國立交通大學

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博 士 論 文

耦合混沌系統的網絡中之同步化與微波變換

Synchronization and Wavelet Transform in Networks of Coupled Chaotic Systems

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摘　要

本論文的目的分成兩個部分。第一部份是研究耦合混沌系統在網格中的全域同步化。第二部份是理論地描述微波變換是如何影響所對應系統的同步化。基於矩陣測度的概念，我們獲得在網絡上全域同步化的穩定性。我們的結果可利用在十分廣義的拓樸連結上。更進一步地，藉由檢驗單一系統的向量場結構，我們就可以決定此系統是否有全域的同步化。不僅如此，我們也獲得對於所有系統全域同步化的耦合強度的精確下界。同步化耦合強度的下界是與耦合矩陣的第二大固有值 λ_2 的絕對值倒數成正比的關係。然而，對於特有的拓樸連結就像是擴散地耦合矩陣，當節點的個數增加時，λ_2 對零點越靠近。總結的來說，為了實現同步化，較大的耦合強度是被要求的。在[48]，魏…等人提出由微波轉換修改拓樸連接。做了這樣的處理後，$\lambda_2=\lambda_2(\alpha)$ 變成隨著微波常數 α 而變。他們還發現一個臨界的微波常數 α_c 可以被選擇使得 $\lambda_2(\alpha_c)$ 遠離零點，而不需要關心節點的個數。這重要地減少了臨界耦合強度的大小。當耦合矩陣是擴散耦合且具有週期與諾曼的邊界條件時，這種現象將被分析地證實。

Synchronization and Wavelet Transform in Networks of Coupled Chaotic Systems

Student: Chin-Lung Li Advisor: Jonq Juang

Department of Applied Mathematics
National Chiao Tung University

Abstract

The purpose of this thesis is two-fold. First, global synchronization in lattices of coupled chaotic systems is studied. Second, how wavelet transforms affect the synchronization of the corresponding systems is theoretically addressed. Based on the concept of matrix measures, global stability of synchronization in networks is obtained. Our results apply to quite general connectivity topology. Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized. In addition, a rigorous lower bound on the coupling strength for global synchronization of all oscillators is also obtained. The lower bound on the coupling strength for synchronization is proportional to the inverse of the magnitude of the second largest eigenvalue λ_2 of the coupling matrix. However, for a typical connectivity topology such as the diffusively coupled matrix, λ_2 moves closer to the origin, as the number of nodes increases. Consequently, a larger coupling strength is required to realize synchronization. In [48], Wei et al, proposed a wavelet transform to alter the connectivity topology. In doing so, $\lambda_2=\lambda_2(\alpha)$ becomes a quantity depending on wavelet parameter α. It is found there that a critical wavelet parameter α_c can be chosen to move $\lambda_2(\alpha_c)$ away from the origin regardless the number of nodes. This in turn greatly reduces the size of the critical coupling strength. Such phenomena are analytically verified when the coupling matrix is diffusively coupled with periodic and Neumann boundary conditions.

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Contents

Chapter 1

Coupled Systems in Lattices

1.1 Introduction and Formulation

Coupled chaotic systems are typically synthesized from simpler, low-dimensional systems to form new and more complex systems. This is often done with the intent of realistically modeling spatially extended systems, with the brief that dominant features of the underlying constituents will be retained. From an applications point of view this building up approach can also be used to create a novel system whose behavior is more flexible or richer than that of the constituents, but whose analysis and/or control remains tractable. These and other motivations have led to numerous studies of coupled systems in a wide range of disciplines. Synchronization has long been of interest in systems of identical or nearly identical coupled subsystems. The phenomenon of synchronization of coupled *chaotic* systems has recently become a topic of great interest, and is the focus of the present work. Systems that display this behavior are temporally chaotic, but spatially ordered or coherent. Here the coherence is of particular type-the dynamics is the same or nearly so for long periods of time for all coupled subsystems or large regions of them. The basic synchronization problem can be framed with the questions, "Will my system ever synchronize and, if so, under what conditions?" During the last few decades the study of networks of dynamical systems has attracted increasing attention [16-18,21-22,24,28-29,30-33,35-36,38-39,41-42,45-46,50-54]. The purpose to connect dynamical systems in networks is to get them to solve problems cooperatively. For instance, such networks are needed for information processing in the brain [17].

A particularly interesting form of dynamical behavior occurs in networks of coupled systems or oscillators when all of the subsystems behave in the same fashion; that is, they all do the same thing at the same time. Such behavior of a network simulates a continuous system that has a uniform movement, models neurons that synchronize, and coupled synchronized lasers and electronic circuit systems. The motion of the systems is described as follows. Let there be m nodes (oscillators). Assume $\mathbf{x}_i$ is the n-dimensional vector of dynamical variables of the ith node. Let the isolated (uncoupling) dynamics be $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t)$ for each node. We assume that $\mathbf{x}_i$ has a chaotic dynamics in the sense that its largest Lyapunov exponent is positive. Let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary function describing the coupling within the components of each node. The connectivity topology, indicating the coupling rules between nodes, is denoted by the coupling matrix $\mathbf{G} = (g_{ij})$. Then the equation of the motion reads

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t) + d\sum_{j=1}^{m} g_{ij}\mathbf{h}(\mathbf{x}_j), i = 1, 2, ..., m, \tag{1.1}$$

where d is a coupling strength. The $m-1$ constraints $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_m$ define the synchronization manifold $\mathfrak{M}$. The sum $\sum_{j=1}^{m} g_{ij} = 0$ is required for the invariance of this synchronization manifold $\mathfrak{M}$. We further assume that 0 is a simple eigenvalue of $\mathbf{G}$. Let $\mathbf{F}(\mathbf{x}, t) = (\mathbf{f}(\mathbf{x}_1, t), \mathbf{f}(\mathbf{x}_2, t), \ldots, \mathbf{f}(\mathbf{x}_m, t))^T$, $\mathbf{H}(\mathbf{x}) = (\mathbf{h}(\mathbf{x}_1), \mathbf{h}(\mathbf{x}_2), \ldots, \mathbf{h}(\mathbf{x}_m))^T$, and

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}, \quad \mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,n} \end{pmatrix}. \tag{1.2}$$

Then (1.1) can be written as the vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) + d(\mathbf{G} \otimes \mathbf{I}_n)\mathbf{H}(\mathbf{x}), \tag{1.3}$$

where $\otimes$ is the Kronecker product(see e.g.,[15]), and $\mathbf{I}_n$ is the $n \times n$ identity matrix. The simplest mode of the coordinated motion between dynamical systems is their

complete synchronization when all cells of the network acquire identical dynamical behavior. Consequently, one asks questions such as: What are the conditions for the stability of the synchronous state, especially with respect to coupling strengths and coupling configurations of the network? Typically, in networks of continuous time oscillators, the synchronous state becomes stable when the coupling strength between the oscillators exceeds a critical value. In this context, a central problem is to find the bounds for the coupling strengths so that the stability of synchronization is guaranteed. It is well-known (see e.g., [4,48,49]) that the lower bound for the coupling strength for synchronization is proportional to the inverse of the magnitude of the second largest eigenvalue λ_2 of the coupling matrix. However, for a typical connectivity topology such as the diffusively coupled matrix, λ_2 moves closer to the origin, as the number of nodes increases. Consequently, a larger coupling strength is required to realize synchronization. As a result, controlling chaos is apparently of great interest and importance [20,34,37,39,48-49]. It is found in [48] that the modification of a tiny fraction of wavelet subspaces of a coupling matrix could lead to a dramatic change in the properties of chaotic synchronization. Specifically, in doing so, $\lambda_2 = \lambda_2(\alpha)$ becomes a quantity depending on wavelet parameter α. It is found there that a critical wavelet parameter α_c can be chosen to move $\lambda_2(\alpha_c)$ away from the origin regardless the number of nodes. This in turn greatly reduces the size of the critical coupling strength.
To be self-contained, we briefly describe such wavelet transform. Let

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}_{n\times n,} \tag{1.4a}$$

be a matrix with the dimension of each block matrix A_{kl} being $2^i \times 2^i$. By an i-scale wavelet operator W [14,48], the matrix A is transformed into $W(A)$ of the form

$$W(A) = \begin{pmatrix} \widetilde{A}_{11} & \cdots & \widetilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \widetilde{A}_{n1} & \cdots & \widetilde{A}_{nn} \end{pmatrix}_{n\times n,} \tag{1.4b}$$

where each entry of $\widetilde{A}_{kl}$ is the average of entries of A_{kl}, $1 \leq k, l \leq n$.
For a given matrix, the above wavelet transform allows a perfect reconstruction (inverse wavelet transform), by which there is nothing to gain: $A = W^{-1}(W(A))$. In [48], a simple operator O_{k} is introduced to attain a desirable coupling matrix. That is,

$$C = W^{-1}(O_{\mathrm{k}}(W(A))) = A + (k-1)W(A) =: A + \alpha W(A), \tag{1.4c}$$

where O_{k} be the multiplication of a scalar factor α on each block matrix $\widetilde{A}_{kl}$. After such reconstruction, the critical strength d_c is again, determined in term of the second largest eigenvalue of C. A numerical simulation of a coupled system of 512 Lorenz oscillators in [48] shows that with $\mathbf{h} = \mathbf{I}_3$ and $\mathbf{G}$ being diffusively coupled with periodic boundary conditions, the critical coupling strength d_c decreases linearly with respect to the increase of α up to a critical value α_c. The smallest d_c is about 6, which is about 10^3 times smaller than the original critical coupling strength, indicating the efficiency of the proposed approach.
To understand how such wavelet transform affects the critical coupling strength, we consider $\mathbf{G}$ to be diffusively coupled with mix boundary conditions. Let such mix boundary conditions be parameterized by a parameter β. Such reconstructed coupling matrix $A_\beta + \alpha W(A_\beta)$ is to be denoted by $\mathbf{G} = \mathbf{G}(\alpha, \beta)$. Let $l = \frac{m}{2^i} \in \mathbb{N}$, where i is a fixed positive integer. Here $\mathbf{G}(\alpha, \beta)$ is an $l \times l$ block matrix of the following form.

$$\mathbf{G}(\alpha,\beta) = \begin{pmatrix} \mathbf{G}_1(\alpha,\beta) & \mathbf{G}_2(\alpha,1) & 0 & \cdots & 0 & \mathbf{G}_2^T(\alpha,\beta) \\ \mathbf{G}_2^T(\alpha,1) & \mathbf{G}_1(\alpha,1) & \mathbf{G}_2(\alpha,1) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{G}_2^T(\alpha,1) & \mathbf{G}_1(\alpha,1) & \mathbf{G}_2(\alpha,1) \\ \mathbf{G}_2(\alpha,\beta) & 0 & \cdots & 0 & \mathbf{G}_2^T(\alpha,1) & \hat{I}\mathbf{G}_1(\alpha,\beta)\hat{I} \end{pmatrix}_{l\times l.} \tag{1.5a}$$

Here

$$\mathbf{G}_1(\alpha,\beta) = \begin{pmatrix} -1-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{2^j\times 2^j} - \frac{\alpha(1+\beta)}{2^{2j}} ee^T$$

$$=: A_1(\beta, 2^j) - \frac{\alpha(1+\beta)}{2^{2j}} ee^T, \tag{1.5b}$$

where $e = (1,1,...,1)^T$, j is a positive integer, $\alpha > 0$ is a (wavelet) scalar factor and $\beta \in \mathbb{R}$ represents a mixed boundary constant. Moreover,

$$\mathbf{G}_2(\alpha,\beta) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \beta & 0 & \cdots & 0 \end{pmatrix}_{2^j\times 2^j} + \frac{\alpha\beta}{2^{2j}} ee^T$$

$$=: A_2(\beta, 2^j) + \frac{\alpha\beta}{2^{2j}} ee^T, \tag{1.5c}$$

$$\hat{\mathbf{I}} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \cdot & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{1.5d}$$

The dimension of $\mathbf{G}(\alpha,\beta)$ is $l2^j \times l2^j$. From here on, we shall call l and j the block and the wavelet dimensions of $\mathbf{G}(\alpha,\beta)$, respectively. $\mathbf{G}(\alpha,\beta)$ is a block circulant matrix (see e.g., [15]) only if $\beta = 1$. It is well-known, see e.g., Theorem 5.6.4 of [15], that for each α the eigenvalues of $\mathbf{G}(\alpha,1)$ consists of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $\mathbf{G}(\alpha,1)$.

