國立交通大學

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碩士論文

On The Largest Eigenvalues

of Bipartite Graphs

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中華民國九十四年六月

二分圖其特徵值的上界

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摘　　要

令 G 為一個圖且 A(G)為 G 的相鄰矩陣。G 的特徵多項式記作 $P_G(x)$，其定義為(xI - A(G))該矩陣的特徵值，其中 I 是單位矩陣。而圖的相鄰矩陣所對應的特徵值視為該圖的特徵值。在本篇論文中，我們將討論二分圖其最大的特徵值。主要而言，對於某幾個類別的二分圖的最大特徵值給一個上界。

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On the Largest Eigenvalues of Bipartite Graphs

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Abstract

Let G be a graph and A(G) be the adjacency matrix of G. The characteristic polynomial of G, denoted by $P_G(x)$, is det (xI - A(G)) where I is the identity matrix. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. In this thesis, we study the largest eigenvalue of bipartite graphs. Mainly, an upper bound for the largest eigenvalues of certain families of bipartite graphs is obtained.

Contents

1 Introduction and Preliminaries

1.1 Motivation

Just as astronomers study stellar spectra to determine the make-up of distant stars, one of the main goal in graph theory is to deduce the principal properties and structure of a graph from a short list of easily computable invariants. The spectral approach, i.e., using the graph spectrum for general graphs is a step in this direction. Indeed, eigenvalues are closely related to almost all major invariants of a graph, linking one extremal property to the other. There is no question that eigenvalues play a central role in understanding a graph, especially starting from the knowledge of discovering the largest eigenvalues of a graph.

Bipartite graphs are known to be a class of most beautiful and useful graphs in graph theory. A tree is a subclass of bipartite graphs. Also, bipartite graphs do have several good properties related to their spectrum. Therefore, it is interested to study their largest eigenvalues.

1.2 Preliminaries

We start with an introduction of graph terms we use in this thesis. Let G = (V,E) be an undirected finite graph without loops or multiple edges, where V is the vertex set of G and E = $\{(x, y) \mid x, y \in V, x \neq y\}$ is the edge set. A graph G = (V,E) is bipartite if V is the union of two disjoint sets such that each edge consists of one vertex from each set. A complete bipartite graph is a bipartite graph whose edge set consists of all pairs having a vertex from each of two disjoint sets covering the vertices. Let $K_{r,s}$ denote the complete bipartite graph with partite sets of sizes r and s. The diameter of a graph G is the largest distance between vertices in it. Let

$\delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G.. The adjacency matrix A(G) of a graph G is a symmetric (0,1) – matrix determined by G with rows and columns indexed by the vertices of G and with entries

$$A(G)_{xy} = \begin{cases} 1, & \text{if x and y are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of G is denoted by $P_G(x)$, i.e.,

$P_G(x)$ = det (xI - A(G)),

where I is the identity matrix. The eigenvalues of a graph are the eigenvalues of it's adjacency matrix. If G is a graph with n vertices, then the eigenvalues of G are the zeros $\lambda_1(G), \cdots, \lambda_n(G)$ of the characteristic polynomial $P_G(x)$ = det (xI - A(G)) $= \prod (x - \lambda_i(G))$. The spectrum is the list of distinct eigenvalues with their respective multiplicities $m_1, \cdots, m_t$; we write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_t \\ m_1 & \cdots & m_t \end{pmatrix}.$$

Since A(G) is symmetric, all of its eigenvalues are real. We assume, without loss of generality, that they are ordered in decreasing order, i.e.,

$\lambda_1(G) \geq \cdots \geq \lambda_n(G)$.

If it is clear which graph is under consideration, we write $\lambda_1 \geq \cdots \geq \lambda_n$ in short. We also refer to $\lambda_1(G)$ and $\lambda_n(G)$ as $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$, respectively.

The basic definitions and facts about the spectra of graphs are given together with a description of some general graph theoretic notions and necessary facts from matrix theory. Before starting our study, we give some preliminary results of matrix theory.

Lemma 1.2.1. [1, Lemma 8.6.9] *If $f(x) = x^T Ax$, where A is a real symmetric matrix, then f attains its maximum and minimum over unit vectors x at eigenvectors of A, where it equals the corresponding.*

Lemma 1.2.2. [1, Theorem 8.6.10] *A real symmetric $n \times n$ matrix has real eigenvalues and n orthonormal eigenvectors.*

Lemma 1.2.3. [2, Theorem 0.1] *The geometric and algebraic multiplicities of an eigenvalue of a symmetric matrix are equal.*

Lemma 1.2.4. [2, Theorem 0.10] *Let A be a real matrix with eigenvalues* $\lambda_1, \cdots, \lambda_n$ *and B be one of its principal submatrices; let B have eigenvalues* $\mu_1, \cdots, \mu_n$. *Then the inequalities* $\lambda_{n-m+1} \leq \mu_i \leq \lambda_i$ *(i = 1, … , m) hold.*

The above inequalities are known as Cauchy's inequalities and the whole lemma is also known as interlacing theorem.

Relating the eigenvalues to other graph parameters requires several results from linear algebra, including the Spectral Theorem and Cayley-Hamilton Theorem for real symmetric matrices. The following lemmas connect the matrix theory with spectra graph theory.

Lemma 1.2.5. [1, Proposition 8.6.6] *The (i,j)th entry of* A^k*counts the number of* v_i, v_j*-walks of length k. The eigenvalues of* A^k *are* $(\lambda_i)^k$.

Lemma 1.2.6. [1, Theorem 8.6.14] *The diameter of a graph is less than its number of distinct eigenvalues.*

Lemma 1.2.7. [1, Lemma 8.6.15] *If H is an induced subgraph of G, then* $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$.

Lemma 1.2.8. [1, Lemma 8.6.16] *For every graph G,* $\delta(G) \leq \lambda_{\max}(G) \leq \Delta(G)$.

Lemma 1.2.9. [1, Theorem 8.6.17] *For every graph G,* $\chi(G) \leq 1 + \lambda_{\max}(G)$.

Next, we give some preliminary results of spectral graph theory for bipartite graphs. They play an important role in the proof of our main results.

