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碩 士 論 文

雙環式網路之退化 L−型
On Degenerate Double-Loop L-Shapes

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中 華 民 國 九 十 四 年 六 月

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摘要

大多數關於雙環式網路之L-型的結果均以四個參數 l, h, p, n來描述，但是這些參數在L-型為退化時，並不well-defined。首先，鄭與黃提出了一個很有效率的演算法來得出雙環式網路的L-型的四個參數，他們的演算法不論L-型是否退化均可執行。之後，陳與黃給了一套規則來定義退化的L-型的四個參數。很不幸的，用上述兩種方法所決定的 l, h, p, n未必一致。在這篇論文中，我們試著了解上述兩種方法所決定的 l, h, p, n所代表的意義及它們之間的關係。

關鍵詞：雙環式網路、L-型、退化

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On Degenerate Double-Loop L-Shapes

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Abstract

Most of the results about the L-shapes of double-loop networks are given in terms of the four parameters ℓ, h, p, n. But these parameters are not well defined in the degenerate case. Recently, Cheng and Hwang gave an efficient algorithm to compute the four parameters ℓ, h, p, n of an L-shape which works for both the regular and the degenerate cases. On the other hand, Chen and Hwang gave a set of rules to determine the four parameters of a degenerate L-shape. Unfortunately, the solutions given by the above two methods do not always coincide. In this thesis, we try to understand their respective meanings and their relations.

Keywords: Double-loop network, L-shape, degenerate.

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1 Introduction

The double-loop network has been well studied (see [7] for a recent survey) as the topology for a communication network or computer network. For example, SONET (synchronous optical network) is a double-loop network. Formally, a double-loop network $DL(N; a, b)$ has N nodes $0, 1, \cdots, N-1$ and $2N$ links, $i \to i+a,\ i \to i+b \pmod N,\ i = 0, 1, \cdots, N-1$. We assume that the weight of each of the $2N$ links is 1 and assume that $\gcd(N, a, b) = 1$ so that the network is strongly connected.

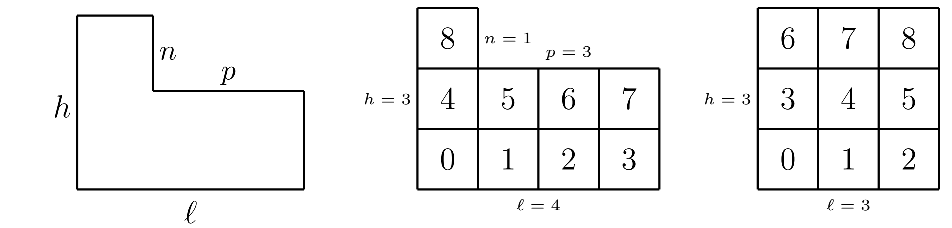
The minimum distance diagram (MDD) of $DL(N; a, b)$ is a diagram with node 0 in cell $(0, 0)$, and node v in cell (i, j) if and only if $ia + jb \equiv v \pmod N$ and $i + j$ is the minimum among all (i', j') satisfying the congruence. Namely, a shortest path from 0 to v is through taking i a-links and j b-links (in any order). Note that in a cell (i, j), i is the column index and j is the row index. An MDD includes every node exactly once (in case of two shortest paths, the convention is to choose the cell with the smaller row index, i.e., the smaller j). Since $DL(N; a, b)$ is clearly node-symmetric, there is no loss of generality in assuming: node 0 is the origin of a path.

Wong and Coppersmith (WC) [9] proved that the MDD of $DL(N; a, b)$ (their proof for $DL(N; 1, h)$ is easily extended to the general case) is always an L-shape which can be characterized by four parameters ℓ, h, p, n (see Fig. 1 (a)). These four parameters are the lengths of four of the six segments on the boundary of the L-shape. Clearly,

$$N = \ell h - pn.$$

In [2], Chen and Hwang showed that necessarily $\ell > n$ and $h \geq p$. Fig. 1 (b) illustrates an MDD with a regular L-shape. Fig. 1 (c) illustrates one with an L-shape degenerate into a rectangle.

Most of the results about the L-shape are given in terms of the four parameters ℓ, h, p, n. But these parameters are not well defined in the degenerate case. Recently, Cheng and Hwang [5] gave an $O(\log N)$-time algorithm to compute the four parameters ℓ, h, p, n of an



(a) The four parameters (b) $a = 1$, $b = 4$ (c) $a = 1$, $b = 3$

Figure 1: Minimum distance diagrams and L-shapes.

L-shape which works for both the regular and the degenerate cases. On the other hand, Chen and Hwang [3] gave a set of rules to determine the four parameters of a degenerate L-shape. Unfortunately, the solutions given by the above two methods do not always coincide. In this thesis, we try to understand their respective meanings and their relations. Since it is also of interest to know when will an L-shape degenerate, in this thesis we give necessary and sufficient conditions depending on N, a, and b only.

2 Necessary and sufficient conditions for degenerate L-shapes

The following five notations will be used throughout this thesis:

$$d = \gcd(N, a),\ d' = \gcd(N, b),\ N' = N/d,\ a' = a/d,\ \text{and } b' = b \bmod N'. \qquad (2.1)$$

Since $\gcd(N, a, b) = 1$, clearly $\gcd(d, d') = 1$. Chen and Hwang [3] proved

Lemma 1 [3] *A degenerate L-shape of height h and width ℓ satisfies one of the following three conditions:*

(1) $hb \not\equiv \ell a \equiv 0 \pmod{N}$.

(2) $\ell a \not\equiv hb \equiv 0 \pmod{N}$.

(3) $\ell a \equiv hb \equiv 0 \pmod{N}$.

