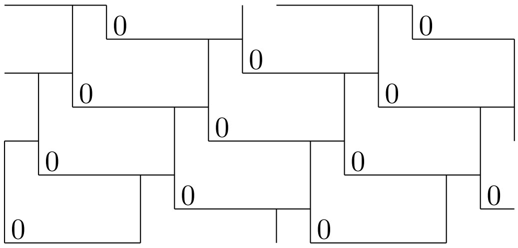


Figure 3: Tessellation of the plane.

The *diameter* $d(N; a, b)$ of a double-loop network $DL(N; a, b)$ is the largest distance between any pair of nodes. It represents the maximum transmission delay between any two nodes. The diameter of a double-loop network $DL(N; a, b)$ can be easily computed from its L-shape(ℓ, h, p, n) by the equation

$$d(N; a, b) = \max\{\ell + h - p,\ \ell + h - n\} - 2. \tag{2.3}$$

In [4], Cheng and Hwang proposed an efficient algorithm to compute the lengths of the L-shape and the diameter of a double-loop network. For completeness of this thesis, we describe their algorithm below.

CHENG-HWANG-ALGORITHM.

Input: $DL(N; a, b)$.

Output: (ℓ, h, p, n) of the L-shape of $DL(N; a, b)$.

Let $d = \gcd(N, a), N' = N/d,\ a' = a/d,$ and $b' = b \mod N$.

Let s_0 be the integer with

$$a's_0 + b' \equiv 0 \pmod{N'},\ 0 \le s_0 < N'.$$

Let $s_{-1} = N'$ and define q_i, s_i, recursively (by the Euclidean algorithm) as follows:

$$\begin{array}{lcll} s_{-1} & = & q_1 s_0 + s_1, & 0 \le s_1 < s_0 \\ s_0 & = & q_2 s_1 + s_2, & 0 \le s_2 < s_1 \\ s_1 & = & q_3 s_2 + s_3, & 0 \le s_3 < s_2 \\ & & \cdots & \\ s_{k-2} & = & q_k s_{k-1} + s_k, & 0 \le s_k < s_{k-1} \\ s_{k-1} & = & q_{k+1} s_k, & 0 = s_{k+1} < s_k. \end{array}$$

Define integers U_i by $U_{-1} = 0$, $U_0 = 1$, and

$$U_{i+1} = q_{i+1} U_i + U_{i-1},\ i = 0, 1, \cdots, k.$$

By induction,

$$s_i U_{i+1} + s_{i+1} U_i = N',\ i = 0, 1, \cdots, k.$$

Regard $s_{-1}/U_{-1} = \infty > x$ for real number x. Since $\{s_i\}_{i=-1}^{k+1}$ and $\{U_i\}_{i=-1}^{k+1}$ are strictly decreasing and increasing, respectively, we have

$$0 = \frac{s_{k+1}}{U_{k+1}} < \frac{s_k}{U_k} < \cdots < \frac{s_0}{U_0} < \frac{s_{-1}}{U_{-1}} = \infty.$$

Let u be the largest odd integer such that $d < \frac{s_u}{U_u}$. Define

$$v = \left\lceil \frac{s_u - dU_u}{s_{u+1} + dU_{u+1}} \right\rceil - 1.$$

Let

$$\ell' = s_u - v s_{u+1},\ h' = U_u + (v+1) U_{u+1},\ p' = s_u - (v+1) s_{u+1},\ n' = U_u + v U_{u+1}.$$

Then

$$(\ell, h, p, n) = (\ell', dh', p', dn'). \tag{2.4}$$

End-of-CHENG-HWANG-ALGORITHM.

3 Some combinatorial results of $MCR(N; a, b)$ and the diameter of $MCR(N; a, \frac{N}{2})$

In a network, the *diameter* is the largest distance between any two nodes and it represents the maximum transmission delay in the network. Recall that $D(N)$ is the smallest diameter of a mixed chordal ring network with N nodes. Let $D(N; a, b)$ denote the *diameter* of $MCR(N; a, b)$. Then

$$D(N) = \min\{D(N; a, b) \mid D(N; a, b) \text{ is the diameter of } MCR(N; a, b)\}.$$

Chen, Hwang and Liu [3] proved that

Lemma 1 [3] $N \leq (D(N)+2)(D(N)+1)/2$.

They also proved $D(N) \geq \sqrt{2N} + o(N)$. Since $N^{1-\epsilon} = o(N)$ for any real number $\epsilon > 0$, it is unclear how good the lower bound $\sqrt{2N} + o(N)$ is. We now sharpen the bound to be

Theorem 2 $D(N) \geq \left\lceil \sqrt{2N} - 3/2 \right\rceil$ *and this bound is tight.*

Proof. By Lemma 1, we have $D(N)^2 + 3D(N) + (2-2N) \geq 0$. Since $D(N)$ is positive, it follows that $D(N) \geq (\sqrt{8N+1} - 3)/2 > \sqrt{2N} - 3/2$. Since $D(N)$ is an integer, we have $D(N) \geq \left\lceil \sqrt{2N} - 3/2 \right\rceil$. This bound is tight since the diameter of $MCR(8;1,3)$ is 3 (see Figure 1) and $\lceil \sqrt{2 \cdot 8} - 3/2 \rceil = 3$. ∎

Chen et al. [3] use the concept of supernodes to transform a mixed chordal ring network $MCR(N;a,b)$ into a double-loop network as follows: Regard each pair of nodes $(2i+1, 2i+1+b)$ as a supernode i^*. Since node $2i+1+b$ is adjacent to node $2i+1+b+a$, there is a link from i^* to $(i+\frac{a+b}{2})^*$ in the corresponding double-loop network. Also, since node $2i+1$ is adjacent to node $2i+1+a$, there is a link from i^* to $(i+\frac{a-b}{2})^*$ in the corresponding double-loop network, too. Chen et al. therefore transformed $MCR(N;a,b)$ into the double-loop network $DL(\frac{N}{2};\frac{a-b}{2},\frac{a+b}{2})$. Note that $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ is isomorphic to $DL(\frac{N}{2};\frac{a-b}{2},\frac{a+b}{2})$. Unless specified otherwise, we transform $MCR(N;a,b)$ into $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$. For example, $MCR(8;1,3)$ in Figure 1(a) is transformed into $DL(4;2,3)$ in Figure 1(b).

It was proved in [3] that

Theorem 3 [3] *$MCR(N;a,b)$ is strongly 2-connected if and only if* $\gcd(N,a,b) = 1$.

The main idea used in the proof of Theorem 3 is to prove that $MCR(N;a,b)$ is strong 2-connected if and only if its corresponding double-loop network $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ is strongly 2-connected. Unfortunately, for some $MCR(N;a,b)$, their corresponding double-loop networks degenerate into single-loop networks (i.e., rings). For example, $MCR(10;1,5)$ in Figure 4 (a) is a legal mixed chordal ring network and is strongly 2-connected, but its corresponding double-loop network $DL(5;3,3)$ (see Figure 4 (b)) degenerates into a single-loop network with multiple links between two adjacent nodes. It is not difficult to see that $DL(5;3,3)$ is not strongly 2-connected; hence the proof of Theorem 3 in [3] has a flaw. We now correct the proof. First a lemma.

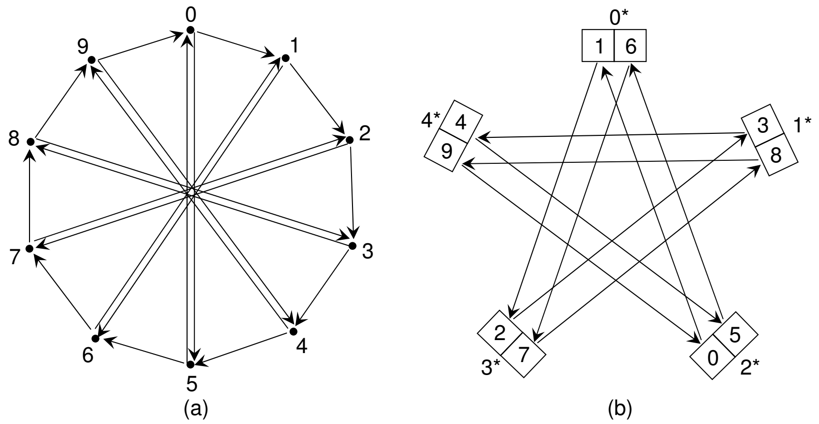


Figure 4: (a) $MCR(10;1,5)$. (b) The corresponding double-loop network $DL(5;3,3)$.

Lemma 4 *Let $MCR(N;a,b)$ be the given mixed chordal ring network. Then $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ is a double-loop network if and only if $b \neq \frac{N}{2}$.*

Proof. $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ degenerates into a single-loop network when $\frac{a+b}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{a-b}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{a+b}{2} \equiv \frac{a-b}{2} \pmod{\frac{N}{2}}$. Since we have assumed that $a \neq b$ and $a+b \neq N$, it is impossible that $\frac{a+b}{2} \equiv 0 \pmod{\frac{N}{2}}$ or $\frac{a-b}{2} \equiv 0 \pmod{\frac{N}{2}}$. Since $\frac{a+b}{2} \equiv \frac{a-b}{2} \pmod{\frac{N}{2}}$ if and only if $b = \frac{N}{2}$, we have this lemma. ∎

Lemma 5 *Let $MCR(N;a,b)$ be the given mixed chordal ring network. Then:*

(i) *If $b \neq \frac{N}{2}$, then $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ is a double-loop network.*

(ii) *$MCR(N;a,\frac{N}{2})$ is itself the double-loop network $DL(N;a,\frac{N}{2})$.*

Proof. By Lemma 4, we have (i). We now prove (ii). $MCR(N; a, \frac{N}{2})$ has $2N$ links of the following types:

$$\begin{aligned}
&i \longrightarrow (i+a) \mod N, \ \forall\, i = 0, 1, 2, \cdots, N-1,\\
&i \longrightarrow (i+\frac{N}{2}) \mod N, \ \forall\, i = 1, 3, 5, \cdots, N-1,\\
&i \longrightarrow (i-\frac{N}{2}) \mod N, \ \forall\, i = 0, 2, 4, \cdots, N-2.
\end{aligned}$$

Since $\frac{N}{2} \equiv -\frac{N}{2} \pmod N$, $(i+\frac{N}{2}) \mod N$ and $(i-\frac{N}{2}) \mod N$ are actually the same node. Thus the $2N$ links of $MCR(N; a, \frac{N}{2})$ are:

$$\begin{aligned}
&i \longrightarrow (i+a) \mod N, \ \forall\, i = 0, 1, 2, \cdots, N-1,\\
&i \longrightarrow (i+\frac{N}{2}) \mod N, \ \forall\, i = 0, 1, 2, \cdots, N-1.
\end{aligned}$$

So $MCR(N; a, \frac{N}{2})$ is itself the double-loop network $DL(N; a, \frac{N}{2})$. ∎

We now reprove Theorem 3.

The proof of Theorem 3. By Lemma 4, we will only prove the case that $b = \frac{N}{2}$; the other cases were proved in [3]. By Lemma 5 (ii), $MCR(N; a, \frac{N}{2})$ is itself the double-loop network $DL(N; a, \frac{N}{2})$. It is well-known that a double-loop network $DL(N; a, b)$ is strongly 2-connected if and only if $\gcd(N, a, b) = 1$. Thus we have this theorem. ∎

Recall that $MCR(N; a, \frac{N}{2})$ is itself the double-loop network $DL(N; a, \frac{N}{2})$. We now derive its diameter.

Theorem 6 $D(N; a, \frac{N}{2}) = \frac{N}{2}$.

Proof. Since $MCR(N; a, \frac{N}{2})$ is the double-loop network $DL(N; a, \frac{N}{2})$, we now use CHENG-HWANG-ALGORITHM to compute its diameter. Since $MCR(N; a, \frac{N}{2})$ is strongly connected, we have $\gcd(N, a, \frac{N}{2}) = 1$ and thus $d = \gcd(N, a) = 1$. By applying the CHENG-HWANG-ALGORITHM, we will have $s_{-1} = N$, $s_0 = \frac{N}{2}$, $s_1 = 0$, $U_{-1} = 0$, $U_0 = 1$, $U_1 = N$,

$u = -1$ and $v = 1$. By (2.4), the L-shape of $DL(N; a, \frac{N}{2})$ is $(\ell, h, p, n) = (\frac{N}{2}, 2, 0, 1)$. By (2.3), $D(N; a, \frac{N}{2}) = \frac{N}{2}$. ∎

Since we have already derived the diameter of $MCR(N; a, \frac{N}{2})$, in the remaining part of this thesis, we only consider $MCR(N; a, b)$ with $b \neq \frac{N}{2}$.

4 The minimum distance diagram of $MCR(N; a, b)$ and its tessellation of the plane

Unless specified otherwise, all nodes are considered taken modular N. As an example, $i - b$ denotes the node $(i - b) \bmod N$. Recall that we regard each pair of nodes $(2i+1, 2i+1+b)$ in $MCR(N; a, b)$ as a supernode i^* and transform $MCR(N; a, b)$ into $DL(\frac{N}{2}; \frac{a+b}{2}, \frac{a-b}{2})$. It is well known that the minimum distance diagram (MDD) of $DL(\frac{N}{2}; \frac{a+b}{2}, \frac{a-b}{2})$ is an L-shape and it tessellates the plane. Suppose the MDD of $DL(\frac{N}{2}; \frac{a+b}{2}, \frac{a-b}{2})$ is the L-shape(ℓ, h, p, n). We now obtain the L-shape of $MCR(N; a, b)$ by replacing each node i^* in the L-shape(ℓ, h, p, n) with the pair of nodes $(2i+1, 2i+1+b)$. In this way, $MCR(N; a, b)$ has the L-shape$(2\ell, h, 2p, n)$.

In the L-shape$(2\ell, h, 2p, n)$ of $MCR(N; a, b)$, $2i + 1$ and $2i + 1 + b$ are adjacent to each other through two chordal links, $2i + 1$ is adjacent to $2i + 1 + a$, and $2i + 1 + b$ is adjacent to $2i + 1 + b + a$. Moreover, if $2i + 1$ is in cell (x, y), then $2i + 1 + b$ is in cell $(x + 1, y)$, $2i + 1 + a$ is in cell $(x + 1, y + 1)$, and $2i + 1 + b + a$ is in cell $(x + 2, y)$. For example, $MCR(100; 27, 61)$ can be transformed into $DL(50; 44, 33)$; Figure 5(a) shows the L-shape of $DL(50; 44, 33)$ and Figure 5(b) shows the L-shape of $MCR(100, 27, 61)$.

It is obvious that L-shape of $MCR(N; a, b)$ tessellates the plane (see Figure 5(b)). In the following, unless specified otherwise, all cells are in the first quadrant. We now prove

Lemma 7 *The node v represented by cell (x, y) is*

$$v = \begin{cases} (1 + \frac{x}{2} \cdot (a + b) + y \cdot (a - b)) \mod N & \text{if } x \text{ is even,} \\ (1 + \frac{x-1}{2} \cdot (a + b) + y \cdot (a - b) + b) \mod N & \text{if } x \text{ is odd.} \end{cases} \quad (4.5)$$

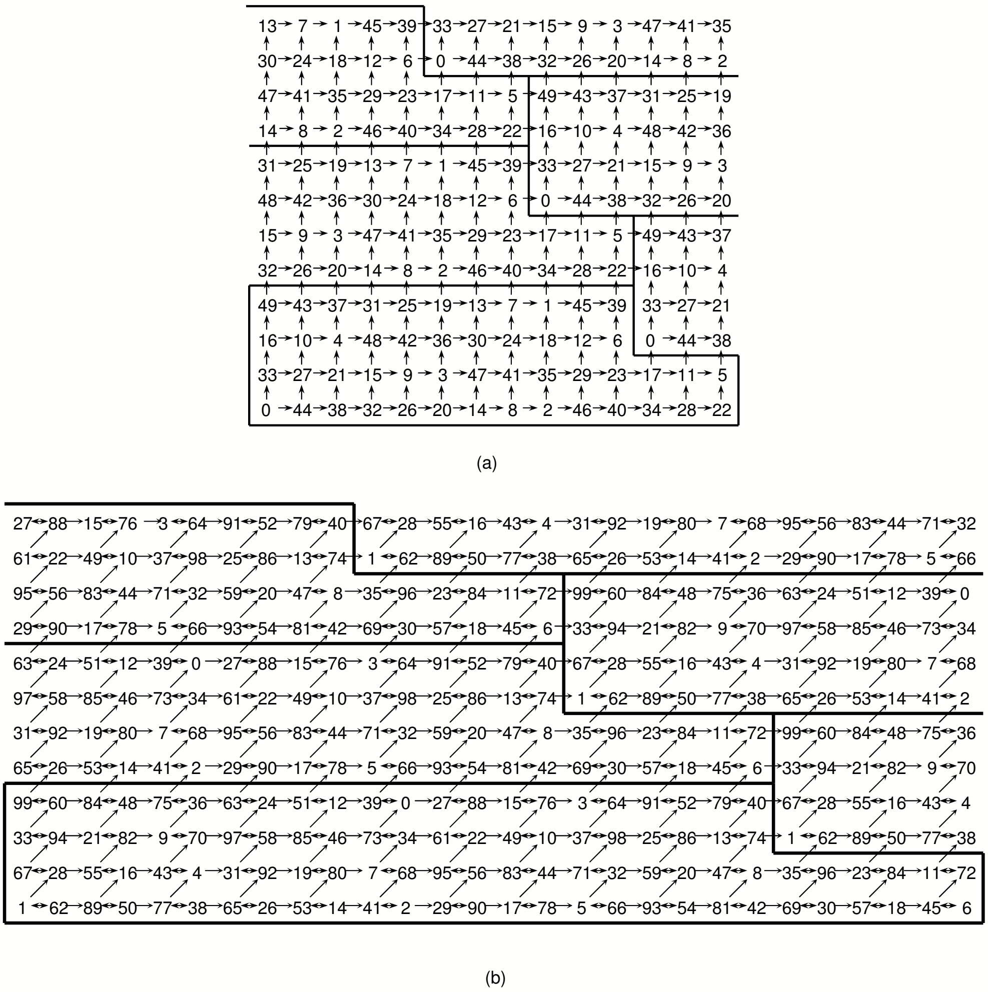


Figure 5: (a) The L-shape of $DL(50; 44, 33)$ and the links between nodes. (b) The L-shape of $MCR(100; 27, 61)$ and the links between nodes.