Lemma 1.2.10. [1, Lemma 8.6.7] *If G is bipartite and* λ *is an eigenvalue of G with multiplicity m, then* – λ *is also an eigenvalue with multiplicity m.*

Lemma 1.2.11. [1, Theorem 8.6.8] *The followings are equivalent statements about a graph G.*
(A) G is bipartite.
(B) The eigenvalues of G occur in pairs λ_i, λ_j *such that* $\lambda_i = \lambda_j$.
(C) $P_G(x)$ *is a polynomial in* x^2.
(D) $\sum_{i=1}^{n} (\lambda_i)^{2t-1} = 0$ *for any positive integer t where* $n = |V(G)|$.

Lemma 1.2.12. [1, Example 8.6.3] $Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$.

Theorem 1.2.13. *Let G be a bipartite graph with partite sets* $U = \{u_1, u_2, ..., u_p\}$ *and* $V = \{v_1, v_2, ..., v_q\}$. *Let the adjacency matrix of G take the form*

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where B is a $p \times q$ *matrix and* B^T *is the transpose of B. If v =* $[x \quad y]^T$ *is an eigenvector of A(G) for eigenvalye* λ, *where* $x^T = [x_1, x_2, ..., x_p]$, $y^T = [y_1, y_2, ..., y_q]$ *then for each i and j*

(a) $\lambda x_i = \sum_{l=1}^{q} B_{il} \cdot y_l$; $\lambda y_j = \sum_{k=1}^{p} B_{kj} \cdot x_k$.

(b) $\lambda^2 x_i = \sum_{k=1}^{p} x_k \mid N(u_i) \cap N(u_k) \mid$; $\lambda^2 y_j = \sum_{l=1}^{q} y_l \mid N(v_j) \cap N(v_l) \mid$.

Proof. (a) Since v is an eigenvector for eigenvalue λ, we compute

$$\lambda v = Av = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \quad By = \lambda x, \quad B^T x = \lambda y.$$

Consider the ith element of By and jth element of $B^T x$, we complete the proof.

(b) From part (a), we have

$$\lambda^2 x_i = \sum_{l=1}^{q} B_{il} \cdot (\lambda y_l) = \sum_{l=1}^{q} B_{il} \cdot (\sum_{k=1}^{p} B_{kl} \cdot x_k) = \sum_{k=1}^{p} x_k \cdot (\sum_{l=1}^{q} B_{il} \cdot B_{kl})$$

$$= \sum_{l=1}^{p} x_k \cdot |\{l \mid B_{il} = B_{kl} = 1\}| = \sum_{k=1}^{p} x_k \cdot |\{l \mid u_i \sim v_l, u_k \sim v_l\}|$$

$$= \sum_{k=1}^{p} x_x \cdot | N(u_i) \cap N(u_k) |.$$

Similarly, $\lambda^2 y_j = \sum_{l=1}^{q} y_l \mid N(v_j) \cap N(v_l) \mid$. ■

1.3 Known results

The study of upper bounds of eigenvalues of trees occurred in [3]-[5]. In [5], it lists the first largest to the seventh largest eigenvalue of trees, and lists the trees which attain these bounds. The main tool of [3]-[5] is the idea of partial eigenvectors.

In this thesis we find the largest eigenvalue of specific subsets of bipartite graphs which have the same partite sets, vertex number and edge number under consideration (see Theorem 2.2.1). In the last part, we list the the first largest eigenvalue to the eleventh largest eigenvalue of bipartite graphs of order 2n.

2 The main results

2.1 Essential Lemmas

In order to prove our main results, the following lemmas are essential. For completeness, we give a proof here.

Lemma 2.1.1. *If G is a graph with largest eigenvalue* λ*, then there exists a unit eigenvector* $x = [x_1, x_2, ..., x_n]^T$, *such that* $x_i \geq 0, \forall 1 \leq i \leq n,$ *where* $n = |V(G)|$.
$x_i = |y_i|$.

Proof. Let $y = [y_1, y_2, ..., y_n]^T$ be a unit eigenvector for λ. Define $x = [x_1, ..., x_n]^T$ by $x_i = |y_i|$. Then $y^T A(G) y = \lambda \geq x^T A(G) x,$ by Lemma 1.2.1. From the definition $x_i = |y_i|$ we have $y^T A(G) y \leq x^T A(G) x,$ so that $\lambda = x^T A(G) x.$ Hence, x is an eigenvector for λ by Lemma 1.2.1. ■

Lemma 2.1.2. *If H is a subgraph of G, then* $\lambda_1(H) \leq \lambda_1(G)$.

Proof. If H is a subgraph of G then there exists an induced subgraph I of G such that $V(H) = V(I)$. From Lemma 2.1.1 there exists a unit eigenvector $x = [x_1, x_2, ..., x_n]^T$ for eigenvalue $\lambda_1(H)$ such that $x_i \geq 0, \forall 1 \leq i \leq n,$ where $n = |V(H)|$. Since $H \subseteq I$, and $x_i \geq 0, \forall 1 \leq i \leq n,$ we have

$$\lambda_1(H) = x^T A(H) x = \sum_{A(H)_{ij}=1} x_i \cdot x_j \leq \sum_{A(I)_{ij}} x_i \cdot x_j = x^T A(I) x.$$

From Lemma 1.2.1 and Lemma 1.2.7, we have $x^T A(I) x \leq \lambda_1(I) \leq \lambda_1(G),$ so that $\lambda_1(H) \leq \lambda_1(G).$ ■

Furthermore, we compute the largest eigenvalue of the special type of bipartite graphs. Let $G(p, q, r, s, t)$ be the bipartite graph with partite sets U=$\{u_1, u_2, ..., u_p\}$ and V= $\{v_1, v_2, ..., v_q\}$, and $E(G(p, q, r, s, t)) = \{(u_i, v_j) \mid 1 \leq i \leq p, 1 \leq j \leq q\} \setminus \bigcup_{k=1}^{t} \{(u_i, v_j) \mid (k-1)r + 1 \leq i \leq kr, (k-1)s + 1 \leq j \leq ks\},$

where p, q, r, s, t are positive integers, and $tr \le p, ts \le q$.

