# 國立交通大學

應用數學系

碩 士 論 文

# 圈分割的極圖
# Extremal Graphs of $C_k$-decomposition

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中 華 民 國 九 十 四 年 六 月

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## 摘要

如果一個圖 $G$ 的邊集合可以分成一些子集合的聯集，而每一個子集合都導出一個 $k$ 圈，圖 $G$ 就稱為有 $k$ 圈分割。很明顯地，如果圖 $G$ 有 $k$ 圈分割，圖 $G$ 一定是一個偶圖，而且 $k$ 會整除圖 $G$ 的邊數。我們稱一個滿足上面兩個條件的圖為 $k$ 充分圖。不難發現，一個 $k$ 充分圖可能沒有 $k$ 圈分割。在論文中的第一部份，將探討一個有 $n$ 個點，是 $r$ 正則且 $k$ 充分，但是卻不存在 $k$ 圈分割的圖。利用直接建構法說明，$r$ 是如何根據 $k$ 和 $n$ 的不同，得到不同的下界。第二部份，探討沒有 $k$ 圈分割的極圖，根據圈大小的不同，也得到不同邊數的下界。

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# Extremal Graphs of $C_k$-decomposition

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**Abstract**

A graph $G$ is said to have a $C_k$-decomposition if $E(G)$ can be partitioned into a collection of subsets each induces a $k$-cycle. Clearly, if $G$ has a $C_k$-decomposition, then $G$ is an even graph of order at least $k$ and $k$ divides $|E(G)|$. The graphs satisfying the above two conditions are called $k$-sufficient. It is not difficult to see that a $k$-sufficient graph may not have a $C_k$-decomposition. In this thesis, at first, we study the $k$-sufficient $r$-regular graphs of order $n$ in which $C_k$-decomposition does not exist. By direct constructions, we show that there are constraints on $r$ with respect to $k$ and $n$. In order to decompose an arbitrary $r$-regular graph of order $n$ into $C_k$'s, $r$ has to be at least $\frac{2t+1}{4t}n$, $\frac{3}{5}n$, $\frac{n}{2}$, and $\frac{n}{2}$ if $k$ is $2t+1$, 4, $2t$, and $n$ respectively. On the second part, we also study the extremal $k$-sufficient graphs which have no $C_k$-decomposition. As a consequence, the following results are obtained: (i) If $n$ is even, then $ex(n; C_3\text{-decomp.}) \geqslant \binom{n}{2} - (n-2) - \epsilon_n$ where $\epsilon_n = 4$ in case that $n \equiv 2 \ or\ 4 \pmod 6$ and $\epsilon_n = 5$ in case that $n \equiv 0 \pmod 6$. (ii) If $n$ is odd, then $ex(n; C_3\text{-decomp.}) \geqslant \binom{n-2}{2} - \epsilon_n$ where $\epsilon_n = 4$ in case that $n \equiv 1 \pmod 6$ and $\epsilon_n = 0$ in case that $n \equiv 3 \ or\ 5 \pmod 6$. (iii) For $k \geqslant 4$, if $n$ is odd, $ex(n; C_k\text{-decomp.}) \geqslant \binom{n}{2} - 2(n-3) - \epsilon_{k,n}$, where $\epsilon_{k,n} \in \{0, 3, 4, 5, ..., k-1, k+1, k+2\}$, such that $\binom{n}{2} - 2(n-3) - \epsilon_{k,n}$ is a multiple of $k$. (iv) For $k \geqslant 4$, if $n$ is even, $ex(n; C_k\text{-decomp.}) \geqslant \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - \epsilon_{k,n}$, where $\epsilon_{k,n} \in \{0, 3, 4, 5, ..., k-1, k+1, k+2\}$, such that $\binom{n}{2} - 2(n-3) - \frac{n-2}{2} - \epsilon_{k,n}$ is a multiple of $k$.

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# 1 Introduction

Graph decomposition is one of the most important topics in the study of graph theory. The main reason is due to the fact that decomposing the complete graph of order $v$ with multiplicity $\lambda$ into a collection of complete subgraphs of order $k$ is equivalent to construct a *balanced incomplete block design(BIBD),* 2-$(v, k, \lambda)$ *design.* By replacing the complete subgraphs of order $k$ with $k$-cycles, we have a $\lambda$-fold $k$-cycle system of order $v$. Both BIBD and cycle system have been utilized in designing experiments with very high efficiency. Therefore, it is interesting to study the graphs which have a $C_k$-decomposition and also the graphs which have no $C_k$-decomposition. We start this thesis with some preliminaries of graph theory.

## 1.1 The Preliminaries in Graph Theory

In this section, we first introduce the terminologies and definitions of graphs. For details, the readers may refer to the book "Introduction to Graph Theory".[10]

A graph $G$ is consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. A *loop* is an edge whose endpoints are equal. *Multiedges* are edges having the same pair of endpoints. A *simple graph* is a graph without loops or multiedges. In this thesis, all the graphs we consider are simple. The size of the vertex set $V(G)$, $|V(G)|$, is called the *order* of $G$. And the size of the edge set $E(G)$, $|E(G)|$, is called the *size* of $G$.