Proof. This lemma follows from the following observations: Cell (0,0) represents node 1. Cell $(0, y)$ represents node $(1 + y \cdot (a - b)) \bmod N$. Also, if cell $(0, y)$ represents node u, then cell (x, y) represents $u + \frac{x}{2} \cdot (b + a)$ if x is even and represents $u + \frac{x-1}{2} \cdot (b + a) + b$ if x is odd. ∎

See Figure 5(b) for an example of this lemma. In this figure, cell (10,3) represents node 39 and cell (11,3) represents node 0. We have

Lemma 8 *Let i be an integer. If cell (x, y) represents node v, then cell $(x - 2pi, y + hi)$ also* *represents node v.*

Proof. Recall that the double-loop network corresponding to $MCR(N;a,b)$ is $DL(\frac{N}{2};\frac{a+b}{2},\frac{a-b}{2})$ and its L-shape is (ℓ,h,p,n). By (2.2), we have $-p\cdot\frac{a+b}{2}+h\cdot\frac{a-b}{2}\equiv 0 \pmod{\frac{N}{2}}$. Hence

$$-p\cdot(a+b)+h\cdot(a-b)\equiv 0 \pmod{N}. \tag{4.6}$$

First suppose x is even. By (4.5), $v=(1+\frac{x}{2}\cdot(a+b)+y\cdot(a-b)) \bmod N$. Since x is even, $x-2pi$ is also even. By (4.5), cell $(x-2pi,y+hi)$ represents the node $(1+\frac{x-2pi}{2}\cdot(a+b)+(y+hi)\cdot(a-b)) \bmod N$, which is $(1+\frac{x}{2}\cdot(a+b)+y\cdot(a-b)+i(-p\cdot(a+b)+h\cdot(a-b))) \bmod N = (v+i(-p\cdot(a+b)+h\cdot(a-b))) \bmod N \overset{(4.6)}{=} v$. The case that x is odd can be proved similarly and we omit the proof. ∎

Before defining the minimum distance diagram (MDD) of $MCR(N;a,b)$, we first define the distance function. Let $\delta(x,y)$ denote the distance (the number of links) from cell $(0,0)$ to cell (x,y). Then for Figure 5(b), $\delta(8,3)=8$ and $\delta(2,7)=14$. The following lemma is obvious and its proof is omitted.

Lemma 9 *(See Figure 6 for an illustration.)*

$$\delta(x,y)=\begin{cases} 2y-1 & \textit{if} \quad 0\le x<2y \textit{ and } x \textit{ is odd},\\ 2y & \textit{if} \quad 0\le x<2y \textit{ and } x \textit{ is even},\\ x & \textit{if} \quad x\ge 2y.\end{cases} \tag{4.7}$$

Again, before defining the MDD of $MCR(N;a,b)$, we discuss the symmetry property of $MCR(N;a,b)$. It is easy to see that in $MCR(N;a,b)$, all odd nodes are symmetric and all even nodes are symmetric. Consider $MCR(12;3,5)$ in Figure 7(a). The distance from node 1 to node 8 is 5, but the distance from node 0 to every node is at most 4. So, in $MCR(N;a,b)$, an odd node may not be symmetric to an even node. To overcome this problem, the following definition and lemma are introduced. Two mixed chordal ring networks $MCR(N;a_1,b_1)$ and $MCR(N;a_2,b_2)$ are *strongly isomorphic*, denoted as $MCR(N;a_1,b_1)\cong MCR(N;a_2,b_2)$, if

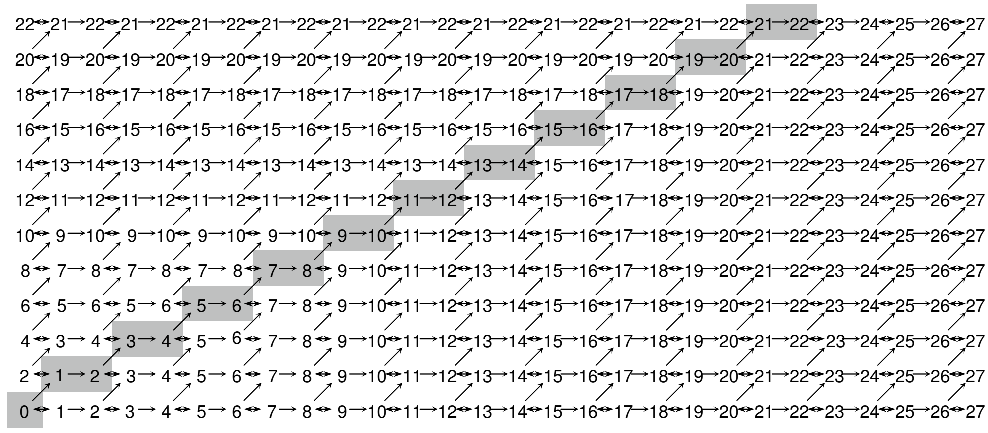


Figure 6: The distance from each cell in the first quadrant to cell (0,0).

there is a bijection Φ from the nodes of $MCR(N; a_1, b_1)$ to the nodes of $MCR(N; a_2, b_2)$ such that either

$$\Phi(i + a_1) = (\Phi(i) + a_2) \mod N \quad \text{for all nod } i,$$

$$\Phi(i + b_1) = (\Phi(i) + b_2) \mod N \quad \text{if } i \text{ is odd and } \Phi(i) \text{ is odd},$$

$$\Phi(i - b_1) = (\Phi(i) - b_2) \mod N \quad \text{if } i \text{ is even and } \Phi(i) \text{ is even},$$

or

$$\Phi(i + a_1) = (\Phi(i) + a_2) \mod N \quad \text{for all node } i,$$

$$\Phi(i + b_1) = (\Phi(i) - b_2) \mod N \quad \text{if } i \text{ is odd and } \Phi(i) \text{ is even},$$

$$\Phi(i - b_1) = (\Phi(i) + b_2) \mod N \quad \text{if } i \text{ is even and } \Phi(i) \text{ is odd}.$$

We now prove

Lemma 10 *$MCR(N; a, b) \cong MCR(N; a, N - b)$. Moreover, node i in $MCR(N; a, b)$ is mapped to node $i + b$ in $MCR(N; a, N - b)$.*

Proof. Define a bijection Φ from the nodes of $MCR(N; a, b)$ to the nodes of $MCR(N; a, N - b)$ as follows:

$$\Phi(i) = (i + b) \mod N. \tag{4.8}$$

Then, for all i, $\Phi(i+a) = ((i+a)+b) \bmod N = ((i+b)+a) \bmod N = (\Phi(i)+a) \bmod N$. Moreover, if i is odd, then $\Phi(i)$ is even and $\Phi(i+b) = ((i+b)+b) \bmod N = ((i+b) - (N-b)) \bmod N = (\Phi(i) - (N-b)) \bmod N$; if i is even, then $\Phi(i)$ is odd and $\Phi(i-b) = ((i-b)+b) \bmod N = ((i+b)+(N-b)) \bmod N = (\Phi(i)+(N-b)) \bmod N$. From the above, $MCR(N; a, b) \cong MCR(N; a, N-b)$. Since b is odd, by (4.8), an even node i $MCR(N; a, b)$ is mapped to an odd node $i+b$ in $MCR(N; a, N-b)$. We have this lemma. ∎

See Figure 7 for an example of Lemma 10. This figure shows that $MCR(12; 3, 5) \cong MCR(12; 3, 7)$; moreover, every even node i in $MCR(12; 3, 5)$ can be regarded as an odd node $i+5$ in $MCR(12; 3, 7)$.

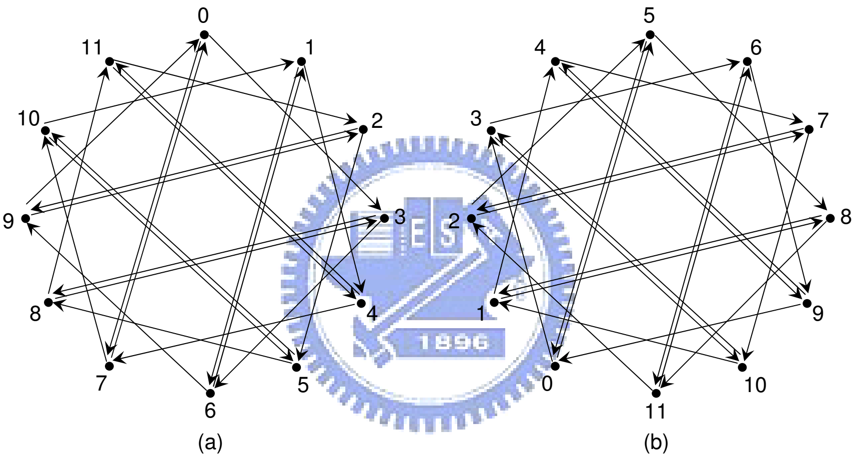


Figure 7: Two isomorphic mixed chordal rings. (a) $MCR(12; 3, 5)$. (b) $MCR(12; 3, 7)$.

The minimum distance diagram (MDD) of $MCR(N; a, b)$ is a diagram with node 1 in cell $(0, 0)$ and node v in cell (x, y) if and only if v is derived by (4.5) and $\delta(x, y)$ is the minimum among all $\delta(x', y')$ where (x', y') represents v. (See Figure 8.) The MDD includes every node of $MCR(N; a, b)$ exactly once. The reason of choosing node 1 instead of node 0 at cell $(0, 0)$ is that: the MDD of $MCR(N; a, b)$ can be converted from the L-shape of $MCR(N; a, b)$ and in the L-shape, node 1 is at cell (0,0).

Note that the L-shape of $MCR(N; a, b)$ may not be its MDD. To see this, consider Figure 5(b). Both cell (27,0) and (21,4) represent the same node – node 6. Cell (27,0) is in the

L-shape and $\delta(27,0) = 27$. However, $\delta(21,4) = 21$. So, cell (27,0) is not in the MDD of $MCR(100;27,61)$.

Now we show how to convert the L-shape of $MCR(N;a,b)$ into its MDD. Assume $\ell \geq h$ in the L-shape$(2\ell,h,2p,n)$ of $MCR(N;a,b)$; the case that $\ell < h$ will be discussed later. Define a strip S (on the cells in the first quadrant) associated with $MCR(N;a,b)$ as follows:

$$S = \{(x,y) \mid 0 \leq y < h,\ 0 \leq x < 2(y+h)\} \cup \{(x,y) \mid y \geq h,\ 2(y-p) \leq x < 2(y+h)\}.$$

For example, the strip S associated with $MCR(100;27,61)$ is shown in Figure 8. Now we prove that

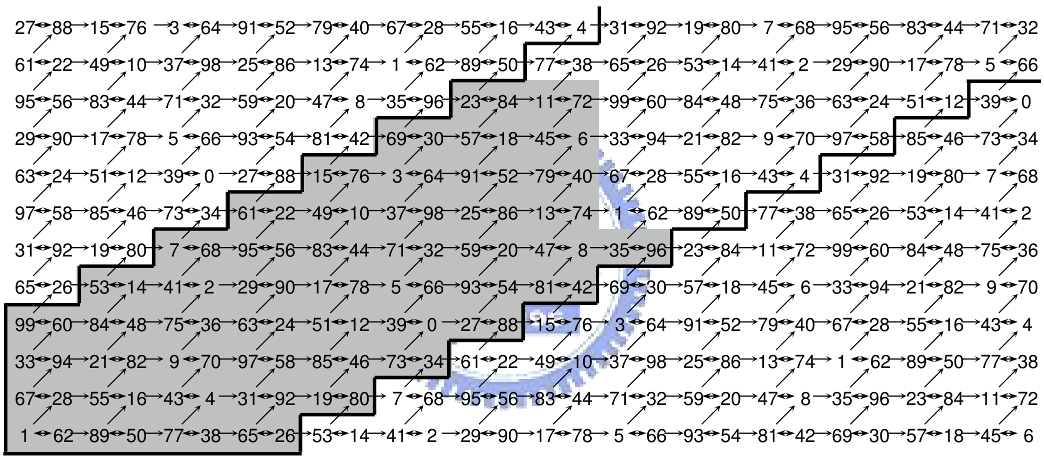


Figure 8: The strip of $MCR(100;27,61)$ and the MDD (shaded) of $MCR(100;27,61)$.

Lemma 11 *The MDD of $MCR(N;a,b)$ is inside the strip S.*

Proof. It is sufficient to prove that if $(x,y) \notin S$ and (x,y) represents node v, then there exists $(x',y') \in S$ such that (x',y') also represents node v and $\delta(x',y') \leq \delta(x,y)$. Since $(x,y) \notin S$, there are two cases:

Case 1. $x \geq 2(y+h)$.

Since $x \geq 2(y+h) > 2y$, by (4.7), $\delta(x,y) = x$. Let $i = \lfloor \frac{x-2(y-p)}{2(h+p)} \rfloor$. Clearly, $i \geq 1$. Since

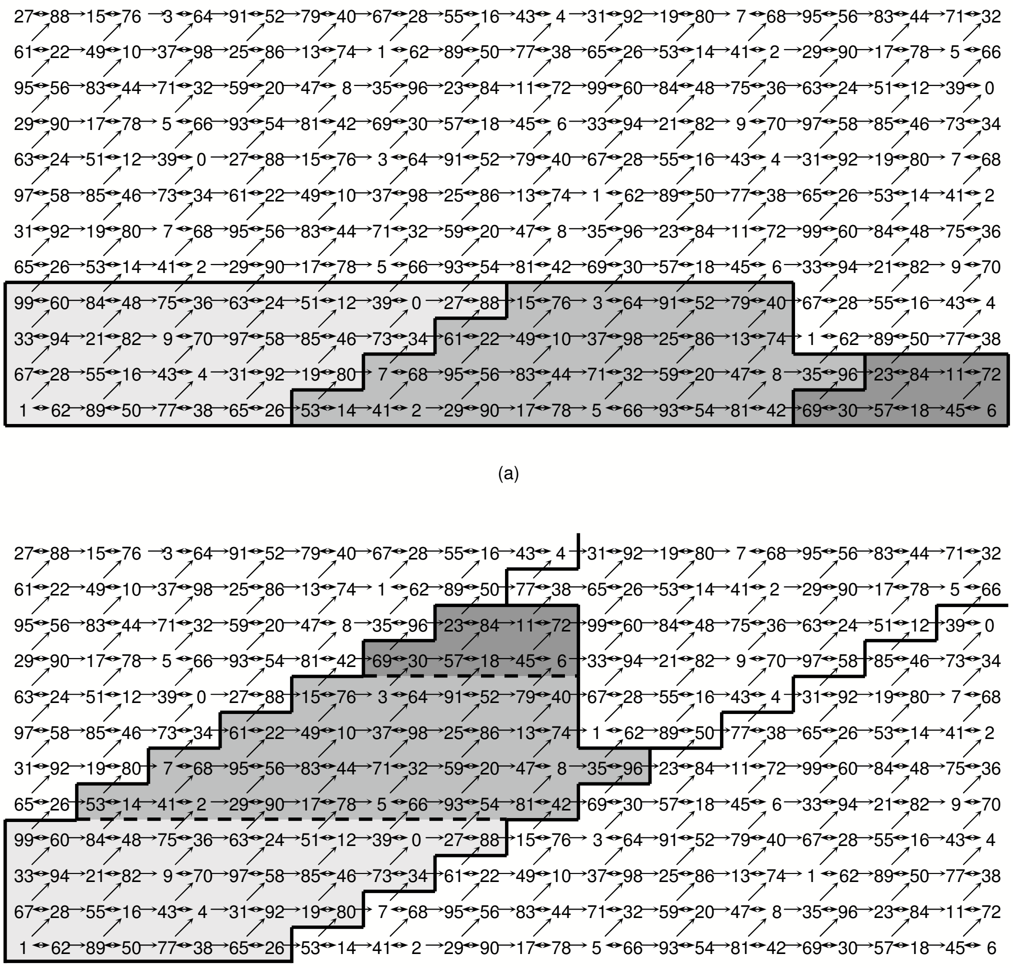


Figure 9: (a) Partitioning the L-shape of $MCR(100;27,61)$ into L_0 (shaded lighter), L_1 (shaded median) and L_2 (shaded deeper). (b) S_0 (shaded lighter), S_1 (shaded median) and S_2 (shaded deeper).

$\frac{x-2(y-p)}{2(h+p)} - 1 < i \le \frac{x-2(y-p)}{2(h+p)}$, we have $x - 2(y+h) < 2i(h+p) \le x - 2(y-p)$. Thus $2(y + hi - p) \le x - 2pi < 2(y + hi + h)$. Let $(x', y') = (x - 2pi, y + hi)$. Then we have

$$2(y' - p) \le x' < 2(y' + h). \tag{4.9}$$

By Lemma 8, cell (x', y') also represents v. Moreover, it is clear that $y' \ge h$ since $i \ge 1$. Therefore $(x', y') \in S$. By (4.9), there are two subcases.

Subcase 1.1. $2(y' - p) \le x' < 2y'$.

By (2.1), $h - p \ge 0$. Since $i \ge 1$, $y' \ge y + h$. Thus $y' - p \ge y + h - p \ge y \ge 0$.

Hence $0 \le x' < 2y'$. By (4.7), $\delta(x', y')$ is either $2y' - 1$ or $2y'$, i.e., $\delta(x', y') \le 2y'$. Since $2(y' - p) \le x'$, $\delta(x', y') \le 2y' \le x' + 2p = x - 2p(i - 1)$. Since $i \ge 1$, $x - 2p(i - 1) \le x = \delta(x, y)$. Thus $\delta(x', y') \le \delta(x, y)$.

Subcase 1.2. $2y' \le x' < 2(y' + h)$.

Then $x' \ge 2y'$. By (4.7), $\delta(x', y') = x'$. Since $i \ge 1$, $x' = x - 2pi \le x = \delta(x, y)$. Thus $\delta(x', y') \le \delta(x, y)$.