Theorem 2.1.3. $$\lambda_1(G(p,q,r,s,t)) = \sqrt{\frac{1}{2}[(pq+rs-2rst)+\sqrt{p^2q^2-2pqrs-4pqrst+4prs^2t+4qr^2st+r^2s^2-4r^2s^2t}]}.$$

Proof. Let A = $A(G(p,q,r,s,t)) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, where B is a (0,1)-matrix such that $B_{ij} = 1$ if and only if $(u_i, v_j) \in E(G(p,q,r,s,t))$, and λ be the largest eigenvalue of $G(p,q,r,s,t)$. Note that

$B_{ij} = 1$ for $1 \le i \le p, ts+1 \le j \le q$, and (1)

$$\sum_{i=1}^{p} B_{ij} = \begin{cases} p-r, & if \quad 1 \le j \le ts, \\ p, & otherwise. \end{cases} \tag{2}$$

From Lemma 2.1.1, there exists an eigenvector $v = [x \quad y]$ for eigenvalue λ such that $x_i \ge 0, y_j \ge 0$ for $1 \le i \le p, 1 \le j \le q$. For the ith coordinate of λx, and the jth coordinate of λy, we have

$$\lambda x_i = (\lambda v)_i = (Av)_i = (\begin{bmatrix} By \\ B^T x \end{bmatrix})_i = (By)_i = \sum_{j=1}^{q} Bij \cdot y_j, \tag{3}$$

$$\lambda y_j = (\lambda v)_{p+j} = (Av)_{p+j} = (\begin{bmatrix} By \\ B^T x \end{bmatrix})_{p+j} = (B^T x)_j = \sum_{i=1}^{p} Bij \cdot x_i.$$

Hence, for $ts+1 \le j \le q$ $\lambda y_j = \sum_{i=1}^{p} Bij \cdot x_i = \sum_{i=1}^{p} 1 \cdot x_i$ (by (1))

$$= \sum_{i=1}^{p} x_i. \tag{4}$$

Define $\omega(x) = \sum_{i=1}^{p} x_i$ and $\omega(y) = \sum_{ji=1}^{q} y_j$, we compute

$$\lambda^2 \omega(x) = \lambda \sum_{i=1}^{p} \lambda x_i = \lambda (\sum_{i=1}^{p} \sum_{j=1}^{q} B_{ij} \cdot y_j) \qquad \text{(by (3))}$$

$$= \lambda (\sum_{j=1}^{q} y_j \sum_{i=1}^{p} B_{ij}) = \lambda ((\sum_{j=1}^{ts} y_j \sum_{i=1}^{p} B_{ij}) + (\sum_{j=ts+1}^{q} y_j \sum_{i=1}^{p} B_{ij}))$$

$$= \lambda ((\sum_{j=1}^{ts} y_j (p-r)) + (\sum_{j=ts+1}^{q} y_j \cdot p)) \qquad \text{(by (2))}$$

$$= \lambda ((\sum_{j=1}^{q} y_j \cdot p) - (r \sum_{j=1}^{ts} y_j)) = \lambda p \omega(y) - \lambda r (\omega(y) - \sum_{j=ts+1}^{q} y_j)$$

$$= \lambda (p-r) \omega(y) + r \sum_{j=ts+1}^{q} \lambda \cdot y_j$$

$$= \lambda (p-r) \omega(y) + r(q-ts) \omega(x), \qquad \text{(by (4))}$$

i.e. $(\lambda^2-(rq-rst))\omega(x)=\lambda(p-r)\omega(y)$.
By the similar steps, we have $(\lambda^2-(sp-rst))\omega(y)=\lambda(q-s)\omega(x)$. Hence,

$$(\lambda^2-(rq-rst))(\lambda^2-(sp-rst))\omega(x)\omega(y)=\lambda^2(p-r)(q-s)\omega(x)\omega(y). \qquad (5)$$

claim : $\omega(x)\omega(y)\neq 0$.
Assume that $\omega(x)=0$. Then, $x_i=0$ for $1\le i\le p$ follows from $x_i\ge 0$ for $1\le i\le p$. Since for $1\le j\le q$, y_j is a linear combination of $x_1,x_2,\cdots,x_p$ (by Theorem 1.2.13 (a)), we have $y_j=0$, for $1\le j\le q$. Hence $v=[x \quad y]^T$ is a zero vector which is a contradiction with to the fact that v is an eigenvector of $G(p,q,r,s,t)$, i.e., $\omega(x)\neq 0$. On the other hand $\omega(y)\neq 0$ by a similar way. Therefore, $\omega(x)\omega(y)\neq 0$.

Since $\omega(x)\omega(y)\neq 0$, we have $(\lambda^2-(rq-rst))(\lambda^2-(sp-rst))=\lambda^2(p-r)(q-s)$ from (5), so $(\lambda^2)^2-(pq+rs-2rst)(\lambda^2)+rs(q-st)(p-rt)=0$. Hence

$$\lambda=\sqrt{\frac{1}{2}[(pq+rs-2rst)+\sqrt{(pq+rs-2rst)^2-4rs(q-st)(p-rt)}}=\sqrt{\frac{1}{2}[(pq+rs-2rst)+\sqrt{p^2q^2-2pqrs-4pqrst+4prs^2t+4qr^2st+r^2s^2-4r^2s^2t}]}.$$ ■

Lemma 2.1.4. *Let* $S=\{G\,|\,G$ *is bipartite with partite sets* $U=\{u_1,u_2,...,u_p\}$ *and* $V=\{v_1,v_2,...,v_q\}$, $|E(G)|=pq-k$, $0\leqq k\leqq p\leqq q$ *and* $d(u_1)\le d(u_2)\le\cdots\le d(u_p)\}$, *and* $T=\{G\in S\,|\,\lambda_1(G)\ge\lambda_1(H),\forall H\in S\}$. *Then there exists a graph* $G\in T$, *such that* $N(u_1)\subseteq N(u_2)\subseteq\cdots\subseteq N(u_p)$.