1.2 Description of the Results

The first results in the thesis are to give another approach to study global synchronization of coupled chaotic systems (1.3). Part of the results in this direction is based on the paper in [27]. Our coupling rules are allowed to be asymmetric and/or some competitive ($g_{ij} < 0$, $i \neq j$) couplings between cells $\mathbf{x}_i$ and $\mathbf{x}_j$ as long as the coupled system is bounded dissipative. In addition, the partial-state coupling in our approach is allowed to have the form satisfying (3.31). Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized. We also obtain a rigorous lower bound on the coupling strength for the global synchronization of all oscillators with coupling configuration satisfying (3.20a), and (3.20b). Finally, the concept of matrix measures is introduced to obtain such global results. The second part of the thesis is to prove analytically that the improvement by wavelet transform as described in section 1.1 is indeed true. Some new phenomena are also discovered via our analysis. The results in this part are re-organized from papers in [25,26]. In the following, we give a detailed description of the results. To understand the effectiveness of wavelet transform, it amounts to studying the eigencurve problem for a class of "perturbed" block circulant matrices. That is,

$$\mathbf{G}(\alpha,\beta)\mathbf{b} = \lambda(\alpha,\beta)\mathbf{b}. \tag{1.6}$$

We prove that for m being a multiple of 4, then

$$\lambda_2(\alpha,1) = \begin{cases} \lambda_1^+(\alpha,1), & 0 \leq \alpha \leq \frac{1}{\sin^2\frac{\pi}{l}}, \\ \lambda_{\frac{l}{2}}^+(\alpha,1) = -2, & \alpha \geq \frac{1}{\sin^2\frac{\pi}{l}}. \end{cases}$$

Let $m = 2l$ be an even number which is not multiple of 4. We show that $\lambda_2(\alpha,1) = \lambda_{[\frac{l}{2}]}^+(\alpha,1)$ for α sufficiently large, where $[\frac{l}{2}]$ = the largest positive integer that is less than or equal to $\frac{n}{2}$. Moreover, we prove that for such m that $\lambda_2(\alpha,1) < -2$, whenever $\alpha > \frac{1}{\sin^2\frac{\pi}{l}}$. With those results above, we get considerable more information than those obtained in [43]. Among other, such result suggests that if the number m of oscillators be even but not a multiple of 4, then the wavelet method works even better. Specifically, it is better in the sense that the corresponding second largest eigenvalue $\lambda_2(\alpha,1)$ is further away from 0, and, hence, gives even smaller critical length. Our second main results of this part are the following. First, the reduced eigenvalue problem for $\mathbf{G}(\alpha, 0)$ is completely solved. Some partial results for the reduced eigenvalue problem of $\mathbf{G}(\alpha, \beta)$ are also obtained. Second, we are then able to understand behavior of $\lambda_2(\alpha, 0)$ and $\lambda_2(\alpha, 1)$ for any j and $l \in \mathbb{N}$.

1.3 Related Work

General approaches to local synchronization of chaotic systems have been proposed, including the master stability function (MSF)- based criteria [3,35-36,38-39,42], originated by Pecora and Carroll [39], and recently the matrix measure approach in [12]. The former computes the Lyapunov exponent of the variational equations, while the latter uses the concept of matrix measures to give criteria on the variation equations. Moreover, local synchronization in a complex network of asymmetrically coupled units was also obtained [11,24] via MSF-based criteria.

Global synchronization of chaotic systems was also intensively studied. The methods include Lyapunov function- based criteria with symmetrical connections [4,6-9,41,50-53] or asymmetrical connections [5,50], and the partial contraction approach [45]. For Lyapunov-based criteria, the partial-state coupling matrix, determining which variables couple the oscillators, is assumed to have the form satisfying (3.20c). While the partial contraction approach needs to verify the contraction of the system, depending on the state variables and time t, which is not a small task. In developing the theory of global synchronization of chaotic systems, one needs to assume the bounded dissipation of the coupled system, that is, all solutions of the coupled system are, in some sense, eventually bounded. Such assumption plays the role of an a priori estimate. However, in obtaining the theory of local synchronization, one dose not need to know the bounded dissipation of the coupled system. Thus, not surprisingly, the criteria in getting local synchronization are composed of a term that describes how chaotic the uncoupled system is and a term that depends on how the configuration of the networks is formed. Some of their work are to be discussed in more details in Chapters 2 and

3. The first analytical work to understand the wavelet transform was done by Shieh, Wei, Wang and Lai et al. [43]. We summary their main results in the following.

Theorem 1.3.1. *Let $N \times N$, $N = 8k$, $k \in N$, be the dimension of the matrix $\mathbf{G}(\alpha, 1)$. Let the dimension of each block matrix in $\mathbf{G}(\alpha, 1)$ is $2^i \times 2^i$. Then the following assertions hold.*

(i) $\rho_i := 2\cos\frac{\pi}{2^i} - 2$ is an eigenvalue of $\mathbf{G}(\alpha, 1)$.

(ii) The second eigenvalue $\lambda_2(\alpha, 1)$ of $\mathbf{G}(\alpha, 1)$ is decreasing in α. Moreover, $\lambda_2(\alpha, 1) = \rho_i$ whenever $\alpha \geq \dfrac{-2^i \rho_i}{4\sin^2 \frac{2^i \pi}{N}}$.

Note that $\mathbf{G}(\alpha, 1)$ is a block circulant matrix (see e.g., [15]). A classical result of a block circulant matrix states that its eigenvalues exactly consist of those of a certain linear combinations of its block matrices. (see e.g., Thm 5.6.4 of [15]). The proof of Theorem 1.3.1 was then reduced to working on the eigenvalues of those linear combinations of block matrices of $\mathbf{G}(\alpha, 1)$.

Chapter 2

Review of Local Synchronization and Global Synchronization

In this chapter, we shall review some of known results for local synchronization and global synchronization in networks of coupled chaotic systems. The local theory includes the *master stability function* (**MSF**), originated by Pecora and Carroll [39], and the matrix measures approach by chen [12]. The theory of global synchronization under review includes Lyapunov function approaches by Belykh [4] and Wu and Chua [53], respectively, and partial contraction approach by W. Wang, and J-J. E. Slotine[45].

2.1 Master Stability Function

In this section, we introduce the master stability function to show the stability condition of local synchronization of coupled system (2.15), which is developed by L. M. Pecora and T. L. Corroll [39]. In determining the stability of the synchronous state, various criteria are possible. The weakest is that the maximum Lyapunov exponent or Floquet exponent be negative. In this respect, we get the variational equation of coupled system (1.3) by letting ξ_i be the variations on the ith node and the collection of variations is $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$. Then,

$$\frac{d\xi}{dt} = [D\mathbf{F}(t) + d(\mathbf{G} \otimes \mathbf{I}_n)D\mathbf{H}(t)]\xi, \tag{2.1}$$

where $D\mathbf{F}$, $D\mathbf{H}$ are the Jacobian matrices of $\mathbf{F}$ and $\mathbf{H}$, respectively. Equation (2.1) is used to calculate Floquet or Lyapunov exponents. We really want to consider only variations ξ which are transverse to the synchronization manifold $\mathfrak{M}$ and $\mathbf{G}$ is a diagonal matrix. Moreover, the Jacobian matrix $D\mathbf{F}$ and $D\mathbf{H}$ are the same for each block, since they are evaluated on the synchronized state. If we rearrange the block diagonalized variational equation in equation (2.1), this leaves us with each block having the form

$$\frac{d\xi_k}{dt} = [D\mathbf{f}(t) + d\lambda_k D\mathbf{h}(t)]\xi_{k,} \tag{2.2}$$

where λ_k is an eigenvalue of $\mathbf{G}$, $k = 1, 2, \ldots, m$. Note that we order the eigenvalues of $\mathbf{G}$ with decreasing order $\lambda_1 = 0 > \lambda_2 \geq \cdots \geq \lambda_m$. For $k = 1$, we have the variational equation for the synchronization manifold $\mathfrak{M}$ ($\lambda_1 = 0$), so we have succeeded in separating that from the other, transverse directions. All other k's correspond to transverse eigenvectors.

Thus, for each k, the form of each block in equation (2.2) is the same with only the scalar multiplier $d\lambda_k$ differing for each. Thus, one can reformulate the above equation as follows,

$$\frac{d\zeta}{dt} = [D\mathbf{f}(t) + (\alpha + i\beta)D\mathbf{h}(t)]\zeta, \tag{2.3}$$

that is the *master stability equation* (**MSE**). This equation depends on the two parameters α and β, and the corresponding largest Floquet or Lyapunov exponent, which is also a function of α and β, represents the *master stability function* (**MSF**). We now give a property of the **MSF** as follows.

Theorem 2.1.1. *If the function $\boldsymbol{h}$ in (1.1) is equal to the identity function, that is $D\boldsymbol{H} = \boldsymbol{I}$, then the Lyapunov exponents $L_i(\alpha, \beta)$ of the* master stability equation *(2.3) are*

$$L_i(\alpha, \beta) = L_i(0, 0) + \alpha, \qquad 1 \leq i \leq n. \tag{2.4}$$

The behavior of the largest Lyapunov exponent with respect to $(\alpha+i\beta)$ fully accounts for linear stability of the synchronization manifold. Namely, the synchronized state (associated with $\lambda_1 = 0$), is stable if all the remaining blocks (associated with λ_i, $i = 2, \cdots, m$) have negative Lyapunov exponents. Moreover, if we suppose that the Lyapunov exponents of (2.3) are in the decreasing order

$$L_1(\alpha,\beta) \geq L_2(\alpha,\beta) \geq \cdots \geq L_n(\alpha,\beta) \quad \text{for } \alpha,\ \beta \in \mathbb{R}. \tag{2.5}$$

Then, the stability condition can be given by

$$L_1(\alpha,\beta) = L_1(0,0) + d\lambda_2 =: L_{max} + d\lambda_2 < 0, \tag{2.6}$$

As a consequence, the second largest eigenvalue λ_2 is dominant in controlling the stability of chaotic synchronization and the critical coupling strength d_c can be determined in terms of λ_2,

$$d_c = \frac{L_{max}}{-\lambda_2}. \tag{2.7}$$

2.2 Matrix Measure Criteria

In this section, another criteria for (local) synchronization is provided by M. Y. Chen [12], which is based on the matrix measure and the Lyapunov converse theorem, analytically. Numerically, one of the local synchronization criteria by computing the Lyapunov exponent of the **MSF** have been introduced. Moreover, the matrix theory can also be used to analyze the stability conditions for the synchronized chaos.

First, we introduce the concept of matrix measure [44]. The matrix measure of matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is

$$\mu_{\cdot}(A) = \lim_{\epsilon\to 0^+} \frac{\|I_n + \epsilon A\| - 1}{\epsilon} \tag{2.8}$$

where $\|\cdot\|$ is the matrix norm, and I_n is the identity matrix.

When matrix norms $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$, $\|A\|_2 = [\lambda_{\max}(A^TA)]^{1/2}$, and $\|A\|_\infty =$

$\max_i \sum_{j=1}^{n} |a_{ij}|$, we can, respectively, obtain the matrix measures

$$\mu_1(A) = \max_j \{a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}|\} \tag{2.9}$$

$$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A^T + A) \tag{2.10}$$

$$\mu_\infty(A) = \max_i \{a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}|\} \tag{2.11}$$

where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue. Let $\Omega = \{1, 2, \infty\}$ denote the set of the above matrix measures. If $\theta \in \Omega$, $\mu_\theta(\cdot)$ is one of the matrix measures $\mu_1(\cdot)$, $\mu_2(\cdot)$, and $\mu_\infty(\cdot)$.

Now, we present a lemma on the manifold $\mathfrak{M}$.

Lemma 2.2.1. *If the n-dimensional linear time-varying systems in (2.2)*

$$\frac{d\xi_k}{dt} = [D\mathbf{f}(t) + d\lambda_k D\mathbf{h}(t)]\xi_{k,} \qquad 2 \leq k \leq m$$

are exponentially stable about its zero solution, then the manifold $\mathfrak{M}$ is exponentially stable for coupled system (1.3).

To assure the zero solution $\mathbf{0}$ of system (2.2) is the exponentially stable, the matrix measure of the matrix is imposed. In the following two Theorems, we assume that $D\mathbf{h}(t)$ is of the following two cases.

1) $D\mathbf{h}(t)$ is a symmetric positive semidefinite matrix when $\theta = 2$.

2) $D\mathbf{h}(t)$ satisfies $(D\mathbf{h}(t))_{ii} \geq \sum_{j=1, j\neq i}^{n} |(D\mathbf{h}(t))_{ij}|$ for $1 \leq i \leq n$ when $\theta \in \Omega \backslash \{2\}$.

Theorem 2.2.2. *If there exists a matrix measure $\mu_\theta(\cdot)$ such that*

$$\int_{t_0}^{\infty} \mu_\theta(D\mathbf{f}(t) + d\lambda_2 D\mathbf{h}(t))\, dt = -\infty, \tag{2.12}$$

then the manifold $\mathfrak{M}$ can be exponentially stable.