If $e = (u, v)$ is an edge of $G$, then $e$ is said to be *incident* to $u$ and $v$. We also say that $u$ and $v$ are *adjacent* to each other. For every $v \in V(G)$, $N(v)$ denotes the neighborhood of $v$, that is, all vertices of $N(v)$ are adjacent to $v$. The *degree* of $v$, $deg(v) = |N(v)|$, is the number of neighborhood of $v$. We denote that $\delta(G)$ is the *minimum degree* of $G$ and $\Delta(G)$ is the *maximum degree* of $G$.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A *k-cycle*, $C_k$, is a cycle of size $k$. A *Hamiltonian graph* is a graph with a spanning cycle, also called a *Hamiltonian cycle* which is denoted by $C_n$ where $n$ is the order of the graph.

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with $n$ vertices is denoted by $K_n$. A graph $G$ is *bipartite* if $V(G)$ is the union of two disjoint independent sets called partite sets of $G$. A graph $G$ is *q-partite* if $V(G)$ can be expressed as the union of $q$ independent sets. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have the sizes $s$ and $t$, the complete bipartite graph is denoted by $K_{s,t}$. If the sets have the same size $n$, the complete bipartite graph is called *balanced*, which is denoted by $K_{n,n}$. Similarly, the complete $q$-partite graph is denoted by $K_{s_1,s_2,\ldots,s_q}$ and the balanced complete $q$-partite graph is denoted by $K_{q(n)}$ where each partite set has $n$ vertices.

An *even graph* is a graph whose degree of vertices are even, and an *odd graph* is a graph whose degree of vertices are odd. A graph is called *r-regular* if all its vertices have the same degree $r$. A *subgraph* of $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *factor* of $G$ is a spanning subgraph of $G$. A *k-factor* is a spanning $k$-regular subgraph. A *matching* of size $k$ in $G$ is a subgraph of $k$ pairwise disjoint edges. If a matching covers all vertice of $G$, then it is a *perfect matching* or *1-factor*.

A graph $G$ is *k-sufficient* if $G$ is an even graph of order at least $k$ and the size of $G$ is a multiple of $k$. A *$C_k$-decomposition* of $G$ is a collection of edge-disjoint $C_k$'s which partition $E(G)$. A graph $G$ is called *$C_k$-decomposable* if $G$ has a $C_k$-decomposition which is denoted by $C_k \mid G$; otherwise, $C_k \nmid G$.

## 1.2 Cycle Systems and Known Results On Cycle Decomposition

If $K_n$ has an $m$-cycle decomposition, i.e., $C_m \mid K_n$, then we refer to this decomposition as an $m$-cycle system of order $n$. The study of cycle system dated back to 1847, Kirkman proved the following result.

**Theorem 1.1.** [6] *A 3-cycle system of the complete graph of order* $n$ *exists if and only if* $n \equiv 1$ *or* $3 \pmod 6$.

Since then, the existence of a $k$-cycle system of order $n$ has attracted quite a few researchers to work on this interesting topic. The following results are well-known now.

**Theorem 1.2.** [9] *Let* $n$ *be an odd integer and* $m$ *be an even integer with* $3 \leqslant m \leqslant n$. *The graph* $K_n$ *can be decomposed into cycles of length* $m$ *whenever* $m$ *divides the number of edges in* $K_n$.

**Theorem 1.3.** [1] *For positive odd integers* $m$ *and* $n$ *with* $3 \leqslant m \leqslant n$, *the graph* $K_n$ *can be decomposed into cycles of length* $m$ *if and only if the number of edges in* $K_n$ *is a multiple of* $m$.

From above three theorems, we can see that the order of complete graph is all odd since the graph must be even. If $n$ is an even integer, then we consider the decomposition of $K_n - I$ where $I$ is a 1-factor of $K_n$.

**Theorem 1.4.** [9] *Let* $n$ *be an even integer and* $m$ *be an odd integer with* $3 \leqslant m \leqslant n$. *The graph* $K_n - I$ *can be decomposed into cycles of length* $m$ *whenever* $m$ *divides the number of edges in* $K_n - I$.

**Theorem 1.5.** [1] *For positive even integers* $m$ *and* $n$ *with* $4 \leqslant m \leqslant n$, *the graph* $K_n - I$ *can be decomposed into cycles of length* $m$ *if and only if the number of edges in* $K_n - I$ *is a multiple of* $m$.

Therefore, it is interesting to know whether $K_n - H$ can be decomposed into $k$-cycles where $H$ is a subgraph of $K_n$ such that $K_n - H$ is $k$-sufficient. The following results deal with the case when $H$ is a 2-regular or 3-regular subgraph.

**Theorem 1.6.** [5] *Let* $F$ *be a 2-regular subgraph of* $K_n$. *There exists a* $C_4$*-decomposition of* $K_n - E(F)$ *if and only if* $n$ *is odd and 4 divides* $|E(K_n) - E(F)|$.

**Theorem 1.7.** [2] *Let* $F$ *be a 2-regular subgraph of* $K_n$. *There exists a* $C_6$*-decomposition of* $K_n - E(F)$ *if and only if* $n$ *is odd and 6 divides* $|E(K_n) - E(F)|$.

**Theorem 1.8.** [8] *Let $U$ be any 2-factor of $K_n$, where n is even. Then there exists a 3-factor $T$ of $K_n$ with $E(U) \subset E(T)$ such that $K_n - E(T)$ admits a hamilton decomposition.*

If we decompose the other kind of graphs, not necessarily be complete graph, then we have different results.