Case 2. $kh \le y < (k + 1)h$, $f(k) \le x < 2(y - p)$, where k is some positive integer and

$$f(k) = \begin{cases} 0 & \textit{if } 2(y - p) - k \cdot 2(h + p) < 0, \\ 2(y - p) - k \cdot 2(h + p) & \textit{if } 2(y - p) - k \cdot 2(h + p) \ge 0. \end{cases}$$

Since $0 \le x < 2(y - p) \le 2y$, by (4.7), $\delta(x, y) = 2y - 1$ if x is odd and $\delta(x, y) = 2y$ if x is even. Let $i = \lceil \frac{2(y-p)-x}{2(h+p)} \rceil$. Clearly, $i \ge 1$. Since $\frac{2(y-p)-x}{2(h+p)} \le i < \frac{2(y-p)-x}{2(h+p)} + 1 = \frac{2(y+h)-x}{2(h+p)}$, we have $2(y-p)-x \le 2i(h+p) < 2(y+h)-x$. Thus $2(y-hi-p) \le x+2pi < 2(y-hi+h)$. Since $i \ge 1$, we have $x + 2pi \ge x \ge 0$. Thus $\max\{2(y - hi - p), 0\} \le x + 2pi < 2(y - hi + h)$. Let $(x', y') = (x + 2pi, y - hi)$. Then we have

$$\max\{2(y' - p), 0\} \le x' < 2(y' + h). \tag{4.10}$$

By Lemma 8, cell (x', y') also represents v. Moreover, since $x \ge f(k) \ge 2(y - p) - k \cdot 2(h + p)$, we have $i < \frac{2(y+h)-x}{2(h+p)} \le \frac{2(y+h)-2(y-p)+k \cdot 2(h+p)}{2(h+p)} = k + 1$. Thus $i \le k$ and $y' = y - hi \ge kh - kh = 0$. Therefore $(x', y') \in S$. By (4.10), there are two subcases.

Subcase 2.1. $\max\{2(y' - p), 0\} \le x' < 2y'$.

Then $0 \le x' < 2y'$. Since $i \ge 1$, we have $y' = y - hi < y$. Note that x' is either odd or even. In the former case, $\delta(x', y') = 2y' - 1$. Thus $\delta(x', y') < 2y - 1$. Since $x = x' - 2pi$ is odd, we have $\delta(x, y) = 2y - 1$. Thus $\delta(x', y') < \delta(x, y)$. In the latter case, $\delta(x', y') = 2y'$. Thus $\delta(x', y') < 2y$. Since $x = x' - 2pi$ is even, we have $\delta(x, y) = 2y$. Thus $\delta(x', y') < \delta(x, y)$.

Subcase 2.2. $2y' \le x' < 2(y' + h)$.

Then $x' \ge 2y'$. By (4.7), $\delta(x', y') = x'$. Since $i \ge 1$, we have $y' + h = y - hi + h \le y$.

Note that x' is either odd or even. In the former case, $x' \leq 2(y'+h)-1$. Thus $x' \leq 2y-1$; hence $\delta(x',y') \leq 2y-1$. Since $x = x'-2pi$ is odd, we have $\delta(x,y) = 2y-1$. Thus $\delta(x',y') \leq \delta(x,y)$. In the latter case, $x' < 2(y'+h)$. Thus $x' < 2y$; hence $\delta(x',y') < 2y$. Since $x = x'-2pi$ is even, we have $\delta(x,y) = 2y$. Thus $\delta(x',y') < \delta(x,y)$.

Case 3. $kh \leq y < (k+1)h$, $0 \leq x < 2(y-p) - k \cdot 2(h+p)$, where k is some positive integer. Since $0 \leq x < 2y$, by (4.7), $\delta(x,y) = 2y-1$ if x is odd and $\delta(x,y) = 2y$ if x is even. Let $i = \lfloor \frac{y}{h} \rfloor$. Clearly, $i = k$ and $i \geq 1$. Thus $0 \leq 2pi \leq x + 2pi < 2(y-p) - i \cdot 2(h+p) + 2pi = 2(y-hi) - 2p \leq 2(y-hi)$. Therefore $0 \leq x+2pi < 2(y-hi)$. Let $(x',y') = (x+2pi, y-hi)$. Then we have

$$0 \leq x' < 2y'. \tag{4.11}$$

By Lemma 8, cell (x',y') also represents v. Moreover, since $kh \leq y < (k+1)h$ and $i = k$, we have $0 \leq y - hi < h$, i.e., $0 \leq y' < h$. By (4.11), $0 \leq x' < 2(y'+h)$. Therefore $(x',y') \in S$. Since $i \geq 1$, we have $y' = y - hi < y$. Note that x' is either odd or even. In the former case, by (4.7), $\delta(x',y') = 2y'-1$. Thus $\delta(x',y') < 2y-1$. Since $x = x'-2pi$ is odd, we have $\delta(x,y) = 2y-1$. Thus $\delta(x',y') \leq \delta(x,y)$. In the latter case, by (4.7), $\delta(x',y') = 2y'$. Thus $\delta(x',y') < 2y$. Since $x = x'-2pi$ is even, we have $\delta(x,y) = 2y$. Thus $\delta(x',y') \leq \delta(x,y)$.

∎

We now convert the L-shape of $MCR(N;a,b)$ to the MDD of $MCR(N;a,b)$. For convenience, let L denote the the L-shape$(2\ell, h, 2p, n)$ of $MCR(N;a,b)$ and set $k = \left\lceil \frac{l-h}{h+p} \right\rceil$ for easy writing. Partition L into $L_0, L_1, \cdots, L_k$ as follows:

$$L_0 = \{(x,y) \in L \mid 0 \leq x < 2(y+h)\},$$

$$L_i = \{(x,y) \in L \mid 2(y+h) + (i-1) \cdot 2(h+p) \leq x < 2(y+h) + i \cdot 2(h+p)\},$$

for $i = 1, 2, \cdots, k$. Derive $S_0, S_1, \cdots, S_k$ inside the strip S as follows:

$$S_0 = \{(x, y) \mid 0 \le y < h,\ 0 \le x < 2(y+h), (x, y) \in L_0\},$$

$$S_i = \{(x, y) \mid ih \le y < (i+1)h,\ 2(y-p) \le x < 2(y+h),\ (x + 2pi, y - hi) \in L_i\},$$

for $i = 1, 2, \cdots, k$. Take $MCR(100; 27, 61)$ as an example again. Figure 9 (a) shows L_0, L_1 and L_2; Figure 9 (b) shows S_0, S_1 and S_2. Note that $\bigcup_{i=0}^{2} S_i$ is the shaded part in Figure 8. We now prove that

Theorem 12 *The MDD of $MCR(N; a, b)$ is $\bigcup_{i=0}^{k} S_i$.*

Proof. Note that $|S_i| = |L_i|$ and if $(x, y) \in L_i$, then $(x - 2pi, y + hi) \in S_i$. Hence all the N nodes in $MCR(N; a, b)$ appear in $\bigcup_{i=0}^{k} S_i$. This theorem now follows from Lemma 11. ∎

It is well known that the MDD of a double-loop network tessellates the plane. We now have

Theorem 13 *The MDD of $MCR(N; a, b)$ tessellates the plane.*

Proof. This theorem follows from the fact that the L-shape of $MCR(N; a, b)$ tessellates the plane and the MDD of $MCR(N; a.b)$ is converted from its L-shape. ∎

The proof of the following lemma is similar to that of Lemma 8 and is therefore omitted.

Lemma 14 *Let i be an integer. If cell (x, y) represents node v, then cell $(x + 2\ell i, y - ni)$ also represents node v.*

Now consider the MDD of $MCR(N; a, b)$ for the case that $\ell < h$. Again, let L denote the L-shape$(2\ell, h, 2p, n)$ of $MCR(N; a, b)$. Set $k = \lceil \frac{h-\ell-1}{\ell+n} \rceil$ for easy writing. Since the arguments of this case is similar to the case that $\ell \ge h$, we will not give proofs for this case. Now, the strip S strip associated $MCR(N; a, b)$ is : (see Figure 11)

$$\begin{aligned} S = \ & \{(x, y) \mid 0 \le x < 2\ell,\ 0 \le y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell\} \ \cup \\ & \{(x, y) \mid x \ge 2\ell,\ \left\lfloor \frac{x}{2} \right\rfloor - n < y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell\}. \end{aligned}$$

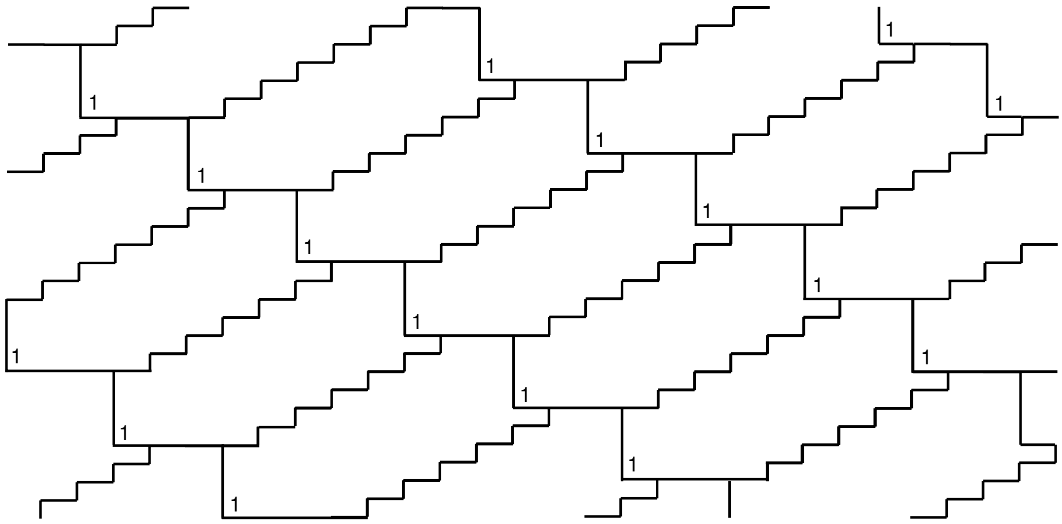


Figure 10: The MDD of $MCR(N;a,b)$ tessellates the plane.

In the proof of Lemma 11, Lemma 14 is used instead of Lemma 8. $L_0, L_1, \cdots, L_k$ are:

$$\begin{aligned} L_0 &= \{(x,y) \in L \mid 0 \le y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell\}, \\ L_i &= \{(x,y) \in L \mid \left\lfloor \frac{x}{2} \right\rfloor + \ell + (i-1)\cdot(\ell+n) < y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell + i\cdot(\ell+n)\}, \end{aligned}$$

for $i = 1, 2, \cdots, k$. $S_0, S_1, \cdots, S_k$ are: (see Figure 12)

$$S_0 = \{(x,y) \mid 0 \le x < 2\ell,\ 0 \le y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell, (x,y) \in L_0\},$$
$$S_i = \{(x,y) \mid i\cdot 2\ell \le x < (i+1)\cdot 2\ell,\ \left\lfloor \frac{x}{2} \right\rfloor - n < y \le \left\lfloor \frac{x}{2} \right\rfloor + \ell,\ (x - 2\ell i, y + ni) \in L_i\},$$

for $i = 1, 2, \cdots, k$. Theorem 12 and Theorem 13 also hold for the case $\ell < h$.

5 The diameter of $MCR(N;a,b)$

In this section, we will propose an algorithm for computing the diameter of $MCR(N;a,b)$. Let $d(u,v)$ be the length of a shortest path from u to v in $MCR(N;a,b)$ and let

$$D_u(N;a,b) = \max\{d(u,v) : v \in \{0, 1, \cdots, N-1\}\}.$$

Recall that $D(N;a,b)$ is the diameter of $MCR(N;a,b)$. We now prove that

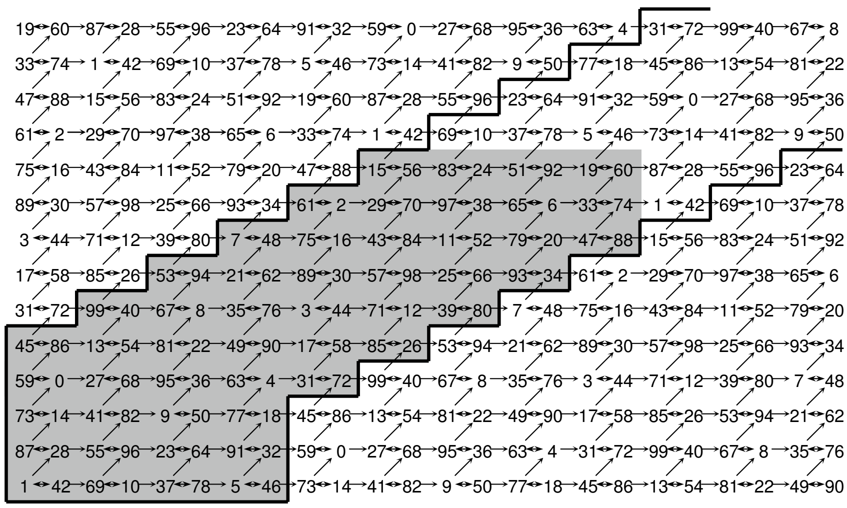


Figure 11: The strip of $MCR(100;27,41)$ and the MDD (shaded) of $MCR(100;27,41)$.

Theorem 15 $D(N;a,b) = \max\{D_1(N;a,b), D_1(N;a,N-b)\}$.

Proof. Clearly $D(N;a,b) = \max\{D_0(N;a,b), D_1(N;a,b)\}$ since in $MCR(N;a,b)$, all even nodes are symmetric and all odd nodes are symmetric. By Lemma 10, $MCR(N;a,b) \cong MCR(N;a,N-b)$ and the bijection from $MCR(N;a,b)$ to $MCR(N;a,N-b)$ is (4.8). Using (4.8), node 0 in $MCR(N;a,b)$ is mapped to node b in $MCR(N;a,N-b)$. Note that b is odd. Thus $D_0(N;a,b) = D_1(N;a,N-b)$ and we have this theorem. ∎

We use the following algorithm to calculate $D_1(N;a,b)$.

CALCULATE-D1.

Input: The L-shape(ℓ,h,p,n) of the corresponding double-loop network of $MCR(N;a,b)$;

Output: $D_1(N;a,b)$.

1. /* the case of $\ell \geq h$ */

if $\ell \geq h$ **then**

begin

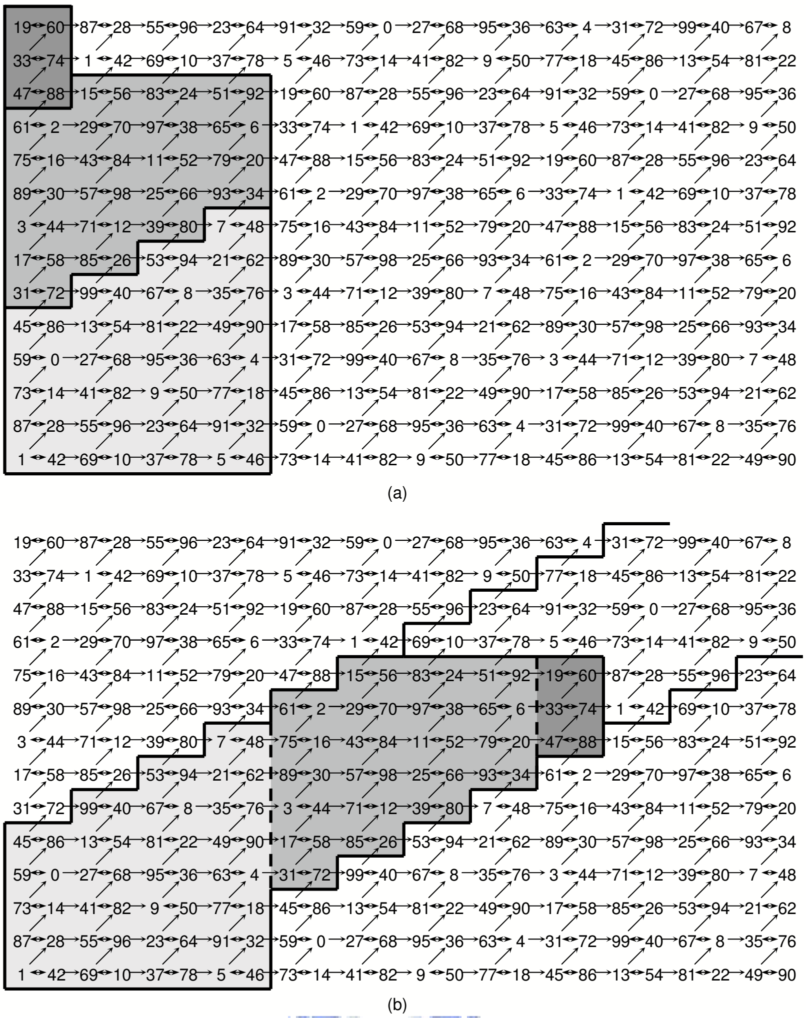


Figure 12: (a) Partitioning the L-shape of $MCR(100; 27, 41)$ into L_0 (shaded lighter), L_1 (shaded median) and L_2 (shaded deeper). (b) S_0 (shaded lighter), S_1 (shaded median) and S_2 (shaded deeper).

$k \leftarrow \left\lceil \frac{\ell - h}{h+p} \right\rceil;$

$r \leftarrow (\ell - h) \bmod (h+p);$

$$D_1(N; a, b) = \begin{cases} 2(k+1)h - 1 & \text{if} \quad r = 0 \\ 2kh + 2r - 1 & \text{if} \quad 0 < r < h - n \\ 2kh + 2(h-n) - 2 & \text{if} \quad h - n \le r < h - n + p \\ 2kh + 2(r-p) - 1 & \text{if} \quad h - n + p \le r < h + p \end{cases}$$

end

2. /* the case of $\ell < h$ */

else

begin

$k \leftarrow \lceil \frac{h-\ell-1}{\ell+n} \rceil$;

$r \leftarrow (h-\ell-1) \bmod (\ell+n)$;

$$D_1(N;a,b) = \begin{cases} 2(k+1)\ell & \textbf{if} \quad r=0 \\ 2k\ell+2r & \textbf{if} \quad 0<r<\ell-p \\ 2k\ell+2(\ell-p)-1 & \textbf{if} \quad \ell-p \le r < \ell-p+n \\ 2k\ell+2(r-n) & \textbf{if} \quad \ell-p+n \le r < \ell+n \end{cases}$$

end

end-of-CALCULATE-D1.