Theorem 2.2.3. *If there exists a diagonal matrix $\mathbf{P} > 0$, a matrix measure $\mu_\theta(\cdot)$, and a constant $\bar{d} < 0$ such that*

$$\int_{t_0}^{\infty} \mu_\theta\left((D\mathbf{f}(t) + \bar{d}D\mathbf{h}(t))^T\mathbf{P} + \mathbf{P}(D\mathbf{f}(t) + \bar{d}D\mathbf{h}(t))\right)\, dt = -\infty, \tag{2.13}$$

then the manifold $\mathfrak{M}$ can be exponentially stable provided that $d\lambda_2 \leq \bar{d}$.

From the above analysis, the criteria given in (2.12)-(2.13) require that $D\mathbf{h}(t)$ must be either a symmetric positive semidefinite matrix or a matrix satisfying $(D\mathbf{h}(t))_{ii} \geq \sum_{j=1,j\neq i}^{n} |(D\mathbf{h}(t))_{ij}|$ for $1 \leq i \leq n$. In the following Theorem, it can be omitted these two conditions for $D\mathbf{h}(t)$.

Theorem 2.2.4. *The stability of the manifold $\mathfrak{M}$ can be transformed into the master stability equation (2.3), and the stability condition is defined as*

$$\int_{t_0}^{\infty} \mu_\theta(D\mathbf{f}(t) + \sigma D\mathbf{h}(t))\, dt = -\infty. \tag{2.14}$$

The "synchronization region" $S \neq \emptyset$ is the set of the parameter σ satisfying (2.14). The manifold $\mathfrak{M}$ is exponentially stable only if $d\lambda_i \in S$ for all $2 \leq i \leq n$.

Based on the concept of matrix measure, this brief provides some simple synchronization criteria of complex dynamical networks. If the coupling matrix and the largest nonzero eigenvalue of the coupling matrix satisfy certain conditions, the stability of the synchronization manifold can be ensured.

2.3 Definitions of Global Synchronization

We assume the system of ordinary differential equations under consideration has a unique solution for all time and for each initial condition. We write $\mathbf{x}(t, \mathbf{x}_0, t_0)$ for the unique solution at time t where $\mathbf{x}_0$ is the initial condition at time t_0. This will sometimes be simplified as $\mathbf{x}(t)$. Let $B_k(\alpha)$ be the ball in $\mathbb{R}^k$ with center at 0 and radius α. We define the system to be synchronized if the trajectories of all the cells approach each other. We define the system to be self-synchronized if the components $x_{i,k}$ of each subsystem $\mathbf{x}_i$ approach each other. Various notions of synchronization and self-synchronization are given in the following.

Definition 2.3.1. *(see e.g., Definition 1 of [53]) Let a ball $B_n(\alpha)$ be given. System (1.3) is uniformly (resp., self-) synchronized if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \leq \delta(\epsilon)$ (resp., $|x_{i,k}(t_0) - x_{j,k}(t_0)| \leq \delta(\epsilon)$), and $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$ for all i, j (resp., i, j, k), then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all $t \geq t_0$ and for all i, j (resp., i, j, k).*

Definition 2.3.2. *(see e.g., Definition 2 of [53]) Let a ball $B_n(\alpha)$ be given. System (1.3) is uniformly asymptotically (resp., self-) synchronized if the followings hold:*

i. It is uniformly synchronized.

ii. There exists a $\delta > 0$ such that for all $\epsilon > 0$ there exists a $t_\epsilon \geq 0$ such that if $\|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)\| \leq \delta$ (resp., $|x_{i,k}(t_0) - x_{j,k}(t_0)| \leq \delta$), and $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$ for all i, j (resp., i, j, k) and $t \geq t_0 + t_\epsilon$, then $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$. (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all i, j (resp., i, j, k).

Definition 2.3.3. *Let a ball $B_n(\alpha)$ be given. System (1.3) is globally (resp., self-) synchronized if for all $\epsilon > 0$, there exists a $t_\epsilon \geq 0$ such that $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ (resp., $|x_{i,k}(t) - x_{j,k}(t)| \leq \epsilon$) for all i, j (resp., i, j, k), all $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$, and all $t \geq t_0 + t_\epsilon$.*

Proposition 2.3.4. *If a system is globally (resp., self-) synchronized, then it is uniformly asymptotically (resp., self-) synchronized.*

Proof. If a system is as assumed, then given $\epsilon > 0$, there exists a t' such that for all i, j and all $\mathbf{x}_i(t_0)$ and $\mathbf{x}_j(t_0) \in B_n(\alpha)$, we have $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \epsilon$ for $t \geq t'$. Letting $t_0 = t'$ and $\delta = \epsilon$, we see immediately that the corresponding system is uniformly synchronized. Obviously, the assumption in Definition 2.3.2.-(ii) can be fulfilled by choosing any $\delta > 0$. The other assertion in the proposition can be similarly proved. $\square$

2.4 Lyapunov Function Approach

2.4.1 Belykh

In the last few years, many researchers try to give criteria for the global (or local) synchronization of coupled chaotic systems. Most of their methods based either on the eigenvalues of the coupling configuration matrix $\mathbf{G}$ or on the Lyapunov exponent of the coupled systems. In order to avoid calculating eigenvalues or Lyapunov exponent to determine global synchronization, the connection graph based stability method is developed by Belykh et al (2004) [4]. This method combines the Lyapunov function approach with graph theoretical reasoning, and elucidates the relation between synchronization and the form of the connected graph (the coupling configuration matrix $\mathbf{G}$). The method can be applied to give a rigorous bound for the coupling strength including in the global, star, diffusive, $2K$-nearest neighbor coupling cases, etc. Moreover, the time-varying coupling configuration matrix $\mathbf{G}(t)$ is also discussed.

In equation (1.3), let $H : \mathbb{R}^{mn} \rightarrow \mathbb{R}^{mn}$ be defined by $\mathbf{H}(\mathbf{x}) = (\mathbf{I}_m \otimes \mathbf{D})\mathbf{x}$, and $\mathbf{G} = (g_{ij}(t))_{m\times m}$ is a time-varying, symmetric matrix with vanishing row-sums, non-negative off-diagonal elements. Then, we have the following equation,

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \mathbf{f}(\mathbf{x}_1, t) \\ \vdots \\ \mathbf{f}(\mathbf{x}_m, t) \end{pmatrix} + d(\mathbf{G} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{D})\mathbf{x} =: \mathbf{F}(\mathbf{x}, t) + d(\mathbf{G} \otimes \mathbf{D})\mathbf{x}, \tag{2.15a}$$

where $\mathbf{D} = \mathrm{diag}(1, \cdots, 1, 0, \cdots 0)$ is a diagonal matrix with k elements equal to 1, $\otimes$ denotes the Kronecker product, and

$$\mathbf{f}(\mathbf{x}_i, t) = \begin{pmatrix} f_1(\mathbf{x}_i, t) \\ \vdots \\ f_n(\mathbf{x}_i, t) \end{pmatrix}. \tag{2.15b}$$

.

Remark 2.4.1. *(i)For all time t, we denoted that the number of the nonzero elements of the off-diagonal elements of* $\mathbf{G}$ *is* $2p$. *(ii) The matrix* $\mathbf{G}$ *is meaningful in the graph theory. It refers to the connected graph with* m *vertices and* p *edges, and if the edge from vertex* i *to vertex* j *exists, then* $g_{j,i}(t) = g_{i,j}(t) > 0$, $1 \le i, j \le m$, *for all time* t.

Before starting the study of the transversal stability of the synchronization manifold $\mathfrak{M}$, we need also one additional assumption on the eventually dissipativeness of coupled system (2.15). Assume that the individual system $\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t)$ is eventually dissipative, i.e. there exists a compact set B which attracts all trajectories of the system from the outside. Therefore, there are no trajectories which go to infinity.

Now, we introduce the notation for the differences $\mathbf{X}_{ij} = \mathbf{x}_j - \mathbf{x}_i$, we obtain the difference equation system as follows,

$$\dot{\mathbf{X}}_{ij} = \left[\int_0^1 D\mathbf{f}(\beta\mathbf{x}_j + (1-\beta)\mathbf{x}_i)\, d\beta\right] \mathbf{X}_{ij} + d\sum_{l=1}^{m}\{g_{jl}\mathbf{D}\mathbf{X}_{jl} - g_{il}\mathbf{D}\mathbf{X}_{il}\}, \tag{2.16}$$

for all $i, j = 1, \ldots, m$. To study the stability of difference equation system (2.16), we introduce the auxiliary system by adding an uncharted matrix A,

$$\dot{\mathbf{X}}_{ij} = \left[\int_0^1 D\mathbf{f}(\beta\mathbf{x}_j + (1-\beta)\mathbf{x}_i)\, d\beta - \mathbf{A}\right] \mathbf{X}_{ij}, \quad i, j = 1, \ldots, m, \tag{2.17}$$

where $\mathbf{A} = \mathrm{diag}(a_1, \cdots, a_k, 0, \cdots, 0)$ is a matrix with $a_i \ge 0$ for $1 \le i \le k$. Moreover, we assume that there exist Lyapunov functions of the form,

$$V_{ij} = \mathbf{X}_{ij}^T \mathbf{H} \mathbf{X}_{ij}, \qquad i, j = 1, \cdots, n. \tag{2.18}$$

where the vector variables $\mathbf{X}_{ij} = \{\mathbf{X}_{ij}^{(1)}, \cdots, \mathbf{X}_{ij}^{(k)}\}$, $\mathbf{H} = \mathrm{diag}(h_1, \ldots, h_k, \mathbf{H}_1)$ with $h_1 > 0, \cdots, h_k > 0$ and the matrix $\mathbf{H}_1$ is positive definite. Furthermore, their derivatives of Lyapunov functions in (2.18) with respect to auxiliary system (2.17) are required to be negative,

$$\dot{V}_{ij} = \mathbf{X}_{ij}^T \mathbf{H} \left[\int_0^1 D\mathbf{f}(\beta\mathbf{x}_j + (1-\beta)\mathbf{x}_i)\, d\beta - \mathbf{A}\right] \mathbf{X}_{ij} < 0, \mathbf{X}_{ij} \ne 0, i, j = 1, \cdots, n. \tag{2.19}$$

Hence, we can study global stability of the synchronization manifold $\mathfrak{M}$ by the following main Theorem.

Theorem 2.4.2. *Under the assumption on the eventual dissipativeness of the individual oscillator system* $\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t)$ *and assumption (2.19), the synchronization manifold* $\mathfrak{M}$ *of coupled system (2.15) is global asymptotically stable if the following inequality holds*

$$d\sum_{l=1}^{p} g_{i_l,j_l}\mathbf{X}_{i_l,j_l}^{(d)^2} > \frac{a_d}{m}\sum_{i=1}^{m-1}\sum_{j>i}^{m}\mathbf{X}_{i,j}^{(d)^2}, \qquad d = 1, \cdots, k \tag{2.20}$$

where the index (i_l, j_l) *is the pair of satisfying* $g_{i_l,j_l} > 0$.

Theorem 2.4.2. indeed gives sufficient conditions for the global synchronization. However, these inequalities in (2.20) are not easily to be applied. To get rid of it and find a rigorous bound of g_{i_l,j_l}, the following theorem is given.

Theorem 2.4.3. *Under the assumption of Theorem 2.4.2, the synchronization manifold* $\mathfrak{M}$ *of coupled system (2.15) is global asymptotically stable if*

$$dg_{i_l,j_l} > \frac{a_l}{m}b_{i_l,j_l}(m,p) \quad for\ l = 1, \cdots, p\ and\ for\ all\ time\ t. \tag{2.21}$$

where $b_{i_l,j_l}(m,p) = \sum_{m_1>m_2;\, i_l,j_l \in P_{m_1m_2}} z(P_{m_1m_2})$ *is the sum of the lengths of all chosen paths* $P_{m_1m_2}$ *which pass through the edge from vertex* i_l *to vertex* j_l.

Several coupling configuration could be given a general rigorous bound of the coupling strength via above two theorems. In the following, some examples are listed.

Example 1. *Suppose the coupled system satisfies the sufficient conditions in the Theorem 2.4.2. Let*

1. *(Global coupling)* $\mathbf{G}$ *has all off-diagonal element nonzero. Then the global synchronization reaches provided* $dg_{i,j} > \frac{a}{m}$ *for all* $i \neq j$.

2. *(Star coupling)* $\mathbf{G} = \begin{pmatrix} -g & g_{12} & g_{13} & \cdots & g_{1m} \\ g_{12} & -g_{12} & 0 & \cdots & 0 \\ g_{13} & 0 & -g_{13} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{1m} & 0 & 0 & \cdots & -g_{1m} \end{pmatrix}$, *where* $g = \sum_{i=2}^{m} g_{1i}$.

   Then the global synchronization reaches provided $dg_{1i} > a\dfrac{2m-3}{m}$ *for all* $i =$

$2, \ldots, m$.

