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碩 士 論 文

Hypercube Variants 的點泛圓性與邊泛圓性之研究

Node-pancyclicity and Edge-pancyclicity of Hypercube Variants

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摘要

Twisted cubes, crossed cubes, Möbius cubes, 以及 locally twisted cubes 都是一些廣泛為人所研究的 hypercube variants。在文獻[2, 1, 5, 10, 6] 裡, 已經分別證明了 twisted cubes, crossed cubes, Möbius cubes, 和 locally twisted cubes具有4-泛圓性, 並且 crossed cubes 同時還具有4-邊泛圓性。應注意的是: 若具有4-邊泛圓性則具有4-點泛圓性; 若具有4-點泛圓性則具有4-泛圓性。我們觀察出文獻[6] 中並沒有提到可以利用其方法來證明一些 hypercube variants 具有4-邊泛圓性; 因此, 在這篇論文中, 我們使用[6] 的方法來證明 Möbius cubes 和 locally twisted cubes 具有 4-邊泛圓性。

關鍵字: 連接網路; 圓嵌入; Hypercube; Crossed cubes; Möbius cubes; Locally twisted cubes; 泛圓性。

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Node-pancyclicity and Edge-pancyclicity of Hypercube Variants

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Abstract

Twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are some of the widely studied hypercube variants. The 4-pancyclicity of twisted cubes, crossed cubes, Möbius cubes, locally twisted cubes and the 4-edge-pancyclicity of crossed cubes are proven in [2, 1, 5, 10, 6] respectively. It should be noted that 4-edge-pancyclicity implies 4-node-pancyclicity which further implies 4-pancyclicity. In this paper, we outline an approach to prove the 4-edge-pancyclicity of some hypercube variants and we prove in particular that Möbius cubes and locally twisted cubes are 4-edge-pancyclic.

Keywords: Interconnection network; Hypercube; Crossed cube; Möbius cube; Locally twisted cube; Pancyclicity.

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1 Introduction

Interconnection networks are essential for parallel and distributed computing. The hypercube is one of the most popular interconnection networks since it has simple structure and is easy to implement. An interconnection network can be represented by a graph $G = (V, E)$, where V is the set of nodes and E is the set of edges of the network. In this paper, we will use graphs and interconnection networks interchangeably.

It has been shown that hypercubes do not achieve the smallest possible diameter for its resources. Therefore, many variants were proposed. The most well-known variants are twisted cubes [7], crossed cubes [4], and Möbius cubes [3]; they have diameters about half of that of a hypercube. Generally, the drawback of these variants is that the labels of some neighboring nodes may differ in as many as $n/2$ bits, where n is the dimension of these hypercube variants (see [8] for details). For example, in the 10-dimensional crossed cube, nodes 0001010101 and 1011111111 are adjacent and they differ in 5 bits. Based on this observation Yang et al. [8] proposed the locally twisted cubes with diameters about half of that of a hypercube, of which the labels of any two neighboring nodes differ in at most two successive bits.

The following terminologies will be used throughout this paper. An ℓ-cycle is a cycle of length ℓ. Let $G = (V, E)$ be a graph and $L \leq |V|$ be a positive integer. G is *L-pancyclic* if for every integer $\ell \in \{L, L+1, \ldots, |V|\}$, G contains an ℓ-cycle. G is *L-node-pancyclic* if for every node $x \in V$ and every integer $\ell \in \{L, L+1, \ldots, |V|\}$, G contains an ℓ-cycle C such that x is in C. G is *L-edge-pancyclic* if for every edge $(x, y) \in E$ and every integer $\ell \in \{L, L+1, \ldots, |V|\}$, G contains an ℓ-cycle C such that (x, y) is in C.

One way to evaluate an interconnection network (a host graph) is to see how well other existing networks (the guest graphs) can be embedded into it. The graph embedding problem asks if a guest graph is a subgraph of a host graph. An important benefit of graph embedding is that we can apply existing algorithms for the guest graphs to the host graph. Cycles (i.e., rings) and trees are commonly used guest graphs. This paper will discuss the cycle-embedding properties of Möbius cubes and locally twisted cubes (these cubes will be defined later).

Twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are superior to hypercubes when the cycle-embedding capability is considered [1, 2, 5, 6, 9, 10]. The 4-pancyclicity of twisted cubes, crossed cubes, Möbius cubes, and locally twisted cubes are proven in [2, 1, 5, 10], respectively. Recently, Fan et al. [6] proved that crossed cubes are not only 4-node-pancyclic but also 4-edge-pancyclic. It should be noted that 4-edge-pancyclicity implies 4-node-pancyclicity (thus the proof in [6] for the 4-node-pancyclicity of crossed cubes is actually redundant) which further implies 4-pancyclic.

In this paper, we outline an approach to prove the 4-edge-pancyclicity of some hypercube variants and we prove in particular that Möbius cubes and locally twisted cubes are 4-edge-pancyclic. We also show how to use our approach to prove that crossed cubes are 4-edge-pancyclic.

This paper is organized as follows. In Section 2, we give some definitions and notations. In Section 3, we outline an approach to prove 4-edge-pancyclicity. In Sections 4, 5, and 6, we prove that locally twisted cubes, crossed cubes, and Möbius cubes are 4-edge-pancyclic. The final section concludes this paper.

2 Preliminaries

Let $G = (V, E)$ be a graph and let $L \leq |V| - 1$ be a positive integer. G is *L-path-connected* if G contains a path of length L between any two distinct nodes. G is *Hamiltonian-connected* if G is $(|V| - 1)$-path-connected.