**Theorem 1.9.** [7] *Let $F$ be a set of $q$ vertex-disjoint cycles with the length of the $j$-th cycle being $s_j$. Then there exists a 2-factor $U \cong F$ of $K_{m,m,m}$, such that $K_{m,m,m} - E(U)$ has a hamilton decomposition if and only if $\sum_{j=1}^{q} s_j = 3m$.*

**Theorem 1.10.** [3] *There exists a maximal set $S$ of $m$ edge-disjoint Hamilton cycles in $K_{n,n}$ if and only if $n/4 < m \leqslant n/2$.*

**Theorem 1.11.** [4] *There exists a maximal set of $m$ hamilton cycles in $K_{n(p)}$, if and only if,*

*1.* $\lceil n(p-1)/4 \rceil \leqslant m \leqslant \lfloor n(p-1)/2 \rfloor$ *and*

*2.* $m > n(p-1)/4$ *if*

*(i)* $n$ *is odd and* $p \equiv 1 \pmod 4$, *or*

*(ii)* $p = 2$, *or*

*(iii)* $n = 1$,

*except possibly if $n = 2m$ and except possibly if $n \geqslant 3$ is odd, $p$ is odd, and $m \leqslant ((n+1)(p-1)-2)/4$.*

On these results, we can see the degree of these regular graphs are larger than $\frac{n}{2}$. It seems that if the degree of a graph $G$ is large enough, then we can decompose $G$ into $k$-cycles as long as the graph is $k$-sufficient. Thus, we are interesting in finding the number $r$ such that an arbitrary $k$-sufficient $r$-regular graph which has a $C_k$-decomposition. For $k = 3$, the following conjecture by Nash-Williams is worth of mentioning first.

**Conjecture**(Nash-Williams). Let $H$ be a subgraph of $K_n (n \neq 9)$ such that $K_n - H$ is 3-sufficient and $\Delta(H) \leqslant \frac{1}{4}(n-1)$. Then $C_3 \mid K_n - H$.

This conjecture is far from being proved at this moment. But, this upper bound on $H$ or equivalently the lower bound on $K_n - H$ plays an important role in decomposition problems. We shall first focus on the situation when $K_n - H$ or the graph $G$ we consider is $r$-regular and $k$-sufficient but $G$ is not able to be decomposed into $k$-cycles even if $G$ is $k$-sufficient. Of course, we are looking for $r$ which is as large as possible. In next section, we shall consider the $r$ with respect to the order of $G$ and show that if $r$ is not large enough, then an arbitrary $k$-sufficient $r$-regular graph does not have a $C_k$-decomposition.

## 2 Lower Bound of degree $r$

Let $G$ be an arbitrary $k$-sufficient $r$-regular graph. It is not difficult to realize that to determine whether $G$ can be decomposed into $k$-cycles or not is not an easy task. Thus, we are interesting in the situation when $G$ is $k$-sufficient and $r$-regular but $G$ has no $C_k$-decomposition. Clearly, we looking for the number "$r$" as large as possible. First, we introduce a couple of definitions.

**Definition 2.1.** Let $G$ be a graph of order $n$ with $V(G) = \{v_0, v_1, ..., v_{n-1}\}$. Given a bijection function $f : V(G) \to \{0, 1, ..., n-1\}$ such that $f(v_i) = i$, $0 \leqslant i \leqslant n-1$. Define the difference of $v_i$ and $v_j$ by $d(i, j) = min\{|j-i|,\ n-|j-i|\}$. A graph $G$ of order $n$ is a difference graph $G[D]$ if $D \subseteq \{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$ and $E(G) = \{(i, i+k) \pmod n \mid for\ all\ k \in D\}$.

**Definition 2.2.** A graph $G$ is a $q$-partite-$K_m$ graph with a difference set $D \subseteq \{1, 2, .., \lfloor \frac{q}{2} \rfloor\}$ if there are $q$ partites $G_0, G_1, ..., G_{q-1}$ in $G$, where each partite $G_i$, $0 \leqslant i \leqslant q-1$, is a complete graph of order $m$. If there are edges between $G_i$ and $G_j$, $0 \leqslant i < j \leqslant q-1$, then the edges between $G_i$ and $G_j$ denoted by $E(G_i, G_j)$ induces a complete bipartite graph $K_{m,m}$. So $V(G) = \bigcup_{i=0}^{q-1} V(G_i)$ and $E(G) = \bigcup_{i=0}^{q-1} E(G_i) \cup \bigcup E(G_i, G_j)$ where $d(i, j) = \{k \mid \forall\ k \in D\}$. The $q$-partite-$K_m$ graph is denoted $G_{q(K_m)}$ with $D \subseteq \{1, 2, .., \lfloor \frac{q}{2} \rfloor\}$. Moreover, let $E_1(G) = \bigcup_{i=0}^{q-1} E(G_i)$ and $E_2(G) = E(G) \setminus E_1(G)$.

Example, let $G$ denote the difference graph $G[D]$. Then, the graph $H_1$ given by $G_{4(K_m)}$ with $D = \{1\}$ and the graph $H_2$ given by $G_{6(K_m)}$ with $D = \{1, 2\}$ are 4-partite-$K_m$ graph and 6-partite-$K_m$ graph respectively. See Figure 1 and Figure 2 as illustration.

**Lemma 2.3.** *If $G$ is a difference graph $G[D]$ of even order $n$, where $D = \{i \mid i\ is\ odd\}$, then $G$ contains no odd cycle.*

**Proof.** In a difference graph, a cycle can be formed by two ways. First, the sum of the difference of the edges in cycle is a multiple of the order of $G$. Second, the sum of the difference of some edges in cycle is equal to the sum of the difference of others.