3. *(Diffusive coupling)*

$$\mathbf{G} = \begin{pmatrix} -(g_{12}+g_{1m}) & g_{12} & 0 & 0 & g_{1m} \\ g_{12} & -(g_{12}+g_{23}) & g_{23} & 0 & 0 \\ 0 & g_{23} & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & g_{m-1,m} \\ g_{1m} & 0 & 0 & g_{m-1,m} & -(g_{1m}+g_{m-1,m}) \end{pmatrix}.$$

Then the global synchronization reaches provided

$$dg_{i,j} > \begin{cases} a\left(\frac{m^2}{24} - \frac{1}{24}\right) & \textit{for odd } m \\ a\left(\frac{m^2}{24} + \frac{1}{12}\right) & \textit{for even } m \end{cases},$$

for all i, j.

4. *(2K-nearest neighbor coupling)* $\mathbf{G}$ *has its off-diagonal elements of the form*

$$dg_{i,j} > \begin{cases} g & \textit{for } 1 \le |j-i| \mod m \le K \\ 0 & \textit{otherwise} \end{cases}.$$

Then the global synchronization reaches provided

$$g > \frac{a}{m}\left(\frac{m}{2K}\right)^3\left(1 + \frac{65}{4}\frac{K}{m}\right).$$

2.4.2 Wu and Chua

In this section, we introduce the Lyapunov's direct method to prove uniformly asymptotical synchronization of coupled system (2.15), which is developed by C. W. Wu, and L. O. Chua [53]. A typical results states that coupled system (2.15) will synchronize if the nonzero eigenvalues of the coupling matrix have real parts that are negative enough. Moreover, sufficient conditions for synchronization for several coupling configurations will be considered.

It will mainly use Lyapunov's direct method to prove uniform asymptotical synchronization of the coupled system in (2.15). We use $d(\mathbf{x})$ to denote a nonnegative real-valued function that measures the distance between the various nodes. We also define the following class of matrices:

- $M_1(k)$ are matrices $\mathbf{M}$ (not necessarily square) with entries in $\mathcal{F}_k$ such that each row of $\mathbf{M}$ contains zeros and exactly one $\alpha \mathbf{I}_k$ and one $-\alpha \mathbf{I}_k$ for some nonzero α.

- $M_2(k)$ are matrices $\mathbf{M}$ in $M_1(k)$ such that for any pair of indices i and j there exist indices $i_1, i_2, \cdots, i_l$ with $i_1 = i$ and $i_l = j$ such that for all $1 \leqq < l$, $\mathbf{M}_(p, i_q) \neq 0$ and $\mathbf{M}_(p, i_{q+1}) \neq 0$ for some p.

In particular, we define $d(\mathbf{x})$ to have the following form:

$$d(\mathbf{x}) = \|\mathbf{Mx}\|^2 = \mathbf{x^T M^T M x}, \mathbf{M} \in M_2(n) \tag{2.22}$$

where $\mathbf{M}$ is an $m \times m$ matrix in $M_2(n)$ (but considered as an $nm \times nm$ real-valued matrix).

Because of the assumptions on $\mathbf{M}$, the crucial property of $d(\mathbf{x})$ is that $d(x) \to 0$ if and only if $\|\mathbf{x}_i - \mathbf{x}_j\| \to 0$ for all i and j. One possible choice for $d(\mathbf{x})$ is

$$d(\mathbf{x}) = \sum_{i=1}^{m-1} \|\mathbf{x}_i - \mathbf{x}_{i+1}\|^2 \tag{2.23}$$

which corresponds to

$$\mathbf{M} = \begin{pmatrix} \mathbf{I} & -\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & -\mathbf{I} \end{pmatrix} \tag{2.24}$$

Definition 2.4.4. *A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to belong to class K if*

1) $\alpha(\cdot)$ is continuous and nondecreasing,

2) $\alpha(0) = 0$,

3) $\alpha(p) > 0$ whenever $p > 0$.

We assume that all Lyapunov functions we consider are continuous. For a Lyapunov Function $V(t, \mathbf{x})$, the generalized (Dini) derivative along the trajectories of the system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$ is defined as

$$D^+V(t, \mathbf{x}) = \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, \mathbf{x} + h\mathbf{f}(\mathbf{x}, t)) - V(t, \mathbf{x})] \tag{2.25}$$

Theorem 2.4.5. *Suppose that D is an open set such that if $\mathbf{X}_i(t_0) \in B_{\alpha^*}$ for all i, then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, locally Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$ and $\mathbf{x} \in D$,*

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are functions in class K. Suppose that there exists $\mu > 0$ such that for all $t \geq t_0$ and $d(\mathbf{x}) \geq \mu$,

$$D^+V(t, \mathbf{x}) \leq -c$$

for some constant $c > 0$ where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of the coupled system in (2.15). If there exists $\delta > 0$ such that $a(\delta) > b(\mu)$, then for each $\mathbf{X}(t_0) \in B_{\alpha^*}$ there exists $t_1 \geq t_0$ such that for all $t \geq t_1$,

$$d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \delta$$

. Furthermore, if $d(\mathbf{x}(t_0)) \leq \mu$, then

$$d(\mathbf{x}(t, \mathbf{x}(t_0), t_0)) \leq \delta$$

for all $t \geq t_0$.

Theorem 2.4.6. *Suppose that D is an open set such that if $\mathbf{x}_i(t_0) \in B_{\alpha^*}$ for all i, then $\mathbf{x}(t, \mathbf{x}(t_0), t_0) \in D$ for all $t \geq t_0$. Suppose that a Lyapunov function $V(t, \mathbf{x})$, locally Lipschitzian in $\mathbf{x}$, exists on $\mathbb{R} \times D$ such that for all $t \geq t_0$, $\mathbf{x} \in D$,*

$$a(d(\mathbf{x})) \leq V(t, \mathbf{x}) \leq b(d(\mathbf{x}))$$

where $a(\cdot)$ and $b(\cdot)$ are in class K, and for all $t \geq t_0$,

$$D^+V(t, \mathbf{x}) \leq -c(d(\mathbf{x}))$$

for some function $c(\cdot)$ in class K where $D^+V(t, \mathbf{x})$ is the generalized derivative of V along the trajectories of the coupled system in (2.15). Then the coupled system in (2.15) is uniformly asymptotically synchronized with respect to α^*.

2.5 Partial Contraction Approach

In this section, we shall describe the partial contraction approach for studying global synchronization of coupled chaotic systems. This approach was given by [45]. method to analyze networks of coupled identical nonlinear oscillators, and study applications to synchronization. Specifically, we use nonlinear partial contraction theory to derive exact and global results on synchronization. The method can be applied to coupled networks of various structures and arbitrary size. For oscillators with positive-definite diffusion coupling, it can be shown that synchronization always occur globally for strong enough coupling strengths, and an explicit upper bound on the corresponding threshold can be computed through eigenvalue analysis.

Basically, a nonlinear time-varying dynamic system will be called contracting if initial conditions or temporary disturbances are forgotten exponentially fast, i.e., if trajectories of the perturbed system return to their nominal behavior with an exponential convergence rate. The concept of partial contraction allows one to extend the applications of contraction analysis to include convergence to behaviors or to specific properties (such as equality of state components, or convergence to a manifold) rather than trajectories. We briefly summarize the basic definitions and main results of Contraction Theory here. Consider a nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{f}$ is an $n \times 1$ vector function. Assuming $\mathbf{f}(\mathbf{x}, t)$ is continuously differentiable, we have

$$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) = 2\delta\mathbf{x}^T\delta\frac{d\mathbf{x}}{dt} = 2\delta\mathbf{x}^T\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\delta\mathbf{x} \leq 2\lambda_{max}\delta\mathbf{x}^T\delta\mathbf{x} \tag{2.26}$$

where $\delta\mathbf{x}$ is a virtual displacement between two neighboring solution trajectories, and $\lambda_{max}(\mathbf{x}, t)$ is the largest eigenvalue of the symmetric part of the Jacobian $\mathbf{J} = \frac{\partial\mathbf{f}}{\partial\mathbf{x}}$. Hence, if $\lambda_{max}(\mathbf{x}, t)$ is uniformly strictly negative, any infinitesimal length $\|\delta\mathbf{x}\|$ converges exponentially to zero. By path integration at fixed time, this implies in turn that all solutions of system (2.26) converge exponentially to a single trajectory, independently of the initial conditions. Note that differential analysis yields global results.

We now introduce the concept of partial contraction, which forms the basis of the contraction analysis. It derives from the very simple but very general result which follows.

Theorem 2.5.1. *Consider a nonlinear system of the form*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$$

and assume that the auxiliary system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$$

is contracting with respect to $\mathbf{y}$*. If a particular solution of the auxiliary* $\mathbf{y}$*-system verifies a smooth specific property, then all trajectories of the original* $\mathbf{x}$*-system verify this property exponentially. The original system is said to be partially contracting.*

Let us now move to networked systems under a very general coupling structure. Consider a coupled system containing m identical nodes

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i), \quad i = 1, \cdots, m, \tag{2.27}$$

where $\mathcal{N}_i$ denotes the set of indices of the active links of elements i. It is equivalent to

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i) - \mathbf{K}_0 \sum_{j=1}^{m} \mathbf{x}_j + \mathbf{K}_0 \sum_{j=1}^{m} \mathbf{x}_j, \quad i = 1, \cdots, m, \tag{2.28}$$

where $\mathbf{K}_0$ is chosen to be a constant symmetric positive definite matrix (we will discuss its function later). Again, construct an auxiliary system

$$\frac{d\mathbf{y}_i}{dt} = \mathbf{f}(\mathbf{y}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}(\mathbf{y}_j - \mathbf{y}_i) - \mathbf{K}_0 \sum_{j=1}^{m} \mathbf{y}_j + \mathbf{K}_0 \sum_{j=1}^{m} \mathbf{x}_j, \quad i = 1, \cdots, m, \tag{2.29}$$

that has a particular solution $\mathbf{y}_1 = \cdots = \mathbf{y}_m = \mathbf{y}_\infty$ with

$$\frac{d\mathbf{y}_\infty}{dt} = \mathbf{f}(\mathbf{y}_\infty, t) - m\mathbf{K}_0\mathbf{y}_\infty + \mathbf{K}_0 \sum_{j=1}^{m} \mathbf{x}_j, \quad i = 1, \cdots, m, \tag{2.30}$$

According to Theorem 2.5.1, if the auxiliary system in (2.29) is contracting, then all system trajectories will verify the independent property $\mathbf{x}_1 = \cdots = \mathbf{x}_m$ exponentially.

Theorem 2.5.2. *Regardless of initial conditions, all the elements within a generally coupled system in (2.27) will reach synchrony or group agreement exponentially if*

1) the network is connected,

2) $\lambda_{max}(\mathbf{J}_{is})$ *is upper bounded,*

3) the couplings are strong enough.

where $\mathbf{J}_{is} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}, t)\right)_s$, *and* $\mathbf{F}_s = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T)$.

Chapter 3

Global Synchronization via Matrix Measures Approach

This chapter contains the main results of the first part of the thesis. In particular, we use matrix measures approach to study global synchronization of coupled chaotic systems. Our coupling rules are allowed to be asymmetric and/or some competitive ($g_{ij} < 0$, $i \neq j$) couplings between cells $\mathbf{x}_i$ and $\mathbf{x}_j$ as long as the coupled system is bounded dissipative. In addition, the partial-state coupling in our approach is allowed to have the form satisfying (3.14). Moreover, by merely checking the structure of the vector field of the single oscillator, we shall be able to determine if the system is globally synchronized. We also obtain a rigorous lower bound on the coupling strength for the global synchronization of all oscillators with coupling configuration satisfying (3.3a), and (3.3b). Part of the results in this direction is bases on the paper [27]. To conclude this section, we define the global synchronization as in the following.

Definition 3.0.3. *(i) System (2.15) is said to be globally synchronized if for any given initial values* $\mathbf{x}_0$ *there exists a* $d = d_{\mathbf{x}_0}$ *such that system (2.15) is synchronized for the initial conditions* $\mathbf{x}_0$*. Here* $d_{\mathbf{x}_0}$ *is a constant depending on* $\mathbf{x}_0$*. (ii) System (2.15) is said to be uniformly, globally synchronized if there exists a* $d = d_1$ *such that system (2.15) is synchronized for all initial values* $\mathbf{x}_0$*.*

3.1 Preliminaries

Chaotic synchronization is a fundamental phenomenon in physical systems with dissipation. In this section, we introduce the concept of the bounded dissipation to coupled system (2.15). Then, we use this concept of the matrix measure theory to achieve the behavior of global synchronization in coupled system (2.15). Hence, we give the definition of bounded dissipation as follows.