The *n-dimensional hypercube* Q_n is a graph with 2^n nodes and $n \cdot (2^{n-1})$ edges such that its nodes are n-tuples with entries in $\{0, 1\}$ and its edges are the pairs of n-tuples that differ in exactly one position. Thus Q_1 is the complete graph with two nodes 0 and 1, and Q_n $(n \geq 2)$ is built from two copies of Q_{n-1} as follows: Let $k \in \{0, 1\}$ and let kQ_{n-1} denote the graph obtained by prefixing the label of each node of one copy of Q_{n-1}

with k; connect each node $0x_{n-1}\dots x_2x_1$ of $0Q_{n-1}$ with the node $1x_{n-1}\dots x_2x_1$ of $1Q_{n-1}$ by an edge.

We now define a generalization of Q_n. The *n-dimensional general cube* GQ_n is defined recursively as follows (see Figure 1). GQ_1 is Q_1, and GQ_n ($n \geq 2$) is built from two GQ_{n-1}'s (not necessarily identical) as follows: Let $k \in \{0,1\}$ and let kGQ_{n-1} denote the graph obtained by prefixing the label of each node of one of the two GQ_{n-1}'s with k; add a perfect matching between $0GQ_{n-1}$ and $1GQ_{n-1}$, i.e., each node in $0GQ_{n-1}$ is adjacent to exactly one node in $1GQ_{n-1}$.

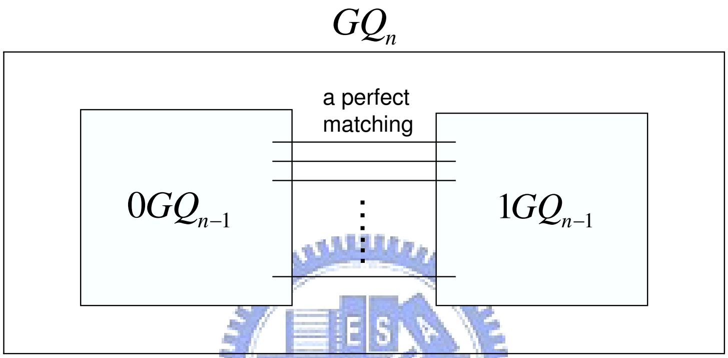

Figure 1: The n-dimensional general cube GQ_n.

We assume conventionality of the node prefixing method kGQ_{n-1} which will be used repeatedly in the definitions of specific hypercube variants late in this paper unless otherwise specified. We will see in the following sections that crossed cubes, Möbius cubes, and locally twisted cubes are the examples of GQ_n. Note that the two GQ_{n-1}'s in GQ_n are not necessarily identical. For instance, for crossed cubes and locally twisted cubes, the two GQ_{n-1}'s are identical; but for Möbius cubes, they are not.

For clarity, let $V(G)$ and $E(G)$ denote the set of nodes and the set of edges of G, respectively. We say that (x,y) is a *matching edge* in GQ_n if $x \in V(0GQ_{n-1})$, $y \in V(1GQ_{n-1})$, and x is matched with y. If (x,y) is a matching edge, then we write $m(x)$ for y and $m(y)$ for x. We say that GQ_n has the *4-cycle property* if for every matching edge (x,y), there exists a matching edge (u,v) such that (x,u,v,y,x) form a 4-cycle in

GQ_n. We say that GQ_n has the *5-cycle property* if for every matching edge (x, y), there exist a matching edge (s, t) and a node $r \in V(0GQ_{n-1})$ such that (x, r, s, t, y, x) form a 5-cycle in GQ_n.

3 4-edge-pancyclicity of general cubes

In this section, we outline an approach to prove 4-edge-pancyclicity. We first give two lemmas.

Lemma 1. *For $n \geq 4$, if both $0GQ_{n-1}$ and $1GQ_{n-1}$ are Hamiltonian-connected, then GQ_n is Hamiltonian-connected.*

Proof. Let x and y be two arbitrary distinct nodes of GQ_n. Then there are four cases.

Case 1. $x \in V(0GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. Since $0GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(p_1, p_2, \ldots, p_{2^{n-1}})$ such that $p_1 = x$ and $p_{2^{n-1}} = y$. Since $1GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(q_1, q_2, \ldots, q_{2^{n-1}})$ such that $q_1 = m(p_1)$ and $q_{2^{n-1}} = m(p_2)$. Hence $(x, q_1, q_2, \ldots, q_{2^{n-1}}, p_2, p_3, \ldots, p_{2^{n-1}-1}, y)$ is a Hamiltonian path between x and y in GQ_n.

Case 2. $x \in V(1GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. The argument is similar to that of Case 1.

Case 3. $x \in V(0GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. Let $z \in V(0GQ_{n-1})$ such that $z \neq x$. Since $0GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(p_1, p_2, \ldots, p_{2^{n-1}})$ such that $p_1 = x$ and $p_{2^{n-1}} = z$. Since $1GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(q_1, q_2, \ldots, q_{2^{n-1}})$ such that $q_1 = m(z)$ and $q_{2^{n-1}} = y$. Hence $(x, p_2, \ldots, p_{2^{n-1}}, q_1, q_2 \ldots, q_{2^{n-1}-1}, y)$ is a Hamiltonian path between x and y in GQ_n.