We now prove

Theorem 2 *The L-shape of $DL(N; a, b)$ is degenerate if and only if one of the following three conditions holds:*

(C1) $d > 1$ *and there exists* $1 \leq i \leq \min\{d, \frac{N}{d} - 1\}$ *such that* $db \equiv ia \pmod{N}$.

(C2) $d' > 1$ *and there exists* $1 \leq j \leq \min\{d' - 1, \frac{N}{d'} - 1\}$ *such that* $d'a \equiv jb \pmod{N}$.

(C3) $d > 1$, $d' > 1$ *and* $d'a \equiv db \equiv 0 \pmod{N}$.

Moreover, (C1) ⇔ (1), (C2) ⇔ (2) and (C3) ⇔ (3). Also, if (C1) holds, then the degenerate L-shape is of height d and width N/d; if (C2) holds, then the degenerate L-shape is of height N/d' and width d'; if (C3) holds, then the degenerate L-shape is of height d and width d'.

Proof. Necessity. Suppose the L-shape is degenerate and is a rectangle of height h and width ℓ. Then by Lemma 1, it satisfies (1) or (2) or (3). We first prove two claims.

Claim 1. If $\ell a \equiv 0 \pmod{N}$, then $h = d$, $\ell = N/d$ and $d > 1$.

Proof of Claim 1. Let $a = \alpha d$ for some integer α. Note that the L-shape being degenerate implies $N = \ell h$. Thus $\ell a \equiv 0 \pmod{N}$ implies $a \equiv 0 \pmod{h}$. Let $a = \beta h$ for some integer β. Then $a = \alpha d = \beta h$. Hence $d = \frac{\beta h}{\alpha}$. Since $1 = \gcd(\alpha, \frac{N}{d}) = \gcd(\alpha, \frac{\ell h}{\frac{\beta h}{\alpha}}) = \gcd(\alpha, \frac{\ell \alpha}{\beta})$, necessarily $\alpha | \beta$. Therefore $\frac{\beta}{\alpha}$ is an integer. Since $d | N$, we have $\frac{\beta}{\alpha} | \ell$. Suppose $\frac{\beta}{\alpha} > 1$. Let $\ell' = \frac{\ell}{\frac{\beta}{\alpha}}$. Then $\ell' < \ell$ and $\ell' a = \frac{\ell}{\frac{\beta}{\alpha}} \beta h = \ell h \alpha = N\alpha \equiv 0 \pmod{N}$. Then row 0 of the L-shape will contain two entries of 0, one at cell (0,0) and the other at cell $(\ell', 0)$, a contradiction to the definition of an L-shape (recall that an MDD includes every node exactly once). Therefore $\frac{\beta}{\alpha} = 1$. Consequently, $h = d$ and $\ell = N/d$. Since $\ell < N$ and $\ell d = N$, clearly $d > 1$. ∎

Claim 2. If the L-shape is degenerate and $hb \equiv 0 \pmod{N}$, then $h = N/d'$, $\ell = d'$ and $d' > 1$.

Proof of Claim 2. Since this proof is similar to that of Claim 1, we omit it. ∎

We now prove the necessity of this theorem. First, assume the L-shape satisfies condition

(1). By Claim 1, we have $d > 1$, $h = d$ and $\ell = N/d$. By the definition of an MDD, hb is the first element in column 0 satisfying

$$hb \equiv ia + jb \pmod{N} \text{ with } i + j \leq h,\ i \geq 0,\ j \geq 0.$$

Therefore $j = 0$ for otherwise $(h - j)b$ would be the first element. Also, $i \geq 1$ for otherwise $hb \equiv 0 \pmod{N}$. Thus $db = hb \equiv ia \pmod{N}$ for $1 \leq i \leq d$. Since $\ell = N/d$, we have $i \leq \frac{N}{d} - 1$. We conclude $db \equiv ia \pmod{N}$ for $1 \leq i \leq \min\{d, \frac{N}{d} - 1\}$, which means (C1) holds. The above discussion also shows that (1) implies (C1), i.e., (1) $\Rightarrow$ (C1).

Next, assume the L-shape satisfies condition (2). Then the argument is similar except at the end we have

$$\ell a \equiv ia + jb \pmod{N} \text{ with } i + j < \ell,\ i \geq 0,\ j \geq 0.$$

The reason for the strict inequality that $i + j < \ell$ is by our construction on tie-breaking in defining the MDD. Thus (C2) holds. So (2) $\Rightarrow$ (C2).

Finally, assume the L-shape satisfies condition (3). By Claim 1, we have $d > 1$, $h = d$ and $\ell = N/d$. By Claim 2, we have $d' > 1$, $h = N/d'$ and $\ell = d'$. Thus $d'a = \ell a \equiv 0 \pmod{N}$ and $db = hb \equiv 0 \pmod{N}$, which means (C3) holds. So (3) $\Rightarrow$ (C3).

Sufficiency. Let the L-shape of $DL(N; a, b)$ be (ℓ, h, p, n). First, assume that (C1) is satisfied. Since $db \equiv ia \pmod{N}$ for $1 \leq i \leq \min\{d, \frac{N}{d} - 1\}$, we have $h \leq d$. On the other hand, $\ell \leq N/d$ since $(N/d)a = N(a/d) \equiv 0 \pmod{N}$. Therefore

$$N = \ell h - pn \leq \ell h \leq (N/d)d = N.$$

Necessarily,

$$\ell = N/d,\ h = d.$$

It follows

$$\ell h = N,$$

i.e., the L-shape is degenerate. Moreover, $\ell a = (N/d)a = N(a/d) \equiv 0 \pmod{N}$; $hb = db \equiv ia \not\equiv 0 \pmod{N}$ since $1 \leq i \leq \ell - 1$. So (C1) $\Rightarrow$ (1).