Proof. Let G_1 be a graph of T. From Lemma 2.1.1, there exists an unit eigenvector $v=[x_1 \ \cdots \ x_p \ \ y_1 \ \cdots \ y_q]^T$ for $\lambda_1(G_1)$ such that $x_i\ge 0, y_j\ge 0$, $\forall 1\le i\le p, 1\le j\le q$. So $\lambda_1(G_1)=\lambda_1(G_1)\cdot(v^T\cdot v)=v^T(\lambda_1(G_1)v)=v^TA(G_1)v$ $=2\sum_{(u_i,v_j)\in E(G_1)}x_i\cdot y_j$. We assume, without loss of generality, that $y_1,y_2,\cdots,y_q$ are ordered in decreasing order , i.e., $y_1\ge y_2\ge\cdots\ge y_q$. Let d_i denote the degree of u_i of G_1. Since $x_i\ge 0, y_j\ge 0, \forall 1\le i\le p, 1\le j\le q$, we have

$$\lambda_1(G_1)=2\sum_{(u_i,v_j)\in E(G_1)}x_i\cdot y_j\le 2[\sum_{i=1}^{p}x_i\cdot(\sum_{j=1}^{d_i}y_j)]. \qquad (1)$$

Let G be a graph of S, such that $E(G)=\bigcup_{i=1}^{p}\{(u_i,v_j)\,|\,1\le j\le d_i\}$.
Note that $N(u_1)\subseteq N(u_2)\subseteq\cdots\subseteq N(u_p)$.

From Lemma 1.2.1, we have

$$\lambda_1(G) \ge v^T A(G)v = 2 \sum_{(u_i,v_j)\in E(G)} x_i \cdot y_j = 2[\sum_{i=1}^{p} x_i \cdot (\sum_{j=1}^{d_i} y_j)]. \qquad (2)$$

Since $G_1 \in T$, we have $\lambda_1(G_1) \ge \lambda_1(G)$, so

$$2[\sum_{i=1}^{p} x_i \cdot (\sum_{j=1}^{d_i} y_j)] \ge \lambda_1(G_1) \ge \lambda_1(G) \ge 2[\sum_{i=1}^{p} x_i \cdot (\sum_{j=1}^{d_i} y_j)], \qquad \text{(by (1) and (2))}$$

So that $\lambda_1(G) = \lambda_1(G_1)$, i.e. G is a graph of T such that $N(u_1) \subseteq N(u_2) \subseteq \cdots \subseteq N(u_p)$. ■

We remark here that if S = {G | G is bipartite with partite sets $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$, | E(G) | = pq - k, 0≦k≦p≦q and $d(u_1) \le \cdots \le d(u_p)$}, and S' = {G | G is bipartite with partite sets $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$, | E(G) | = pq - k, 0≦k≦p≦q and $N(u_1) \subseteq \cdots \subseteq N(u_p)$}. Lemma 2.1.4 says that the greatest largest eigenvalue of graphs of S is equal to the greatest largest eigenvalue of graphs of S'. It is an important result to help us to prove the main theorem in the next section.

2.2 Main Theorem

Theorem 2.2.1. *Let $S_{(p,q;k)}$ = { G | G is bipartite with partite sets X = $\{u_1 \ \cdots \ u_p\}$ and Y =$\{v_1 \ \cdots \ v_q\}$, | E(G) | = pq - k , 0≦k≦p≦q }. Then for each H in $S_{(p,q;k)}$,*

$$\lambda_1(H) \le \sqrt{\frac{1}{2}[(pq-k) + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}]}.$$

Proof.

Let S = { G | G is bipartite with partite sets X = $\{u_1 \ \cdots \ u_p\}$and Y = $\{v_1 \ \cdots \ v_q\}$ | E(G) | = pq-k , 0≦k≦p≦q and $N(u_1) \subseteq \cdots \subseteq N(u_p)$}. By the result of Lemma 2.1.4 we just prove that for each H in S,

$$\lambda_1(H) \le \sqrt{\frac{1}{2}[(pq-k) + \sqrt{p^2q^2 - 6pqk + 4pk + 4qk^2 - 3k^2}]}.$$

Note that $\sqrt{\frac{1}{2}[(pq-k)+\sqrt{p^2q^2-6pqk+4pk+4qk^2-3k^2}]}$ is the largest zero of equation $x^4-(pq-k)x^2+k(p-k)(q-1)$. Hence, $\forall H\in S, \lambda=\lambda_1(H)$, $\lambda^4-(pq-k)\lambda^2+k(p-k)(q-1)\le 0$ implies that

$\lambda \le \sqrt{\frac{1}{2}[(pq-k)+\sqrt{p^2q^2-6pqk+4pk+4qk^2-3k^2}]}$.

So, we claim that $\forall H\in S, \lambda=\lambda_1(H)$, $\lambda^4-(pq-k)\lambda^2+k(p-k)(q-1)\le 0$.

Given $H\in S$. Let $\lambda_1(H)=\lambda\ne 0$, $A(H)=\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, and $v=[x \quad y]^T$ be an eigenvector for eigenvalue λ, where $x_i\ge 0$ and $y_j\ge 0$, $\forall 1\le i\le p, 1\le j\le q$.
Let s be the positive integer such that $d(u_1)\le\cdots\le d(u_s)<q$, and $d(u_{s+1})=\cdots=d(u_p)=q$. From Theorem 1.2.13 (b), we have

$\lambda^2 x_i = \sum_{k=1}^{p} x_k\,|N(u_i)\cap N(u_k)| \le \sum_{k=1}^{p} x_k\,|N(u_i)| = d(u_i)w(x)$, where $w(x)=\sum_{k=1}^{p} x_k$.

So, $\lambda^2\sum_{i=1}^{s} x_i \le (\sum_{i=1}^{s} d(u_i))w(x) = (qs-k)w(x)$. (1)

For $s+1\le i\le p$, $\lambda x_i = \sum_{l}^{q} B_{il}\cdot y_l = w(y)$ (by Theorem 1.2.13 (a)), i.e., $x_{s+1} = \cdots = x_p$, where $w(y)=\sum_{j=1}^{q} y_j$.

Claim 1 : $x_i\ne 0$, $\forall s+1\le i\le p$.
Suppose not . Then $\lambda\cdot 0 = w(y)$. This implies for each $j\in\{1,\cdots,q\}$, $y_j=0$.
From Theorem 1.2.13 (a), we have $\lambda x_i = \sum_{l=1}^{q} B_{il}\cdot y_l = 0, \forall i$. This is a contradiction, since v is a nonzero vector.