Case 4. $x \in V(1GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. The argument is similar to that of Case 3. ∎

Lemma 2. *For $n \geq 4$, if both $0GQ_{n-1}$ and $1GQ_{n-1}$ are Hamiltonian-connected and $(2^{n-1} - 2)$-path-connected, then GQ_n is $(2^n - 2)$-path-connected.*

Proof. Let x and y be two arbitrary distinct nodes of GQ_n. Then there are four cases.

Case 1. $x \in V(0GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. Since $0GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(p_1, p_2, \ldots, p_{2^{n-1}})$ such that $p_1 = x$ and $p_{2^{n-1}} = y$. Since $1GQ_{n-1}$ is $(2^{n-1} - 2)$-path-connected, it has a path $(q_1, q_2, \ldots, q_{2^{n-1}-1})$ of length $2^{n-1} - 2$ such that $q_1 = m(p_1)$ and $q_{2^{n-1}-1} = m(p_2)$. Hence $(x, q_1, q_2, \ldots, q_{2^{n-1}-1}, p_2, p_3, \ldots, p_{2^{n-1}-1}, y)$ is a path of length $2^n - 2$ between x and y in GQ_n.

Case 2. $x \in V(1GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. The argument is similar to that of Case 1.

Case 3. $x \in V(0GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. Let $z \in V(0GQ_{n-1})$ such that $z \neq x$. Since $0GQ_{n-1}$ is Hamiltonian-connected, it has a Hamiltonian path $(p_1, p_2, \ldots, p_{2^{n-1}})$ such that $p_1 = x$ and $p_{2^{n-1}} = z$. Since $1GQ_{n-1}$ is $(2^{n-1} - 2)$-path-connected, it has a path $(q_1, q_2, \ldots, q_{2^{n-1}-1})$ of length $2^{n-1} - 2$ such that $q_1 = m(z)$ and $q_{2^{n-1}-1} = y$. Hence $(x, p_2, \ldots, p_{2^{n-1}}, q_1, q_2 \ldots, q_{2^{n-1}-2}, y)$ is a path of length $2^n - 2$ between x and y in GQ_n.

Case 4. $x \in V(1GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. The argument is similar to that of Case 3. ∎

We now outline an approach to prove the 4-edge-pancyclicity of GQ_n.

Theorem 3. *For $n \geq 4$, if all the GQ_3's in GQ_n are 4-edge-pancyclic, Hamiltonian-connected, and $(2^3 - 2)$-path-connected, and if GQ_n has both the 4-cycle and the 5-cycle properties, then GQ_n is 4-edge-pancyclic.*

Proof. This theorem follows from Lemma 1, Lemma 2, and the following claim.

Claim. *For $n \geq 4$, if both $0GQ_{n-1}$ and $1GQ_{n-1}$ are 4-edge-pancyclic, Hamiltonian-connected, and $(2^{n-1} - 2)$-path-connected, and if GQ_n has both the 4-cycle property and the 5-cycle property, then GQ_n is 4-edge-pancyclic.*

We now prove the claim. Let (x, y) be an arbitrary edge of $E(GQ_n)$ and let $\ell \in \{4, 5, \ldots, 2^n\}$. There are four cases.

Case 1. $x \in V(0GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. Then there are three subcases.

Subcase 1.1. $4 \le \ell \le 2^{n-1}$. Since $0GQ_{n-1}$ is 4-edge-pancyclic, there exists an ℓ-cycle that contains (x, y) in $0GQ_{n-1}$, hence in GQ_n.

Subcase 1.2. $\ell = 2^{n-1}+1$. Let $u = m(x)$ and $v = m(y)$. Since $1GQ_{n-1}$ is $(2^{n-1}-2)$-path-connected, it has a path $(p_1, p_2, \ldots, p_{2^{n-1}-1})$ of length $2^{n-1}-2$ such that $p_1 = v$ and $p_{2^{n-1}-1} = u$. Thus $(x, y, p_1, p_2, \ldots, p_{2^{n-1}-1}, x)$ is a $(2^{n-1}+1)$-cycle in GQ_n that contains (x, y).

Subcase 1.3. $2^{n-1}+2 \le \ell \le 2^n$. Since $0GQ_{n-1}$ is 4-edge-pancyclic and (x, y) is an edge in $0GQ_{n-1}$, there exists a 2^{n-1}-cycle $C = (p_1, p_2, \ldots, p_{2^{n-1}}, p_1)$ in $0GQ_{n-1}$ such that $p_1 = x$ and $p_2 = y$. Note that $1 \le \ell - 2^{n-1} - 1 \le 2^{n-1} - 1$. Let $(p_1, p_2, \ldots, p_{\ell-2^{n-1}})$ be the path of length $\ell - 2^{n-1} - 1$ in C. Set $w = p_{\ell-2^{n-1}}$ for easy writing. Let $u = m(x)$ and $v = m(w)$. Then $u, v \in V(1GQ_{n-1})$. Since $1GQ_{n-1}$ is Hamiltonian-connected, there is a path $(q_1, q_2, \ldots, q_{2^{n-1}})$ of length $2^{n-1}-1$ in $1GQ_{n-1}$ such that $q_1 = v$ and $q_{2^{n-1}} = u$. Thus $(p_1, p_2, \ldots, p_{\ell-2^{n-1}}, q_1, q_2, \ldots, q_{2^{n-1}}, p_1)$ is a cycle of length $(\ell-2^{n-1}-1)+1+(2^{n-1}-1)+1 = \ell$ in GQ_n that contains (x, y).