Definition 3.1.1. *(i) A system of n ordinary differential equations is called bounded dissipative provided that for any $r > 0$ and for any initial conditions $\mathbf{x}_0$ in $B_n(r)$, there exists a time $t^* \geq t_0$ and α_r such that $\|\mathbf{x}(t)\| \leq \alpha_r$ for all $t \geq t^*$. (ii) If, in addition, α_r is independent of r, then the system is said to be uniformly bounded dissipative with respect to α_r.*

To prove global synchronization of coupled chaotic systems, one needs to assume bounded dissipation of the system, which plays the role of an a priori estimate. Without such an a priori estimate, as in the case of Rössler system, the global synchronization is much more difficult to obtain. Only local synchronization was reported numerically in literature (see e.g., [4]). We remark that in certain cases of the Rössler system, the trajectory of each oscillator grows unbounded yet approaches each other (see e.g., [4]). An interesting question in this direction is how bounded dissipation of the coupled system is related to the uncoupled dynamics and its connectivity topology. Not much general theorems have been provided so far. In the case that **G** is diffusively coupled with periodic boundary conditions or zero-flux and **D** satisfies (3.3c), it was shown in [5] that bounded dissipation of the single oscillator implies that of the coupled chaotic oscillators. Moreover, the absorbing domain of the coupled system is a topological product of the absorbing domain of each individual system. Moreover, it often requires to construct an approximate Lyapunov function to prove the bounded dissipation of the system. The following proposition gives the type of Lyapunov functions that would ensure the bounded dissipation of the system.

Proposition 3.1.2. *Let a system of n ordinary differential equations be given. Let V be a continuous real-valued function $V : \mathbb{R}^n \to \mathbb{R}^+$ so that V is strictly decreasing along* *the solution of the system on* $\mathbb{R}^n - \Gamma$*, where* Γ *is homeomorphic to an open ball in* $\mathbb{R}^n$*. Suppose*

$$\lim_{\|\mathbf{x}\|\to\infty} V(\mathbf{x}) = \infty. \tag{3.1}$$

Then the system is bounded dissipative.

Proof. For any $\mathbf{x}_0 \in \mathbb{R}^n$, we first prove that $\mathbf{x}(t)$ must enter Γ at a certain time. Otherwise, the values of V at the points of the ω-limit set of $\mathbf{x}(t)$ must be the same, a contradiction. The contradiction comes from the facts that the ω-limit set is closed and invariant and V is strictly decreasing along the solution trajectory, which stays in $\mathbb{R}^n - \Gamma$. We then find a ball $B_n(r)$ so that $B_n(r) \supset \Gamma$. Let $k_1 = \max_{\mathbf{x}\in\bar{B}_n(r)} V(\mathbf{x})$, and $B_n(\alpha_r)$ be a ball satisfying $V(\mathbf{x}) > k_2$ whenever $\mathbf{x} \in \mathbb{R}^n - B_n(\alpha_r)$, where $k_2 > k_1$. Then we conclude that if $\mathbf{x}_0 \in B_n(r)$, $\mathbf{x}(t)$ stays in $B_n(\alpha_r)$ for all time t. We just complete the proof of the proposition. □

In our derivation of synchronization of system(3.1), we need the concept of matrix measure. For the completeness and ease of references, we also recall the following definition of the matrix measure and its properties (see e.g., [44]).

Theorem 3.1.3. *(see e.g., 3.5.32 of [44]) Consider the differential equation* $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{v}(t)$*,* $t \geq 0$*, where* $\mathbf{x}(t) \in \mathbb{R}^n$*,* $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$*, and* $\mathbf{A}(t), \mathbf{v}(t)$ *are piecewise-continuous. Let* $\|\cdot\|_i$ *be a norm on* $\mathbb{R}^n$*, and* $\|\cdot\|_i$*,* μ_i *denote, respectively, the corresponding induced norm and matrix measure on* $\mathbb{R}^{n\times n}$*. Then whenever* $t \geq t_0 \geq 0$*, we have*

$$\begin{aligned} \|\mathbf{x}(t_0)\| \exp\left\{\int_{t_0}^{t} -\mu_i(-\mathbf{A}(s))ds\right\} - \int_{t_0}^{t} \exp\left\{\int_{s}^{t} -\mu_i(-\mathbf{A}(\tau))d\tau\right\} \|\mathbf{v}(s)\|ds \leq \|\mathbf{x}(t)\| \\ \leq \|\mathbf{x}(t_0)\| \exp\left\{\int_{t_0}^{t} \mu_i(\mathbf{A}(s))ds\right\} + \int_{t_0}^{t} \exp\left\{\int_{s}^{t} \mu_i(\mathbf{A}(\tau))d\tau\right\} \|\mathbf{v}(s)\|ds. \end{aligned} \tag{3.2}$$

3.2 Global synchronization results

Our main result in the first part of the thesis is contained in this section. We begin with imposing conditions on coupling matrices $\mathbf{G}$ and $\mathbf{D}$. We assume that the coupling matrix $\mathbf{G}$ satisfies the following:

$$\text{(i) } \lambda = 0 \text{ is a simple eigenvalue of } \mathbf{G} \text{ and } \mathbf{e} = [1, 1, \ldots, 1]^T_{1\times m} \text{ is its corresponding eigenvector.} \tag{3.3a}$$

$$\text{(ii) All nonzero eigenvalues of } \mathbf{G} \text{ have negative real part.} \tag{3.3b}$$

We further assume that the matrix $\mathbf{D}$ is, without loss of generality, of the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n.} \tag{3.3c}$$

The index k, $1 \leq k \leq n$, means that the first k components of the subsystem are coupled. If $k \neq n$, then the system is said to be partial-state coupled. Otherwise, it is said to be full-state coupled.

From time to time, we will refer system (3.1) as coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$.

To study the synchronization of such system, we permute the state variables in the following way:

$$\tilde{\mathbf{x}}_i = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m,i} \end{pmatrix}, \text{ and } \tilde{\mathbf{x}} = \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{pmatrix}. \tag{3.4}$$

Then (3.1) can be written as

$$\dot{\tilde{\mathbf{x}}} = \begin{pmatrix} \tilde{\mathbf{f}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ \tilde{\mathbf{f}}_n(\tilde{\mathbf{x}}, t) \end{pmatrix} + d(\mathbf{D} \otimes \mathbf{G})\tilde{\mathbf{x}} =: \tilde{\mathbf{F}}(\tilde{\mathbf{x}}, t) + d(\mathbf{D} \otimes \mathbf{G})\tilde{\mathbf{x}}, \tag{3.5a}$$

where

$$\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}}, t) = \begin{pmatrix} f_i(\mathbf{x}_1, t) \\ \vdots \\ f_i(\mathbf{x}_m, t) \end{pmatrix}. \tag{3.5b}$$

Note that such reformulation is certainly not new (see e.g., [29, 53]). From here on, we will treat $\tilde{}$ as a function that takes $\mathbf{x}$ into $\tilde{\mathbf{x}}$ or $\mathbf{x}_i$ into $\tilde{\mathbf{x}}_{i.}$ A transformation of coordinates of $\tilde{\mathbf{x}}$ is then to be applied to (3.4) so as to decompose the synchronous manifold. The problem of synchronization of (2.15), and hence (3.5) is then equivalent to proving the asymptotical stability of reduced system (see (3.8)). To study the synchronization of (2.15), we first make a coordinate change to decompose the synchronous subspace. Let $\mathbf{A}$ be an $m \times m$ matrix of the form

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}_{m \times m} =: \begin{pmatrix} \mathbf{C} \\ \mathbf{e}^T \end{pmatrix}, \tag{3.6a}$$

where $\mathbf{e}$ is given as in (3.3a). It is then easy to see that $\mathbf{CC}^T$ is invertible and that

$$\mathbf{A}^{-1} = \left(\mathbf{C}^T(\mathbf{CC}^T)^{-1} | \frac{\mathbf{e}}{m}\right). \tag{3.6b}$$

Setting

$$\mathbf{E} = \mathbf{I}_n \otimes \mathbf{A}, \tag{3.6c}$$

we see that

$$\begin{aligned}
\mathbf{E}(\mathbf{D}\otimes\mathbf{G})\mathbf{E}^{-1} &= (\mathbf{I}_n\otimes\mathbf{A})(\mathbf{D}\otimes\mathbf{G})(\mathbf{I}_n\otimes\mathbf{A}^{-1}) \\
&= \mathbf{D}\otimes\mathbf{A}\mathbf{G}\mathbf{A}^{-1} = \mathbf{D}\otimes\begin{pmatrix} \mathbf{C}\mathbf{G}\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} & \mathbf{0} \\ * & 0 \end{pmatrix} \\
&=: \mathbf{D}\otimes\begin{pmatrix} \bar{\mathbf{G}} & \mathbf{0} \\ * & 0 \end{pmatrix}. \qquad (3.6d)
\end{aligned}$$

We remark, via (3.6d), that $\sigma(\mathbf{G})-\{0\}=\sigma(\bar{\mathbf{G}})$, where $\sigma(\mathbf{A})$ is the spectrum of matrix $\mathbf{A}$. Multiplying $\mathbf{E}$ to the both side of equation (3.5a), we get

$$\begin{aligned}
\dot{\tilde{\mathbf{y}}} =: \mathbf{E}\dot{\tilde{\mathbf{x}}} &= \mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) + d\mathbf{E}(\mathbf{D}\otimes\mathbf{G})\mathbf{E}^{-1}\tilde{\mathbf{y}} \\
&= \mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t) + d(\,\mathbf{D}\otimes\begin{pmatrix} \bar{\mathbf{G}} & \mathbf{0} \\ * & 0 \end{pmatrix})\tilde{\mathbf{y}}. \qquad (3.7)
\end{aligned}$$

Let $\tilde{\mathbf{y}} = \begin{pmatrix} \tilde{\mathbf{y}}_1 \\ \vdots \\ \tilde{\mathbf{y}}_n \end{pmatrix}.$ Then $\tilde{\mathbf{y}}_i = \begin{pmatrix} x_{1,i}-x_{2,i} \\ \vdots \\ x_{m-1,i}-x_{m,i} \\ \sum_{j=1}^m x_{j,i} \end{pmatrix}.$ Setting $\tilde{\mathbf{y}}_i = \begin{pmatrix} \bar{\mathbf{y}}_i \\ \sum_{j=1}^m x_{j,i} \end{pmatrix},$ and $\bar{\mathbf{y}} = \begin{pmatrix} \bar{\mathbf{y}}_1 \\ \vdots \\ \bar{\mathbf{y}}_n \end{pmatrix},$ we have that the dynamics of $\bar{\mathbf{y}}$ is satisfied by the following equation

$$\dot{\bar{\mathbf{y}}} = d(\mathbf{D}\otimes\bar{\mathbf{G}})\bar{\mathbf{y}} + \bar{\mathbf{F}}(\bar{\mathbf{y}},t). \qquad (3.8)$$

Here $\bar{\mathbf{F}}$ is obtained from $\mathbf{E}\tilde{\mathbf{F}}(\mathbf{E}^{-1}\tilde{\mathbf{y}},t)$ accordingly.

The task of obtaining the global synchronization of system (2.15) is now reduced to showing that the origin is globally and asymptotically stable with respect to system (3.8). To this end, the space $\bar{\mathbf{y}}$ is broken into two parts $\bar{\mathbf{y}}_c$, the coupled space, and $\bar{\mathbf{y}}_u$, the uncoupled space.

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_u \end{pmatrix}, \text{ and } \bar{\mathbf{F}}(\bar{\mathbf{y}},t) = \begin{pmatrix} \bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) \\ \bar{\mathbf{F}}_u(\bar{\mathbf{y}},t) \end{pmatrix}, \text{ respectively.} \qquad (3.9)$$

Here $\bar{\mathbf{y}}_c = \begin{pmatrix} \bar{\mathbf{y}}_1 \\ \vdots \\ \bar{\mathbf{y}}_k \end{pmatrix}$, and $\bar{\mathbf{y}}_u = \begin{pmatrix} \bar{\mathbf{y}}_{k+1} \\ \vdots \\ \bar{\mathbf{y}}_n \end{pmatrix}$. The dynamics on the coupled space with respect to the linear part is under the influence of $\bar{\mathbf{G}}$, which is asymptotically stable. The dynamics of the nonlinear part on coupled space can then be controlled by choosing large coupling strength. In short, this part of the dynamics is easy to contain. In fact, the larger k, the number of state variables being coupled, gives the better chance of the synchronization of the coupled system. On the other hand, the uncoupled space has no stable matrix $\bar{\mathbf{G}}$ to play with. Thus, its corresponding vector field $\bar{\mathbf{F}}_u(\bar{\mathbf{y}}, t)$ must have a certain structure to make the trajectory stay closer to the origin as time progresses. As we shall explain latter.