For $s+1\le i\le p$,

$$\begin{aligned}
\lambda^2 x_i &= \sum_{k=1}^{p} x_k\,|N(u_i)\cap N(u_k)| \qquad (\text{Theorem 1.2.13 (b)})\\
&= \sum_{k=1}^{s} x_k\,|N(u_i)\cap N(u_k)| + \sum_{k=s+1}^{p} x_k\,|N(u_i)\cap N(u_k)|\\
&= \sum_{k=1}^{s} x_k\cdot d(u_k) + \sum_{k=s+1}^{p} x_k\cdot q\\
&= \sum_{k=1}^{s} x_k\cdot d(u_k) + (q(p-s))x_i
\end{aligned}$$

i.e. $(\lambda^2-q(p-s))x_i = \sum_{k=1}^{s} x_k\cdot d(u_k) \le \sum_{k=1}^{s} x_k(q-1) \le \frac{(qs-k)(q-1)}{\lambda^2}w(x)$ (by (1)).

So, $(\lambda^2-q(p-s))\sum_{i=s+1}^{p} x_i = (\lambda^2-q(p-s))(p-s)x_i \le \frac{(qs-k)(q-1)(p-s)}{\lambda^2}w(x)$. (2)

Claim 2 : $(\lambda^2 - q(p-s)) \geq 0$

For $s+1 \leq i \leq p$, $(\lambda^2 - q(p-s))x_i = \sum_{k=1}^{s} x_k \cdot d(u_k) \geq 0$. So, $(\lambda^2 - q(p-s)) \geq 0$

follows from Claim 1 : $x_i \neq 0, \forall s+1 \leq i \leq p$.

From Claim 2 and (1), we have

$$(\lambda^2 - q(p-s))\sum_{i=1}^{s} x_i \leq \frac{(\lambda^2 - q(p-s))(qs-k)}{\lambda^2} w(x). \tag{3}$$

From (2) and (3), we have

$$(\lambda^2 - q(p-s))w(x) \leq \frac{(qs-k)(q-1)(p-s)}{\lambda^2} w(x) + \frac{(\lambda^2 - q(p-s))(qs-k)}{\lambda^2} w(x).$$

From claim 1, we have $w(x) > 0$. So,

$$(\lambda^2 - q(p-s)) \leq \frac{(qs-k)(q-1)(p-s)}{\lambda^2} + \frac{(\lambda^2 - q(p-s))(qs-k)}{\lambda^2}$$
$$= \frac{(qs-k)(\lambda^2 - p + s)}{\lambda^2}.$$

So $\lambda^4 - (pq-k)\lambda^2 \leq (qs-k)(s-p) \leq (qk-k)(k-p) = k(q-1)(k-p)$, i.e. $\lambda^4 - (pq-k)\lambda^2 + k(q-1)(p-k) \leq 0$, and Theorem 2.2.1 is proved. ■

From Theorem 2.1.3, and Theorem 2.2.1, we know that the largest eigenvalue of $G(p,q,k,1,1)$ is the greatest largest eigenvalue of graphs of $S_{(p,q;k)}$. Let $S_{(n;m)}$ be the set defined as $S_{(n;m)}$ = { G | G is bipartite with |V(G)| = n and |E(G)| = m }. What is the greatest largest eigenvalue of graphs of $S_{(n;m)}$? Unfortunately, it may not have the general form as the description of Theorem 2.2.1. Next, we try to resolve this problem.

Here, we consider the case |V(G)| = 2n and |E(G)| = n^2 - k. Note that,

$$S_{(2n;n^2-k)} = \bigcup_{a=0}^{\lfloor\sqrt{k}\rfloor} S_{(n-a,n+a;k-a^2)}.$$

Let $\alpha = \max\{\lambda_1(G) \mid G \in S_{(2n;n^2-k)}\}$ and

$$\alpha_a = \max\{\lambda_1(G) \mid G \in S_{(n-a,n+a;k-a^2)}\}.$$

Then $\alpha = \max\{\alpha_a \mid 0 \leq a \leq \lfloor\sqrt{k}\rfloor\}$.

From Theorem 2.2.1, we have

$$\alpha_a = \sqrt{\frac{1}{2}[(pq-l)+\sqrt{p^2q^2-6pql+4pl+4ql^2-3l^2}]},$$

where p = n-a , q = n+a and $l = k-a^2$.

By direction computation, we have

$$\alpha_a = \sqrt{\frac{1}{2}[(n^2-k)+\sqrt{f_{(n,k)}(a)}]}, \text{ where } f_{(n,k)}(a) = n^4 + (4a^2-6k)n^2$$
$$+4(k-a^2)(k-a^2+1)n + (k-a^2)(-4a^3+9a^2+(4k-4)a-3k)+4a^4.$$

Define $M_{n,k} = \max\{a \mid \alpha = \alpha_a\}$. Note that $f_{(n,k)}(a+1) - f_{(n,k)}(a) = 4[(2a+1)n^2 + 2(2a+1)(a^2+a-k)n + (5a^4+2a^3+(1-6k)a^2+k^2)]$.

Lemma 2.2.2.

(1) $M_{n,k} \geq 1$

(2) $M_{n,k} = \lfloor \sqrt{k} \rfloor$, *if n is large enough.*

Proof.

(1) It follows from $f_{(n,k)}(1) - f_{(n,k)}(0) = 4(n-k)^2 \geq 0$.

(2) If a and k are fixed, then $f_{(n,k)}(a+1) - f_{(n,k)}(a) = c_1n^2 + c_2n + c_3$, where $c_1 = 8a+4 > 0$. Hence $M_{n,k} = \lfloor \sqrt{k} \rfloor$ for large enough n. ■

2.3 The Ordering of Bipartite Graphs w.r.t the Largest Eigenvalue

Theorem 2.3.1. *Let G be a bipartite graph with 2n vertices. Then* $\lambda_1(G) \leq n$, *and equality holds if and only if G =* $K_{n,n}$.

Proof. If G is a bipartite graph with 2n vertices, then $G \subseteq K_{p,q}$ for some p + q = 2n, $1 \leq p \leq n$. From Lemma 2.1.2 and Lemma 1.2.12, we have $\lambda_1(G) \leq \lambda_1(K_{p,q}) = \sqrt{pq}$. So that $\lambda_1(G) \leq \sqrt{pq} = \sqrt{p(2n-p)} = \sqrt{-(p-n)^2+n^2} \leq n = \lambda_1(K_{n,n})$. ■

Excluding the bipartite graphs satisfying equality in Theorem 2.3.1 leads to the following result.