Case 2. $x \in V(1GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. The argument is similar to that of Case 1.

Case 3. $x \in V(0GQ_{n-1})$ and $y \in V(1GQ_{n-1})$. Then there are four subcases.

Subcase 3.1. $\ell \in \{4, 5\}$. Since GQ_n has the 4-cycle property and the 5-cycle property, there exists a cycle of length ℓ in GQ_n that contains (x, y).

Subcase 3.2. $6 \le \ell \le 2^{n-1}+2$. Since GQ_n has the 4-cycle property, there exist $u \in V(0GQ_{n-1})$ and $v \in V(1GQ_{n-1})$ such that (x, u, v, y, x) form a 4-cycle in GQ_n. Let $m = \ell - 2$. Then $4 \le m \le 2^{n-1}$. Since $0GQ_{n-1}$ is 4-edge-pancyclic, there exists a m-cycle $(p_1, p_2, \ldots, p_m, p_1)$ in $0GQ_{n-1}$ such that $p_1 = x$ and $p_m = u$. Thus $(x, p_2, \ldots, p_m, v, y, x)$ is an $(m+2)$-cycle (i.e., an ℓ-cycle) in GQ_n that contains (x, y).

Subcase 3.3. $\ell = 2^{n-1}+3$. Since GQ_n has the 4-cycle property, there exist $u \in V(0GQ_{n-1})$ and $v \in V(1GQ_{n-1})$ such that (x, u, v, y, x) form a 4-cycle in GQ_n. Since $0GQ_{n-1}$ is 4-edge-pancyclic, there exists a $(2^{n-1}-1)$-cycle $(p_1, p_2, \ldots, p_{2^{n-1}-1}, p_1)$ in

$0GQ_{n-1}$ such that $p_1 = x$ and $p_{2^{n-1}-1} = u$. Since $1GQ_{n-1}$ is 4-edge-pancyclic, there exists a 4-cycle $(q_1, q_2, q_3, q_4, q_1)$ in $1GQ_{n-1}$ such that $q_1 = v$ and $q_4 = y$. Thus $(p_1, p_2, \ldots, p_{2^{n-1}-1}, q_1, q_2, q_3, q_4, p_1)$ is a $(2^{n-1}+3)$-cycle in GQ_n that contains (x, y).

Subcase 3.4. $2^{n-1} + 4 \leq \ell \leq 2^n$. Since GQ_n has the 4-cycle property, there exist $u \in V(0GQ_{n-1})$ and $v \in V(1GQ_{n-1})$ such that (x, u, v, y, x) form a 4-cycle in GQ_n. Since $0GQ_{n-1}$ is 4-edge-pancyclic, there exists a 2^{n-1}-cycle $(p_1, p_2, \ldots, p_{2^{n-1}}, p_1)$ in $0GQ_{n-1}$ such that $p_1 = x$ and $p_{2^{n-1}} = u$. Let $m = \ell - 2^{n-1}$. Then $4 \leq m \leq 2^{n-1}$. Since $1GQ_{n-1}$ is 4-edge-pancyclic, there exists a m-cycle $(q_1, q_2, \ldots, q_m, q_1)$ in $1GQ_{n-1}$ such that $q_1 = v$ and $q_m = y$. Thus $(p_1, p_2, \ldots, p_{2^{n-1}}, q_1, q_2, \ldots, q_m)$ is a cycle of length $(2^{n-1}-1)+(m-1)+2 = m + 2^{n-1} = \ell$ in GQ_n that contains (x, y).

Case 4. $x \in V(1GQ_{n-1})$ and $y \in V(0GQ_{n-1})$. The argument is similar to that of Case 3. ∎

4 Pancyclicity of locally twisted cubes

The purpose of this section is to use Theorem 3 to prove that locally twisted cubes are 4-edge-pancyclic.

The *n-dimensional locally twisted cube* LTQ_n is defined recursively as follow. LTQ_1 is Q_1, and LTQ_2 is the graph consisting of four nodes labelled with 00, 01, 10, and 11, respectively, and connected by the four edges (00, 01) (00, 10), (01, 11), and (10, 11). LTQ_n $(n \geq 3)$ is built from two identical LTQ_{n-1}'s as follows: connect each node $0x_{n-1}x_{n-2}\ldots x_1$ of $0LTQ_{n-1}$ with the node $1(x_{n-1}+x_1)x_{n-2}\ldots x_1$ of $1LTQ_{n-1}$ by an edge, where $'+'$ means the modulo 2 addition operation. See Figures 2 and 3 for examples.

Before going any further, we work out the adjacency relation of LTQ_n. For convenience, $\overline{x}_i$ denotes the complement of x_i.

Lemma 4. *For every $x = x_n x_{n-1} \ldots x_1 \in V(LTQ_n)$, the n nodes $y_1, y_2, \ldots, y_n$ adjacent*

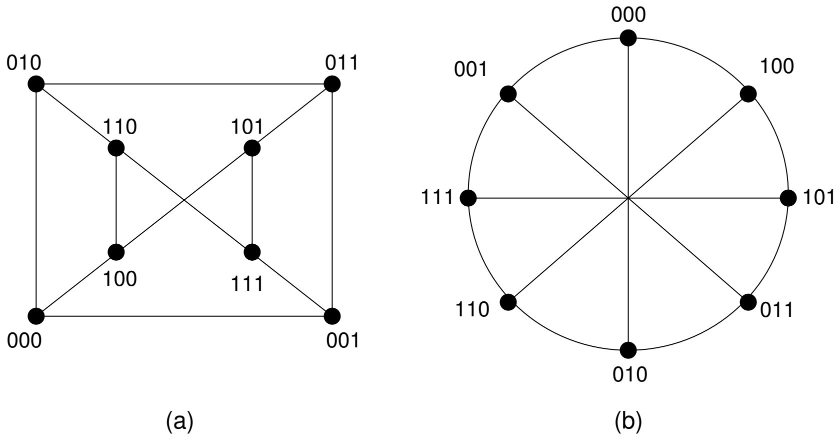

Figure 2: (a) LTQ_3. (b) A symmetric drawing of LTQ_3.