Theorem 16 *Algorithm CALCULATE-D1 is correct and its time complexity is $O(1)$.*

Proof.

Set $m = \ell - p$ and $q = h - n$ for easy writing. Note that the MDD of $MCR(N;a,b)$ is $\bigcup_{i=0}^{k} S_i$ and the L-shape L of $MCR(N;a,b)$ is $\bigcup_{i=0}^{k} L_i$. Also note that S_i corresponds to L_i. Let $\mathcal{A} = (\mathcal{A}_x, \mathcal{A}_y)$ be the cell left to $\mathcal{B}$, where $\mathcal{B} = (\mathcal{B}_x, \mathcal{B}_y)$ is the rightmost of uppermost cells in the MDD. Also, let $\mathcal{U} = (\mathcal{U}_x, \mathcal{U}_y)$ be the cell left to $\mathcal{V}$, where $\mathcal{V} = (\mathcal{V}_x, \mathcal{V}_y)$ is the uppermost of rightmost cells in the MDD. Let $\mathcal{A}', \mathcal{B}', \mathcal{U}', \mathcal{V}'$ denote the cells in L corresponding to $\mathcal{A}, \mathcal{B}, \mathcal{U}, \mathcal{V}$, respectively. It is not difficult to see that

$$D_1(N;a,b) = \max\{\delta(\mathcal{A}), \delta(\mathcal{B}), \delta(\mathcal{U}), \delta(\mathcal{V})\}. \tag{5.12}$$

First consider the case that $\ell \ge h$; see Figure 13 for an illustration. Since $k = \left\lceil \frac{\ell-h}{h+p} \right\rceil$ and $r = (\ell-h) \bmod (h+p)$, we have

$$\ell = \begin{cases} h + k(h+p) & if \quad r = 0, \\ h + (k-1)(h+p) + r & if \quad 0 < r < h+p. \end{cases} \tag{5.13}$$

It is not difficult to see that

$$\text{if cell } (x,y)\in L_k,\ \text{ then cell } (x-2kp,\,y+kh)\in S_k. \tag{5.14}$$

There are four cases:

Case 1. $r=0$.

Then $\mathcal{A}',\mathcal{B}',\mathcal{U}',\mathcal{V}'\in L_k$ and $\mathcal{A}'=(2m-2,h-1)$, $\mathcal{B}'=(2m-1,h-1)$, $\mathcal{U}'=(2\ell-2,q-1)$, $\mathcal{V}'=(2\ell-1,q-1)$. By (5.14), $\mathcal{A}=(2m-2-2kp,h-1+kh)$, $\mathcal{B}=(2m-1-2kp,h-1+kh)$, $\mathcal{U}=(2\ell-2-2kp,q-1+kh)$, $\mathcal{V}=(2\ell-1-2kp,q-1+kh)$. Since $\mathcal{A}_x-2\mathcal{A}_y=2m-2h-2k(h+p)\stackrel{(5.13)}{=}2m-2h-2(\ell-h)=2m-2\ell\le 0$, by (4.7),

$$\delta(\mathcal{A})=2\mathcal{A}_y=2(k+1)h-2.$$

Again, $\mathcal{B}_x-2\mathcal{B}_y=2m-2h+1-2k(h+p)\stackrel{(5.13)}{=}2m-2h+1-2(\ell-h)=2m-2\ell+1$. Note that either $\ell>m$ or $\ell=m$. In the former case, $\mathcal{B}_x-2\mathcal{B}_y<0$; thus by (4.7), $\delta(\mathcal{B})=2\mathcal{B}_y-1=2(k+1)h-3$. In the latter case, $\mathcal{B}_x-2\mathcal{B}_y>0$; thus by (4.7), $\delta(\mathcal{B})=\mathcal{B}_x=2(m-kp)-1=2(\ell-kp)-1\stackrel{(5.13)}{=}2(k+1)h-1$. Hence

$$\delta(\mathcal{B})=\begin{cases}2(k+1)h-3 & \text{if } \ell>m,\\ 2(k+1)h-1 & \text{if } \ell=m.\end{cases}$$

Since $\mathcal{U}_x-2\mathcal{U}_y=2\ell-2q-2k(h+p)\stackrel{(5.13)}{=}2\ell-2q-2(\ell-h)=2h-2q\le 0$, by (4.7),

$$\delta(\mathcal{U})=\mathcal{U}_x=2(\ell-kp)-2\stackrel{(5.13)}{=}2(k+1)h-2.$$

Since $\mathcal{V}_x-2\mathcal{V}_y=2\ell-2q+1-2k(h+p)\stackrel{(5.13)}{=}2\ell-2q+1-2(\ell-h)=2h-2q+1>0$, by (4.7),

$$\delta(\mathcal{V})=\mathcal{V}_x=2(\ell-kp)-1\stackrel{(5.13)}{=}2(k+1)h-1.$$

By (5.12),

$$D_1(N;a,b)=2(k+1)h-1.$$

Case 2. $0 < r < h - n$.

Then $\mathcal{A}', \mathcal{B}' \in L_k$ and $\mathcal{U}', \mathcal{V}' \in L_{k-1}$. Moreover, $\mathcal{A}' = (2\ell-2, r-1)$, $\mathcal{B}' = (2\ell-1, r-1)$, $\mathcal{U}' = (2\ell-2, q-1)$, $\mathcal{V}' = (2\ell-1, q-1)$. By (5.14), $\mathcal{A} = (2\ell-2-2kp, r-1+kh)$, $\mathcal{B} = (2\ell-1-2kp, r-1+kh)$, $\mathcal{U} = (2\ell-2-2(k-1)p, q-1+(k-1)h)$, $\mathcal{V} = (2\ell-1-2(k-1)p, q-1+(k-1)h)$. Since $\mathcal{A}_x - 2\mathcal{A}_y = 2\ell - 2r - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2r - 2(\ell+p-r) = -2p \leq 0$. By (4.7),

$$\delta(\mathcal{A}) = 2\mathcal{A}_y = 2kh + 2r - 2.$$

Again, $\mathcal{B}_x - 2\mathcal{B}_y = 2\ell - 2r + 1 - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2r + 1 - 2(\ell+p-r) = 1-2p$. If $p > 0$, then $\mathcal{B}_x - 2\mathcal{B}_y < 0$; thus by (4.7), $\delta(\mathcal{B}) = 2\mathcal{B}_y - 1 = 2kh + 2r - 3$. If $p = 0$, then $\mathcal{B}_x - 2\mathcal{B}_y > 0$; thus by (4.7), $\delta(\mathcal{B}) = \mathcal{B}_x = 2\ell - 1 - 2kp = 2\ell - 1 \overset{(5.13)}{=} 2kh + 2r - 1$. Hence

$$\delta(\mathcal{B}) = \begin{cases} 2kh + 2r - 3 & \text{if } p > 0, \\ 2kh + 2r - 1 & \text{if } p = 0. \end{cases}$$

Since $\mathcal{U}_x - 2\mathcal{U}_y = 2\ell - 2q - 2(k-1)(p+h) \overset{(5.13)}{=} 2\ell - 2q - 2(\ell - h - r) = 2(h-q+r) > 0$, by (4.7),

$$\delta(\mathcal{U}) = \mathcal{U}_x = 2(\ell - (k-1)p) - 2 \overset{(5.13)}{=} 2kh + 2r - 2.$$

Since $\mathcal{V}_x - 2\mathcal{V}_y = 2\ell - 2q + 1 - 2(k-1)(h+p) \overset{(5.13)}{=} 2\ell - 2q + 1 - 2(\ell - h - r) = 2h - 2q + 2r + 1 > 0$, by (4.7),

$$\delta(\mathcal{V}) = \mathcal{V}_x = 2(\ell - (k-1)p) - 1 \overset{(5.13)}{=} 2kh + 2r - 1.$$

By (5.12),

$$D_1(N; a, b) = 2kh + 2r - 1.$$

Case 3. $h - n \leq r < h - n + p$.

Then $\mathcal{A}', \mathcal{B}' \in L_k$ and $\mathcal{U}', \mathcal{V}' \in L_{k-1}$. Moreover, $\mathcal{A}' = (2\ell-2, q-1)$, $\mathcal{B}' = (2\ell-1, q-1)$, $\mathcal{U}' = (2h+(k-1)\cdot 2(h+p)+2(q-1)-2, q-1)$, $\mathcal{V}' = (2h+(k-1)\cdot 2(h+p)+2(q-1)-1, q-1)$.

By (5.14), $\mathcal{A} = (2\ell - 2 - 2kp, q - 1 + kh)$, $\mathcal{B} = (2\ell - 1 - 2kp, q - 1 + kh)$, $\mathcal{U} = (2kh + 2(q-1) - 2, q - 1 + (k-1)h)$, $\mathcal{V} = (2kh + 2(q-1) - 1, q - 1 + (k-1)h)$. Since $r < h - n + p$, we have $r - p - q < 0$. Since $\mathcal{A}_x - 2\mathcal{A}_y = 2\ell - 2q - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2q - 2(\ell + p - r) = 2(r - p - q) < 0$, by (4.7),

$$\delta(\mathcal{A}) = 2\mathcal{A}_y = 2kh + 2q - 2.$$

Since $\mathcal{B}_x - 2\mathcal{B}_y = 2\ell - 2q + 1 - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2q + 1 - 2(\ell + p - r) = 2(r - p - q) + 1 < 0$, by (4.7),

$$\delta(\mathcal{B}) = 2\mathcal{B}_y - 1 = 2kh + 2q - 3.$$

Since $\mathcal{U}_x - 2\mathcal{U}_y = 2h - 2 \geq 0$, by (4.7),

$$\delta(\mathcal{U}) = \mathcal{U}_x = 2kh + 2q - 4.$$

Since $\mathcal{V}_x - 2\mathcal{V}_y = 2h - 1 > 0$, by (4.7),

$$\delta(\mathcal{V}) = \mathcal{V}_x = 2kh + 2q - 3.$$

By (5.12),

$$D_1(N; a, b) = 2kh + 2q - 2 = 2kh + 2(h - n) - 2.$$

Case 4. $h - n + p \leq r < h + p$.

Then $\mathcal{A}', \mathcal{B}', \mathcal{U}', \mathcal{V}' \in L_k$ and $\mathcal{A}' = (2m - 2, r - p - 1)$, $\mathcal{B}' = (2m - 1, r - p - 1)$, $\mathcal{U}' = (2\ell - 2, q - 1)$, $\mathcal{V}' = (2\ell - 1, q - 1)$. By (5.14), $\mathcal{A} = (2m - 2 - 2kp, r - p - 1 + kh)$, $\mathcal{B} = (2m - 1 - 2kp, r - p - 1 + kh)$, $\mathcal{U} = (2\ell - 2 - 2kp, q - 1 + kh)$, $\mathcal{V} = (2\ell - 1 - 2kp, q - 1 + kh)$. Since $\mathcal{A}_x - 2\mathcal{A}_y = 2m - 2r + 2p - 2k(h+p) \overset{(5.13)}{=} 2m - 2r + 2p - 2(\ell + p - r) = 2m - 2\ell \leq 0$, by (4.7),

$$\delta(\mathcal{A}) = 2\mathcal{A}_y = 2kh + 2(r - p) - 2.$$

Again, $\mathcal{B}_x - 2\mathcal{B}_y = 2m - 2r + 2p + 1 - 2k(h+p) \overset{(5.13)}{=} 2m - 2r + 2p + 1 - 2(\ell + p - r) = 2m - 2\ell + 1$. If $\ell > m$, then $\mathcal{B}_x - 2\mathcal{B}_y < 0$; thus by (4.7), $\delta(\mathcal{B}) = 2\mathcal{B}_y - 1 = 2kh + 2(r - p) - 3$. If

$\ell = m$, then $\mathcal{B}_x - 2\mathcal{B}_y > 0$; thus by (4.7), $\delta(\mathcal{B}) = \mathcal{B}_x = 2(m-kp)-1 = 2(\ell-kp)-1 \overset{(5.13)}{=} 2kh + 2(r-p) - 1$. Hence

$$\delta(\mathcal{B}) = \begin{cases} 2kh + 2(r-p) - 3 & \text{if } \ell > m, \\ 2kh + 2(r-p) - 1 & \text{if } \ell = m. \end{cases}$$

Since $r \geq h - n + p$, we have $r - p - q \geq 0$. Since $\mathcal{U}_x - 2\mathcal{U}_y = 2\ell - 2q - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2q - 2(\ell + p - r) = 2(r-p-q) \geq 0$, by (4.7),

$$\delta(\mathcal{U}) = \mathcal{U}_x = 2(\ell - kp) - 2 \overset{(5.13)}{=} 2kh + 2(r-p) - 2.$$

Since $\mathcal{V}_x - 2\mathcal{V}_y = 2\ell - 2q + 1 - 2k(h+p) \overset{(5.13)}{=} 2\ell - 2q + 1 - 2(\ell + p - r) = 2(r-p-q) + 1 > 0$, by (4.7),

$$\delta(\mathcal{V}) = \mathcal{V}_x = 2(\ell - kp) - 1 \overset{(5.13)}{=} 2kh + 2(r-p) - 1.$$

By (5.12),

$$D_1(N; a, b) = 2kh + 2(r-p) - 1.$$

Next consider the case that $\ell < h$. See Figure 14 for an illustration. Since $k = \lceil \frac{h-\ell-1}{\ell+n} \rceil$ and $r = (h - \ell - 1) \bmod (\ell + n)$, we have

$$h = \begin{cases} l + 1 + k(l+n) & \text{if } r = 0, \\ l + 1 + (k-1)(l+n) + r & \text{if } 0 < r < \ell + n. \end{cases} \tag{5.15}$$

It is not difficult to see that

$$\text{if cell } (x, y) \in L_k, \text{ then cell } (x + 2k\ell, y - kn) \in S_k. \tag{5.16}$$

There are four cases:

Case 1. $r = 0$.

Then $\mathcal{A}', \mathcal{B}', \mathcal{U}', \mathcal{V}' \in L_k$ and $\mathcal{A}' = (2m-2, h-1)$, $\mathcal{B}' = (2m-1, h-1)$, $\mathcal{U}' = (2\ell-2, q-1)$, $\mathcal{V}' = (2\ell-1, q-1)$. By (5.16), $\mathcal{A} = (2m - 2 + 2k\ell, h - 1 - kn)$,

$\mathcal{B} = (2m-1+2k\ell, h-1-kn), \mathcal{U} = (2\ell-2+2k\ell, q-1-kn), \mathcal{V} = (2\ell-1+2k\ell, q-1-kn)$.

By similar arguments as that in the case of $\ell \geq h$, we have

$$\begin{aligned}
\delta(\mathcal{A}) &= 2\mathcal{A}_y = 2(k+1)\ell, \\
\delta(\mathcal{B}) &= 2\mathcal{B}_y - 1 = 2(k+1)\ell - 1, \\
\delta(\mathcal{U}) &= \begin{cases} \mathcal{U}_x = 2(k+1)\ell - 2 & \text{if } h > q, \\ 2\mathcal{U}_y = 2(k+1)\ell & \text{if } h = q. \end{cases} \\
\delta(\mathcal{V}) &= \begin{cases} \mathcal{V}_x = 2(k+1)\ell - 1 & \text{if } h > q, \\ 2\mathcal{V}_y = 2(k+1)\ell & \text{if } h = q. \end{cases}
\end{aligned}$$

By (5.12),

$$D_1(N; a, b) = 2(k+1)\ell.$$

Case 2. $0 < r < \ell - p$.

Then $\mathcal{A}', \mathcal{B}' \in L_{k-1}$ and $\mathcal{U}', \mathcal{V}' \in L_k$. Moreover, $\mathcal{A}' = (2m-2, h-1)$, $\mathcal{B}' = (2m-1, h-1)$, $\mathcal{U}' = (2r-2, h-1)$, $\mathcal{V}' = (2r-1, h-1)$. By (5.16), $\mathcal{A} = (2m-2+2(k-1)\ell, h-1-(k-1)n)$, $\mathcal{B} = (2m-1+2(k-1)\ell, h-1-(k-1)n)$, $\mathcal{U} = (2r-2+2k\ell, h-1-kn)$, $\mathcal{V} = (2r-1+2k\ell, h-1-kn)$. By similar arguments as that in the case of $\ell \geq h$, we have

$$\begin{aligned}
\delta(\mathcal{A}) &= 2\mathcal{A}_y = 2k\ell + 2r, \\
\delta(\mathcal{B}) &= 2\mathcal{B}_y - 1 = 2k\ell + 2r - 1, \\
\delta(\mathcal{U}) &= \begin{cases} \mathcal{U}_x = 2k\ell + 2r - 2 & \text{if } n > 0, \\ 2\mathcal{U}_y = 2k\ell + 2r & \text{if } n = 0. \end{cases} \\
\delta(\mathcal{V}) &= \begin{cases} \mathcal{V}_x = 2k\ell + 2r - 1 & \text{if } n > 0, \\ 2\mathcal{V}_y = 2k\ell + 2r & \text{if } n = 0. \end{cases}
\end{aligned}$$

By (5.12),

$$D_1(N; a, b) = 2k\ell + 2r.$$

Case 3. $\ell - p \leq r < \ell - p + n$.

Then $\mathcal{A}', \mathcal{B}' \in L_{k-1}$, $\mathcal{U}', \mathcal{V}' \in L_k$ and $\mathcal{A}' = (2m-2, \ell+1+(k-1)(\ell+n)+(m-1)-1)$,

$\mathcal{B}' = (2m-1, \ell+1+(k-1)(\ell+n)+(m-1)-1), \mathcal{U}' = (2m-2, h-1), \mathcal{V}' = (2m-1, h-1)$. By (5.16), $\mathcal{A} = (2m-2+2(k-1)\ell, k\ell+m-1)$, $\mathcal{B} = (2m-1+2(k-1)\ell, k\ell+m-1)$, $\mathcal{U} = (2m-2+2k\ell, h-1-kn)$, $\mathcal{V} = (2m-1+2k\ell, h-1-kn)$. By similar arguments as that in the case of $\ell \geq h$, we have

$$\delta(\mathcal{A}) = 2k\ell+2m-2,\ \delta(\mathcal{B}) = 2k\ell+2m-3,\ \delta(\mathcal{U}) = 2k\ell+2m-2, \text{ and } \delta(\mathcal{V}) = 2k\ell+2m-1.$$

By (5.12),

$$D_1(N; a, b) = 2k\ell + 2m - 1 = 2k\ell + 2(\ell - p) - 1.$$

Case 4. $\ell - p + n \leq r < \ell + n$.