Theorem 2.3.2. *Let G be a bipartite graph with 2n vertices, where* $n \geq 2$ *and*

$G \neq K_{n,n}$. *Then* $\lambda_1(G) \le \sqrt{n^2-1}$, *and equality holds if and only if* $G = K_{n-1,n+1}$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}$, then $G \subseteq G(n,n,1,1,1)$ or $G \subseteq K_{p,q}$, for some p + q = 2n , $1 \le p \le n-1$. So

$\lambda_1(G) \le \max\{\lambda_1(G(n,n,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-1\}\}$.

We compute $\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-1\}\}$

$= \max\{\sqrt{pq} \mid p+q=2n, 1 \le p \le n-1\}\}$ (by Lemma 1.2.12)

$= \max\{\sqrt{-(p-n)^2+n^2} \mid 1 \le p \le n-1\}\} = \sqrt{-((n-1)-n)^2+n^2} = \sqrt{n^2-1}$,

and $\lambda_1(G(n,n,1,1,1))$

$= \sqrt{\frac{1}{2}[(n^2+1-2)+\sqrt{n^4-2n^2-4n^2+4n+4n+1-4}]}$ (by Theorem 2.1.3)

$= \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{n^4-6n^2+8n-3}]}$.

Since $4n^2-8n+4 = 4(n-1)^2 \ge 0$, we have $\lambda_1(G(n,n,1,1,1))$

$= \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{n^4-6n^2+8n-3}]}$

$\le \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{(n^4-6n^2+8n-3)+(4n^2-8n+4)}]}$

$= \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{n^4-2n^2+1}]} = \sqrt{\frac{1}{2}[(n^2-1)+(n^2-1)]} = \sqrt{n^2-1}$

$= \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-1\}\}$. So that

$\lambda_1(G) \le \max\{\lambda_1(G(n,n,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-1\}\}$

$= \sqrt{n^2-1} = \lambda_1(K_{n-1,n+1})$.

Note that this result says that if $G \neq K_{n,n}$, then $\lambda_1(G) \le \sqrt{n^2-1} < n$, i.e., the equality in the Theorem 2.3.1 holds if and only if G = $K_{n,n}$. So, we complete the proof Theorem 2.3.1. ∎

Theorem 2.3.3. *Let G be a bipartite graph with 2n vertices, where* $n \ge 3, G \neq K_{n,n}$ *and* $G \neq K_{n-1,n+1}$. *Then*

$\lambda_1(G) \le \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{n^4-6n^2+8n-3}]}$, *and equality holds if and only if G =* $G(n,n,1,1,1)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}$ and $G \neq K_{n-1,n+1}$, then

$G \subseteq G(n,n,1,1,1)$, or $G \subseteq G(n-1,n+1,1,1,1)$ or $G \subseteq K_{p,q}$ for some p + q = 2n , $1 \le p \le n-2$. So $\lambda_1(G) \le \max\{\lambda_1(G(n,n,1,1,1)), \lambda_1(G(n-1,n+1,1,1,1)),$ $\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-2\}\}$. We compute $\lambda_1(G(n,n,1,1,1))$ $= \sqrt{\frac{1}{2}[(n^2-1)+\sqrt{n^4-6n^2+8n-3}]}$, and $\lambda_1(G(n-1,n+1,1,1,1))$, $= \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]}$, (by Theorem 2.1.3)

and $\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-2\} = \lambda_1(K_{n-2}, n+2) = \sqrt{n^2-4}$.

For $n \ge 3$, we have $2n^2-7 \ge 0$, and $8n^2+8n-52 \ge 0$, so

$$\lambda_1(G(n-1,n+1,1,1,1)) = \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]}$$

$$\le \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{(n^4-8n^2+8n+4)+(2n^2-7)}]}$$

$$= \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-6n^2+8n-3}]},$$ and

$$\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-2\} = \lambda_1(K_{n-2}, n+2) = \sqrt{n^2-4}$$

$$= \sqrt{\frac{1}{2}[n^2-1+\sqrt{n^4-14n^2+49}]}$$

$$\le \sqrt{\frac{1}{2}[n^2-1+\sqrt{(n^4-14n^2+49)+(8n^2+8n-52)}]}$$

$$= \sqrt{\frac{1}{2}[n^2-1+\sqrt{n^4-6n^2+8n-3}]}.$$

Hence $\lambda_1(G) \le \max\{\lambda_1(G(n,n,1,1,1)), \lambda_1(G(n-1,n+1,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid$ $p+q=2n, 1 \le p \le n-2\}\} = \lambda_1(G(n,n,1,1,1)) = \sqrt{\frac{1}{2}[n^2-1+\sqrt{n^4-6n^2+8n-3}]}$.

Note that this result says that if $G \ne K_{n,n}$ and $G \ne K_{n-1,n+1}$, then $\lambda_1(G) \le$ $\sqrt{\frac{1}{2}[n^2-1+\sqrt{n^4-6n^2+8n-3}]} < \sqrt{n^2-1}$, for $n \ge 2$, i.e., the equality in the Theorem 2.3.2 holds if and only if G = $K_{n-1,n+1}$. So, we complete the proof Theorem 2.3.2. ■

Theorem 2.3.4. *Let G be a bipartite graph with 2n vertices, where* $n \ge 3$, $G \ne K_{n,n}$, $G \ne K_{n-1,n+1}$ *and* $G \ne G(n,n,1,1,1)$. *Then* $\lambda_1(G) \le \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]}$, *and equality holds if and only if G =* $G(n-1,n+1,1,1,1)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$ and $G \neq G(n,n,1,1,1)$, then $G \subseteq G(n,n,2,1,1)$ or $G \subseteq G(n,n,1,1,2)$ or $G \subseteq G(n-1,n+1,1,1,1)$ or $G \subseteq K_{p,q}$ for some p + q = 2n, $1 \leq p \leq n-2$. So,

$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,2,1,1)), \lambda_1(G(n,n,1,1,2)), \lambda_1(G(n-1,n+1,1,1,1))$

$\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-2\}\} = \max\{\sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]}, \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}}, \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]},$

$\sqrt{n^2-4}\} = \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]}$, for $n \geq 3$.

Note that this result says that if $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, and $G \neq G(n,n,1,1,1)$, then

$\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]} < \sqrt{\frac{1}{2}[n^2-1+\sqrt{n^4-6n^2+8n-3}]}$

for $n \geq 3$, i.e., the equality in the Theorem 2.3.3 holds if and only if $G = G(n,n,1,1,1)$.