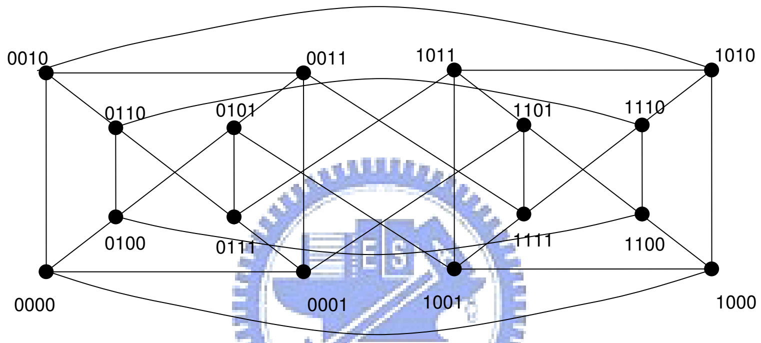

Figure 3: LTQ_4.

to x are:

$$
\begin{aligned}
y_1 &= x_n x_{n-1} x_{n-2} \ldots x_3 x_2 \overline{x}_1, \\
y_2 &= x_n x_{n-1} x_{n-2} \ldots x_3 \overline{x}_2 x_1, \\
y_3 &= x_n x_{n-1} x_{n-2} \ldots \overline{x}_3 (x_2 + x_1) x_1, \\
&\vdots \\
y_{n-1} &= x_n \overline{x}_{n-1} (x_{n-2} + x_1) \ldots x_3 x_2 x_1, \\
y_n &= \overline{x}_n (x_{n-1} + x_1) x_{n-2} \ldots x_3 x_2 x_1.
\end{aligned}
$$

Proof. By the definition of LTQ_n, $(x, y_n) \in E(LTQ_n)$. $(x, y_1) \in E(LTQ_n)$ because $(x_1, \overline{x}_1) \in E(LTQ_1)$ and LTQ_n is built from LTQ_1. Similarly, $(x, y_2) \in E(LTQ_n)$ because $(x_2 x_1, \overline{x}_2 x_1) \in E(LTQ_2)$ and LTQ_n is built from LTQ_2. For $3 \leq i \leq n-1$, $(x, y_i) \in E(LTQ_n)$ because $(x_i x_{i-1} x_{i-2} \ldots x_1, \overline{x}_i (x_{i-1} + x_1) x_{i-2} \ldots x_1) \in E(LTQ_i)$ and LTQ_n is built from LTQ_i. ∎

It is not difficult to see that: for each n, there is only one type of LTQ_n. Thus for $n \geq 4$, all the LTQ_3's in LTQ_n are identical. We are now ready to prove that locally twisted cubes satisfy Theorem 3.

Theorem 5. *LTQ_3 is 4-edge-pancyclic, Hamiltonian-connected, and (2^3-2)-path-connected. For $n \geq 4$, LTQ_n has both the 4-cycle property and the 5-cycle property.*

Proof. In [10], it was proven that LTQ_n is Hamiltonian-connected and $(2^n - 2)$-path-connected for $n \geq 3$. Thus LTQ_3 is Hamiltonian-connected and $(2^3 - 2)$-path-connected. We now prove that LTQ_3 is 4-edge-pancyclic. Since LTQ_3 is node-symmetric (see Figure 2(b)), it suffices to consider the edge $(x, y) \in \{(000, 001), (000, 010)\}$. The cycles of lengths from 4 to 8 containing $(000, 001)$ (underlined) are listed as follows:

$$\text{length } 4: \ \underline{000, 001}, 011, 010, 000;$$
$$\text{length } 5: \ \underline{000, 001}, 111, 101, 100, 000;$$
$$\text{length } 6: \ \underline{000, 001}, 011, 010, 110, 100, 000;$$
$$\text{length } 7: \ \underline{000, 001}, 011, 101, 111, 110, 100, 000;$$
$$\text{length } 8: \ \underline{000, 001}, 111, 110, 010, 011, 101, 100, 000.$$

The cycles of lengths from 4 to 8 containing $(000, 010)$ (underlined) are listed as follows:

$$\text{length } 4: \ \underline{000, 010}, 110, 100, 000;$$
$$\text{length } 5: \ \underline{000, 010}, 110, 111, 001, 000;$$
$$\text{length } 6: \ \underline{000, 010}, 110, 111, 101, 100, 000;$$
$$\text{length } 7: \ \underline{000, 010}, 110, 100, 101, 111, 001, 000;$$
$$\text{length } 8: \ \underline{000, 010}, 110, 111, 001, 011, 101, 100, 000.$$

Thus LTQ_3 is 4-edge-pancyclic.