Then $\mathcal{A}', \mathcal{B}', \mathcal{U}', \mathcal{V}' \in L_k$ and $\mathcal{A}' = (2m-2, h-1)$, $\mathcal{B}' = (2m-1, h-1)$, $\mathcal{U}' = (2r-2n-2, q-1)$, $\mathcal{V}' = (2r-2n-1, q-1)$. By (5.16), $\mathcal{A} = (2m-2+2k\ell, h-1-kn)$, $\mathcal{B} = (2m-1+2k\ell, h-1-kn)$, $\mathcal{U} = (2r-2n-2+2k\ell, q-1-kn)$, $\mathcal{V} = (2r-2n-1+2k\ell, q-1-kn)$. By similar arguments as that in the case of $\ell \geq h$, we have

$$\delta(\mathcal{A}) = 2k\ell + 2(r-n),\ \delta(\mathcal{B}) = 2k\ell + 2(r-n) - 1,\ \delta(\mathcal{V}) = 2k\ell + 2(r-n) - 1, \text{ and}$$

$$\delta(\mathcal{U}) = \begin{cases} 2k\ell + 2(r-n) - 2 & \text{if } h > q, \\ 2k\ell + 2(r-n) & \text{if } h = q. \end{cases}$$

By (5.12),

$$D_1(N; a, b) = 2k\ell + 2(r-n).$$

From the above discussion, Algorithm CALCULATE-D1 is correct. Since all steps in Algorithm CALCULATE-D1 take constant time, its time complexity is $O(1)$. ∎

We now propose an algorithm to compute the diameter of $MCR(N; a, b)$.

DIAMETER-OF-MCR.

Input: A mixed chordal ring network $MCR(N; a, b)$.

Output: The diameter $D(N;a,b)$ of $MCR(N;a,b)$.

1. if $b = \frac{N}{2}$ **then** return $\frac{N}{2}$ and stop the algorithm;

2. else

begin

use CHENG-HWANG-ALGORITHM to derive (ℓ, h, p, n) of $DL(\frac{N}{2}; \frac{a+b}{2}, \frac{a-b}{2})$;

use CALCULATE-D1 to derive $D_1(N;a,b)$ from (ℓ, h, p, n);

use CHENG-HWANG-ALGORITHM to derive (ℓ, h, p, n) of $DL(\frac{N}{2}; \frac{a-b}{2}, \frac{a+b}{2})$;

use CALCULATE-D1 to derive $D_1(N;a,N-b)$ from (ℓ, h, p, n);

$D(N;a,b) = \max\{D_1(N;a,b), D_1(N;a,N-b)\}$;

end

end-of-DIAMETER-OF-MCR.

Theorem 17 *Algorithm DIAMETER-OF-MCR is correct and takes* $O(\log N)$ *time.*

Proof. Note that the double-loop network corresponding to $MCR(N;a,N-b)$ is $DL(\frac{N}{2};\frac{a-b}{2}, \frac{a+b}{2})$. So, by Theorem 15 and Theorem 16, Algorithm DIAMETER-OF-MCR is correct. Since CHENG-HWANG-ALGORITHM takes $O(\log N)$ time and CALCULATE-D1 takes $O(1)$ time, the time complexity of DIAMETER-OF-MCR is $O(\log N)$. ∎

We now use an example to show how DIAMETER-OF-MCR is executed. Consider the mixed chordal ring network $MCR(100;27,61)$. By CHENG-HWANG-ALGORITHM, the (ℓ, h, p, n) of $DL(\frac{100}{2}; \frac{27+61}{2}, \frac{27-61}{2})$ is (14,4,3,2). Thus input to CALCULATE-D1 is (14,4,3,2). So $k = \lceil \frac{14-4}{4+3} \rceil = 2$ and $r = (14-4) \bmod (4+3) = 3$. Since $h - n \le r < h - n + p$, $D_1(100;27,61) = 2kh + 2(h-n) - 2 = 18$. Also, by CHENG-HWANG-ALGORITHM, the (ℓ, h, p, n) of $DL(\frac{100}{2}; \frac{27-61}{2}, \frac{27+61}{2})$ is (4,14,2,3). Thus input to CALCULATE-D1 is (4,14,2,3). So $k = \lceil \frac{14-4-1}{4+3} \rceil = 2$ and $r = (14-4-1) \bmod (4+3) = 2$. Since $\ell - p \le r < \ell - p + n$, $D_1(100;27,100-61) = 19$. Hence $D(100;27,61) = D_1(100;27,100-61) = 19$.

6 Some experimental results

In this section, we list $D(N)$ and the parameters a and b so that $D(N; a, b) = D(N)$ for $N \leq 5000$. Note that N must be even. Note also that no $MCR(N; a, b)$ satisfying $\gcd(N, a, b) = 1$, $a \neq b$, and $a + b \neq N$ when $N = 4$. So there are $5000/2 - 1 = 2499$ possible N's. Among these 2499 N's, 2471 of them (about 98.88%) have their $D(N)$ achievable by choosing $a = 1$. Among these 1299 N's, only 28 of them (about 1.12%) have their $D(N)$ not achievable by choosing $a = 1$; see Table 1. (In the following two tables, lb is the abbreviation for lower bound and D-1 means the smallest diameter that can be achieved by choosing $a = 1$.) And among these 28 special N's, 24 of them have their N equal to 2 times two consecutive odd integers, i.e., $N = 2(2k-1)(2k+1)$ for some k; for these 24 N's, $D(N)$ is achievable by choosing $a = 2k - 1$ and $b = 2k + 1$; see also Table 1. The remaining four N's are 1320, 2250, 2280 and 4914; their $D(N)$ and the parameters a and b so that $D(N; a, b) = D(N)$ are given in Table 2. We summarize the results for $N \leq 5000$ below.

the percentage of N's whose $D(N)$ are achievable by choosing $a = 1$	about 98.88%
the percentage of N's whose $D(N)$ are not achievable by choosing $a = 1$	about 1.12%

the percentage of N's whose $D(N)$ are achievable by choosing $a = 1$	about 98.88%
the percentage of N's whose $D(N)$ are not achievable by choosing $a = 1$ and $N = 2(2k-1)(2k+1)$	about 0.96%
the percentage of N's whose $D(N)$ are not achievable by choosing $a = 1$ and $N \neq 2(2k-1)(2k+1)$	about 0.16%

N	lb for $D(N)$	$D(N)$	a	b	D-1	a	b
$30 = 2 \cdot 3 \cdot 5$	7	7	3	5	8	1	5
$70 = 2 \cdot 5 \cdot 7$	11	11	5	7	12	1	9
$126 = 2 \cdot 7 \cdot 9$	15	15	7	9	16	1	11
$198 = 2 \cdot 9 \cdot 11$	19	19	9	11	20	1	17
$286 = 2 \cdot 11 \cdot 13$	23	23	11	13	24	1	21
$390 = 2 \cdot 13 \cdot 15$	27	27	13	15	28	1	21
$510 = 2 \cdot 15 \cdot 17$	31	31	15	17	32	1	29
$646 = 2 \cdot 17 \cdot 19$	35	35	17	19	36	1	33
$798 = 2 \cdot 19 \cdot 21$	39	39	19	21	40	1	37
$966 = 2 \cdot 21 \cdot 23$	43	43	21	23	44	1	41
$1150 = 2 \cdot 23 \cdot 25$	47	47	23	25	48	1	39
1320	50	51	3	95	52	1	365
$1350 = 2 \cdot 25 \cdot 27$	51	51	25	27	52	1	49
$1566 = 2 \cdot 27 \cdot 29$	55	55	27	29	56	1	53
$1798 = 2 \cdot 29 \cdot 31$	59	59	29	31	60	1	57
$2046 = 2 \cdot 31 \cdot 33$	63	63	31	33	64	1	61
2250	66	67	3	65	68	1	57
2280	67	67	3	625	68	1	309
$2310 = 2 \cdot 33 \cdot 35$	67	67	33	35	68	1	57
$2590 = 2 \cdot 35 \cdot 37$	71	71	35	37	72	1	69
$2886 = 2 \cdot 37 \cdot 39$	75	75	37	39	76	1	73
$3198 = 2 \cdot 39 \cdot 41$	79	79	39	41	80	1	77
$3526 = 2 \cdot 41 \cdot 43$	83	83	41	43	84	1	81
$3870 = 2 \cdot 43 \cdot 45$	87	87	43	45	88	1	85
$4230 = 2 \cdot 45 \cdot 47$	91	91	45	47	92	1	89
$4606 = 2 \cdot 47 \cdot 49$	95	95	47	49	96	1	83
4914	98	99	3	581	100	1	87
$4998 = 2 \cdot 49 \cdot 51$	99	99	49	51	100	1	97

Table 1: The N's whose $D(N)$ are not achievable by choosing $a = 1$.

N	lb for $D(N)$	$D(N)$	a	b	D-1	a	b
1320	50	51	3	95	52	1	365
2250	66	67	3	65	68	1	57
2280	67	67	3	625	68	1	309
4914	98	99	3	581	100	1	87

Table 2: The N's whose $D(N)$ are not achievable by choosing $a = 1$ and $N \neq 2(2k-1)(2k+1)$.

N	$D(N)$	a	b
2	1	1	1
4	NULL	NULL	NULL
6	3	1	3
8	3	1	3
10	4	1	3
12	4	1	3
14	5	1	3
16	6	1	3
18	5	1	5
20	6	1	5
22	6	1	5
24	6	1	5
26	7	1	7
28	7	1	5
30	7	3	5
32	7	1	7
34	8	1	5
36	8	1	15
38	8	1	7
40	8	1	7
42	9	1	9
44	10	1	5
46	9	1	7
48	10	1	7
50	9	1	9
52	10	1	7
54	10	1	7
56	10	1	21
58	10	1	9
60	10	1	9
62	11	1	11
64	11	1	19
66	11	1	25
68	11	1	9
70	11	5	7
72	11	1	11
74	12	1	11
76	12	1	9
78	12	1	9
80	12	1	35
82	12	1	11
84	12	1	11
86	13	1	9
88	14	1	9
90	13	1	33
92	14	1	11
94	13	1	11
96	14	1	9
98	13	1	13
100	14	1	13

N	$D(N)$	a	b
102	14	1	39
104	14	1	11
106	14	1	11
108	14	1	45
110	14	1	13
112	14	1	13
114	15	1	15
116	15	1	11
118	15	1	27
120	15	1	33
122	15	1	51
124	15	1	13
126	15	7	9
128	15	1	15
130	16	1	15
132	16	1	39
134	16	1	29
136	16	1	13
138	16	1	13
140	16	1	63
142	16	1	15
144	16	1	15
146	17	1	17
148	18	1	11
150	17	1	13
152	18	1	13
154	17	1	47
156	18	1	15
158	17	1	15
160	18	1	15
162	17	1	17
164	18	1	13
166	18	1	31
168	18	1	45
170	18	1	39
172	18	1	15
174	18	1	15
176	18	1	77
178	18	1	17
180	18	1	17
182	19	1	19
184	19	1	51
186	19	1	33
188	19	1	15
190	19	1	41
192	19	1	69
194	19	1	85
196	19	1	17
198	19	9	11
200	19	1	19

N	$D(N)$	a	b
202	20	1	15
204	20	1	15
206	20	1	47
208	20	1	37
210	20	1	39
212	20	1	17
214	20	1	17
216	20	1	99
218	20	1	19
220	20	1	19
222	21	1	21
224	22	1	21
226	21	1	69
228	22	1	17
230	21	1	17
232	22	1	15
234	21	1	69
236	22	1	19
238	21	1	19
240	22	1	19
242	21	1	21
244	22	1	21
246	22	1	17
248	22	1	17
250	22	1	67
252	22	1	47
254	22	1	45
256	22	1	19
258	22	1	19
260	22	1	117
262	22	1	21
264	22	1	21
266	23	1	23
268	23	1	61
270	23	1	97
272	23	1	41
274	23	1	43
276	23	1	19
278	23	1	65
280	23	1	87
282	23	1	127
284	23	1	21
286	23	11	13
288	23	1	23
290	24	1	23
292	24	1	81
294	24	1	19
296	24	1	19
298	24	1	45
300	24	1	47

N	$D(N)$	a	b
302	24	1	107
304	24	1	21
306	24	1	21
308	24	1	143
310	24	1	23
312	24	1	23
314	25	1	19
316	26	1	17
318	25	1	69
320	26	1	57
322	25	1	133
324	26	1	21
326	25	1	21
328	26	1	21
330	25	1	117
332	26	1	19
334	25	1	23
336	26	1	23
338	25	1	25
340	26	1	25
342	26	1	123
344	26	1	45
346	26	1	21
348	26	1	21
350	26	1	55
352	26	1	53
354	26	1	75
356	26	1	23
358	26	1	23
360	26	1	165
362	26	1	25
364	26	1	25
366	27	1	27
368	27	1	21
370	27	1	115
372	27	1	87
374	27	1	49
376	27	1	51
378	27	1	57
380	27	1	23
382	27	1	87
384	27	1	117
386	27	1	177
388	27	1	25
390	27	13	15
392	27	1	27
394	28	1	27
396	28	1	75
398	28	1	93
400	28	1	91

N	$D(N)$	a	b
402	28	1	23
404	28	1	23
406	28	1	55
408	28	1	93
410	28	1	89
412	28	1	25
414	28	1	25
416	28	1	195
418	28	1	27
420	28	1	27
422	29	1	29
424	30	1	23
426	29	1	23
428	30	1	23
430	29	1	91
432	30	1	21
434	29	1	189
436	30	1	25
438	29	1	25
440	30	1	25
442	29	1	139
444	30	1	27
446	29	1	27
448	30	1	23
450	29	1	29
452	30	1	29
454	30	1	81
456	30	1	99
458	30	1	53
460	30	1	55
462	30	1	25
464	30	1	25
466	30	1	61
468	30	1	111
470	30	1	89
472	30	1	27
474	30	1	27
476	30	1	221
478	30	1	29
480	30	1	29
482	31	1	31
484	31	1	105
486	31	1	201
488	31	1	25
490	31	1	145
492	31	1	57
494	31	1	59
496	31	1	65
498	31	1	87
500	31	1	27

N	$D(N)$	a	b
502	31	1	119
504	31	1	177
506	31	1	235
508	31	1	29
510	31	15	17
512	31	1	31
514	32	1	25
516	32	1	135
518	32	1	121
520	32	1	165
522	32	1	93
524	32	1	71
526	32	1	27
528	32	1	27
530	32	1	155
532	32	1	93
534	32	1	99
536	32	1	29
538	32	1	29
540	32	1	255
542	32	1	31
544	32	1	31
546	33	1	33
548	34	1	33
550	33	1	115
552	34	1	27
554	33	1	27
556	34	1	27
558	33	1	177
560	34	1	65
562	33	1	157
564	34	1	25
566	33	1	29
568	34	1	29
570	33	1	177
572	34	1	31
574	33	1	31
576	34	1	31
578	33	1	33
580	34	1	27
582	34	1	27
584	34	1	93
586	34	1	111
588	34	1	61
590	34	1	63
592	34	1	155
594	34	1	29
596	34	1	29
598	34	1	111
600	34	1	107

N	$D(N)$	a	b
602	34	1	187
604	34	1	31
606	34	1	31
608	34	1	285
610	34	1	33
612	34	1	33
614	35	1	35
616	35	1	163
618	35	1	131
620	35	1	97
622	35	1	255
624	35	1	29
626	35	1	65
628	35	1	67
630	35	1	73
632	35	1	187
634	35	1	101
636	35	1	31
638	35	1	149
640	35	1	203
642	35	1	301
644	35	1	33
646	35	17	19
648	35	1	35
650	36	1	35
652	36	1	29
654	36	1	29
656	36	1	117
658	36	1	103
660	36	1	193
662	36	1	79
664	36	1	69
666	36	1	31
668	36	1	31
670	36	1	185
672	36	1	107
674	36	1	141
676	36	1	33
678	36	1	33
680	36	1	323
682	36	1	35
684	36	1	35
686	37	1	37
688	38	1	27
690	37	1	123
692	38	1	65
694	37	1	105
696	38	1	31
698	37	1	31
700	38	1	31

N	$D(N)$	a	b
702	37	1	249
704	38	1	73
706	37	1	105
708	38	1	33
710	37	1	33
712	38	1	29
714	37	1	249
716	38	1	35
718	37	1	35
720	38	1	35
722	37	1	37
724	38	1	37
726	38	1	255
728	38	1	31
730	38	1	31
732	38	1	153
734	38	1	69
736	38	1	71
738	38	1	273
740	38	1	79
742	38	1	33
744	38	1	33
746	38	1	111
748	38	1	117
750	38	1	159
752	38	1	35
754	38	1	35
756	38	1	357
758	38	1	37
760	38	1	37
762	39	1	39
764	39	1	105
766	39	1	145
768	39	1	201
770	39	1	161
772	39	1	123
774	39	1	327
776	39	1	33
778	39	1	75
780	39	1	81
782	39	1	287
784	39	1	329
786	39	1	123
788	39	1	35
790	39	1	189
792	39	1	249
794	39	1	375
796	39	1	37
798	39	19	21
800	39	1	39

N	$D(N)$	a	b
802	40	1	39
804	40	1	139
806	40	1	109
808	40	1	33
810	40	1	33
812	40	1	127
814	40	1	121
816	40	1	87
818	40	1	77
820	40	1	79
822	40	1	35
824	40	1	35
826	40	1	125
828	40	1	219
830	40	1	159
832	40	1	37
834	40	1	37
836	40	1	399
838	40	1	39
840	40	1	39
842	41	1	33
844	42	1	33
846	41	1	295
848	42	1	135
850	41	1	115
852	42	1	31
854	41	1	127
856	42	1	35
858	41	1	35
860	42	1	35
862	41	1	195
864	42	1	139
866	41	1	119
868	42	1	37
870	41	1	37
872	42	1	37
874	41	1	279
876	42	1	33
878	41	1	39
880	42	1	39
882	41	1	41
884	42	1	41
886	42	1	155
888	42	1	115
890	42	1	117
892	42	1	35
894	42	1	35
896	42	1	77
898	42	1	79
900	42	1	273