The proof of (C2) is similar to that of (C1). Finally, assume that (C3) is satisfied. Then since $d'a \equiv db \equiv 0 \pmod{N}$, we have $\ell \leq d'$ and $h \leq d$. Since $d|N$, $d'|N$ and $\gcd(d, d') = 1$, we have $d'd \leq N$. Therefore

$$N = \ell h - pn \leq \ell h \leq d'd \leq N.$$

Necessarily,

$$\ell = d', \ h = d.$$

It follows

$$\ell h = N,$$

i.e., the L-shape is degenerate. Moreover, $\ell a = d'a \equiv 0 \pmod{N}$; $hb = db \equiv 0 \pmod{N}$. So (C3) $\Rightarrow$ (3). ∎

Remarks. From the proof of Theorem 2, when an L-shape(ℓ, h, p, n) degenerates into a rectangle, it is reasonable to set ℓ to the width and h to the height of the rectangle. Moreover, it is reasonable to set $p = 0$ or $n = 0$ since $N = \ell h - pn$ and $\ell h = N$ hold simultaneously.

3 Strongly isomorphic double-loop networks and degenerate L-shapes

The following property was proved in [1].

Lemma 3 [1] *If α and β are integers, not both zero, then there exist integers x and y such that $y\alpha + x\beta = \gcd(\alpha, \beta)$ and $\gcd(x, \gcd(\alpha, \beta)) = 1$.*

Let $DL(N; a, b)$ be a double-loop network. Then

Lemma 4 *There exists an integer x such that $\gcd(x, N) = 1$ and $ax \equiv d \pmod{N}$.*

Proof. Since $\gcd(N, a) = d$, by Lemma 3, there exist integers x and y such that $yN + xa = d$ and $\gcd(x, d) = 1$. Hence $ax \equiv d \pmod{N}$. Moreover, $y(N/d) + x(a/d) = 1$ implies $\gcd(x, N/d) = 1$. It follows that $\gcd(x, N) = \gcd(x, (N/d)d) = 1$. Hence the lemma. ∎

Two double-loop networks $DL(N; a, b)$ and $DL(N; a', b')$ are *strongly isomorphic* if there exists a z prime to N such that $a' \equiv az,\ b' \equiv bz \pmod N$ or $a' \equiv bz,\ b' \equiv az \pmod N$ [8]. It is well known that two strongly isomorphic double-loop networks realize the same L-shape. The following property greatly simplifies the proofs in the remaining sections.

Theorem 5 *Let x be an integer such that $\gcd(x, N) = 1$ and $ax \equiv d \pmod N$. Let $b'' = bx \pmod N$. Then $DL(N; a, b)$ and $DL(N; d, b'')$ are strongly isomorphic.*

Proof. This theorem follows from Lemma 4. ∎

In the following, we characterize a degenerate L-shape by the four independent parameters ℓ, h, p, n. Set

$$m = \ell - p,\ \ q = h - n$$

for convenience; see Fig. 2(a). Then

Lemma 6 *For a degenerate L-shape, at least one of m, n, p, q is zero and at most two of m, n, p, q are zero. Moreover, it is impossible that both m and p, both n and q, or both m and q are zero.*

Proof. It is obvious that at least one of m, n, p, q is zero. Since $\ell = m + p$ and $h = n + q$, if more than two of m, n, p, q are zero, then $\ell = 0$ or $h = 0$ will happen, which is impossible. Suppose two of m, n, p, q are zero. If both m and p (n and q) are zero, then $\ell = m + p = 0$ ($h = n + q = 0$), which is impossible. If both m and q are zero, then $\ell = p$, $h = n$, and then $N = \ell h - pn = 0$, which is also impossible. Hence the lemma. ∎

Corollary 7 *There are only seven possible ways to view a degenerate L-shape. We define these shapes by identifying the parameters which are set to zero:* **(S1)**: *only $m = 0$,* **(S2)**: *only $n = 0$,* **(S3)**: *only $p = 0$,* **(S4)**: *only $q = 0$,* **(S5)**: *$m = 0$ and $n = 0$,* **(S6)**: *$p = 0$ and $q = 0$,* **(S7)**: *$n = 0$ and $p = 0$.*