Now, $D = \{i \mid i\ is\ odd\}$ and $n$ is even. Since the odd sum of odd integers is not an even integer, the proof follows. ∎

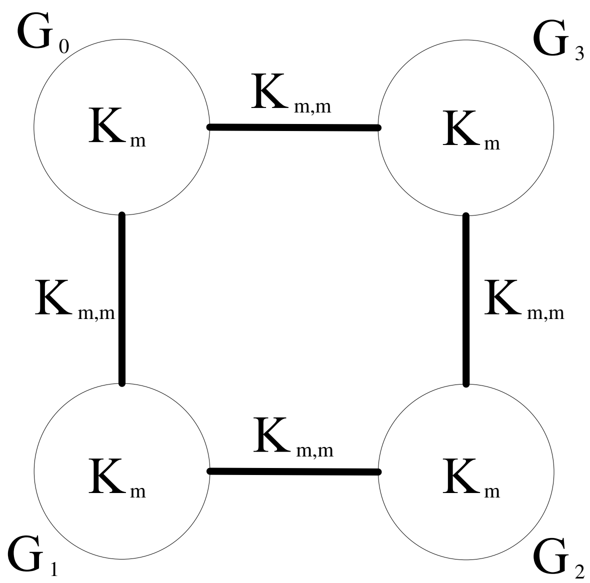


Figure 1: $G_{4(K_m)}$ with $D = \{1\}$.

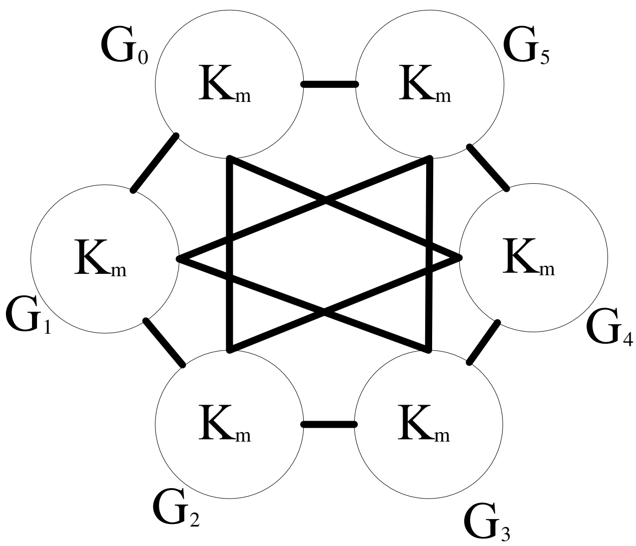


Figure 2: Each edge is $K_{m,m}$. This graph is $G_{6(K_m)}$ with $D = \{1, 2\}$.

From this fact, the following result is easy to see.

**Corollary 2.4.** *If $G = G_{2q(K_m)}$ with $D = \{i \mid i \text{ is odd}\}$, then $E_2(G)$ contains no odd cycle.*

**Lemma 2.5.** *Consider $G = G_{q(K_m)}$ and suppose $E_2(G)$ contains no $C_{2k+1}$. If $2k \times |E_1(G)| < |E_2(G)|$, then $G$ is not $C_{2k+1}$-decomposable.*

**Proof.** Suppose $G$ is $C_{2k+1}$-decomposable. Since $E_2(G)$ contains no $C_{2k+1}$, we must use at least one edge in $E_1(G)$ and at most $2k$ edges in $E_2(G)$ to form a $C_{2k+1}$. Thus, $2k \times |E_1(G)| \geqslant |E_2(G)|$, a contradiction. ∎

Now, we start the constructions with odd cycles decomposition.

**Proposition 2.6.** There is a family of $(2k+1)$-sufficient $r$-regular graphs of order $n$ which have no $C_{2k+1}$-decomposition, where $r = \frac{2k+1}{4k}n - 1$.

**Proof.** Let $G = G_{4k(K_m)}$ with $D = \{i \mid i \text{ is odd}\}$, where $m = (2k+1)(2t+1)$ for any nonnegative integer $t$. For example $k = 1$, $G$ is given by Figure 1

First, we claim that $G$ is $(2k+1)$-sufficient. Since for all $v \in V(G)$, $deg(v) = (m-1) + m \times 2k \equiv 0 \pmod 2$, so $G$ is an even graph. Since $|E(G)| = |E_1(G)| + |E_2(G)| = \frac{m(m-1)}{2} \times 4k + m^2 \times k \times 4k \equiv 0 \pmod m \equiv 0 \pmod{2k+1}$, so the size of $G$ is a multiple of $2k+1$.

Next, $E_2(G)$ contains no $C_{2k+1}$ by Corollary 2.4. And $G$ is not $C_{2k+1}$-decomposable since $2k \times |E_2(G)| = 2k \times \frac{m(m-1)}{2} \times 4k = 4k^2 m(m-1) < 4k^2 m^2 = |E_2(G)|$ (by Lemma 2.5).