N	$D(N)$	a	b
902	42	1	87
904	42	1	85
906	42	1	37
908	42	1	37
910	42	1	125
912	42	1	237
914	42	1	173
916	42	1	39
918	42	1	39
920	42	1	437
922	42	1	41
924	42	1	41
926	43	1	43
928	43	1	35
930	43	1	363
932	43	1	141
934	43	1	179
936	43	1	123
938	43	1	203
940	43	1	195
942	43	1	81
944	43	1	37
946	43	1	89
948	43	1	207
950	43	1	123
952	43	1	413
954	43	1	165
956	43	1	39
958	43	1	227
960	43	1	333
962	43	1	457
964	43	1	41
966	43	21	23
968	43	1	43
970	44	1	43
972	44	1	147
974	44	1	233
976	44	1	157
978	44	1	171
980	44	1	37
982	44	1	37
984	44	1	213
986	44	1	95
988	44	1	85
990	44	1	87
992	44	1	235
994	44	1	39
996	44	1	39
998	44	1	135
1000	44	1	173

N	$D(N)$	a	b
1002	44	1	393
1004	44	1	41
1006	44	1	41
1008	44	1	483
1010	44	1	43
1012	44	1	43
1014	45	1	45
1016	46	1	37
1018	45	1	37
1020	46	1	37
1022	45	1	287
1024	46	1	81
1026	45	1	141
1028	46	1	99
1030	45	1	195
1032	46	1	35
1034	45	1	39
1036	46	1	39
1038	45	1	237
1040	46	1	143
1042	45	1	141
1044	46	1	41
1046	45	1	41
1048	46	1	41
1050	45	1	333
1052	46	1	43
1054	45	1	43
1056	46	1	37
1058	45	1	45
1060	46	1	45
1062	46	1	169
1064	46	1	243
1066	46	1	127
1068	46	1	129
1070	46	1	147
1072	46	1	39
1074	46	1	39
1076	46	1	87
1078	46	1	327
1080	46	1	95
1082	46	1	93
1084	46	1	131
1086	46	1	41
1088	46	1	41
1090	46	1	317
1092	46	1	191
1094	46	1	227
1096	46	1	43
1098	46	1	43
1100	46	1	525

N	$D(N)$	a	b
1102	46	1	45
1104	46	1	45
1106	47	1	47
1108	47	1	127
1110	47	1	263
1112	47	1	39
1114	47	1	311
1116	47	1	133
1118	47	1	485
1120	47	1	145
1122	47	1	245
1124	47	1	89
1126	47	1	91
1128	47	1	41
1130	47	1	345
1132	47	1	245
1134	47	1	137
1136	47	1	319
1138	47	1	183
1140	47	1	43
1142	47	1	275
1144	47	1	367
1146	47	1	547
1148	47	1	45
1150	47	23	25
1152	47	1	47
1154	48	1	47
1156	48	1	353
1158	48	1	277
1160	48	1	133
1162	48	1	135
1164	48	1	153
1166	48	1	139
1168	48	1	41
1170	48	1	41
1172	48	1	103
1174	48	1	93
1176	48	1	95
1178	48	1	135
1180	48	1	155
1182	48	1	43
1184	48	1	43
1186	48	1	229
1188	48	1	191
1190	48	1	249
1192	48	1	45
1194	48	1	45
1196	48	1	575
1198	48	1	47
1200	48	1	47

N	$D(N)$	a	b
1202	49	1	49
1204	50	1	37
1206	49	1	231
1208	50	1	41
1210	49	1	41
1212	50	1	41
1214	49	1	141
1216	50	1	91
1218	49	1	213
1220	50	1	105
1222	49	1	477
1224	50	1	43
1226	49	1	43
1228	50	1	39
1230	49	1	141
1232	50	1	229
1234	49	1	361
1236	50	1	45
1238	49	1	45
1240	50	1	45
1242	49	1	429
1244	50	1	47
1246	49	1	47
1248	50	1	47
1250	49	1	49
1252	50	1	41
1254	50	1	435
1256	50	1	165
1258	50	1	241
1260	50	1	549
1262	50	1	203
1264	50	1	221
1266	50	1	147
1268	50	1	43
1270	50	1	43
1272	50	1	333
1274	50	1	103
1276	50	1	101
1278	50	1	345
1280	50	1	331
1282	50	1	45
1284	50	1	45
1286	50	1	269
1288	50	1	205
1290	50	1	249
1292	50	1	47
1294	50	1	47
1296	50	1	621
1298	50	1	49
1300	50	1	49

N	$D(N)$	a	b
1302	51	1	51
1304	51	1	269
1306	51	1	419
1308	51	1	141
1310	51	1	195
1312	51	1	43
1314	51	1	369
1316	51	1	159
1318	51	1	245
1320	51	3	95
1322	51	1	97
1324	51	1	99
1326	51	1	105
1328	51	1	45
1330	51	1	395
1332	51	1	429
1334	51	1	159
1336	51	1	197
1338	51	1	213
1340	51	1	47
1342	51	1	321
1344	51	1	429
1346	51	1	645
1348	51	1	49
1350	51	25	27
1352	51	1	51
1354	52	1	43
1356	52	1	43
1358	52	1	327
1360	52	1	515
1362	52	1	145
1364	52	1	147
1366	52	1	165
1368	52	1	163
1370	52	1	157
1372	52	1	45
1374	52	1	45
1376	52	1	101
1378	52	1	103
1380	52	1	475
1382	52	1	149
1384	52	1	165
1386	52	1	47
1388	52	1	47
1390	52	1	205
1392	52	1	333
1394	52	1	267
1396	52	1	49
1398	52	1	49
1400	52	1	675

N	$D(N)$	a	b
1402	52	1	51
1404	52	1	51
1406	53	1	53
1408	54	1	53
1410	53	1	145
1412	54	1	147
1414	53	1	273
1416	54	1	41
1418	53	1	45
1420	54	1	45
1422	53	1	163
1424	54	1	115
1426	53	1	227
1428	54	1	113
1430	53	1	295
1432	54	1	47
1434	53	1	47
1436	54	1	47
1438	53	1	155
1440	54	1	43
1442	53	1	609
1444	54	1	49
1446	53	1	49
1448	54	1	49
1450	53	1	467
1452	54	1	51
1454	53	1	51
1456	54	1	51
1458	53	1	53
1460	54	1	53
1462	54	1	45
1464	54	1	45
1466	54	1	283
1468	54	1	151
1470	54	1	153
1472	54	1	171
1474	54	1	157
1476	54	1	235
1478	54	1	101
1480	54	1	47
1482	54	1	47
1484	54	1	111
1486	54	1	109
1488	54	1	153
1490	54	1	173
1492	54	1	361
1494	54	1	49
1496	54	1	49
1498	54	1	223
1500	54	1	363

N	$D(N)$	a	b
1502	54	1	517
1504	54	1	51
1506	54	1	51
1508	54	1	725
1510	54	1	53
1512	54	1	53
1514	55	1	55
1516	55	1	365
1518	55	1	315
1520	55	1	263
1522	55	1	209
1524	55	1	177
1526	55	1	225
1528	55	1	47
1530	55	1	423
1532	55	1	455
1534	55	1	283
1536	55	1	105
1538	55	1	107
1540	55	1	113
1542	55	1	639
1544	55	1	49
1546	55	1	159
1548	55	1	537
1550	55	1	573
1552	55	1	215
1554	55	1	267
1556	55	1	51
1558	55	1	377
1560	55	1	537
1562	55	1	751
1564	55	1	53
1566	55	27	29
1568	55	1	55
1570	56	1	55
1572	56	1	153
1574	56	1	47
1576	56	1	47
1578	56	1	273
1580	56	1	203
1582	56	1	303
1584	56	1	459
1586	56	1	165
1588	56	1	257
1590	56	1	285
1592	56	1	49
1594	56	1	49
1596	56	1	111
1598	56	1	297
1600	56	1	473

N	$D(N)$	a	b
1602	56	1	285
1604	56	1	165
1606	56	1	51
1608	56	1	51
1610	56	1	223
1612	56	1	277
1614	56	1	333
1616	56	1	53
1618	56	1	53
1620	56	1	783
1622	56	1	55
1624	56	1	55
1626	57	1	57
1628	58	1	57
1630	57	1	303
1632	58	1	157
1634	57	1	159
1636	58	1	195
1638	57	1	369
1640	58	1	49
1642	57	1	49
1644	58	1	45
1646	57	1	245
1648	58	1	121
1650	57	1	267
1652	58	1	113
1654	57	1	321
1656	58	1	51
1658	57	1	51
1660	58	1	51
1662	57	1	177
1664	58	1	307
1666	57	1	721
1668	58	1	47
1670	57	1	53
1672	58	1	53
1674	57	1	537
1676	58	1	55
1678	57	1	55
1680	58	1	55
1682	57	1	57
1684	58	1	57
1686	58	1	251
1688	58	1	491
1690	58	1	49
1692	58	1	49
1694	58	1	163
1696	58	1	165
1698	58	1	183
1700	58	1	181

N	$D(N)$	a	b
1702	58	1	175
1704	58	1	109
1706	58	1	111
1708	58	1	51
1710	58	1	51
1712	58	1	117
1714	58	1	305
1716	58	1	167
1718	58	1	183
1720	58	1	179
1722	58	1	53
1724	58	1	53
1726	58	1	237
1728	58	1	299
1730	58	1	359
1732	58	1	55
1734	58	1	55
1736	58	1	837
1738	58	1	57
1740	58	1	57
1742	59	1	59
1744	59	1	451
1746	59	1	337
1748	59	1	163
1750	59	1	361
1752	59	1	227
1754	59	1	243
1756	59	1	169
1758	59	1	427
1760	59	1	51
1762	59	1	485
1764	59	1	497
1766	59	1	113
1768	59	1	115
1770	59	1	121
1772	59	1	551
1774	59	1	733
1776	59	1	53
1778	59	1	173
1780	59	1	401
1782	59	1	229
1784	59	1	245
1786	59	1	289
1788	59	1	55
1790	59	1	431
1792	59	1	579
1794	59	1	865
1796	59	1	57
1798	59	29	31
1800	59	1	59

N	$D(N)$	a	b
1802	60	1	59
1804	60	1	281
1806	60	1	371
1808	60	1	349
1810	60	1	51
1812	60	1	51
1814	60	1	171
1816	60	1	189
1818	60	1	175
1820	60	1	399
1822	60	1	667
1824	60	1	443
1826	60	1	127
1828	60	1	53
1830	60	1	53
1832	60	1	317
1834	60	1	171
1836	60	1	191
1838	60	1	403
1840	60	1	179
1842	60	1	55
1844	60	1	55
1846	60	1	383
1848	60	1	299
1850	60	1	359
1852	60	1	57
1854	60	1	57
1856	60	1	899
1858	60	1	59
1860	60	1	59
1862	61	1	51
1864	62	1	51
1866	61	1	639
1868	62	1	199
1870	61	1	517
1872	62	1	257
1874	61	1	195
1876	62	1	175
1878	61	1	177
1880	62	1	53
1882	61	1	53
1884	62	1	53
1886	61	1	259
1888	62	1	49
1890	61	1	327
1892	62	1	121
1894	61	1	295
1896	62	1	55
1898	61	1	55
1900	62	1	55

N	$D(N)$	a	b
1902	61	1	417
1904	62	1	371
1906	61	1	537
1908	62	1	57
1910	61	1	57
1912	62	1	51
1914	61	1	657
1916	62	1	59
1918	61	1	59
1920	62	1	59
1922	61	1	61
1924	62	1	61
1926	62	1	663
1928	62	1	171
1930	62	1	233
1932	62	1	201
1934	62	1	53
1936	62	1	53
1938	62	1	249
1940	62	1	293
1942	62	1	183
1944	62	1	267
1946	62	1	117
1948	62	1	119
1950	62	1	693
1952	62	1	55
1954	62	1	55
1956	62	1	471
1958	62	1	305
1960	62	1	593
1962	62	1	183
1964	62	1	189
1966	62	1	57
1968	62	1	57
1970	62	1	555
1972	62	1	317
1974	62	1	381
1976	62	1	59
1978	62	1	59
1980	62	1	957
1982	62	1	61
1984	62	1	61
1986	63	1	63
1988	63	1	53
1990	63	1	643
1992	63	1	513
1994	63	1	341
1996	63	1	177
1998	63	1	579
2000	63	1	257

N	$D(N)$	a	b
2002	63	1	451
2004	63	1	195
2006	63	1	303
2008	63	1	55
2010	63	1	777
2012	63	1	121
2014	63	1	123
2016	63	1	129
2018	63	1	429
2020	63	1	307
2022	63	1	849
2024	63	1	57
2026	63	1	195
2028	63	1	459
2030	63	1	247
2032	63	1	591
2034	63	1	327
2036	63	1	59
2038	63	1	495
2040	63	1	657
2042	63	1	987
2044	63	1	61
2046	63	31	33
2048	63	1	63
2050	64	1	63
2052	64	1	175
2054	64	1	303
2056	64	1	399
2058	64	1	333
2060	64	1	597
2062	64	1	55
2064	64	1	55
2066	64	1	201
2068	64	1	199
2070	64	1	193
2072	64	1	313
2074	64	1	323
2076	64	1	135
2078	64	1	125
2080	64	1	57
2082	64	1	57
2084	64	1	335
2086	64	1	185
2088	64	1	201
2090	64	1	197
2092	64	1	271
2094	64	1	59
2096	64	1	59
2098	64	1	405
2100	64	1	543

N	$D(N)$	a	b
2102	64	1	677
2104	64	1	61
2106	64	1	61
2108	64	1	1023
2110	64	1	63
2112	64	1	63
2114	65	1	65
2116	66	1	51
2118	65	1	55
2120	66	1	55
2122	65	1	181
2124	66	1	173
2126	65	1	599
2128	66	1	207
2130	65	1	187
2132	66	1	189
2134	65	1	199
2136	66	1	57
2138	65	1	57
2140	66	1	57
2142	65	1	623
2144	66	1	129
2146	65	1	345
2148	66	1	53
2150	65	1	337
2152	66	1	59
2154	65	1	59
2156	66	1	59
2158	65	1	471
2160	66	1	417
2162	65	1	317
2164	66	1	61
2166	65	1	61
2168	66	1	61
2170	65	1	703
2172	66	1	55
2174	65	1	63
2176	66	1	63
2178	65	1	65
2180	66	1	65
2182	66	1	375
2184	66	1	283
2186	66	1	451
2188	66	1	251
2190	66	1	305
2192	66	1	187
2194	66	1	57
2196	66	1	57
2198	66	1	193
2200	66	1	285

N	$D(N)$	a	b
2202	66	1	343
2204	66	1	125
2206	66	1	127
2208	66	1	355
2210	66	1	135
2212	66	1	59
2214	66	1	59
2216	66	1	189
2218	66	1	209
2220	66	1	715
2222	66	1	197
2224	66	1	571
2226	66	1	61
2228	66	1	61
2230	66	1	327
2232	66	1	573
2234	66	1	459
2236	66	1	63
2238	66	1	63
2240	66	1	1085
2242	66	1	65
2244	66	1	65
2246	67	1	67
2248	67	1	579
2250	67	3	65
2252	67	1	57
2254	67	1	889
2256	67	1	771
2258	67	1	259
2260	67	1	213
2262	67	1	1035
2264	67	1	195
2266	67	1	517
2268	67	1	495
2270	67	1	345
2272	67	1	59
2274	67	1	129
2276	67	1	131
2278	67	1	137
2280	67	3	625
2282	67	1	483
2284	67	1	345
2286	67	1	195
2288	67	1	61
2290	67	1	695
2292	67	1	261
2294	67	1	277
2296	67	1	973
2298	67	1	393
2300	67	1	63

N	$D(N)$	a	b
2302	67	1	557
2304	67	1	789
2306	67	1	1117
2308	67	1	65
2310	67	33	35
2312	67	1	67
2314	68	1	67
2316	68	1	363
2318	68	1	321
2320	68	1	189
2322	68	1	399
2324	68	1	219
2326	68	1	283
2328	68	1	267
2330	68	1	59
2332	68	1	59
2334	68	1	201
2336	68	1	355
2338	68	1	497
2340	68	1	657
2342	68	1	143
2344	68	1	133
2346	68	1	135
2348	68	1	61
2350	68	1	61
2352	68	1	573
2354	68	1	539
2356	68	1	201
2358	68	1	207
2360	68	1	285
2362	68	1	63
2364	68	1	63
2366	68	1	349
2368	68	1	405
2370	68	1	489
2372	68	1	65
2374	68	1	65
2376	68	1	1155
2378	68	1	67
2380	68	1	67
2382	69	1	69
2384	70	1	69
2386	69	1	463
2388	70	1	59
2390	69	1	59
2392	70	1	55
2394	69	1	195
2396	70	1	519
2398	69	1	649
2400	70	1	231

N	$D(N)$	a	b
2402	69	1	213
2404	70	1	129
2406	69	1	261
2408	70	1	61
2410	69	1	61
2412	70	1	61
2414	69	1	825
2416	70	1	137
2418	69	1	393
2420	70	1	337
2422	69	1	741
2424	70	1	57
2426	69	1	63
2428	70	1	63
2430	69	1	789
2432	70	1	277
2434	69	1	339
2436	70	1	65
2438	69	1	65
2440	70	1	65
2442	69	1	789
2444	70	1	67
2446	69	1	67
2448	70	1	59
2450	69	1	69
2452	70	1	69
2454	70	1	397
2456	70	1	193
2458	70	1	477
2460	70	1	265
2462	70	1	341
2464	70	1	453
2466	70	1	199
2468	70	1	201
2470	70	1	61
2472	70	1	61
2474	70	1	211
2476	70	1	299
2478	70	1	133
2480	70	1	135
2482	70	1	779
2484	70	1	143
2486	70	1	141
2488	70	1	63
2490	70	1	63
2492	70	1	203
2494	70	1	219
2496	70	1	215
2498	70	1	271
2500	70	1	609