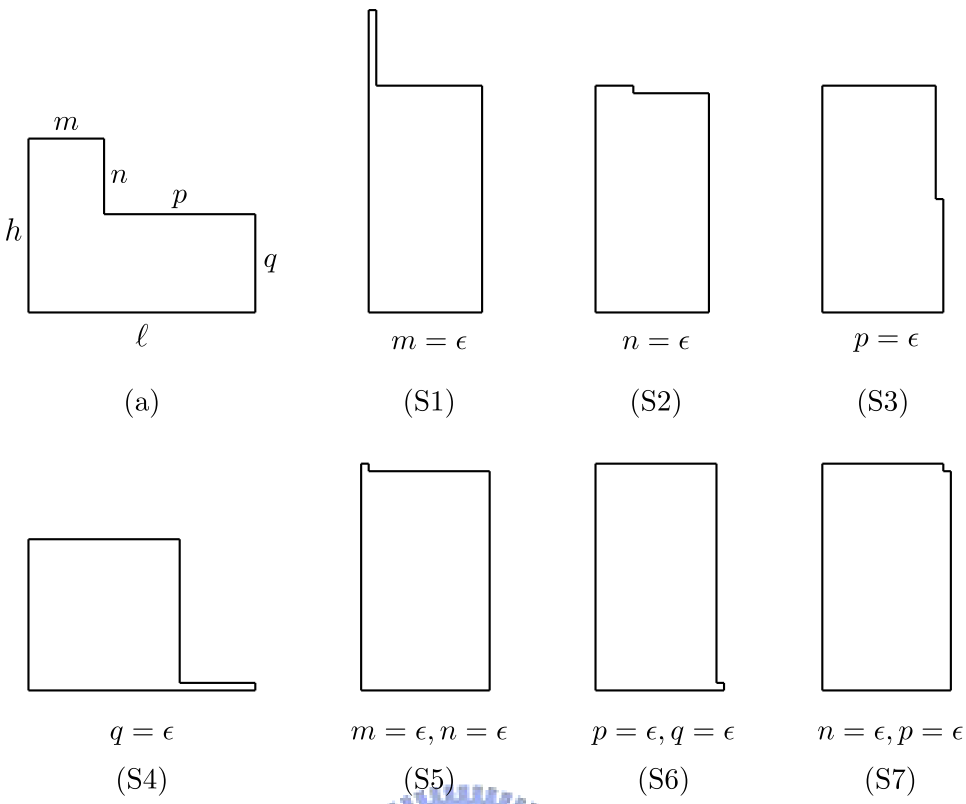


Figure 2: The ways to degenerate an L-shape.

By Corollary 7, there are seven ways to view a degenerate L-shape as the product of a limiting process operated on a regular L-shape. Fig. 2 (S2), (S3), (S5), (S6) and (S7) show five processes of shrinking a subrectangle with a side (or two sides) of length approaching zero; Fig. 2 (S1) and (S4) show two processes of cutting off a subrectangle with a side of length approaching ℓ or h. When $\epsilon = 0$, they all represent the same rectangle. But the different underlying process can induce different values of (ℓ, h, p, n).

Fiol, Yebra, Alegre, and Valero [6] pointed out that an L-shape, regular or degenerate, always tessellates the plane. Then $(\ell, -n)$ and $(-p, h)$ are simply two independent vectors characterizing the distribution of the nodes labelled by 0 (will be referred to as the 0-nodes) as seen by the equations:

$$\begin{array}{rcrcll} \ell a & - & nb & \equiv & 0 & \pmod{N} \\ -pa & + & hb & \equiv & 0 & \pmod{N}. \end{array} \tag{3.2}$$

Note that $(\ell, -n)$ is a vector in the fourth quadrant, and $(-p, h)$ one in the second. But there are other choices of two independent vectors.

4 Cheng-Hwang's algorithm

Cheng and Hwang [5] gave an algorithm (CH-ALGO in short) to solve for (ℓ, h, p, n) for $DL(N; a, b)$. The algorithm works regardless whether the L-shape is regular or not and the obtained (ℓ, h, p, n) satisfy the basic congruence equations in (3.2). For completeness, we give a brief review of this algorithm (note that the weight of each link in the given double-loop network is assumed to be 1).

CHENG-HWANG-ALGORITHM.

Input: $DL(N; a, b)$.

Output: (ℓ, h, p, n) of the L-shape of $DL(N; a, b)$.

Let d, d', N', a' and b' be defined as in (2.1).

Let s_0 be the integer with

$$a's_0 + b' \equiv 0 \pmod{N'}, \; 0 \le s_0 < N'.$$

Let $s_{-1} = N'$ and define q_i, s_i, recursively (by the Euclidean algorithm) as follows:

$$\begin{array}{lcll} s_{-1} & = & q_1 s_0 + s_1, & 0 \le s_1 < s_0 \\ s_0 & = & q_2 s_1 + s_2, & 0 \le s_2 < s_1 \\ s_1 & = & q_3 s_2 + s_3, & 0 \le s_3 < s_2 \\ & & \cdots & \\ s_{k-2} & = & q_k s_{k-1} + s_k, & 0 \le s_k < s_{k-1} \\ s_{k-1} & = & q_{k+1} s_k, & 0 = s_{k+1} < s_k. \end{array} \tag{4.3}$$

Define integers U_i by $U_{-1} = 0$, $U_0 = 1$, and

$$U_{i+1} = q_{i+1} U_i + U_{i-1}, \; i = 0, 1, \cdots, k. \tag{4.4}$$

By induction,

$$s_i U_{i+1} + s_{i+1} U_i = N', \; i = 0, 1, \cdots, k. \tag{4.5}$$

Regard $s_{-1}/U_{-1} = \infty > x$ for real number x. Since $\{s_i\}_{i=-1}^{k+1}$ and $\{U_i\}_{i=-1}^{k+1}$ are strictly decreasing and increasing, respectively, we have

$$0 = \frac{s_{k+1}}{U_{k+1}} < \frac{s_k}{U_k} < \cdots < \frac{s_0}{U_0} < \frac{s_{-1}}{U_{-1}} = \infty.$$

Let u be the largest odd integer such that $d < \frac{s_u}{U_u}$. Define

$$v = \left\lceil \frac{s_u - dU_u}{s_{u+1} + dU_{u+1}} \right\rceil - 1.$$

Let

$$\ell' = s_u - vs_{u+1},\ h' = U_u + (v+1)U_{u+1},\ p' = s_u - (v+1)s_{u+1},\ n' = U_u + vU_{u+1}.$$

Then

$$(\ell, h, p, n) = (\ell', dh', p', dn').$$

End-of-CHENG-HWANG-ALGORITHM.

Now we characterize the (ℓ, h, p, n) obtained by CH-ALGO when $DL(N; a, b)$ has a degenerate L-shape. By Theorem 5, it suffices to consider the case that $a|N$. Since $a|N$, CH-ALGO derives

$$d = a,\ d' = \gcd(N, b),\ N' = N/d = N/a,\ a' = 1,\ b' = b \bmod N',\ s_{-1} = N'.$$

So we have

Lemma 8 $s_i \equiv (-1)^i U_i s_0 \pmod{N'}$ for $1 \le i \le k+1$.