Hence $G$ is a $(2k+1)$-sufficient $r$-regular graph which has no $C_{2k+1}$-decomposition, where $r = deg(v) = (2k+1)m - 1 = \frac{2k+1}{4k}n - 1$. ∎

**Corollary 2.7.** *If every $(2k+1)$-sufficient $r$-regular graphs of order $n$ have $C_{2k+1}$-decompositions, then $r$ has to be at least $\frac{2k+1}{4k}n$.*

**Proof.** By the direct construction of Proposition 2.6, there is a family of $(2k+1)$-sufficient $r$-regular graphs of order $n$ which have no $C_{2k+1}$-decomposition where $r = \frac{2k+1}{4k}n - 1$. So if we want to decompose every $(2k+1)$-sufficient $r$-regular graphs of order $n$, then $r$ has to be at least $\frac{2k+1}{4k}n$. ∎

Besides the construction of Proposition 2.6, there are another two family of graphs which satisfy such conditions.

First, let $H$ be a balanced complete bipartite graph of order $4t$. Consider $G$ is a graph of order $4t$ where $V(G) = V(H)$ and $E(G) = E(H) \cup \bigcup_i E(C_i)$ where $C_i$ belongs to partite set. We can choose these $C_i$ properly, such that the minimum degree of $G$ is as large as possible and $G$ is $(2k+1)$-sufficient, but $\sum_i |E(C_i)| < \frac{4t^2}{2k}$. Then $G$ is not $C_{2k+1}$-decomposable by a similar idea of Lemma 2.5.

Second, let $G$ be a difference graph $G[D]$ of even order. Choose $D = A \cup B$ where $A = \{i \mid i \ is \ odd\}$ and $B \subset \{j \mid j \ is \ even\}$, but $|A| < \frac{|B|}{2k}$. Then $G$ is not $C_{2k+1}$-decomposable by a similar idea of Lemma 2.5.

**Proposition 2.8.** There is a family of 4-sufficient $r$-regular graphs of order $n$ which have no $C_4$-decomposition, where $r = \frac{3}{5}n - 1$.

**Proof.** Let $G = G_{5(K_m)}$ with $D = \{1\}$, where $m = 8t + 3$ for any nonnegative integer $t$. Clearly, for all $v \in V(G)$, $deg(v) = (m-1) + 2m \equiv 0 \pmod 2$, and $|E(G)| = \frac{m(m-1)}{2} \times 5 + m^2 \times 5 = 480t^2 + 340t + 60 \equiv 0 \pmod 4$ . Thus, $G$ is 4-sufficient.

For all $i$, $0 \leqslant i \leqslant 4$, the size of $E(G_i, G_{i+1})$ is odd, but it is impossible to use an odd number of edges in $E(G_i, G_{i+1})$ and some edges in $E_1(G)$ to form a $C_4$. Thus, $G$ is not $C_4$-decomposable.

Hence $G$ is a 4-sufficient $r$-regular graph which has no $C_4$-decomposition, where $r = deg(v) = 3m - 1 = \frac{3}{5}n - 1$. ∎

**Corollary 2.9.** *If every 4-sufficient $r$-regular graphs of order $n$ have $C_4$-decompositions, then $r$ has to be at least $\frac{3}{5}n$.*

For even cycles, the lower bound we obtain is not as good as those we found for odd cycles.

**Proposition 2.10.** There is a family of $2k$-sufficient $r$-regular graphs of order $n$ which have no $C_{2k}$-decomposition, where $r = \frac{n}{2} - 1$.

**Proof.** Let $G_1$ and $G_2$ be the complete graphs of order $4kt+1$ respectively for any nonnegative integer $t$. Let $G$ be the graph of order $2(4kt+1)$ where $V(G) = V(G_1) \cup V(G_2)$. Choose $E(G) = E(G_1) \cup E(G_2) \cup \{(x_1, y_1), (x_2, y_2)\} \setminus \{(x_1, x_2), (y_1, y_2)\}$ where $x_1,\ x_2 \in V(G_1)$ and $y_1,\ y_2 \in V(G_2)$. (see Figure 3)

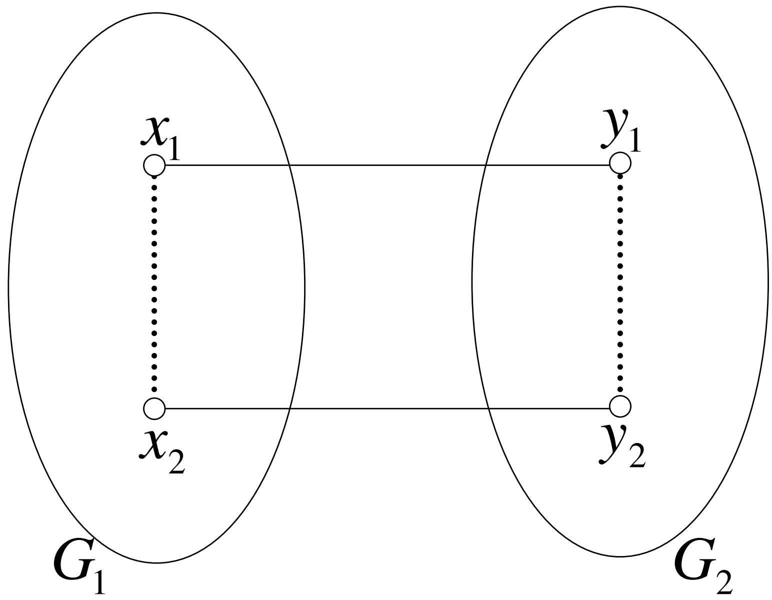


Figure 3: $2k$-sufficient $r$-regular graph which has no $C_{2k}$-decomposition.



For all $v \in V(G)$, $deg(v) = (4kt+1) - 1 \equiv 0 \pmod 2$. $|E(G)| = (4kt+1) \times 4kt \equiv 0 \pmod{2k}$. So $G$ is $2k$-sufficient.