Now, assume that $\bar{\mathbf{F}}_c(\bar{\mathbf{y}}, t)$ satisfies a dual-Lipschitz condition with a dual-Lipschitz constant b_1. That is,

$$\|\bar{\mathbf{F}}_c(\bar{\mathbf{y}}, t)\| \le b_1 \|\bar{\mathbf{y}}\| \tag{3.10a}$$

whenever $\bar{\mathbf{y}}$ in the ball $B_{(m-1)n}(\alpha)$, and for all time t. Since the estimate in the right-hand side of (3.10a) depends on the whole space $\bar{\mathbf{y}}$, condition (3.10a) is a mild assumption provided that coupled system is bounded dissipative. Write $\bar{\mathbf{F}}_u(\bar{\mathbf{y}}, t)$ as

$$\begin{aligned} \bar{\mathbf{F}}_u(\bar{\mathbf{y}}, t) &= \mathbf{U}(t)\bar{\mathbf{y}}_u + (\bar{\mathbf{F}}_u(\bar{\mathbf{y}}, t) - \mathbf{U}(t)\bar{\mathbf{y}}_u) \\ &=: \mathbf{U}(t)\bar{\mathbf{y}}_u + \bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t). \end{aligned} \tag{3.10b}$$

Assume that $\mathbf{U}(t)$ is a block diagonal matrix of the form $\mathbf{U}(t) = \text{diag}(\mathbf{U}_1(t), \cdots, \mathbf{U}_l(t))$ where $\mathbf{U}_j(t)$, $j = 1, \ldots, l$, are matrices of size $(m-1)k_j \times (m-1)k_j$. Here $\sum_{j=1}^{l} k_j = n-k$, and $k_j \in \mathbb{N}$. We assume further that the followings hold.

(i) The matrix measures $\mu_i(\mathbf{U}_j(t))$ are less than $-\gamma$ for all t and all j, where $\gamma > 0$. (3.10c)

(ii) Let $\bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t) = \begin{pmatrix} \mathbf{R}_{u1}(\bar{\mathbf{y}}, t) \\ \vdots \\ \mathbf{R}_{ul}(\bar{\mathbf{y}}, t) \end{pmatrix}$. Then $\mathbf{R}_{uj}(\bar{\mathbf{y}}, t)$, $j = 1, \ldots, l$, satisfy a strong dual-Lipschitz condition with a strong dual-Lipschitz constant b_2. Specifically, let $\bar{\mathbf{y}}_u = \begin{pmatrix} \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{ul} \end{pmatrix}$, written in accordance with the block structure of $\mathbf{U}(t)$. Then we assume that

$$\|\mathbf{R}_{uj}(\bar{\mathbf{y}}, t)\| \leq b_2 \left\| \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_{u1} \\ \vdots \\ \bar{\mathbf{y}}_{u\,j-1} \end{pmatrix} \right\| \tag{3.10d}$$

whenever $\bar{\mathbf{y}}$ in the ball $B_{(m-1)n}(\alpha)$, and for all $j = 1, \ldots, l$ and all time t.

Specifically, we break the vector field $\bar{\mathbf{F}}_u$ into (time dependent) linear part $\mathbf{U}(t)\bar{\mathbf{y}}_u$ and nonlinear part $\bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t)$. We will further break $\mathbf{U}(t)$ into certain block diagonal form if necessary. Note that the form (3.10b) can always be achieved. Since the remainder term $\bar{\mathbf{R}}$ still depends on the whole space $\bar{\mathbf{y}}$. To take control of the dynamics on the linear part, we assume that the matrix measure of each diagonal block $\mathbf{U}_j(t)$ is negative. As to contain corresponding dynamics on the nonlinear part, we assume that the (3.10d) holds. Note that though the nonlinear terms $\mathbf{R}_{uj}(\bar{\mathbf{y}}, t)$ could possibly depend on the whole space, their norm estimate are required to depend only on the coupled space and uncoupled subspaces with their indexes proceeding j. In this set up, the nonlinear dynamics on uncoupled space can be iteratively controlled by choosing large coupling strength. We also remark that if (3.10c) and (3.10d) are satisfied for l, the number of diagonal blocks, being one, then we do not need to further break $\mathbf{U}(t)$. Such further breaking is needed only if (3.10c) and (3.10d) are not satisfied. The proof in the following theorem gives exactly how the above strategy can be realized.

Theorem 3.2.1. *Let* $\mathbf{G}$ *and* $\mathbf{D}$ *be given as in (3.3). Assume that* $\bar{\mathbf{F}}$ *satisfies (3.10a-d), and system (3.8) is uniformly bounded dissipative with respect to* α*. Let* $\lambda_1 =$

$\max\{\lambda_j|\lambda_j \in Re(\sigma(\bar{\mathbf{G}}))\}$. *If*

$$d > \frac{cb_1}{-\lambda_1 + \epsilon}\left(1 + (\frac{b_2}{\gamma})^2\right)^{\frac{l}{2}} =: d_c, \tag{3.11}$$

where $\epsilon \geq 0$ *and* c *is some constant depending on* $\mathbf{G}$ *and* ϵ*, then* $\lim_{t\to\infty} \bar{\mathbf{y}}(t) = 0$.

Proof. Since system (3.8) is uniformly bounded dissipative with respect to α, without loss of generality, we may assume that $\|\bar{\mathbf{y}}(t)\| \leq \alpha$ for all time $t \geq t_0$. Using (3.10b), we write (3.8) as

$$\begin{pmatrix} \dot{\bar{\mathbf{y}}}_c \\ \dot{\bar{\mathbf{y}}}_u \end{pmatrix} = \begin{pmatrix} d(\mathbf{I}_k \otimes \bar{\mathbf{G}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}(t) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}}_c \\ \bar{\mathbf{y}}_u \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{F}}_c(\bar{\mathbf{y}}, t) \\ \bar{\mathbf{R}}_u(\bar{\mathbf{y}}, t) \end{pmatrix}. \tag{3.12a}$$

Applying the variation of constant formula to (3.12a) on $\bar{\mathbf{y}}_c$, we get

$$\bar{\mathbf{y}}_c(t) = e^{(t-t_0)d(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\bar{\mathbf{y}}_c(t_0) + \int_{t_0}^{t} e^{(t-s)d(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\bar{\mathbf{F}}_c(\bar{\mathbf{y}}(s), s)ds.$$

Let $\lambda_1 = \max\{\, \lambda_j|\lambda_j \in Re(\, \sigma(\mathbf{G}) - \{0\}\,)\,\}$. Then

$$\|e^{td(\mathbf{I}_k\otimes\bar{\mathbf{G}})}\| \leq ce^{td\nu} \tag{3.12b}$$

for $\nu = \lambda_1 + \epsilon$ and some constant c. Here $0 < \epsilon < -\lambda_1$. Thus,

$$\begin{aligned} \|\bar{\mathbf{y}}_c(t)\| &\leq ce^{(t-t_0)d\nu}\|\bar{\mathbf{y}}_c(t_0)\| + cb_1\int_{t_0}^{t} e^{d(t-s)\nu}\|\bar{\mathbf{y}}(s)\|ds \\ &\leq ce^{(t-t_0)d\nu}\alpha + \frac{\alpha}{d}\frac{cb_1}{|\nu|} =: ce^{(t-t_0)d\nu}\alpha + \frac{\alpha}{d}c_0. \end{aligned}$$

Let $\delta > 1$, we see that

$$\|\bar{\mathbf{y}}_c(t)\| \leq \frac{\alpha}{d}c_0\delta, \tag{3.13a}$$

whenever $t \geq t_{0,1}$ for some $t_{0,1} > 0$. We then apply Theorem 3.1.3 on $\bar{\mathbf{y}}_{u1}$ and the resulting inequality is

$$\|\bar{\mathbf{y}}_{u1}(t)\| \leq \|\bar{\mathbf{y}}_{u1}(t_{0,1})\| \exp\left\{\int_{t_{0,1}}^{t} \mu_i(\mathbf{U}_1(s))ds\right\} + \int_{t_{0,1}}^{t} \exp\left\{\int_{s}^{t} \mu_i(\mathbf{U}_1(\tau))d\tau\right\} \|\mathbf{R}_{u1}(\bar{\mathbf{y}}(s), s)\|ds.$$

It then follows from (3.10c-d) and (3.13a) that

$$\|\bar{\mathbf{y}}_{u1}(t)\| \leq \alpha e^{-\gamma(t-t_{0,1})} + \frac{\alpha}{d}\frac{b_2}{\gamma}c_0\delta \leq \frac{\alpha}{d}\frac{b_2}{\gamma}c_0\delta^2 =: \frac{\alpha}{d}c_1\delta^2, \tag{3.13b}$$

whenever $t \geq t_{1,1}$ for some $t_{1,1} \geq t_{0,1}$. Inductively, we get

$$\|\bar{\mathbf{y}}_{uj}(t)\| \leq \frac{\alpha}{d}\left(\frac{b_2}{\gamma}\sqrt{\sum_{i=0}^{j-1} c_i^2}\right)\delta^{j+1} =: \frac{\alpha}{d}c_j\delta^{j+1}, \quad j = 2, \ldots, l, \tag{3.13c}$$

whenever $t \geq t_{j,1}(\geq t_{j-1,1})$. Letting $t_{l,1} = t_1$ and summing up (3.13a), (3.13b) and (3.13c), we get

$$\|\bar{\mathbf{y}}(t)\| = \sqrt{\sum_{j=1}^{l} \|\bar{\mathbf{y}}_{uj}(t)\|^2 + \|\bar{\mathbf{y}}_c(t)\|^2} \leq \frac{\alpha}{d}\left(1 + (\frac{b_2}{\gamma})^2\right)^{\frac{l}{2}} \frac{cb_1}{|\nu|}\delta^{l+1} =: h\alpha,$$

whenever $t \geq t_1$. Choosing $d > \left(1 + (\frac{b_2}{\gamma})^2\right)^{\frac{l}{2}} \frac{cb_1}{|\nu|}\delta^{l+1}$, we see that the contraction factor h is strictly less than 1, and $\|\bar{\mathbf{y}}(t)\|$ contracts as time progresses. To complete the proof of the theorem, we note that $\delta > 1$ can be made arbitrary close to 1. Consequently, if $d > \left(1 + (\frac{b_2}{\gamma})^2\right)^{\frac{l}{2}} \frac{cb_1}{|\nu|}$, then h can still be made to be less than 1. □

Remark 3.2.2. *(i) In case that* $\bar{\mathbf{G}}$ *is symmetric, then* c *and* ϵ *can be chosen to be one and zero, respectively. (ii)* b_1 *and* b_2 *could possibly depend on* α*. (iii) If system (3.8) is only bounded dissipative, then the estimate in (3.11) is still valid. However, in this case,* b_1 *and* b_2 *depend not only on* α *but also on* $\mathbf{x}_0$*.*

Corollary 3.2.3. *Suppose* $\bar{\mathbf{F}}$ *and* $\mathbf{G}$ *are given as in Theorem 3.2.1. Let*

$$\mathbf{D} = \begin{pmatrix} \bar{\mathbf{D}}_{k\times k} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}_{n\times n,} \quad \textit{where } Re(\,\sigma(\bar{\mathbf{D}})\,) > 0. \tag{3.14a}$$

Assume, in addition, that either $\sigma(\mathbf{G})$ *or* $\sigma(\bar{\mathbf{D}})$ *has no complex eigenvalue.*
Then assertions in Theorem 3.2.1 still hold true, except d_c *needs to be replaced by*

$$d_c = \frac{c\, b_1}{|\nu| \min\{Re(\,\sigma(\bar{\mathbf{D}})\,)\}} \left(1 + (\frac{b_2}{\gamma})^2\right)^{\frac{l}{2}}. \tag{3.14b}$$

Proof. Assumption on $\mathbf{D}$ is to ensure that (3.29b) is still valid. Other parts of the proof are similar to those in Theorem 3.2.1 and are thus omitted. □

We next turn our attention to finding conditions on the nonlinearities $f_i(\mathbf{u}, t)$, $i = 1, \ldots, n$, $\mathbf{u} \in \mathbb{R}^n$, so that assumptions (3.10a-d) are satisfied. To this end, we need the following notations. Let $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}$ be given as in (3.4). Define

$$[\tilde{\mathbf{x}}_i]^- = \begin{pmatrix} x_{1,i} \\ \vdots \\ x_{m-1,i} \end{pmatrix}, \text{ and } [\tilde{\mathbf{x}}]^- = \begin{pmatrix} [\tilde{\mathbf{x}}_1]^- \\ \vdots \\ [\tilde{\mathbf{x}}_n]^- \end{pmatrix}. \tag{3.15}$$

We then break $\tilde{\mathbf{F}}$ as given in (3.5a) into two parts so that the breaking is in consistent with $\bar{\mathbf{y}}$ in (3.9). Specifically, we shall write

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, t) = \begin{pmatrix} \tilde{\mathbf{F}}_c(\tilde{\mathbf{x}}, t) \\ \tilde{\mathbf{F}}_u(\tilde{\mathbf{x}}, t) \end{pmatrix}. \tag{3.16}$$

We are now in the position to state the following propositions.