We now prove that LTQ_n has the 4-cycle property and the 5-cycle property. Let (x, y) be an arbitrary matching edge of LTQ_n and let $x = 0x_{n-1}x_{n-2} \dots x_2x_1$. By the definition of LTQ_n, $y = 1(x_{n-1} + x_1)x_{n-2} \dots x_2x_1$.

First consider the 4-cycle property. Let $u = 0x_{n-1}x_{n-2} \dots \overline{x}_2x_1$ and $v = 1(x_{n-1} + x_1)x_{n-2} \dots \overline{x}_2x_1$. By Lemma 4, $\{(x, u), (u, v), (v, y)\} \subseteq E(LTQ_n)$. Hence (x, u, v, y, x) is a 4-cycle in LTQ_n that contains (x, y). Now consider the 5-cycle property. If $x_1 = 0$, let $r = 0\overline{x}_{n-1}x_{n-2} \dots x_20$, $s = 0\overline{x}_{n-1}x_{n-2} \dots x_21$, and $t = 1x_{n-1}x_{n-2} \dots x_21$; otherwise, if $x_1 = 1$, let $r = 0x_{n-1}x_{n-2} \dots x_20$, $s = 0\overline{x}_{n-1}x_{n-2} \dots x_20$, and $t = 1\overline{x}_{n-1}x_{n-2} \dots x_20$. By Lemma 4, $\{(x, r), (r, s), (s, t), (t, y)\} \subseteq E(LTQ_n)$. Hence (x, r, s, t, y, x) is a 5-cycle in LTQ_n that contains (x, y). ∎

It was proven in [10] that LTQ_n is 4-pancyclic. We now strengthen this result.

Theorem 6. *For $n \geq 2$, LTQ_n is 4-edge-pancyclic.*

Proof. Clearly, this theorem holds when $n = 2$. By Theorem 5, this theorem holds when $n = 3$. For $n \geq 4$, this theorem follows from Theorem 3 and Theorem 5. ∎

The following corollary is obvious.

Corollary 7. *For $n \geq 2$, LTQ_n is 4-node-pancyclic.*

5 Pancyclicity of crossed cubes

We first give the definition of crossed cubes. Two binary strings $x = x_2x_1$ and $y = y_2y_1$ of length two are said to be *pair related* (denoted by $x \sim y$) if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. The *n-dimensional crossed cube* CQ_n is defined recursively as follows. CQ_1 is Q_1, and CQ_2 is the graph consisting of four nodes labelled with 00, 01, 10 and 11, respectively, and connected by the four edges (00, 01) (00, 10), (01, 11), and (10, 11). CQ_n ($n \geq 3$) is built from two identical CQ_{n-1}'s as follows: connect each node $0x_{n-1} \dots x_2x_1$ of $0CQ_{n-1}$ with the node $1y_{n-1} \dots y_2y_1$ of $1CQ_{n-1}$ by an edge if and only if

(1) $x_{n-1} = y_{n-1}$ if n is even, and

(2) $x_{2i}x_{2i-1} \sim y_{2i}y_{2i-1}$ for $1 \leq i < \lceil n/2 \rceil$.

In [6], Fan et al. have proven that crossed cubes are 4-edge-pancyclic. We now show how to use Theorem 3 to obtain this result. It is not difficult to see that: for each n, there is only one type of CQ_n. Thus for $n \geq 4$, all the CQ_3's in CQ_n are identical. We are now ready to prove that crossed cubes satisfy Theorem 3.

Theorem 8. *CQ_3 is 4-edge-pancyclic, Hamiltonian-connected, and $(2^{3-1}-2)$-path-connected. For $n \geq 4$, CQ_n has both the 4-cycle property and the 5-cycle property.*

Since the proof for each condition in this theorem can be found in [6], we omit the proof. We have the following theorem.

Theorem 9. **[6]** *For $n \geq 2$, CQ_n is 4-edge-pancyclic.*

Proof. Clearly, this theorem holds when $n = 2$. By Theorem 8, this theorem holds when $n = 3$. For $n \geq 4$, this theorem follows from Theorem 3 and Theorem 8. ∎

By Theorem 9, it is obvious that for $n \geq 2$, CQ_n is 4-node-pancyclic and 4-pancyclic.

6 Pancyclicity of Möbius cubes

In this section, we show how to use Theorem 3 to prove that Möbius cubes are 4-edge-pancyclic.

The *n-dimensional Möbius cube MQ_n* is defined recursively as follow (see Figures 4 and 5):

(1) MQ_1 is Q_1.

(2) There are two types of MQ_2: one is named 0-MQ_2 and the other, 1-MQ_2. 0-MQ_2 is the graph consisting of four nodes labelled with 00, 01, 10, and 11, respectively, and connected by the four edges (00, 01), (00, 10), (01, 11), and (10, 11). 1-MQ_2

has the same nodes as 0-MQ_2, but connected by the four edges (00, 01) (00, 11), (01, 10), and (10, 11).

(3) For $n \geq 3$, there are two types of MQ_n: 0-MQ_n and 1-MQ_n. Both 0-MQ_n and 1-MQ_n are built from $0MQ_{n-1}$ and $1MQ_{n-1}$ with the MQ_{n-1} in $0MQ_{n-1}$ being 0-MQ_{n-1} and the MQ_{n-1} in $1MQ_{n-1}$ being 1-MQ_{n-1}. In 0-MQ_n, each node $0x_{n-1}x_{n-2}\ldots x_1$ of $0MQ_{n-1}$ is connected with the node $1x_{n-1}x_{n-2}\ldots x_1$ of $1MQ_{n-1}$; while in 1-MQ_n, each node $0x_{n-1}x_{n-2}\ldots x_1$ of $0MQ_{n-1}$ is connected with the node $1\overline{x}_{n-1}\overline{x}_{n-2}\ldots\overline{x}_1$ of $1MQ_{n-1}$.