Suppose that $G$ is $C_{2k}$-decomposable, then $(x_1, y_1)$ and $(x_2, y_2)$ belong to a $C_{2k}$ in $G$. Next, we want to use $(x_1, y_1)$, $(x_2, y_2)$, $q$ edges in $E(G_1) \setminus (x_1, x_2)$ where $1 \leqslant q \leqslant 2k-3$, and $2k-2-q$ edges in $E(G_2) \setminus (y_1, y_2)$ to form a $C_{2k}$. Since $G$ is $C_{2k}$-decomposable and $|E(G_1) \setminus (x_1, x_2)| = |E(G_2) \setminus (y_1, y_2)| = \frac{(4kt+1)\times 4kt}{2} - 1 \equiv -1 \pmod{2k} \equiv 2k-1 \pmod{2k}$, so we must choose $q$ edges properly such that $q \equiv 2k-2-q \equiv 2k-1 \pmod{2k}$, it is a contradiction to $1 \leqslant q \leqslant 2k-3$. Thus, $G$ is not $C_{2k}$-decomposable.

Hence $G$ is a $2k$-sufficient $r$-regular graph which has no $C_{2k}$-decomposition, where $r = deg(v) = (4kt+1) - 1 = \frac{n}{2} - 1$. ∎

**Corollary 2.11.** *If every $2k$-sufficient $r$-regular graphs of order $n$ have $C_{2k}$-decompositions, then $r$ has to be at least $\frac{n}{2}$.*

**Proof.** This follows immediately from Proposition 2.10. ∎

**Corollary 2.12.** *If every $n$-sufficient $r$-regular graphs of order $n$ have $C_n$-decompositions, then $r$ has to be at least $\frac{n}{2}$.*

**Proof.** The construction of this proof is the same as Proposition 2.10 except the order of $G_1$ and $G_2$ are $2t+1$ for any nonnegative integer $t$. ∎

So, we conclude this section with a table for "$r$" in which $G$ is an arbitrary $r$-regular $k$-sufficient graph but $G$ has no $C_k$-decomposition.

Table 1: The lower bound of $r$.

| | $C_3$ | $C_4$ | $C_5$ | $C_6$ | ... | $C_k$, $k$ is odd | $C_k$, $k$ is even | $C_n$ |
|---|---|---|---|---|---|---|---|---|
| r | $\frac{3}{4}n-1$ | $\frac{3}{5}n-1$ | $\frac{5}{8}n-1$ | $\frac{n}{2}-1$ | ... | $\frac{k}{2(k-1)}n-1$ | $\frac{n}{2}-1$ | $\frac{n}{2}-1$ |

# 3 Lower Bound of $ex(n\ ;\ C_k$-$decomp.)$

Let $F$ be a given graph. Then we define $ex(n;F) = max\{|E(G)| \mid |V(G)| = n,\ but\ G$ *contains no subgraph which induces* $F\}$. We call the graph $G$ of order $n$ an extremal graph of $F$ if $G$ contains no subgraph which induces $F$ and $|E(G)| = ex(n;F)$. In this section, we will study a new topic "extremal graph of $C_k$-decomposition."

**Definition 3.1.** We define $ex(n;\ C_k\text{-}decomp.) = max\{|E(G)| \mid |V(G)| = n,\ G\ is\ k-sufficient,\ but\ C_k \nmid G\}$. We call the graph $G$ of order $n$ an extremal graph of $C_k$-decomposition if $G$ satisfies the followings: $G$ is $k$-sufficient, $G$ is not $C_k$-decomposable, and $|E(G)| = ex(n;\ C_k\text{-}decomp.)$.

In what follows, we obtain the lower bound of $ex(n;\ C_k\text{-}decomp.)$.

**Lemma 3.2.** *Let $G$ be a $k$-sufficient graph. If there is an edge $e$ in $G$, but $e$ does not lie in any $k$-cycle in $G$, then $G$ is not $C_k$-decomposable.*

Although the idea of Lemma 3.2 is very simple, it is very useful in proving the following results.

**Theorem 3.3.** *If $n$ is even, then $ex(n;C_3\text{-}decomp.) \geqslant \binom{n}{2} - (n-2) - \epsilon_n$ where*

$$\begin{cases} \epsilon_n = 5 & if\ n \equiv 0 \pmod 6 \\ \epsilon_n = 4 & if\ n \equiv 2\ or\ 4 \pmod 6;\ and \end{cases}$$

*if $n$ is odd, then $ex(n;C_3\text{-}decomp.) \geqslant \binom{n-2}{2} - \epsilon_n$ where*

$$\begin{cases} \epsilon_n = 4 & if\ n \equiv 1 \pmod 6 \\ \epsilon_n = 0 & if\ n \equiv 3\ or\ 5 \pmod 6. \end{cases}$$