So, we complete the proof Theorem 2.3.3. ■

Theorem 2.3.5. *Let G be a bipartite graph with 2n vertices, where* $n \geq 3, G \neq K_{n,n}$, $G \neq K_{n-1,n+1}$, $G \neq G(n,n,1,1,1)$ *and* $G \neq G(n-1,n+1,1,1,1)$. *Then* $\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]}$, *and equality holds if and only if G =* $G(n,n,2,1,1)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq G(n,n,1,1,1)$ and $G \neq G(n-1,n+1,1,1,1)$, then $G \subseteq G(n,n,2,1,1)$ or $G \subseteq G(n,n,1,1,2)$ or $G \subseteq G(n-1,n+1,2,1,1)$ or $G \subseteq G(n-1,n+1,1,2,1)$ or $G \subseteq G(n-1,n+1,1,1,2)$ or $G \subseteq K_{p,q}$ for some p + q = 2n, $1 \leq p \leq n-2$. So,

$$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,2,1,1)), \lambda_1(G(n,n,1,1,2)), \lambda_1(G(n-1,n+1,2,1,1)), \lambda_1(G(n-1,n+1,1,2,1)), \lambda_1(G(n-1,n+1,1,1,2)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-2\}\}.$$

From Theorem 2.2.1 we have $\lambda_1(G(n,n,2,1,1)) \geq \lambda_1(G(n,n,1,1,2))$, $\lambda_1(G(n-1,n+1,2,1,1)) \geq \lambda_1(G(n-1,n+1,1,2,1))$, and $\lambda_1(G(n-1,n+1,2,1,1)) \geq \lambda_1(G(n-1,n+1,1,1,2))$. So

$$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,2,1,1)), \lambda_1(G(n-1,n+1,2,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-2\}\}$$

$$= \max\{\sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]},$$

$$\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]}, \sqrt{n^2-4}\}$$

$$=\sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]}, \text{ for } n\geq 3.$$

Note that this result says that if $G\neq K_{n,n}, G\neq K_{n-1,n+1}, G\neq G(n,n,1,1,1)$ and $G\neq G(n-1,n+1,1,1,1)$, then

$$\lambda_1(G)\leq \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]} < \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-8n^2+8n+4}]}$$

for $n\geq 3$, i.e., the equality in the Theorem 2.3.4 holds if and only if $G=G(n-1,n+1,1,1,1)$. So, we complete the proof Theorem 2.3.4. ■

Theorem 2.3.6. *Let G be a bipartite graph with 2n vertices, where* $n\geq 4, G\neq K_{n,n}$, $G\neq K_{n-1,n+1}$, $G\neq G(n,n,1,1,1)$, $G\neq G(n-1,n+1,1,1,1)$ *and* $G\neq G(n,n,2,1,1)$. *Then* $\lambda_1(G)\leq \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}]}$, *and equality holds if and only if G =* $G(n,n,1,1,2)$.

Proof. If G is a bipartite graph with 2n vertices, $G\neq K_{n,n}, G\neq K_{n-1,n+1}$, $G\neq G(n,n,1,1,1), G\neq G(n-1,n+1,1,1,1)$ and $G\neq G(n,n,2,1,1)$), then

$$\lambda_1(G)\leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n,n,1,1,2)), \lambda_1(G(n-1,n+1,2,1,1)),$$
$$\max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1\leq p\leq n-2\}\} \qquad \text{(by Theorem 2.2.1)}$$

$$=\max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]},$$

$$\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}]}, \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]},$$

$$\sqrt{n^2-4}\}$$

$$=\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}]}, \text{ for } n\geq 4.$$

Note that this result says that if $G\neq K_{n,n}, G\neq K_{n-1,n+1}, G\neq G(n,n,1,1,1)$, $G\neq G(n-1,n+1,1,1,1)$ and $G\neq G(n,n,2,1,1)$, then $\lambda_1(G)\leq$

$$\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}]} < \sqrt{\frac{1}{2}[(n^2-2)+\sqrt{n^4-12n^2+24n-12}]}$$

for $n\geq 3$, i.e., the equality in the Theorem 2.3.5 holds if and only if $G=G(n,n,2,1,1)$. So, we complete the proof Theorem 2.3.5. ■

Theorem 2.3.7. *Let G be a bipartite graph with 2n vertices, where* $n\geq 6$, $G\neq K_{n,n}$,

$G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1)$ *and* $G \neq G(n,n,1,1,2)$. *Then* $\lambda_1(G) \leq \sqrt{n^2-4}$, *and equality holds if and only if* $G = K_{n-2,n+2}$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1)$ and $G \neq G(n,n,1,1,2)$, then $\lambda_1(G) \leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n-1,n+1,2,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-2\}\}$ (by Theorem 2.2.1)

$$= \max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]}, \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]}, \sqrt{n^2-4}\}$$

$= \sqrt{n^2-4}$, for $n \geq 6$.

Note that this result says that if $G \neq K_{n,n}, G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1)$, $G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1)$ and $G \neq G(n,n,1,1,2)$, then $\lambda_1(G) \leq \sqrt{n^2-4} < \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-10n^2+16n-7}]}$, for $n \geq 4$, i.e., the equality in the Theorem 2.3.6 holds if and only if $G = G(n,n,1,1,2)$. So, we complete the proof Theorem 2.3.6. ■

Theorem 2.3.8. *Let G be a bipartite graph with 2n vertices, where* $n \geq 6, G \neq K_{n,n}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$ *and* $G \neq K_{n-2,n+2}$. *Then*

$\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]}$, *and equality holds if and only if G =* $G(n-1,n+1,2,1,1)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$ and $G \neq K_{n-2,n+2}$, then

$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n-1,n+1,2,1,1)), \lambda_1(G(n-2,n+2,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-3\}\}$ (by Theorem 2.2.1)

$$= \max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]}, \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]}, \sqrt{\frac{1}{2}[(n^2-5)+\sqrt{n^4-14n^2+8n+37}]}, \sqrt{n^2-9}\} = \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]}, \text{ for } n \geq 6.$$