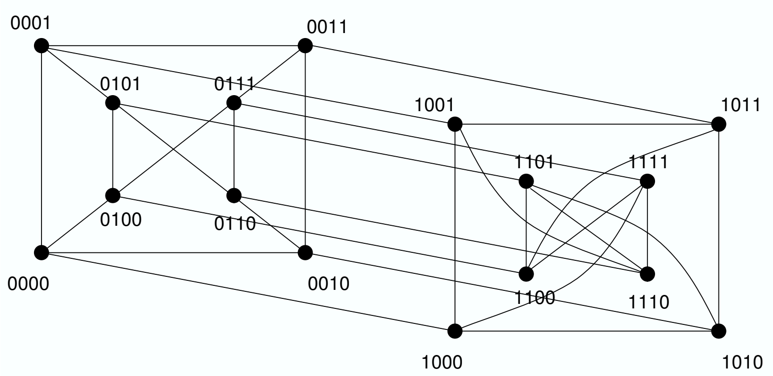

Figure 4: 0-MQ_4.

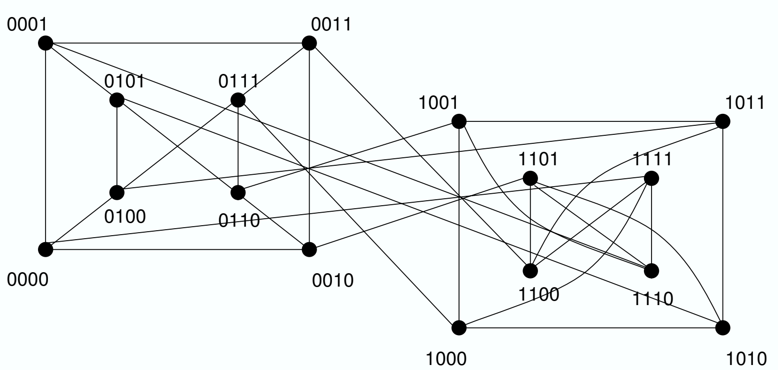

Figure 5: 1-MQ_4.

Before going any further, we work out the adjacency relation of MQ_n.

Lemma 10. *For every* $x = x_n x_{n-1} \dots x_2 x_1 \in V(MQ_n)$*, the* n *nodes* $y_1, y_2, \dots, y_n$ *adjacent to* x *are as follows. For* $1 \le i \le n-1$*,*

$$y_i \;=\; \begin{cases} x_n x_{n-1} \dots x_{i+1} \overline{x}_i x_{i-1} \dots x_1 & \text{if } x_{i+1} = 0, \\ x_n x_{n-1} \dots x_{i+1} \overline{x}_i \overline{x}_{i-1} \dots \overline{x}_1 & \text{if } x_{i+1} = 1. \end{cases}$$

For 0-MQ_n*,* $y_n = \overline{x}_n x_{n-1} \dots x_1$*; for* 1-MQ_n*,* $y_n = \overline{x}_n \overline{x}_{n-1} \dots \overline{x}_1$*.*

Proof. This lemma follows from the definition of Möbius cubes given in [3]. ∎

It is not difficult to see that: for each n, there are two types of MQ_n: the 0-MQ_n and the 1-MQ_n. Thus for $n \ge 4$, all the MQ_3's in MQ_n are either 0-MQ_3 or 1-MQ_3. We are now ready to prove that Möbius cubes satisfy Theorem 3.

Theorem 11. *Both the 0-*MQ_3 *and the 1-*MQ_3 *are 4-edge-pancyclic, Hamiltonian-connected, and* (2^3-2)*-path-connected. For* $n \ge 4$*,* MQ_n *has both the 4-cycle property and the 5-cycle property.*

Proof. From Figures 2, 6, and 7, both 0-MQ_3 and 1-MQ_3 are isomorphic to LTQ_3. Thus by Theorem 5, both 0-MQ_3 and 1-MQ_3 are 4-edge-pancyclic, Hamiltonian-connected, and (2^3-2)-path-connected.

We now prove that MQ_n has the 4-cycle property and the 5-cycle property. Let (x, y) be an arbitrary matching edge of MQ_n and let $x = 0x_{n-1}x_{n-2} \dots x_2 x_1$. By the definition of MQ_n, $y = 1x_{n-1}x_{n-2} \dots x_2 x_1$ if this MQ_n is 0-MQ_n and $y = 1\overline{x}_{n-1}\overline{x}_{n-2} \dots \overline{x}_2 \overline{x}_1$ if this MQ_n is 1-MQ_n.

First consider the 4-cycle property. Let $u = x_n x_{n-1} \dots x_2 \overline{x}_1$. If this MQ_n is 0-MQ_n, then let $v = \overline{x}_n x_{n-1} \dots x_2 \overline{x}_1$; otherwise, if this MQ_n is 1-MQ_n, then let $v = \overline{x}_n \overline{x}_{n-1} \dots \overline{x}_2 x_1$. By Lemma 10, $\{(x, u), (u, v), (v, y)\} \subseteq E(MQ_{n)})$. Hence (x, u, v, y, x) is a 4-cycle in MQ_n that contains (x, y).