**Proof.** Let $H$ be the complete graph of order $n-2$. Suppose $V(H) = \{v_0, v_1, ..., v_{n-3}\}$, and choose $V(H_1) = \{v_0, v_1, ..., v_{\frac{n}{2}-2}\}$ and $V(H_2) = V(H) \setminus V(H_1)$. If $n$ is even, let $G$ be the graph of order $n$ where $V(G) = V(H) \cup \{x, y\}$ and $E(G) = E(H) \cup (x,y) \cup \bigcup_{u \in V(H_1)}(x,u) \cup \bigcup_{v \in V(H_2)}(y,v) \setminus E(C_{\epsilon_n})$ where $E(C_{\epsilon_n}) \subseteq E(H)$. Choose $\epsilon_n = 5$ if $n \equiv 0 \pmod 6$, and $\epsilon_n = 4$ if $n \equiv 2\ or\ 4 \pmod 6$(see Figure 4). If $n$ is odd, let $G$ be the graph of order $n$ where $V(G) = V(H) \cup \{x, y\}$ and $E(G) = E(H) \cup \{(x,z_1),(x,z_2),(y,z_1),(y,z_2)\} \setminus E(C_{\epsilon_n})$ where $E(C_{\epsilon_n}) \subseteq E(H)$ and $z_1,\ z_2 \in V(H)$. Choose $\epsilon_n = 4$ if $n \equiv 1 \pmod 6$, and

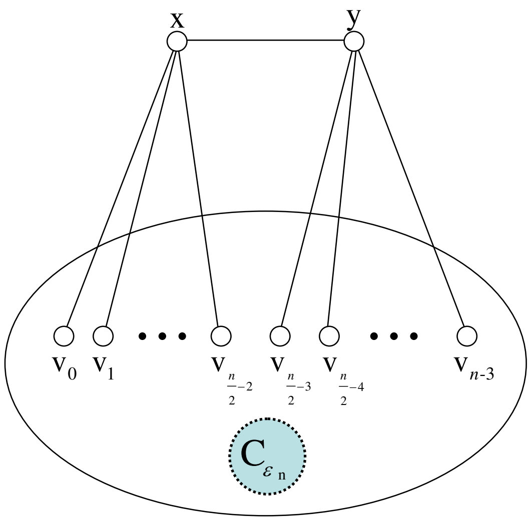


Figure 4: $\epsilon_n = 4$ or 5 if $n$ is even.

$\epsilon_n = 0$ if $n \equiv 3$ *or* $5 \pmod 6$ (see Figure 5). Now, we delete a $C_{\epsilon_n}$ from $K_{n-2}$ to make the graph 3-sufficient.

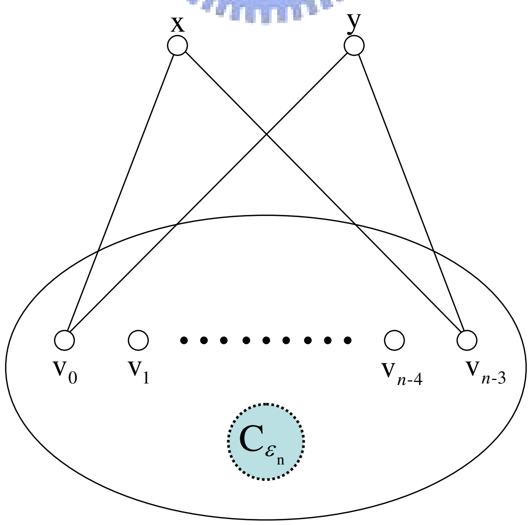


Figure 5: $\epsilon_n = 4$ or 0 if $n$ is odd.

Since $(x, y)$ does not lie in any 3-cycle in $G$, then $G$ is not $C_3$-decomposable by Lemma 3.2. So $G$ is a 3-sufficient graph $G$ with $|E(G)| = \binom{n}{2} - 2 \times \frac{n-2}{2} - \epsilon_n = \binom{n}{2} - (n-2) - \epsilon_n$ if $n$ is even, and $|E(G)| = \binom{n-2}{2} - \epsilon_n$ if $n$ is odd, but $G$ has no $C_3$-decomposition. Hence $ex(n; C_3\text{-}decomp.) \geqslant \binom{n}{2} - (n-2) - \epsilon_n$ if $n$ is even, and $ex(n; C_3\text{-}decomp.) \geqslant \binom{n-2}{2} - \epsilon_n$ if $n$ is odd. ∎

**Theorem 3.4.** *If $n$ is odd, then $ex(n; C_4\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \epsilon_n$ where*

$$\begin{cases} \epsilon_n = 0 & if\ n \equiv 1 \pmod 8 \\ \epsilon_n = 3 & if\ n \equiv 3 \pmod 8 \\ \epsilon_n = 6 & if\ n \equiv 5 \pmod 8 \\ \epsilon_n = 5 & if\ n \equiv 7 \pmod 8; \end{cases}$$

*if $n$ is even, then $ex(n; C_4\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - 3$.*

**Proof.** Let $H$ be the complete graph of order $n-2$. If $n$ is odd, let $G$ be the graph of order $n$ where $V(G) = V(H) \cup \{x, y\}$ and $E(G) = E(H) \cup \{(x, y), (y, z), (z, x)\} \setminus E(C_{\epsilon_n})$ where $z \in V(H)$, $E(C_{\epsilon_n}) \subseteq E(H)$ (see Figure 6). Choose $\epsilon_n = 0\ if\ n \equiv 1 \pmod 8$, $\epsilon_n = 3\ if\ n \equiv 3 \pmod 8$, $\epsilon_n = 6\ if\ n \equiv 5 \pmod 8$, and $\epsilon_n = 5\ if\ n \equiv 7 \pmod 8$. If $n$ is even, let $G$ be the graph of order $n$ where $V(G) = V(H) \cup \{x, y\}$ and $E(G) = E(H) \cup \{(x, y), (y, z), (z, x)\} \setminus \{F \cup E(C_{\epsilon_n})\}$ where $z \in V(H)$, $F$ is a perfect matching of $H$, and $E(C_{\epsilon_n}) \subseteq E(H)$. Choose $\epsilon_n = 3\ if\ n\ is\ even$. (see Figure 7) Finally, we delete a $C_{\epsilon_n}$ to make the graph $G$ 4-sufficient.