Proposition 3.2.4. *Suppose that $f_i(\mathbf{x},t)$, $i=1,2,\ldots,k$ satisfy a Lipschitz condition in $B_n(\frac{\alpha}{2})$ with a Lipschitz constant b_1. That is*

$$|f_i(\mathbf{u},t) - f_i(\mathbf{v},t)| \leq \frac{b_1}{k}\,\|\mathbf{u}-\mathbf{v}\|, i=1,2,\ldots,k, \tag{3.17}$$

for all $\mathbf{u}$, $\mathbf{v}$ in $B_n(\frac{\alpha}{2})$ and all time t. Then (3.10a) holds true.

Proof. Note that $\mathbf{E}\tilde{\mathbf{F}}(\tilde{\mathbf{x}},t) = \begin{pmatrix} \mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t) \\ \vdots \\ \mathbf{A}\tilde{\mathbf{f}}_n(\tilde{\mathbf{x}},t) \end{pmatrix}_,$ where $\mathbf{A}$ is given as in (3.23a), and so

$$[\mathbf{A}\tilde{\mathbf{f}}_i(\tilde{\mathbf{x}},t)]^- = \begin{pmatrix} f_i(\mathbf{x}_1,t) - f_i(\mathbf{x}_2,t) \\ \vdots \\ f_i(\mathbf{x}_{m-1},t) - f_i(\mathbf{x}_m,t) \end{pmatrix}_, \quad i=1,2,\ldots,n. \tag{3.18}$$

Since

$$\bar{\mathbf{F}}_c(\bar{\mathbf{y}},t) = \begin{pmatrix} [\mathbf{A}\tilde{\mathbf{f}}_1(\tilde{\mathbf{x}},t)]^- \\ \vdots \\ [\mathbf{A}\tilde{\mathbf{f}}_k(\tilde{\mathbf{x}},t)]^- \end{pmatrix}_,$$

we conclude that (3.10a) holds. □

From the above proposition, we see that the nonlinearities on the corresponding coupled space are only assumed to be Lipchitz. The following proposition is very useful in the sense that by checking how each component f_i of the nonlinearity $\mathbf{f}$ is formed, one would then be able to conclude whether (3.10c-d) are satisfied.

Proposition 3.2.5. *Let $\mathbf{u}=(u_1,\ldots,u_n)^T$ and $\mathbf{v}=(v_1,\ldots,v_n)^T$ be vectors in $B_n(\frac{\alpha}{2})$. Let $w_p = \sum_{i=0}^{p} k_i$, $p=1,\ldots,l$, where $k_0=k$, the dimension of coupled space, and $k_1,\ldots,k_l$ and l are given as in (3.10c). Write $f_{w_{p-1}+i}(\mathbf{u},t) - f_{w_{p-1}+i}(\mathbf{v},t)$, $i=1,\ldots,k_p$, as*

$$f_{w_{p-1}+i}(\mathbf{u},t) - f_{w_{p-1}+i}(\mathbf{v},t)$$
$$= \sum_{j=1}^{k_p} q_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u},\mathbf{v},t)(u_{w_{p-1}+j} - v_{w_{p-1}+j}) + r_{w_{p-1}+i}(\mathbf{u},\mathbf{v},t). \tag{3.19a}$$

We further assume that the followings are true.

(i) For $p = 1,\ldots,l$, let $\mathbf{Q}_{\mathbf{u},\mathbf{v},p} = (q_{w_{p-1}+i,w_{p-1}+j}(\mathbf{u},\mathbf{v},t))$, where $1 \le i,j \le k_p$. Then $\mu_*(\mathbf{V}_p) < -\gamma$ for all p, $\mathbf{u},\mathbf{v}$ in $B_n(\frac{\alpha}{2})$ and all time t, where $* = 1,2,\infty$. (3.19*b*)

(ii) Let $\mathbf{r}_p = \big(r_{w_{p-1}+1}(\mathbf{u},\mathbf{v},t),\ldots,r_{w_p}(\mathbf{u},\mathbf{v},t)\big)^T$. We have that

$$\|\mathbf{r}_p\| \le b_2 \left\| \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_{w_{p-1}} - v_{w_{p-1}} \end{pmatrix} \right\| \tag{3.19c}$$

for all p, $\mathbf{u},\mathbf{v}$ in $B_n(\frac{\alpha}{2})$ and all time t.

Then (3.10c) and (3.10d) hold true for $* = 1,2,\infty$.

Proof. Since $r_i(\mathbf{u},\mathbf{v},t)$ depend on whole space, $f_i(\mathbf{u},t) - f_i(\mathbf{v},t)$ can always be written as the form in (3.19a). Using (3.19a) and (3.18), we have that the matrices $\mathbf{U}_p(t)$ in the linear part of $\bar{\mathbf{F}}_u(\bar{\mathbf{y}},t)$ take the form

$$\mathbf{U}_p(t) = \sum_{w=1}^{m-1} \mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t) \otimes \mathbf{D}_w, \tag{3.20}$$

where $\mathbf{x}_w$ are given as in (1.2), and

$$(\mathbf{D}_w)_{ij} = \begin{cases} 1 & i = j = w, \\ 0 & otherwise, \end{cases} \quad 1 \le i,j \le m-1.$$

It then follows from (2.9-2.11), and (3.20) that $\mu_*(\mathbf{U}_p(t)) < -\gamma$ for $* = 1$ or ∞. For $* = 2$, we have that

$$\begin{aligned}
&\bigcup_{w=1}^{m-1} \sigma\{\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t) + \left(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)\right)^T\} \\
&\qquad\qquad = \sigma\left\{\sum_{w=1}^{m-1}\left(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)\otimes\mathbf{D}_w + \left(\mathbf{Q}_{\mathbf{x}_w,\mathbf{x}_{w+1},p}(t)\right)^T\otimes\mathbf{D}_w\right)\right\} \\
&\qquad\qquad = \sigma\left(\mathbf{U}_p(t) + \mathbf{U}_p^T(t)\right),
\end{aligned}$$

where $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$. We remark that the first equality above can be verified by the definition of eigenvalues due to the structure of $\mathbf{U}_p(t)$. It then follows from (2.11) that $\mu_2(\mathbf{U}_p(t)) < -\gamma$. The remainder of the proof is similar to that of Proposition 3.2.4, and is thus omitted. □

Remark 3.2.6. *The upshot of Proposition 3.2.5 is that by only checking the "structure" of the vector field* $\mathbf{f}$ *of the single oscillator, one should be able to determine if our main result can be applied. To be precise, we begin with saving notations by setting* $\mathbf{f}$ *as* $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), \dots, f_n(\mathbf{x}, t))^T$. *We then check the form of the difference of "uncoupled" part of dynamics. That is, we write* $f_i(\mathbf{u}, t) - f_i(\mathbf{v}, t)$ *in the form of (3.19a) with* $i = k + 1, \dots, n$. *If (3.19b, c) can be satisfied, then* $l = 1$ *gets the job done. Otherwise, we further break the uncoupled states into a set of smaller pieces to see if the resulting (3.19b, c) are satisfied.*

We are now ready to state the main theorems of the paper.

Theorem 3.2.7. *Assume that system (2.15) is (resp., uniformly) bounded dissipative. Let the coupling matrices* $\mathbf{G}$ *and* $\mathbf{D}$ *satisfy (3.3) and the nonlinearities* $f_i(\mathbf{x}, t)$, $i = 1, 2, \dots, n$, *satisfy (3.17) and (3.19). Suppose* d *is greater than* d_c, *as given in (3.11). Then system (2.15) is (resp., uniformly,) globally synchronized.*

Proof. The proof is direct consequences of Propositions 3.2.4 and 3.2.5, and Theorem 3.2.1. □

Remark 3.2.8. *From here on, we will refer the assumptions in Theorem 3.2.7 as synchronization hypotheses.*

Theorem 3.2.9. *The coupled system* $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}, t))$*, given as in Corollary 3.2.3, is also (resp., uniformly,) globally synchronized provided that its coupled system is (resp., uniformly) bounded dissipative and that* d *is greater than* d_c*. Here* d_c *is given in (3.14b).*

3.3 Applications

To see the effectiveness of our main results, we consider four examples in this section. These are coupled Lorenz equations [4,29], coupled chaotic walk system [56], coupled Duffing oscillators [54] and coupled Lorenz-like system.

3.3.1 Coupled Lorenz System

We shall begin with Lorenz equations. Let $\mathbf{x} = (x_1, x_2, x_3)^T$,

$$\begin{aligned} \mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}) &= (\sigma(x_2 - x_1),\ rx_1 - x_2 - x_1x_3,\ -bx_3 + x_1x_2)^T \\ &=: (f_1(\mathbf{x}),\ f_2(\mathbf{x}),\ f_3(\mathbf{x}))^T. \end{aligned}$$

Here $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$. In the following cases (a), (b), (c) and (d), $\mathbf{G}$ denotes the diffusive coupling with zero flux and $\mathbf{D}$ is, respectively, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. For the first three cases, it was shown in [8] that such coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ have the topological product of an absorbing domain

$$B = \{x_1^2 + x_2^2 + (x_3 - r - \sigma)^2 < \frac{b^2(r + \sigma)^2}{4(b - 1)} =: \beta\}. \tag{3.21}$$

Hence, in each case, we will concentrate on the illustration of how our main results may or may not be applied.

(a) Let $\mathbf{D} = \mathbf{D}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ For the corresponding "coupled" nonlinearity f_1, we get that

$$|f_1(\mathbf{u}) - f_1(\mathbf{v})| = \sigma|(u_2 - v_2) - (u_1 - v_1)| \leq \sqrt{2}\sigma\|\mathbf{u} - \mathbf{v}\|.$$

Hence, condition (3.10a) is satisfied. For the corresponding "uncoupled" nonlinearities f_2 and f_3, we see that

$$\begin{aligned} f_2(\mathbf{u}) - f_2(\mathbf{v}) &= (-u_2 - u_1u_3 + ru_1) - (-v_2 - v_1v_3 + rv_1) \\ &= [-(u_2 - v_2) - u_1(u_3 - v_3)] + (r - v_3)(u_1 - v_1) \end{aligned} \tag{3.22a}$$

and

$$\begin{aligned} f_3(\mathbf{u}) - f_3(\mathbf{v}) &= (u_1u_2 - bu_3) - (v_1v_2 - bv_3) \\ &= [u_1(u_2 - v_2) - b(u_3 - v_3)] + v_2(u_1 - v_1). \end{aligned} \tag{3.22b}$$

Writing (3.22a,b) in the vector form, we get

$$\begin{aligned} \begin{pmatrix} f_2(\mathbf{u}) - sf_2(\mathbf{v}) \\ f_3(\mathbf{u}) - f_3(\mathbf{v}) \end{pmatrix} &= \begin{pmatrix} -1 & -u_1(t) \\ u_1(t) & -b \end{pmatrix} \begin{pmatrix} u_2 - v_2 \\ u_3 - v_3 \end{pmatrix} + \begin{pmatrix} (r - v_3)(u_1 - v_1) \\ v_2(u_1 - v_1) \end{pmatrix} \\ &=: \mathbf{Q}_{\mathbf{u},\mathbf{v},1}(t) \begin{pmatrix} u_2 - v_2 \\ u_3 - v_3 \end{pmatrix} + \mathbf{r}_1. \end{aligned} \tag{3.22c}$$

Clearly, $\mu_2(\mathbf{Q}_{\mathbf{u},\mathbf{v},1}(t)) = \max\{-1, -b\} = -1 < 0$, and $\|\mathbf{r}_1\| \leq (\sigma + \sqrt{\beta}) \cdot |u_1 - v_1|$, where its estimate depends only on coupled space. Hence, conditions (3.19b,c) are satisfied.

(b) Let $\mathbf{D} = \mathbf{D}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. As in the case (a), the corresponding "coupled" nonlinearity f_2 is clearly Lipschitz on the absorbing domain. For the corresponding "uncoupled" nonlinearities f_1 and f_3, we get

$$f_1(\mathbf{u}) - f_1(\mathbf{v}) = [-\sigma(u_1 - v_1)] + \sigma(u_2 - v_2),$$
$$f_3(\mathbf{u}) - f_3(\mathbf{v}) = [-b(u_3 - v_3)] + u_1(u_2 - v_2) + v_2(u_1 - v_1).$$

If $l = 1$ is chosen, then (3.19c) is violated. For in the case, the norm estimate in the right hand side of (3.19c) can only depend on $u_2 - v_2$. Now, if we choose $l = 2$ and pick the space of the first diagonal block being the one associated with the nonlinearity f_1, then $\mathbf{Q}_{\mathbf{u},\mathbf{v},1} = (-\sigma)$ and $r_1 = \sigma(u_2 - v_2)$. Consequently, (3.19b) and (3.19c) are satisfied with $p = 1$. For $p = 2$, we have $\mathbf{Q}_{\mathbf{u},\mathbf{v},2} = (-b)$ and $r_2 = u_1(u_2 - v_2) + v_2(u_1 - v_1)$, which depends only on the coupled space and the preceding uncoupled space. Thus, r_2 can also be satisfied with (3.19c).