Now consider the 5-cycle property. Let $s = 0\overline{x}_{n-1}\overline{x}_{n-2} \dots \overline{x}_2 \overline{x}_1$ and choose r and t according to the following rules:

1. If this MQ_n is 0-MQ_n and $x_{n-1} = 0$, then let $r = 0\overline{x}_{n-1}x_{n-2}\dots x_2x_1$ and $t = 1\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_2\overline{x}_1$.

2. If this MQ_n is 0-MQ_n and $x_{n-1} = 1$, then let $r = 0x_{n-1}\overline{x}_{n-2}\dots\overline{x}_2\overline{x}_1$ and $t = 1\overline{x}_{n-1}\overline{x}_{n-2}\dots\overline{x}_2\overline{x}_1$.

3. If this MQ_n is 1-MQ_n and $x_{n-1} = 0$, then let $r = 0\overline{x}_{n-1}x_{n-2}\dots x_2x_1$ and $t = 1x_{n-1}x_{n-2}\dots x_2x_1$.

4. If this MQ_n is 1-MQ_n and $x_{n-1} = 1$, then let $r = 0x_{n-1}\overline{x}_{n-2}\dots\overline{x}_2\overline{x}_1$ and $t = 1x_{n-1}x_{n-2}\dots x_2x_1$.

By Lemma 10, $\{(x,r),(r,s),(s,t),(t,y)\} \subseteq E(MQ_n)$. Hence (x,r,s,t,y,x) is a 5-cycle in MQ_n that contains (x,y). ∎

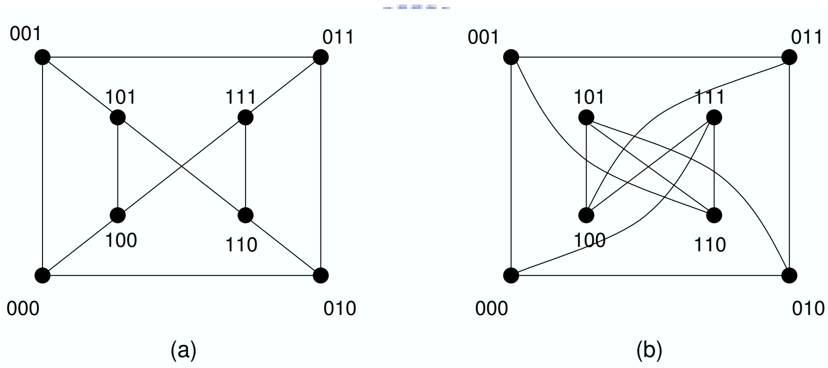

Figure 6: (a) 0-MQ_3. (b) 1-MQ_3.

It was proven in [5] that MQ_n is 4-pancyclic. We now strengthen this result.

Theorem 12. *For $n \geq 2$, MQ_n is 4-edge-pancyclic.*

Proof. Clearly, this theorem holds when $n = 2$. By Theorem 11, this theorem holds when $n = 3$. For $n \geq 4$, this theorem follows from Theorem 3 and Theorem 11. ∎

The following corollary is obvious.

Corollary 13. *For $n \geq 2$, MQ_n is 4-node-pancyclic.*

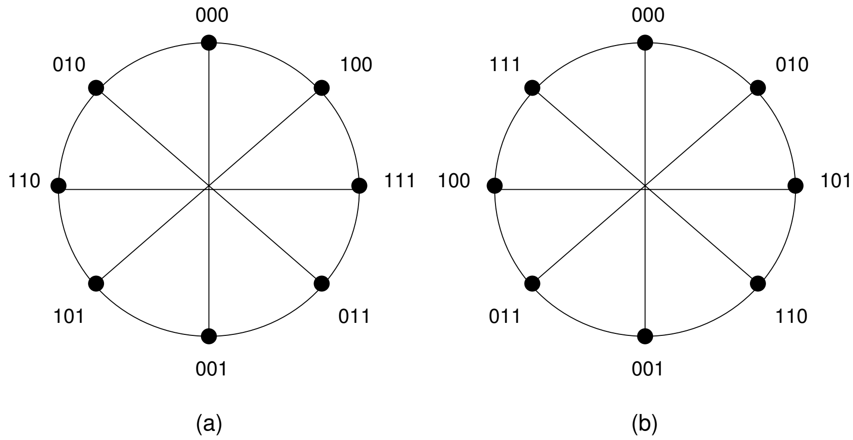

Figure 7: (a) A symmetric drawing of 0-MQ_3. (b) A symmetric drawing of 1-MQ_3.

7 Concluding remarks

In this paper, we outline an approach to prove the 4-edge-pancyclicity (hence 4-node-pancyclicity and 4-pancyclicity) of some hypercube variants. We prove in particular that locally twisted cubes and Möbius cubes are 4-edge-pancyclic. We also show how to use our approach to prove the 4-edge-pancyclicity of crossed cubes. It remains open whether twisted cubes are 4-node-pancyclic and 4-edge-pancyclic. We now summarize known results on the pancyclicity properties of various hypercube variants in Table 1 (in this table, "pan" means pancyclic and "loc twisted" means locally twisted).

Table 1: Pancyclicity of hypercube variations.

cubes	4-pan	4-node-pan	4-edge-pan
twisted	[2]	unknown	unknown
crossed	[1]	[6]	[6]
Möbius	[5]	this paper	this paper
loc twisted	[10]	this paper	this paper

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