N	$D(N)$	a	b
2502	70	1	65
2504	70	1	65
2506	70	1	349
2508	70	1	431
2510	70	1	489
2512	70	1	67
2514	70	1	67
2516	70	1	1221
2518	70	1	69
2520	70	1	69
2522	71	1	71
2524	71	1	333
2526	71	1	305
2528	71	1	409
2530	71	1	1095
2532	71	1	61
2534	71	1	273
2536	71	1	753
2538	71	1	289
2540	71	1	205
2542	71	1	1161
2544	71	1	217
2546	71	1	345
2548	71	1	483
2550	71	1	617
2552	71	1	63
2554	71	1	139
2556	71	1	145
2558	71	1	483
2560	71	1	345
2562	71	1	553
2564	71	1	529
2566	71	1	209
2568	71	1	65
2570	71	1	765
2572	71	1	279
2574	71	1	955
2576	71	1	1113
2578	71	1	419
2580	71	1	67
2582	71	1	629
2584	71	1	839
2586	71	1	1255
2588	71	1	69
2590	71	35	37
2592	71	1	71
2594	72	1	61
2596	72	1	817
2598	72	1	631
2600	72	1	343

N	$D(N)$	a	b
2602	72	1	421
2604	72	1	333
2606	72	1	363
2608	72	1	205
2610	72	1	207
2612	72	1	225
2614	72	1	63
2616	72	1	63
2618	72	1	567
2620	72	1	335
2622	72	1	497
2624	72	1	151
2626	72	1	141
2628	72	1	143
2630	72	1	347
2632	72	1	65
2634	72	1	65
2636	72	1	227
2638	72	1	485
2640	72	1	215
2642	72	1	303
2644	72	1	681
2646	72	1	67
2648	72	1	67
2650	72	1	367
2652	72	1	431
2654	72	1	515
2656	72	1	69
2658	72	1	69
2660	72	1	1295
2662	72	1	71
2664	72	1	71
2666	73	1	73
2668	74	1	73
2670	73	1	1053
2672	74	1	275
2674	73	1	521
2676	74	1	63
2678	73	1	63
2680	74	1	63
2682	73	1	231
2684	74	1	59
2686	73	1	213
2688	74	1	217
2690	73	1	275
2692	74	1	139
2694	73	1	309
2696	74	1	65
2698	73	1	65
2700	74	1	65

N	$D(N)$	a	b
2702	73	1	639
2704	74	1	197
2706	73	1	357
2708	74	1	231
2710	73	1	213
2712	74	1	67
2714	73	1	67
2716	74	1	61
2718	73	1	933
2720	74	1	293
2722	73	1	377
2724	74	1	69
2726	73	1	69
2728	74	1	69
2730	73	1	933
2732	74	1	71
2734	73	1	71
2736	74	1	71
2738	73	1	73
2740	74	1	63
2742	74	1	63
2744	74	1	773
2746	74	1	315
2748	74	1	207
2750	74	1	283
2752	74	1	237
2754	74	1	1041
2756	74	1	299
2758	74	1	509
2760	74	1	747
2762	74	1	65
2764	74	1	65
2766	74	1	363
2768	74	1	141
2770	74	1	143
2772	74	1	975
2774	74	1	151
2776	74	1	149
2778	74	1	657
2780	74	1	67
2782	74	1	67
2784	74	1	603
2786	74	1	219
2788	74	1	225
2790	74	1	513
2792	74	1	357
2794	74	1	69
2796	74	1	69
2798	74	1	543
2800	74	1	453

N	$D(N)$	a	b
2802	74	1	1107
2804	74	1	71
2806	74	1	71
2808	74	1	1365
2810	74	1	73
2812	74	1	73
2814	75	1	75
2816	75	1	323
2818	75	1	579
2820	75	1	687
2822	75	1	345
2824	75	1	363
2826	75	1	1221
2828	75	1	65
2830	75	1	307
2832	75	1	843
2834	75	1	483
2836	75	1	231
2838	75	1	1305
2840	75	1	555
2842	75	1	383
2844	75	1	537
2846	75	1	145
2848	75	1	67
2850	75	1	153
2852	75	1	395
2854	75	1	541
2856	75	1	387
2858	75	1	619
2860	75	1	555
2862	75	1	231
2864	75	1	69
2866	75	1	293
2868	75	1	309
2870	75	1	367
2872	75	1	811
2874	75	1	465
2876	75	1	71
2878	75	1	699
2880	75	1	933
2882	75	1	1401
2884	75	1	73
2886	75	37	39
2888	75	1	75
2890	76	1	75
2892	76	1	65
2894	76	1	65
2896	76	1	211
2898	76	1	471
2900	76	1	393

N	$D(N)$	a	b
2902	76	1	373
2904	76	1	315
2906	76	1	217
2908	76	1	219
2910	76	1	237
2912	76	1	235
2914	76	1	67
2916	76	1	67
2918	76	1	385
2920	76	1	357
2922	76	1	159
2924	76	1	149
2926	76	1	151
2928	76	1	499
2930	76	1	397
2932	76	1	69
2934	76	1	69
2936	76	1	237
2938	76	1	233
2940	76	1	873
2942	76	1	317
2944	76	1	301
2946	76	1	71
2948	76	1	71
2950	76	1	833
2952	76	1	717
2954	76	1	605
2956	76	1	73
2958	76	1	73
2960	76	1	1443
2962	76	1	75
2964	76	1	75
2966	77	1	77
2968	78	1	77
2970	77	1	563
2972	78	1	289
2974	77	1	403
2976	78	1	401
2978	77	1	217
2980	78	1	67
2982	77	1	67
2984	78	1	67
2986	77	1	223
2988	78	1	225
2990	77	1	235
2992	78	1	63
2994	77	1	293
2996	78	1	163
2998	77	1	323
3000	78	1	69

N	$D(N)$	a	b
3002	77	1	69
3004	78	1	69
3006	77	1	439
3008	78	1	221
3010	77	1	395
3012	78	1	211
3014	77	1	227
3016	78	1	71
3018	77	1	71
3020	78	1	71
3022	77	1	679
3024	78	1	65
3026	77	1	877
3028	78	1	73
3030	77	1	73
3032	78	1	73
3034	77	1	987
3036	78	1	75
3038	77	1	75
3040	78	1	75
3042	77	1	77
3044	78	1	77
3046	78	1	521
3048	78	1	67
3050	78	1	67
3052	78	1	625
3054	78	1	297
3056	78	1	401
3058	78	1	391
3060	78	1	223
3062	78	1	225
3064	78	1	243
3066	78	1	229
3068	78	1	897
3070	78	1	69
3072	78	1	69
3074	78	1	149
3076	78	1	151
3078	78	1	975
3080	78	1	159
3082	78	1	157
3084	78	1	377
3086	78	1	405
3088	78	1	71
3090	78	1	71
3092	78	1	527
3094	78	1	233
3096	78	1	303
3098	78	1	555
3100	78	1	319

N	$D(N)$	a	b
3102	78	1	73
3104	78	1	73
3106	78	1	607
3108	78	1	759
3110	78	1	639
3112	78	1	75
3114	78	1	75
3116	78	1	1517
3118	78	1	77
3120	78	1	77
3122	79	1	79
3124	79	1	371
3126	79	1	609
3128	79	1	337
3130	79	1	459
3132	79	1	381
3134	79	1	687
3136	79	1	305
3138	79	1	1149
3140	79	1	69
3142	79	1	565
3144	79	1	231
3146	79	1	509
3148	79	1	581
3150	79	1	921
3152	79	1	1231
3154	79	1	1001
3156	79	1	153
3158	79	1	155
3160	79	1	71
3162	79	1	1233
3164	79	1	917
3166	79	1	1379
3168	79	1	849
3170	79	1	231
3172	79	1	1251
3174	79	1	1317
3176	79	1	73
3178	79	1	311
3180	79	1	699
3182	79	1	389
3184	79	1	465
3186	79	1	543
3188	79	1	75
3190	79	1	779
3192	79	1	1089
3194	79	1	1555
3196	79	1	77
3198	79	39	41
3200	79	1	79

N	$D(N)$	a	b
3202	80	1	79
3204	80	1	597
3206	80	1	781
3208	80	1	69
3210	80	1	69
3212	80	1	225
3214	80	1	391
3216	80	1	255
3218	80	1	313
3220	80	1	521
3222	80	1	885
3224	80	1	299
3226	80	1	237
3228	80	1	867
3230	80	1	71
3232	80	1	71
3234	80	1	567
3236	80	1	167
3238	80	1	157
3240	80	1	159
3242	80	1	385
3244	80	1	417
3246	80	1	567
3248	80	1	73
3250	80	1	73
3252	80	1	237
3254	80	1	243
3256	80	1	335
3258	80	1	897
3260	80	1	319
3262	80	1	75
3264	80	1	75
3266	80	1	477
3268	80	1	557
3270	80	1	639
3272	80	1	77
3274	80	1	77
3276	80	1	1599
3278	80	1	79
3280	80	1	79
3282	81	1	81
3284	82	1	65
3286	81	1	383
3288	82	1	147
3290	81	1	373
3292	82	1	261
3294	81	1	1401
3296	82	1	229
3298	81	1	231
3300	82	1	71

N	$D(N)$	a	b
3302	81	1	71
3304	82	1	71
3306	81	1	249
3308	82	1	241
3310	81	1	307
3312	82	1	155
3314	81	1	341
3316	82	1	67
3318	81	1	1467
3320	82	1	73
3322	81	1	73
3324	82	1	73
3326	81	1	395
3328	82	1	773
3330	81	1	543
3332	82	1	251
3334	81	1	249
3336	82	1	75
3338	81	1	75
3340	82	1	75
3342	81	1	753
3344	82	1	541
3346	81	1	1421
3348	82	1	69
3350	81	1	77
3352	82	1	77
3354	81	1	1089
3356	82	1	79
3358	81	1	79
3360	82	1	79
3362	81	1	81
3364	82	1	81
3366	82	1	547
3368	82	1	469
3370	82	1	823
3372	82	1	71
3374	82	1	71
3376	82	1	315
3378	82	1	383
3380	82	1	743
3382	82	1	235
3384	82	1	237
3386	82	1	255
3388	82	1	253
3390	82	1	247
3392	82	1	349
3394	82	1	73
3396	82	1	73
3398	82	1	159
3400	82	1	437

N	$D(N)$	a	b
3402	82	1	167
3404	82	1	165
3406	82	1	397
3408	82	1	635
3410	82	1	405
3412	82	1	75
3414	82	1	75
3416	82	1	251
3418	82	1	317
3420	82	1	1033
3422	82	1	771
3424	82	1	333
3426	82	1	77
3428	82	1	77
3430	82	1	503
3432	82	1	587
3434	82	1	669
3436	82	1	79
3438	82	1	79
3440	82	1	1677
3442	82	1	81
3444	82	1	81
3446	83	1	83
3448	83	1	883
3450	83	1	355
3452	83	1	561
3454	83	1	481
3456	83	1	589
3458	83	1	505
3460	83	1	235
3462	83	1	323
3464	83	1	339
3466	83	1	807
3468	83	1	73
3470	83	1	1607
3472	83	1	253
3474	83	1	403
3476	83	1	957
3478	83	1	727
3480	83	1	715
3482	83	1	161
3484	83	1	163
3486	83	1	169
3488	83	1	75
3490	83	1	973
3492	83	1	407
3494	83	1	651
3496	83	1	939
3498	83	1	245
3500	83	1	715

N	$D(N)$	a	b
3502	83	1	1451
3504	83	1	77
3506	83	1	341
3508	83	1	769
3510	83	1	427
3512	83	1	491
3514	83	1	573
3516	83	1	79
3518	83	1	857
3520	83	1	1147
3522	83	1	1717
3524	83	1	81
3526	83	41	43
3528	83	1	83
3530	84	1	83
3532	84	1	561
3534	84	1	315
3536	84	1	691
3538	84	1	403
3540	84	1	73
3542	84	1	73
3544	84	1	779
3546	84	1	347
3548	84	1	241
3550	84	1	243
3552	84	1	261
3554	84	1	247
3556	84	1	317
3558	84	1	567
3560	84	1	413
3562	84	1	75
3564	84	1	75
3566	84	1	175
3568	84	1	165
3570	84	1	167
3572	84	1	405
3574	84	1	749
3576	84	1	435
3578	84	1	243
3580	84	1	77
3582	84	1	77
3584	84	1	251
3586	84	1	833
3588	84	1	349
3590	84	1	333
3592	84	1	437
3594	84	1	79
3596	84	1	79
3598	84	1	503
3600	84	1	587

N	$D(N)$	a	b
3602	84	1	1227
3604	84	1	81
3606	84	1	81
3608	84	1	1763
3610	84	1	83
3612	84	1	83
3614	85	1	73
3616	86	1	73
3618	85	1	615
3620	86	1	237
3622	85	1	643
3624	86	1	69
3626	85	1	395
3628	86	1	271
3630	85	1	975
3632	86	1	231
3634	85	1	267
3636	86	1	75
3638	85	1	75
3640	86	1	75
3642	85	1	1077
3644	86	1	161
3646	85	1	325
3648	86	1	179
3650	85	1	355
3652	86	1	177
3654	85	1	801
3656	86	1	71
3658	85	1	77
3660	86	1	77
3662	85	1	437
3664	86	1	233
3666	85	1	249
3668	86	1	327
3670	85	1	715
3672	86	1	79
3674	85	1	79
3676	86	1	79
3678	85	1	417
3680	86	1	437
3682	85	1	1589
3684	86	1	81
3686	85	1	81
3688	86	1	73
3690	85	1	1257
3692	86	1	83
3694	85	1	83
3696	86	1	83
3698	85	1	85
3700	86	1	85

N	$D(N)$	a	b
3702	86	1	1263
3704	86	1	761
3706	86	1	759
3708	86	1	1467
3710	86	1	605
3712	86	1	75
3714	86	1	75
3716	86	1	273
3718	86	1	405
3720	86	1	345
3722	86	1	1151
3724	86	1	453
3726	86	1	255
3728	86	1	579
3730	86	1	443
3732	86	1	363
3734	86	1	77
3736	86	1	77
3738	86	1	1305
3740	86	1	175
3742	86	1	173
3744	86	1	549
3746	86	1	447
3748	86	1	641
3750	86	1	435
3752	86	1	79
3754	86	1	79
3756	86	1	261
3758	86	1	335
3760	86	1	1227
3762	86	1	351
3764	86	1	921
3766	86	1	81
3768	86	1	81
3770	86	1	525
3772	86	1	613
3774	86	1	771
3776	86	1	83
3778	86	1	83
3780	86	1	1845
3782	86	1	85
3784	86	1	85
3786	87	1	87
3788	87	1	75
3790	87	1	1235
3792	87	1	969
3794	87	1	369
3796	87	1	453
3798	87	1	531
3800	87	1	833

N	$D(N)$	a	b
3802	87	1	337
3804	87	1	249
3806	87	1	353
3808	87	1	1353
3810	87	1	885
3812	87	1	77
3814	87	1	679
3816	87	1	711
3818	87	1	445
3820	87	1	1065
3822	87	1	801
3824	87	1	169
3826	87	1	171
3828	87	1	177
3830	87	1	933
3832	87	1	79
3834	87	1	1071
3836	87	1	445
3838	87	1	713
3840	87	1	1785
3842	87	1	267
3844	87	1	597
3846	87	1	1611
3848	87	1	81
3850	87	1	1165
3852	87	1	1257
3854	87	1	1433
3856	87	1	537
3858	87	1	627
3860	87	1	83
3862	87	1	945
3864	87	1	1257
3866	87	1	1887
3868	87	1	85
3870	87	43	45
3872	87	1	87
3874	88	1	87
3876	88	1	329
3878	88	1	947
3880	88	1	463
3882	88	1	421
3884	88	1	247
3886	88	1	1447
3888	88	1	77
3890	88	1	77
3892	88	1	345
3894	88	1	253
3896	88	1	255
3898	88	1	273
3900	88	1	271

N	$D(N)$	a	b
3902	88	1	265
3904	88	1	455
3906	88	1	761
3908	88	1	819
3910	88	1	79
3912	88	1	79
3914	88	1	173
3916	88	1	175
3918	88	1	819
3920	88	1	427
3922	88	1	455
3924	88	1	667
3926	88	1	257
3928	88	1	81
3930	88	1	81
3932	88	1	639
3934	88	1	367
3936	88	1	771
3938	88	1	351
3940	88	1	1009
3942	88	1	83
3944	88	1	83
3946	88	1	773
3948	88	1	1011
3950	88	1	809
3952	88	1	85
3954	88	1	85
3956	88	1	1935
3958	88	1	87
3960	88	1	87
3962	89	1	89
3964	90	1	77
3966	89	1	77
3968	90	1	77
3970	89	1	649
3972	90	1	163
3974	89	1	431
3976	90	1	371
3978	89	1	253
3980	90	1	73
3982	89	1	605
3984	90	1	279
3986	89	1	259
3988	90	1	79
3990	89	1	79
3992	90	1	79
3994	89	1	339
3996	90	1	171
3998	89	1	373
4000	90	1	185

N	$D(N)$	a	b
4002	89	1	1193
4004	90	1	177
4006	89	1	957
4008	90	1	81
4010	89	1	81
4012	90	1	75
4014	89	1	559
4016	90	1	713
4018	89	1	263
4020	90	1	247
4022	89	1	613
4024	90	1	83
4026	89	1	83
4028	90	1	83
4030	89	1	439
4032	90	1	749
4034	89	1	1141
4036	90	1	85
4038	89	1	85
4040	90	1	85
4042	89	1	1319
4044	90	1	77
4046	89	1	87
4048	90	1	87
4050	89	1	89
4052	90	1	89
4054	90	1	691
4056	90	1	413
4058	90	1	519
4060	90	1	431
4062	90	1	379
4064	90	1	755
4066	90	1	347
4068	90	1	79
4070	90	1	79
4072	90	1	259
4074	90	1	261
4076	90	1	279
4078	90	1	265
4080	90	1	435
4082	90	1	607
4084	90	1	381
4086	90	1	569
4088	90	1	173
4090	90	1	81
4092	90	1	81
4094	90	1	183
4096	90	1	181
4098	90	1	435
4100	90	1	467