Proof. By (4.3) and (4.4), $s_1 = s_{-1} - q_1 s_0 = N' - U_1 s_0$, $s_2 = s_0 - q_2 s_1 = s_0 - q_2(N' - U_1 s_0) = -q_2 N' + (1 + q_2 U_1)s_0 = -q_2 N' + U_2 s_0$. Thus $s_1 \equiv (-1)^1 U_1 s_0 \pmod{N'}$ and $s_2 \equiv (-1)^2 U_2 s_0 \pmod{N'}$. We prove the general case by induction on i. Assume this lemma holds for $i \le t$. By (4.3), $s_{t+1} = s_{t-1} - q_{t+1} s_t$. By (4.4), $U_{t+1} = U_{t-1} + q_{t+1} U_t$. Thus by induction,

$$s_{t+1} \equiv (-1)^{t-1} U_{t-1} s_0 - q_{t+1}(-1)^t U_t s_0 \pmod{N'}.$$

Since $(-1)^{t-1} U_{t-1} s_0 - q_{t+1}(-1)^t U_t s_0 = (-1)^{t+1}(U_{t-1} + q_{t+1} U_t)s_0 = (-1)^{t+1} U_{t+1} s_0$, we have $s_{t+1} \equiv (-1)^{t+1} U_{t+1} s_0 \pmod{N'}$. Hence the lemma. ∎

Theorem 9 *If $DL(N;a,b)$ satisfies*

(C1), then CH-ALGO derives an L-shape of shape (S2) with $(\ell,h,p,n)=(N',d,i,0)$;

(C2), then CH-ALGO derives an L-shape of shape

(S1) with $(\ell,h,p,n)=(d',j+\left\lceil\frac{d'-j}{\frac{N}{d'}}\right\rceil\frac{N}{d'},d',j+(\left\lceil\frac{d'-j}{\frac{N}{d'}}\right\rceil-1)\frac{N}{d'})$ if $j<\frac{N}{2d'}$;

(S3) with $(\ell,h,p,n)=(d',\frac{N}{d'},0,j)$ if $j\geq\frac{N}{2d'}$;

(C3), then CH-ALGO derives an L-shape of shape

(S1) with $(\ell,h,p,n)=(d',\lceil\frac{d'}{d}\rceil d,d',(\lceil\frac{d'}{d}\rceil-1)d)$ if $d<d'$;

(S5) with $(\ell,h,p,n)=(d',d,d',0)$ if $d>d'$.

Proof. First suppose $DL(N;a,b)$ satisfies (C1). Then there exists $1\leq i\leq\ \min\{d,N'-1\}$ such that $db\equiv ia\pmod N$. Since $a=d$, we have $b\equiv i\pmod{N'}$. Since $b'=b\pmod{N'}$ and $1\leq i\leq N'-1$, it follows that

$$b'=i.$$

Since $a's_0+b'=s_0+b'\equiv 0\pmod{N'}$ and $0\leq s_0<N'$,

$$s_0=N'-b'.$$

By (4.4), $U_1=q_1$. By (4.3), $s_{-1}=q_1s_0+s_1$ and $q_1\geq 1$. So

$$\frac{s_1}{U_1}=\frac{s_1}{q_1}=\frac{s_{-1}}{q_1}-s_0=N'(\frac{1}{q_1}-1)+b'\leq b'=i\leq d.$$

Therefore $u=-1$. Since $b'=i\leq d$, we have $N'\leq(N'-b')+d$; therefore $\left\lceil\frac{N'}{(N'-b')+d}\right\rceil=1$. Thus $v=\left\lceil\frac{s_{-1}-dU_{-1}}{s_0+dU_0}\right\rceil-1=\left\lceil\frac{N'}{(N'-b')+d}\right\rceil-1=0$. Hence $m=s_0=N'-b'>0$, $n=d(U_{-1}+vU_0)=0$, $p=s_{-1}-(v+1)s_0=b'=i>0$, $q=dU_0=d>0$. Thus the L-shape is of shape (S2) and

$$(\ell,h,p,n)=(N',d,i,0).$$

Now suppose $DL(N;a,b)$ satisfies (C2). Then $DL(N;a,b)$ does not satisfy (C3). Hence $N>dd'$. Assume that

$$N=dd'N''.$$

Then $N'' > 1$. By Theorem 2, there exists $1 \le j \le \min\{d'-1, N/d'-1\}$ such that $d'a \equiv jb \pmod{N}$. Since $d = a$, we have $d'd \equiv jb \pmod{N}$. Since $\gcd(N, b) = d'$ and $N = dd'N''$, it follows that $d|j$. Let $j = dj'$. Then $d'd \equiv dj'b \pmod{dN'}$, which implies $d' \equiv j'b \pmod{N'}$. Thus

$$d' \equiv j'b' \pmod{N'}.$$

Since $\gcd(N', b') = \gcd(N', b) = \gcd(N, b)$ and $\gcd(N, b) = d'$,

$$\gcd(N', b') = d'.$$

Since $a's_0 + b' = s_0 + b' \equiv 0 \pmod{N'}$ and $0 \le s_0 < N'$,

$$s_0 = N' - b'.$$

Since $s_k = \gcd(s_{-1}, s_0) = \gcd(N', N' - b') = \gcd(N', b')$,

$$s_k = d'.$$

By (4.5), $s_kU_{k+1} + s_{k+1}U_k = N'$. Since $s_k = d'$ and $s_{k+1} = 0$, it follows that $d'U_{k+1} = d'N''$. Thus

$$U_{k+1} = N''.$$

By Lemma 8, $s_k \equiv (-1)^kU_ks_0 \equiv (-1)^kU_k(N' - b') \equiv (-1)^{k+1}U_kb' \pmod{N'}$. Since k is either odd or even, there are two cases:

Case 1. k is odd.