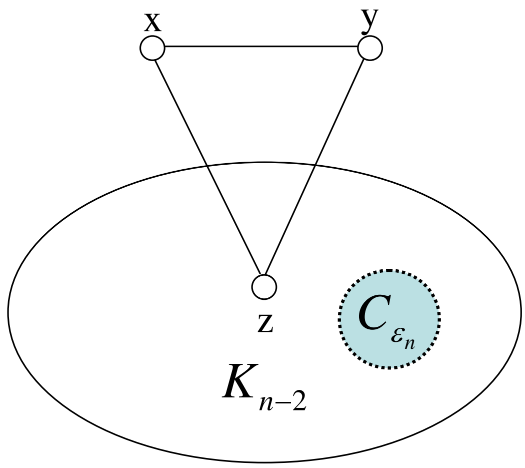


Figure 6: $K_{n-2}$ may minus a $C_{\epsilon_n}$.

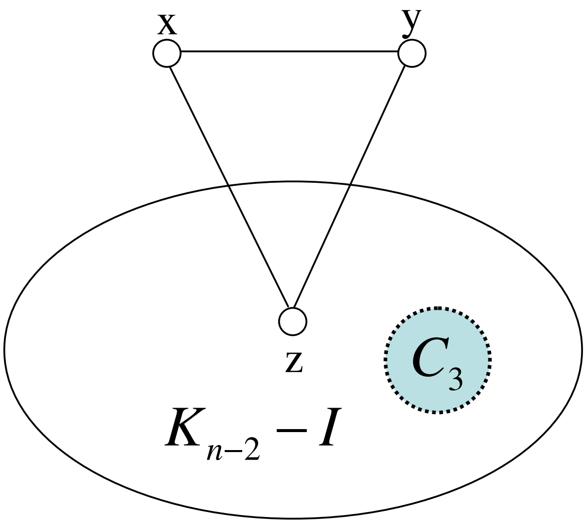


Figure 7: $K_{n-2} - I$ minus a $C_3$.

Since $(x, y)$ does not lie in any 4-cycle in $G$, then $G$ is not $C_4$-decomposable by Lemma 3.2. So $G$ is a 4-sufficient graph $G$ with $|E(G)| = \binom{n}{2} - 2(n-3) - \epsilon_n$ if $n$ is odd, and $|E(G)| = \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - 3$ if $n$ is even,but $G$ has no $C_k$-decomposition. Hence $ex(n; C_4\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \epsilon_n$ if $n$ is odd, and $ex(n; C_4\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - 3$ if $n$ is even. ∎

Similar to the constructions of Theorem 3.4, we can also construct graphs which can not be decomposed into $C_k$, $k \geqslant 5$. Therefore, a lower bound for $ex(n; C_k\text{-}decomp.)$ we have.

**Theorem 3.5.** *For $k \geqslant 5$, if $n$ is odd, $ex(n; C_k\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \epsilon_{k,n}$, where $\epsilon_{k,n} \in \{0, 3, 4, 5, ..., k-1, k+1, k+2\}$, such that the size of the graph is a multiple of $k$. If $n$ is even, $ex(n; C_k\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - \epsilon_{k,n}$, where $\epsilon_{k,n} \in \{0, 3, 4, 5, ..., k-1, k+1, k+2\}$, such that the size of the graph is a multiple of $k$.*

Clearly, the above construction also works for the case on $C_n$-decomposition.

**Theorem 3.6.** *For $n \geqslant 9$, if $n$ is odd $ex(n; C_n\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - 6$. If $n$ is even $ex(n; C_n\text{-}decomp.) \geqslant \binom{n}{2} - 2(n-3) - \frac{n-2}{2} - 7$.*

To summarize this section, we use the following two tables to depict the study of this topic.

Table 2: The lower bound of $ex(n; C_k\text{-}decomp.)$ if $n$ is odd.

| | $C_3$ | $C_k, k \geqslant 4$ | $C_n$ |
|---|---|---|---|
| $ex(n;\, C_k\text{-}decomp.)$ | $\binom{n}{2} - (n+2) - \epsilon_n$ | $\binom{n}{2} - 2(n-3) - \epsilon_{k,n}$ | $\binom{n}{2} - 2(n-3) - 6$ |

Table 3: The lower bound of $ex(n; C_k\text{-}decomp.)$ if $n$ is even.

| | $C_3$ | $C_k, k \geqslant 4$ | $C_n$ |
|---|---|---|---|
| $ex(n;\, C_k\text{-}decomp.)$ | $\binom{n-2}{2} - \epsilon_n$ | $\binom{n}{2} - 2(n-3) - \frac{n-2}{2} - \epsilon_{k,n}$ | $\binom{n}{2} - 2(n-3) - 7$ |

# 4 Conclusion

From the results obtained in thesis, we have quite a few examples of showing a $C_k$-decomposition is not possible. But, for those graphs, say with large $r$ in degree or with large size, it is not known whether we can decompose them into $k$-cycles. We shall work on those decompositions in the future. If possible, we would like to prove that the bounds are sharp, especially those bounds on sizes.

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