N	$D(N)$	a	b
4102	90	1	961
4104	90	1	261
4106	90	1	281
4108	90	1	83
4110	90	1	83
4112	90	1	349
4114	90	1	627
4116	90	1	843
4118	90	1	365
4120	90	1	525
4122	90	1	85
4124	90	1	85
4126	90	1	845
4128	90	1	1053
4130	90	1	809
4132	90	1	87
4134	90	1	87
4136	90	1	2021
4138	90	1	89
4140	90	1	89
4142	91	1	91
4144	91	1	581
4146	91	1	1677
4148	91	1	79
4150	91	1	1765
4152	91	1	531
4154	91	1	423
4156	91	1	1229
4158	91	1	1503
4160	91	1	355
4162	91	1	371
4164	91	1	285
4166	91	1	511
4168	91	1	267
4170	91	1	975
4172	91	1	81
4174	91	1	445
4176	91	1	621
4178	91	1	679
4180	91	1	1133
4182	91	1	177
4184	91	1	179
4186	91	1	185
4188	91	1	477
4190	91	1	625
4192	91	1	83
4194	91	1	1161
4196	91	1	611
4198	91	1	267
4200	91	1	1965

N	$D(N)$	a	b
4202	91	1	895
4204	91	1	655
4206	91	1	357
4208	91	1	85
4210	91	1	1255
4212	91	1	1437
4214	91	1	537
4216	91	1	1219
4218	91	1	717
4220	91	1	87
4222	91	1	1031
4224	91	1	1437
4226	91	1	2065
4228	91	1	89
4230	91	45	47
4232	91	1	91
4234	92	1	91
4236	92	1	675
4238	92	1	619
4240	92	1	347
4242	92	1	723
4244	92	1	443
4246	92	1	543
4248	92	1	261
4250	92	1	463
4252	92	1	81
4254	92	1	81
4256	92	1	661
4258	92	1	485
4260	92	1	1305
4262	92	1	273
4264	92	1	349
4266	92	1	1179
4268	92	1	455
4270	92	1	635
4272	92	1	435
4274	92	1	83
4276	92	1	83
4278	92	1	183
4280	92	1	525
4282	92	1	447
4284	92	1	639
4286	92	1	455
4288	92	1	701
4290	92	1	1005
4292	92	1	85
4294	92	1	85
4296	92	1	915
4298	92	1	381
4300	92	1	365

N	$D(N)$	a	b
4302	92	1	1557
4304	92	1	1051
4306	92	1	87
4308	92	1	87
4310	92	1	1217
4312	92	1	733
4314	92	1	843
4316	92	1	89
4318	92	1	89
4320	92	1	2115
4322	92	1	91
4324	92	1	91
4326	93	1	93
4328	94	1	93
4330	93	1	603
4332	94	1	81
4334	93	1	81
4336	94	1	81
4338	93	1	705
4340	94	1	297
4342	93	1	473
4344	94	1	265
4346	93	1	267
4348	94	1	369
4350	93	1	663
4352	94	1	77
4354	93	1	285
4356	94	1	83
4358	93	1	83
4360	94	1	83
4362	93	1	357
4364	94	1	195
4366	93	1	387
4368	94	1	193
4370	93	1	445
4372	94	1	185
4374	93	1	981
4376	94	1	85
4378	93	1	85
4380	94	1	85
4382	93	1	1267
4384	94	1	79
4386	93	1	285
4388	94	1	389
4390	93	1	667
4392	94	1	87
4394	93	1	87
4396	94	1	87
4398	93	1	477
4400	94	1	897

N	$D(N)$	a	b
4402	93	1	641
4404	94	1	89
4406	93	1	89
4408	94	1	89
4410	93	1	1437
4412	94	1	91
4414	93	1	91
4416	94	1	81
4418	93	1	93
4420	94	1	93
4422	94	1	689
4424	94	1	719
4426	94	1	541
4428	94	1	435
4430	94	1	453
4432	94	1	265
4434	94	1	361
4436	94	1	1357
4438	94	1	1201
4440	94	1	83
4442	94	1	83
4444	94	1	273
4446	94	1	291
4448	94	1	289
4450	94	1	283
4452	94	1	1043
4454	94	1	463
4456	94	1	395
4458	94	1	181
4460	94	1	183
4462	94	1	85
4464	94	1	85
4466	94	1	189
4468	94	1	455
4470	94	1	857
4472	94	1	485
4474	94	1	467
4476	94	1	275
4478	94	1	291
4480	94	1	87
4482	94	1	87
4484	94	1	367
4486	94	1	823
4488	94	1	383
4490	94	1	487
4492	94	1	551
4494	94	1	89
4496	94	1	89
4498	94	1	655
4500	94	1	767

N	$D(N)$	a	b
4502	94	1	1467
4504	94	1	91
4506	94	1	91
4508	94	1	2205
4510	94	1	93
4512	94	1	93
4514	95	1	95
4516	95	1	1081
4518	95	1	923
4520	95	1	737
4522	95	1	1827
4524	95	1	83
4526	95	1	1017
4528	95	1	463
4530	95	1	445
4532	95	1	271
4534	95	1	1387
4536	95	1	385
4538	95	1	1203
4540	95	1	277
4542	95	1	1693
4544	95	1	289
4546	95	1	1065
4548	95	1	85
4550	95	1	483
4552	95	1	679
4554	95	1	871
4556	95	1	185
4558	95	1	187
4560	95	1	193
4562	95	1	1457
4564	95	1	495
4566	95	1	679
4568	95	1	87
4570	95	1	1263
4572	95	1	641
4574	95	1	281
4576	95	1	1215
4578	95	1	973
4580	95	1	895
4582	95	1	375
4584	95	1	89
4586	95	1	467
4588	95	1	1029
4590	95	1	563
4592	95	1	1953
4594	95	1	751
4596	95	1	91
4598	95	1	1127
4600	95	1	1503

N	$D(N)$	a	b
4602	95	1	2251
4604	95	1	93
4606	95	47	49
4608	95	1	95
4610	96	1	95
4612	96	1	1473
4614	96	1	361
4616	96	1	725
4618	96	1	753
4620	96	1	687
4622	96	1	565
4624	96	1	493
4626	96	1	473
4628	96	1	851
4630	96	1	377
4632	96	1	85
4634	96	1	85
4636	96	1	297
4638	96	1	283
4640	96	1	363
4642	96	1	1285
4644	96	1	493
4646	96	1	485
4648	96	1	889
4650	96	1	1313
4652	96	1	199
4654	96	1	87
4656	96	1	87
4658	96	1	893
4660	96	1	693
4662	96	1	497
4664	96	1	1027
4666	96	1	279
4668	96	1	299
4670	96	1	1655
4672	96	1	89
4674	96	1	89
4676	96	1	399
4678	96	1	843
4680	96	1	383
4682	96	1	859
4684	96	1	477
4686	96	1	91
4688	96	1	91
4690	96	1	685
4692	96	1	767
4694	96	1	957
4696	96	1	93
4698	96	1	93
4700	96	1	2303

N	$D(N)$	a	b
4702	96	1	95
4704	96	1	95
4706	97	1	97
4708	98	1	97
4710	97	1	903
4712	98	1	461
4714	97	1	659
4716	98	1	85
4718	97	1	85
4720	98	1	85
4722	97	1	801
4724	98	1	307
4726	97	1	1105
4728	98	1	385
4730	97	1	303
4732	98	1	267
4734	97	1	285
4736	98	1	289
4738	97	1	1007
4740	98	1	81
4742	97	1	87
4744	98	1	87
4746	97	1	405
4748	98	1	201
4750	97	1	1303
4752	98	1	193
4754	97	1	467
4756	98	1	309
4758	97	1	1809
4760	98	1	89
4762	97	1	89
4764	98	1	89
4766	97	1	285
4768	98	1	269
4770	97	1	813
4772	98	1	83
4774	97	1	937
4776	98	1	91
4778	97	1	91
4780	98	1	91
4782	97	1	1053
4784	98	1	915
4786	97	1	671
4788	98	1	93
4790	97	1	93
4792	98	1	93
4794	97	1	1629
4796	98	1	95
4798	97	1	95
4800	98	1	95

N	$D(N)$	a	b
4802	97	1	97
4804	98	1	85
4806	98	1	1635
4808	98	1	659
4810	98	1	979
4812	98	1	513
4814	98	1	471
4816	98	1	411
4818	98	1	591
4820	98	1	279
4822	98	1	491
4824	98	1	309
4826	98	1	395
4828	98	1	87
4830	98	1	87
4832	98	1	757
4834	98	1	291
4836	98	1	945
4838	98	1	505
4840	98	1	413
4842	98	1	1839
4844	98	1	189
4846	98	1	191
4848	98	1	1443
4850	98	1	89
4852	98	1	89
4854	98	1	1263
4856	98	1	477
4858	98	1	505
4860	98	1	1401
4862	98	1	1723
4864	98	1	829
4866	98	1	291
4868	98	1	91
4870	98	1	91
4872	98	1	1245
4874	98	1	1319
4876	98	1	397
4878	98	1	879
4880	98	1	499
4882	98	1	93
4884	98	1	93
4886	98	1	685
4888	98	1	797
4890	98	1	999
4892	98	1	95
4894	98	1	95
4896	98	1	2397
4898	98	1	97
4900	98	1	97

N	$D(N)$	a	b
4902	99	1	99
4904	99	1	371
4906	99	1	961
4908	99	1	471
4910	99	1	419
4912	99	1	803
4914	99	3	581
4916	99	1	87
4918	99	1	1103
4920	99	1	501
4922	99	1	387
4924	99	1	285
4926	99	1	1509
4928	99	1	557
4930	99	1	1309
4932	99	1	303
4934	99	1	631
4936	99	1	1915
4938	99	1	1815
4940	99	1	89
4942	99	1	1939
4944	99	1	1611
4946	99	1	193
4948	99	1	195
4950	99	1	201
4952	99	1	1717
4954	99	1	679
4956	99	1	2193
4958	99	1	1217
4960	99	1	91
4962	99	1	1923
4964	99	1	725
4966	99	1	303
4968	99	1	1317
4970	99	1	1071
4972	99	1	389
4974	99	1	405
4976	99	1	93
4978	99	1	489
4980	99	1	1119
4982	99	1	609
4984	99	1	2149
4986	99	1	813
4988	99	1	95
4990	99	1	1221
4992	99	1	1629
4994	99	1	2445
4996	99	1	97
4998	99	49	51
5000	99	1	99

7 Concluding remarks

In [3], Chen, Hwang and Liu proposed the mixed chordal ring network which is very comparable to the double-loop network. They derived both the upper and the lower bounds for the diameter of a mixed chordal ring network. They also proposed the necessary and sufficient conditions for a mixed chordal ring to be strongly connected and strongly 2-connected. However, their proof for the strongly 2-connected case has a flaw. In this thesis, we correct the flaw and propose an $O(\log N)$-time algorithm to derive the exact value of the diameter of a mixed chordal ring network.

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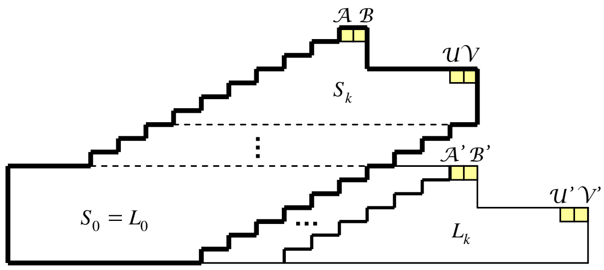
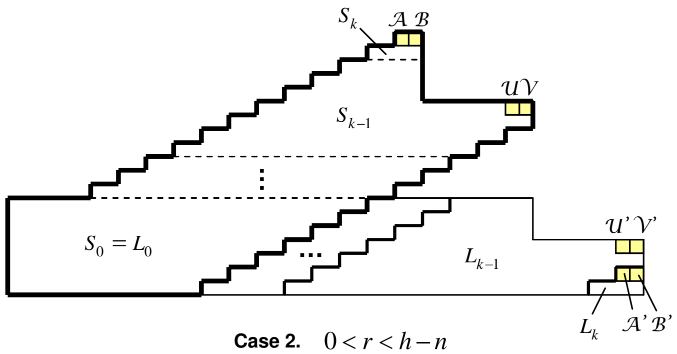
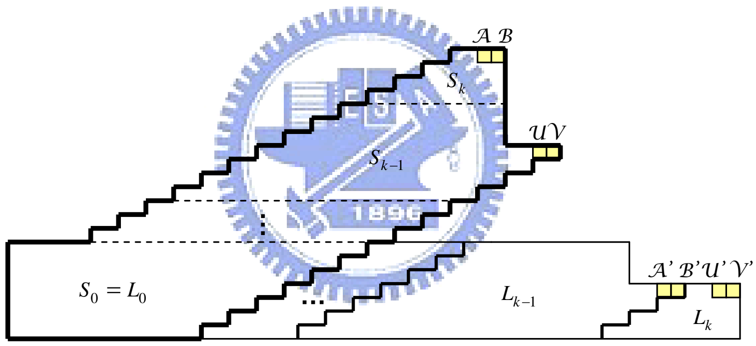


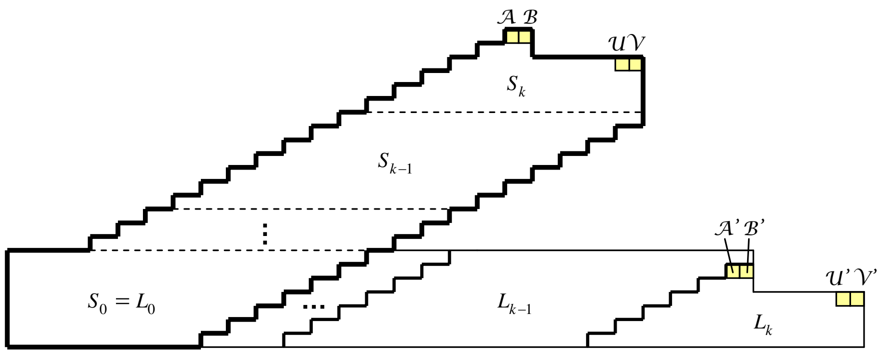
Case 1. $r = 0$



Case 2. $0 < r < h - n$



Case 3. $h - n \le r < h - n + p$



Case 4. $h - n + p \le r < h + p$

Figure 13: The proof of Theorem 16 for the case that $\ell \geq h$.

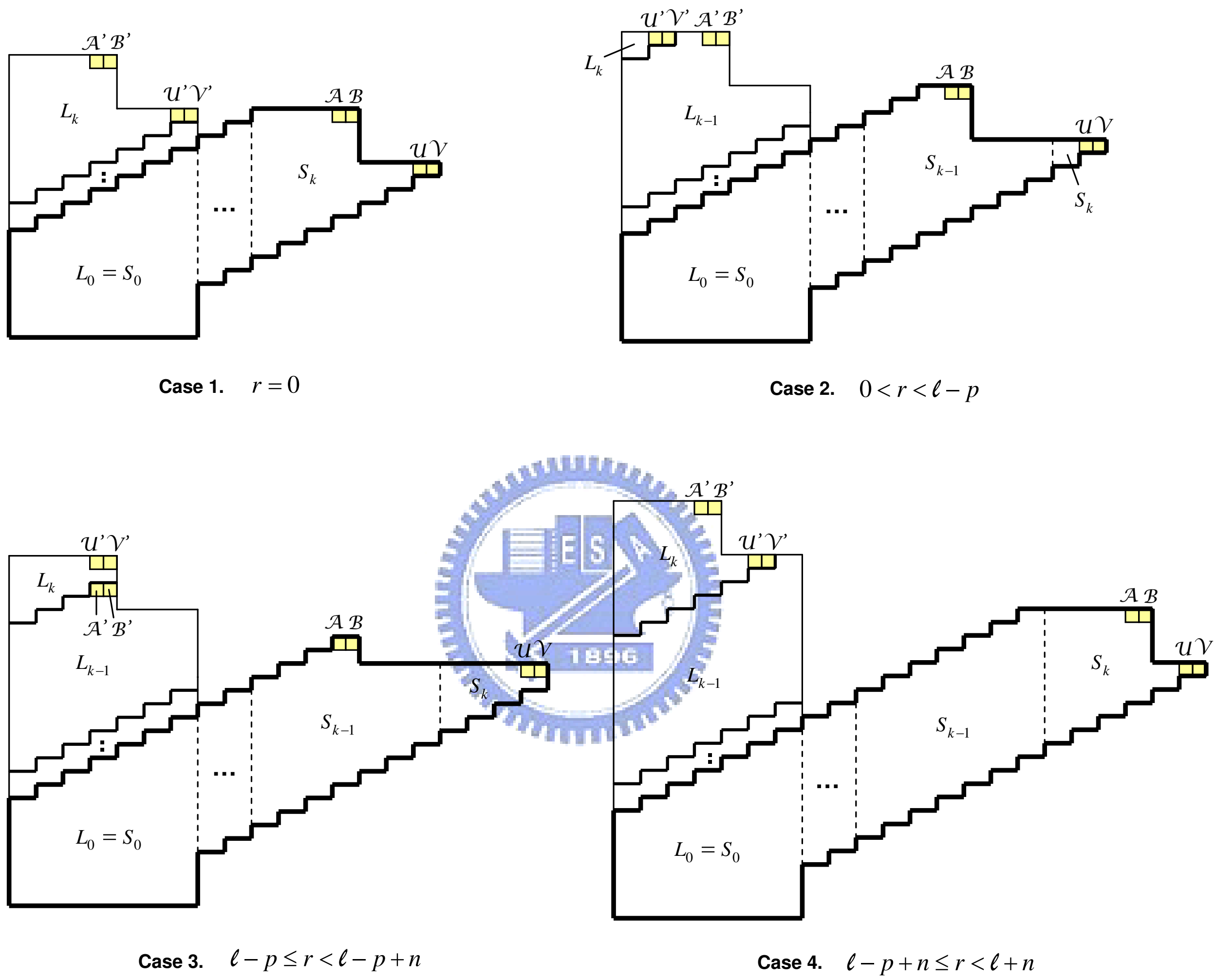


Figure 14: The proof of Theorem 16 for the case that $\ell < h$.