Then $s_k \equiv U_kb' \pmod{N'}$. Since $s_k = d' \equiv j'b' \pmod{N'}$, we have $U_kb' \equiv j'b' \pmod{N'}$. Thus

$$(U_k - j')b' \equiv 0 \pmod{N'}.$$

Since $U_k < U_{k+1}$ and $U_{k+1} = N''$, we therefore have $U_k < N''$. Since $j < N/d'$, we then have $j' < N''$. Since $(U_k - j')b' \equiv 0 \pmod{N'}$, $\gcd(N', b') = d'$, $U_k < N''$ and $j' < N''$, it follows that

$$U_k = j'.$$

Then $\frac{s_k}{U_k} = \frac{d'}{j'} > d$. Hence $u = k$. Since $dU_k = dj' = j$ and $dU_{k+1} = dN'' = \frac{N}{d'}$,

$$v + 1 = \left\lceil \frac{s_k - dU_k}{s_{k+1} + dU_{k+1}} \right\rceil = \left\lceil \frac{d' - j}{\frac{N}{d'}} \right\rceil.$$

Thus $m = s_{k+1} = 0$, $n = d(j' + vN'') = j + v\frac{N}{d'} > 0$, $p = s_k - (v+1)s_{k+1} = d' > 0$, $q = dU_{k+1} = \frac{N}{d'} > 0$. So the L-shape is of shape (S1) and

$$(\ell, h, p, n) = (d', j + \left\lceil \frac{d' - j}{\frac{N}{d'}} \right\rceil \frac{N}{d'}, d', j + (\left\lceil \frac{d' - j}{\frac{N}{d'}} \right\rceil - 1)\frac{N}{d'}).$$

Since k is odd and $\{U_i\}_{i=-1}^{k+1}$ are strictly increasing, we clearly have $U_{k-1} \geq 1$. Since $q_{k+1} \geq 2$, by (4.4),

$$j = dj' = dU_k = d\frac{(U_{k+1} - U_{k-1})}{q_{k+1}} < d\frac{U_{k+1}}{2} = d\frac{N''}{2} = \frac{N}{2d'}.$$

Case 2. k is even.

Then $s_k \equiv -U_k b' \pmod{N'}$. Since $s_k = d' \equiv j'b' \pmod{N'}$, we have $-U_k b' \equiv j'b' \pmod{N'}$. Thus

$$(U_k + j')b' \equiv 0 \pmod{N'}.$$

Since $U_k < U_{k+1}$ and $U_{k+1} = N''$, we therefore have $U_k < N''$. Since $j < N/d'$, we then have $j' < N''$. Since $(U_k + j')b' \equiv 0 \pmod{N'}$, $\gcd(N', b') = d'$, $U_k < N''$ and $j' < N''$, it follows that

$$U_k = N'' - j'.$$

By (4.3), (4.4) and by the fact that $q_{k+1} \geq 2$ and $d' > j$,

$$\begin{aligned} s_{k-1} - dU_{k-1} &= q_{k+1}s_k - d(U_{k+1} - q_{k+1}U_k) \\ &= q_{k+1}d' - d(N'' - q_{k+1}(N'' - j')) \\ &= q_{k+1}(d' + \frac{N}{d'} - j) - \frac{N}{d'} > 0. \end{aligned}$$

In other words, $\frac{s_{k-1}}{U_{k-1}} > d$. Hence $u = k - 1$. Since $dU_k = d(N'' - j') = \frac{N}{d'} - j$,

$$v + 1 = \left\lceil \frac{s_{k-1} - dU_{k-1}}{s_k + dU_k} \right\rceil = \left\lceil \frac{q_{k+1}(d' + \frac{N}{d'} - j) - \frac{N}{d'}}{d' + \frac{N}{d'} - j} \right\rceil = \left\lceil q_{k+1} - \frac{\frac{N}{d'}}{d' + \frac{N}{d'} - j} \right\rceil = q_{k+1}.$$

Thus $m = s_k = d' > 0,\ n = d(U_{k-1} + (q_{k+1} - 1)U_k) = d(U_{k+1} - U_k) = d(N'' - (N'' - j')) = j > 0,\ p = s_{k-1} - q_{k+1}s_k = s_{k+1} = 0,\ q = dU_k = \frac{N}{d'} - j > 0$. So the L-shape is of shape (S3) and

$$(\ell, h, p, n) = (d', \frac{N}{d'}, 0, j).$$

Note that since $U_{k-1} \geq 0$ and $q_{k+1} \geq 2$,

$$j = dj' = d(N'' - U_k) = \frac{N}{d'} - d\frac{(U_{k+1} - U_{k-1})}{q_{k+1}} \geq \frac{N}{d'} - d\frac{N''}{2} \geq \frac{N}{d'} - \frac{N}{2d'} = \frac{N}{2d'}.$$

Note that when k is even, we have $j \geq \frac{N}{2d'}$. This implies that if $j < \frac{N}{2d'}$, then k is odd, which means Case 1 occurs. Therefore CH-ALGO derives an L-shape of shape (S1) if $j < \frac{N}{2d'}$ and an L-shape of shape (S3) if $j \geq \frac{N}{2d'}$.

Finally, suppose $DL(N; a, b)$ satisfies (C3). By Theorem 2, $N = dd'$; thus $N' = d'$. Since $db \equiv 0 \pmod{N}$, we have $b \equiv 0 \pmod{N'}$. Since $b' = b \bmod N'$, it follows that $b' = 0$. Therefore $s_0 = 0$ and

$$\frac{s_0}{U_0} = \frac{0}{1} < d.$$

Hence $u = -1$ and $v = \lceil \frac{N'}{d} \rceil - 1 = \lceil \frac{d'}{d} \rceil - 1$. Since $d \neq d'$, there are two cases:

Case 1. $d < d'$.