Note that this result says that if $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq G(n,n,1,1,1)$, $G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$ and $G \neq K_{n-2,n+2}$, then

$$\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]} < \sqrt{n^2-4},$$

for $n \geq 6$, i.e., the equality in the Theorem 2.3.7 holds if and only if $G = K_{n-2,n+2}$.
So, we complete the proof Theorem 2.3.7. ■

Theorem 2.3.9. *Let G be a bipartite graph with 2n vertices, where* $n \geq 7, G \neq K_{n,n}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$, $G \neq K_{n-2,n+2}$ *and* $G \neq G(n-1,n+1,2,1,1)$. *Then*

$$\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n-7}]},$$ *and equality holds if and only if G =* $G(n-1,n+1,1,2,1)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$, $G \neq K_{n-2,n+2}$ and $G \neq G(n-1,n+1,2,1,1)$, then

$$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n-1,n+1,1,2,1)), \lambda_1(G(n-1,n+1,1,1,2)), \lambda_1(G(n-2,n+2,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-3\}\}$$

(by Theorem 2.2.1)

$$= \max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]}, \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n-7}]}, \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]}, \sqrt{\frac{1}{2}[(n^2-5)+\sqrt{n^4-14n^2+8n+37}]}, \sqrt{n^2-9}\}$$

$$= \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n-7}]}, \text{ for } n \geq 7.$$

Note that this result says that if $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq G(n,n,1,1,1)$, $G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2), G \neq K_{n-2,n+2}$ and $G \neq G(n-1,n+1,2,1,1)$, then $\lambda_1(G) \leq$

$$\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n-7}]} < \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n+9}]},$$

for $n \geq 6$, i.e., the equality in the Theorem 2.3.8 holds if and only if $G = G(n-1,n+1,2,1,1)$. So, we complete the proof Theorem 2.3.8. ■

Theorem 2.3.10. *Let G be a bipartite graph with 2n vertices, where* $n \geq 7$,

$G \neq K_{n,n}, G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1)$, $G \neq G(n,n,1,1,2), G \neq K_{n-2,n+2}, G \neq G(n-1,n+1,2,1,1)$ and $G \neq G(n-1,n+1,1,2,1)$. Then $\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]}$, and equality holds if and only if $G = G(n-1,n+1,1,1,2)$.

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$, $G \neq K_{n-2,n+2}, G \neq G(n-1,n+1,2,1,1)$ and $G \neq G(n-1,n+1,1,2,1)$, then

$$\lambda_1(G) \leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n-1,n+1,1,1,2)), \lambda_1(G(n-2,n+2,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \leq p \leq n-3\}\} \quad \text{(by Theorem 2.2.1)}$$

$$= \max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]}, \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]}, \sqrt{\frac{1}{2}[(n^2-5)+\sqrt{n^4-14n^2+8n+37}]}, \sqrt{n^2-9}\}$$

$$= \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]}, \text{ for } n \geq 7.$$

Note that this result says that if $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq G(n,n,1,1,1)$, $G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2), G \neq K_{n-2,n+2}$, $G \neq G(n-1,n+1,2,1,1)$ and $G \neq G(n-1,n+1,1,2,1)$, then $\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]} < \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-14n^2+24n-7}]}$, for $n \geq 7$, i.e., the equality in the Theorem 2.3.9 holds if and only if $G = G(n-1,n+1,1,2,1)$. So, we complete the proof Theorem 2.3.9. ■

Theorem 2.3.11. *Let G be a bipartite graph with 2n vertices, where $n \geq 7, G \neq K_{n,n}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$, $G \neq K_{n-2,n+2}, G \neq G(n-1,n+1,2,1,1), G \neq G(n-1,n+1,1,2,1)$ and $G \neq G(n-1,n+1,1,1,2)$. Then $\lambda_1(G) \leq \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]} = \lambda_1(G(n,n,3,1,1))$.*

Proof. If G is a bipartite graph with 2n vertices, $G \neq K_{n,n}, G \neq K_{n-1,n+1}$, $G \neq K_{n-1,n+1}, G \neq G(n,n,1,1,1), G \neq G(n-1,n+1,1,1,1), G \neq (n,n,2,1,1), G \neq G(n,n,1,1,2)$, $G \neq K_{n-2,n+2}, G \neq G(n-1,n+1,2,1,1), G \neq G(n-1,n+1,1,2,1)$ and $G \neq G(n-1,n+1,1,1,2)$, then $\lambda_1(G) \leq \max\{\lambda_1(G(n,n,3,1,1)), \lambda_1(G(n-1,n+1,3,1,1))$,

$\lambda_1(G(n-2,n+2,1,1,1)), \max\{\lambda_1(K_{p,q}) \mid p+q=2n, 1 \le p \le n-3\}\}$

(by Theorem 2.2.1)

$$= \max\{\sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]},$$

$$\sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-16n^2+48n+16}]}, \sqrt{\frac{1}{2}[(n^2-5)+\sqrt{n^4-14n^2+8n+37}]},$$

$$\sqrt{n^2-9}\} = \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]}, \text{ for } n \ge 7.$$

Note that this result says that if $G \ne K_{n,n}, G \ne K_{n-1,n+1}, G \ne G(n,n,1,1,1)$, $G \ne G(n-1,n+1,1,1,1), G \ne (n,n,2,1,1), G \ne G(n,n,1,1,2), G \ne K_{n-2,n+2}, G \ne$ $G(n-1,n+1,2,1,1), G \ne G(n-1,n+1,1,2,1)$ and $G \ne G(n-1,n+1,1,1,2)$, then $\lambda_1(G) \le$

$$= \sqrt{\frac{1}{2}[(n^2-3)+\sqrt{n^4-18n^2+48n-27}]} < \sqrt{\frac{1}{2}[(n^2-4)+\sqrt{n^4-12n^2+16n+4}]},$$

for $n \ge 7$, i.e., the equality in the Theorem 2.3.10 holds if and only if $G = G(n-1,n+1,1,1,2)$. So, we complete the proof Theorem 2.3.10. ■

Through a detail calculation, we are able to find an upper bound for the largest eigenvalues of certain bipartite graphs. Clearly, as shown in this thesis, these bounds are in fact sharp bounds. But, at this moment we are not able to find a more general expicit form for the upper bounds of the largest eigenvalues of general bipartite graphs, it is even harder when a general graph is considered. It will be very useful if we can also determine the second largest eigenvalues of certain bipartite graphs in the near future.

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