Then $v > 0$. So $m = s_0 = 0,\ n = d(U_{-1} + vU_0) = dv > 0,\ p = s_{-1} - s_0 = d' > 0,\ q = dU_0 = d > 0$. Thus the L-shape is of shape (S1) with

$$(\ell, h, p, n) = (d', \left\lceil \frac{d'}{d} \right\rceil d, d', (\left\lceil \frac{d'}{d} \right\rceil - 1)d).$$

Case 2. $d > d'$.

Then $v = 0$. So $m = s_0 = 0,\ n = d(U_{-1} + vU_0) = 0,\ p = s_{-1} - s_0 = N' - 0 = d' > 0,\ q = dU_0 = d > 0$. Thus the L-shape is of shape (S5) with

$$(\ell, h, p, n) = (d', d, d', 0).$$

∎

5 Chen-Hwang's rule

Chen and Hwang [3] gave a set of rules (CH-RULE in short) to determine the parameters ℓ, h, p, n for a degenerate L-shape. Their rules always set ℓ to the width and h to the height of the degenerate L-shape. We now briefly describe their rules.

CHEN-HWANG-RULE.

(i) Suppose $hb \not\equiv \ell a \equiv 0 \pmod{N}$. Let the zero immediately above the L-shape occurs at column j. Then

$$p = \ell - j, \; n = 0.$$

(ii) Suppose $\ell a \not\equiv hb \equiv 0 \pmod{N}$. Let the zero immediately to the right of the L-shape occurs at row i. Then

$$p = 0, \; n = h - i.$$

(iii) Suppose $\ell a \equiv hb \equiv 0 \pmod{N}$. If $h > \ell$, follow rule (i); otherwise, follow rule (ii).

End-of-CHEN-HWANG-RULE.

The ℓ, h, p, n chosen by CH-RULE satisfy the basic congruence equations in (3.2). Fig. 3 illustrates these rules.

			0		
14	2	5	8	11	
7	10	13	1	4	
0	3	6	9	12	0

$(\ell, h, p, n) = (5, 3, 2, 0)$
(a) rule (i)

0					
10	14	3	7	11	0
5	9	13	2	6	
0	4	8	12	1	

$(\ell, h, p, n) = (5, 3, 0, 1)$
(b) rule (ii)

0					
10	13	1	4	7	
5	8	11	14	2	
0	3	6	9	12	0

$(\ell, h, p, n) = (5, 3, 0, 3)$
(c) rule (iii)

Figure 3: The (ℓ, h, p, n) determined by CH-RULE.

W now characterize the (ℓ, h, p, n) obtained by CH-RULE when $DL(N; a, b)$ has a degenerate L-shape.

Theorem 10 *If $DL(N;a,b)$ satisfies*

(C1), then CH-RULE derives an L-shape of shape (S2) with $(\ell,h,p,n)=(N',d,i,0)$*;*

(C2), then CH-RULE derives an L-shape of shape (S3) with $(\ell,h,p,n)=(d',\frac{N}{d'},0,j)$*;*

(C3), then CH-RULE derives an L-shape of shape

(S6) with $(\ell,h,p,n)=(d',d,0,d)$ *if* $d<d'$*;*

(S5) with $(\ell,h,p,n)=(d',d,d',0)$ *if* $d>d'$*.*

Proof. First, suppose $DL(N;a,b)$ satisfies (C1). Then there exists $1\le i\le \min\{d,N'-1\}$ such that $db\equiv ia \pmod N$. By Theorem 2, $\ell=N'$, $h=d$; also, (C1) $\Rightarrow$ (1). So $hb\not\equiv \ell a\equiv 0 \pmod N$. Let the zero immediately above the L-shape occurs at column j. Since $\ell a\equiv 0 \pmod N$, $j=\ell-i$. So CH-RULE follows rule (i) and sets $p=\ell-j=i$, $n=0$. Then $m=\ell-p=j>0,\ n=0,\ p=i>0,\ q=h-n=h>0$; the L-shape is of shape (S2).

Next, suppose $DL(N;a,b)$ satisfies (C2). Then there exists $1\le j\le \min\{d',\frac{N}{d'}-1\}$ such that $d'a\equiv jb \pmod N$. By Theorem 2, $\ell=d'$ and $h=N/d'$; also, (C2) $\Rightarrow$ (2). So $\ell a\not\equiv hb\equiv 0 \pmod N$. Let the zero immediately to the right of L-shape occurs at row i. we have $i=h-j$. So CH-RULE follows rule (ii) and sets $p=0$, $n=h-i=j$. Then $m=\ell-p=\ell>0,\ n=j>0,\ p=0,\ q=h-n=N/d'-j>0$; the L-shape is of shape (S3).

Finally, suppose $DL(N;a,b)$ satisfies (C3). By Theorem 2, (C3) $\Rightarrow$ (3). So $\ell a\equiv hb\equiv 0 \pmod N$. Let the zero immediately above the L-shape occurs at column j and to the right of L-shape occurs at row i. Then $i=j=0$. Suppose $d<d'$. Then $h<\ell$. So CH-RULE follows rule (ii) and sets $p=0$, $n=h-i=h=d$. Thus $m=\ell-p=\ell>0,\ n=d>0,\ p=0,\ q=h-n=0$; the L-shape is of shape (S6). Suppose $d>d'$. Then $h>\ell$. So CH-RULE follows rule (i) and sets $p=\ell-j=\ell=d'$, $n=0$. Then $m=\ell-p=0,\ n=0,\ p=d'>0,\ q=h-n=d>0$; the L-shape is of shape (S5). ∎

6 The relations between CH-ALGO and CH-RULE

Both CH-ALGO and CH-RULE determine the four parameters ℓ, h, p, n for a degenerate L-shape. Unfortunately, the solution of (ℓ, h, p, n) using CH-RULE [3] does not always coincide with the values given by the CH-ALGO. For the example in Fig. 3 (b), the solution of the CH-RULE is

$$(\ell, h, p, n) = (5, 3, 0, 1)$$

and the solution of the CH-ALGO is

$$(\ell, h, p, n) = (5, 7, 5, 4)$$

(see Fig. 4). In this section, we will explain the relations between the two sets of solutions.

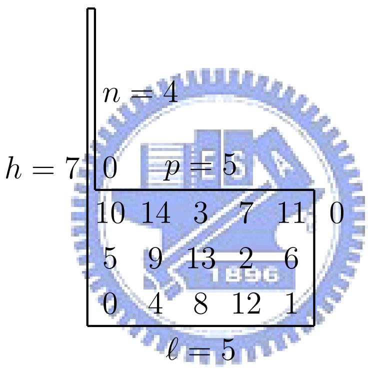


Figure 4: An alternative representation of the L-shape in Fig. 3 (b).

From Theorem 9 and Theorem 10, we know that CH-ALGO will not derive an L-shape of shape (S4) or (S6) or (S7) and CH-RULE will not derive an L-shape of shape (S1) or (S4) or (S7). We now further explain the reason below. CH-ALGO will not derive an L-shape of shape (S4) or (S6) because it always has $q = h - n = dU_{u+1} > 0$ (recall that $\{U_i\}_{i=-1}^{k+1}$ is strictly increasing, $U_{-1} = 0$ and $u \geq -1$). Also, CH-ALGO will not derive an L-shape of shape (S7) since if $n = d(U_u + vU_{u+1}) = 0$, then $u = -1$ and $v = 0$ and therefore $p = s_u - (v+1)s_{u+1} = s_{-1} - s_0 > 0$, a contradiction to the assumption that the L-shape is of shape (S7). CH-RULE will not derive an L-shape of shape (S1) or (S4) since it always sets ℓ to the width and h to the height of the degenerate L-shape. Also CH-RULE will not derive an L-shape of shape (S7) since it always has n and p not both zero. We now summarize the results of Theorem 9 and Theorem 10 in Table 1 and compare the degenerate shapes derived by CH-ALGO and CH-RULE in Table 2.

The following three corollaries follow from Theorem 9 and Theorem 10.

Corollary 11 *CH-ALGO and CH-RULE derive the same shape when $DL(N;a,b)$ satisfies (C1), satisfies (C2) and $j \geq \frac{N}{2d'}$ or satisfies (C3) and $d > d'$. CH-ALGO and CH-RULE derive different shapes when $DL(N;a,b)$ satisfies (C2) and $j < \frac{N}{2d'}$ or satisfies (C3) and $d < d'$.*

Let $(\hat{\ell}, \hat{h}, \hat{p}, \hat{n})$ denote the solution of CH-ALGO and $(\dot{\ell}, \dot{h}, \dot{p}, \dot{n})$, the solution of CH-RULE. Corollary 12 and Corollary 13 show that when the two sets of solutions are different, one can be obtained from the other.

Corollary 12 *If $DL(N;a,b)$ satisfies (C2) and $j < \frac{N}{2d'}$, then*

$$\hat{\ell} = \hat{p} = \dot{\ell},\ \hat{h} = \dot{n} + \left\lceil \frac{\dot{\ell} - \dot{n}}{\dot{h}} \right\rceil \dot{h},\ \hat{n} = \dot{n} + (\left\lceil \frac{\dot{\ell} - \dot{n}}{\dot{h}} \right\rceil - 1)\dot{h},$$

and

$$\dot{\ell} = \hat{\ell},\ \dot{h} = \hat{h} - \hat{n},\ \dot{p} = 0,\ \dot{n} = j.$$

Corollary 13 *If $DL(N;a,b)$ satisfies (C3) and $d < d'$, then*

$$\hat{\ell} = \hat{p} = \dot{\ell},\ \hat{h} = \left\lceil \frac{\dot{\ell}}{\dot{h}} \right\rceil \dot{h},\ \hat{n} = (\left\lceil \frac{\dot{\ell}}{\dot{h}} \right\rceil - 1)\dot{h},$$

and

$$\dot{\ell} = \hat{\ell},\ \dot{h} = \dot{n} = \hat{h} - \hat{n},\ \dot{p} = 0.$$

Table 1: The shapes derived by CH-ALGO and CH-RULE.

shape	S1	S2	S3	S4	S5	S6	S7
CH-ALGO	v	v	v		v		
CH-RULE		v	v		v	v	

Table 2: The comparison between CH-ALGO and CH-RULE.

condition	C1	C2		C3	
		$j < \frac{N}{2d'}$	$j \geq \frac{N}{2d'}$	$d < d'$	$d > d'$
CH-ALGO	S2	S1	S3	S1	S5
CH-RULE	S2	S3	S3	S6	S5
consistent	yes	no	yes	no	yes

7 Concluding remarks

Most of the results about the L-shapes of double-loop networks are given in terms of the four parameters ℓ, h, p, n. For example, the diameter of a double-loop network can be easily computed from its L-shape by the equation $\max\{\ell + h - p,\ \ell + h - n\} - 2$. In [4], Chen, Hwang and Liu transformed a mixed chordal ring network into a double-loop network and derived an upper bound for the diameter of a mixed chordal ring network from the L-shape of its corresponding double-loop network (see [4] for details). However, the parameters ℓ, h, p, n are not well defined in the degenerate case. In particular, both Cheng-Hwang [5] and Chen-Hwang [3] determine the four parameters for a degenerate L-shape. Unfortunately, the solutions given by the above two methods do not always coincide. In this paper, we have explored the respective meanings and the relations between these two sets of solutions.

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