(c) For illustration, we also consider $\mathbf{D} = \mathbf{D}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In this case, the uncoupled nonlinearities of f_1 and f_2 both contain the terms x_2 and x_1. The only feasible choice to break the uncoupled space is not to do any breaking. That is, pick $l = 1$. Otherwise, (3.19c) is isolated. For $l = 1$, we have that $\mathbf{Q}_{\mathbf{u},\mathbf{v},1} = \begin{pmatrix} -\sigma & \sigma \\ r - u_3(t) & -1 \end{pmatrix}$. For such $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}$, its matrix measure can not stay negative for all time. An indicated, see e.g., [29], synchronization fails for this type of partial coupling.

(d) Let $\mathbf{D} = \mathbf{D}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. To apply Theorem 3.2.9, we first note that for $\mathbf{D} = \mathbf{D}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the corresponding coupled system $(\mathbf{D}_5, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ is indeed globally synchronized, and hence, so is the system $(\mathbf{D}_4, \mathbf{G}, \mathbf{F}(\mathbf{x}))$. Note that bounded dissipation of the system $(\mathbf{D}_4, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ can be verified similarly as in [29].

(e) The work that are most related to ours are those in [4,5]. While their estimates for d_c seems to be sharper than ours, which we shall illustrate in case (f), their connectivity topology requires that off-diagonal entries be nonnegative. We only assume our connectivity topology satisfies (3.3a,b). Consider for instant the following matrix:

$$\mathbf{G} = \begin{pmatrix} -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 2 \\ 2 & -1 & -3 & 2 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

Such $\mathbf{G}$ has some negative off-diagonal entries and satisfy (3.3a,b). In fact, the eigenvalues of $\mathbf{G}$ are 0, $-1 \pm \sqrt{5}i$, and -6. Clearly, applying our results, we see immediately that coupled systems $(\mathbf{D}_i, \mathbf{G}, \mathbf{F}(\mathbf{x}))$, $i = 1, 2, 4$ are globally synchronized. Numerical results (see Figure 3.1) indeed confirm synchronization of such connectivity topology. We remark that by constructing the Lyapunov function as given in [29], one would be able to show the bounded dissipation of the coupled system with this particular connectivity topology.

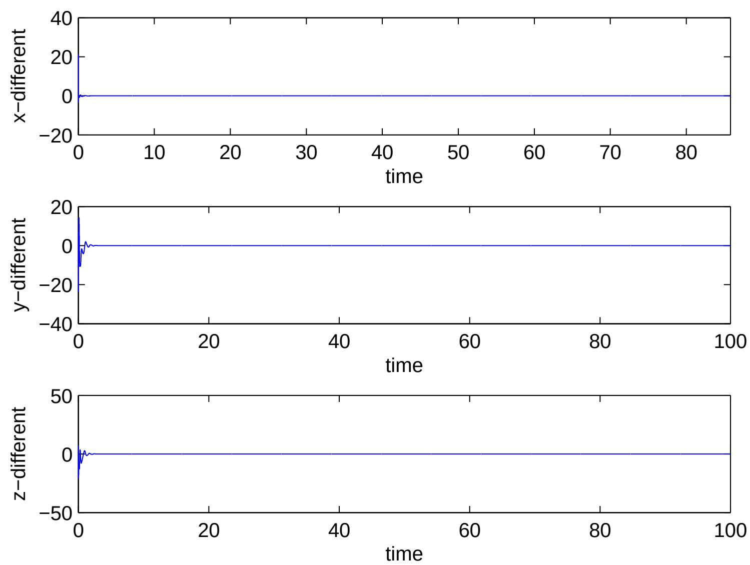

Figure 3.1: The difference of each component of two coupled oscillators in case (e).

(f) In this part, we shall compute the lower bound for the global synchronization for case (a) by using our method, those obtained in [4] and MSF, respectively. To compute d_c, given in (3.11), we note that $\bar{\mathbf{G}} = \mathbf{C}\mathbf{G}\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} = \mathbf{C}(\mathbf{C}^T\mathbf{C})\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} = \mathbf{C}\mathbf{C}^T$. Since $\bar{\mathbf{G}}$ is symmetric, c and ϵ, given as in (3.12b), can be chosen to be 1, and 0, respectively. Consequently,

$$d_c = \frac{\sqrt{2}\sigma\sqrt{1+\beta+2\sigma\sqrt{\beta}+\sigma^2}}{4\sin^2(\frac{\pi}{2n})}. \tag{3.32}$$

Here $4\sin^2(\frac{\pi}{2n}) = |\lambda_1|$. Applying Theorem 3.2.9, we see that coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ is uniformly, globally synchronized provided that the coupling strength d is greater than d_c. For $n = 4$, $d_c \approx 1189$. In [53], the bound $\bar{d}_c$ for threshold of uniformly global synchronization is

$$\bar{d}_c = \begin{cases} \frac{a}{8}n^2 & \text{if n is even} \\ \frac{a}{8}(n^2-1) & \text{if n is odd} \end{cases}.$$

Here $a = \frac{b(b+1)(r+\sigma)^2}{16(b-1)} - \sigma$. For $n = 4$, $\bar{d}_c \approx 1039$, which is slightly better than d_c.

Using the MSF-criteria, we numerically (see Figure 3.2) compute the maximum Lyapunov exponent of the variational equations with respect to the parameter α. We have in this example that if

$$\alpha = d\lambda_1 < -7.778, \tag{4.4}$$

then its maximum Lyapunov exponent is negative. Here $\lambda_1 = -4\sin^2\frac{\pi}{8}$ is the largest nonzero eigenvalues of $\mathbf{G}$. Hence if $d > \frac{-7.778}{\lambda_1} \approx 13.3$, then local synchronization of the coupled system $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ can be realized.

3.3.2 Coupled Chaotic Walks System

For the second example, we consider the subsystem (see e.g., [12]) of chaotic walks. That is,

$$\dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1) = (f_1(\mathbf{x}_1), \ldots, f_n(\mathbf{x}_1))^T, \tag{3.24a}$$

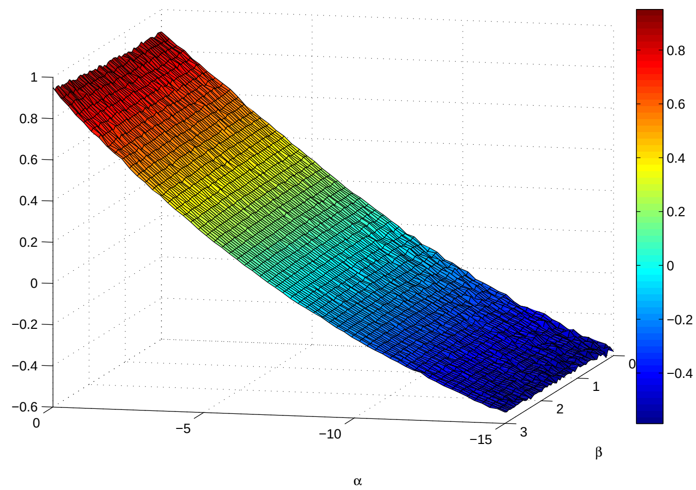

Figure 3.2: The vertical axis denotes the maximum Lyapunov exponent of the variational equations. While the horizontal axis represents $\alpha = d\lambda$.

where

$$f_i(\mathbf{x}_1) = \sin(x_{1,k}) - bx_{1,i}, \;\; i = 1, 2, \ldots, n, \text{ and}$$
$$k = (i \mod n) + 1. \tag{3.24b}$$

Note that in [56], it was demonstrated numerically that subsystems (3.24a) exhibits hyperchaos. We next show that the coupled system (2.15) with the nonlinearities given as in (3.24b) is bounded dissipative provided that $\mathbf{G}$ is a negative semidefinite matrix, and $\mathbf{D}$ is given as in (3.3c). To this end, we introduce a Lyapunov function of the form

$$V(\mathbf{x}) = \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{x_{j,i}^2}{2}.$$

By taking the time derivative of V along solutions of (2.15), one obtains

$$
\begin{aligned}
\frac{dV}{dt} &= \sum_{j=1}^{m}\sum_{i=1}^{n} x_{j,i}(\sin(x_{j,k}) - bx_{j,i}) + d\sum_{j=1}^{k} < \mathbf{x}_j, \mathbf{G}\mathbf{x}_j > \\
&\le \sum_{j=1}^{m}\sum_{i=1}^{n} -bx_{j,i}^2 + |x_{j,i}| =: b_{m,n}.
\end{aligned}
$$

Suppose $\sum_{j=1}^{m}\sum_{i=1}^{n} x_{j,i}^2 \ge mnc_0^2$, where $c_0 > 0$ satisfying

$$
-bc_0^2 + c_0 < -\frac{1}{2b}(mn-1). \tag{3.25}
$$

Then, we may assume, without loss of generality, that $|x_{1,1}| \ge c_0$. Now,

$$
\begin{aligned}
b_{m,n} &= -bx_{1,1}^2 + |x_{1,1}| + \left[\left(\sum_{j=1}^{m}\sum_{i=1}^{n} -bx_{j,i}^2 + |x_{j,i}|\right) + bx_{1,1}^2 - |x_{1,1}|\right] \\
&< -\frac{1}{2b}(mn-1) + \frac{1}{4b}(mn-1) = -\frac{1}{4b}(mn-1) < 0.
\end{aligned}
$$

We have used (3.25) and the fact that $\max(-bx^2 + |x|) = \frac{1}{4b}$ to justify the above inequality. It then follows from Proposition 3.1.2 that the coupled chaotic walk is bounded dissipative as claimed. Noting that the permutation symmetry of equation (3.23), we only consider the case that the matrix $\mathbf{D}$ satisfying (3.3d) with $k = 1$. Letting $l = n - k = n - 1$, we see that $\mathbf{Q}_{\mathbf{u},\mathbf{v},p} = -b$, $p = 1, 2, \ldots, l$. Thus, their matrix measure $\mu_i(\mathbf{Q}_{\mathbf{u},\mathbf{v},p}) = -b < 0$. Moreover, the corresponding remaining terms $\mathbf{r}_p$ satisfy (3.19c). Thus, system (2.15) is globally synchronized. In summary, we have our results in the following.

Theorem 3.3.1. *Let* $\mathbf{f}(\mathbf{x})$ *be given as in (3.23) and* $\mathbf{G}$ *be a symmetry matrix satisfying (3.3a, 3.3b). Let* $\mathbf{D}$ *be a matrix satisfying (3.14a). Then the coupled system* $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ *is globally synchronized provided that* d *is chosen sufficiently large.*

Proof. To complete the proof of the theorem, it suffices to show that the coupled system (2.15) is bounded dissipative. Writing the first k components of the coupled system, we get

$$\dot{\mathbf{z}}_k := \begin{pmatrix} \dot{\tilde{\mathbf{x}}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_k \end{pmatrix} = \begin{pmatrix} -b\tilde{\mathbf{x}}_1 + \tilde{\mathbf{g}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ -b\tilde{\mathbf{x}}_k + \tilde{\mathbf{g}}_k(\tilde{\mathbf{x}}, t) \end{pmatrix} + d(\bar{\mathbf{D}} \otimes \mathbf{G}) \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_k \end{pmatrix}, \tag{3.26}$$

where the components of $\tilde{\mathbf{g}}_i(\tilde{\mathbf{x}}, t)$ have the form of $\sin(*)$. Applying the variation of constant formula to (3.26), we see that

$$\mathbf{z}_k(t) = e^{(-b\mathbf{I}+d\bar{\mathbf{D}}\otimes\mathbf{G})t}\mathbf{z}_k(0) + \int_0^t e^{(-b\mathbf{I}+d\bar{\mathbf{D}}\otimes\mathbf{G})(t-s)}\bar{\mathbf{G}}(\mathbf{x}, s)ds,$$

where $\bar{\mathbf{G}}(\mathbf{x}, t) = \begin{pmatrix} \tilde{\mathbf{g}}_1(\tilde{\mathbf{x}}, t) \\ \vdots \\ \tilde{\mathbf{g}}_k(\tilde{\mathbf{x}}, t) \end{pmatrix}.$ Now,

$$\begin{aligned} \|\mathbf{z}_k(t)\| &\le c_0 e^{-\frac{b}{2}t}\|\mathbf{z}_k(0)\| + c_0\sqrt{mk}\int_0^t e^{-\frac{b}{2}(t-s)}ds \\ &\le c_0 e^{-\frac{b}{2}t}\|\mathbf{z}_k(0)\| + \alpha, \end{aligned}$$

for some constant $c_0 > 0$ and $\alpha = 2\frac{c_0}{b}\sqrt{mk}$. Similarly, we have $\|\tilde{\mathbf{x}}_{k+i}(t)\| \le c_0 e^{-\frac{b}{2}t}\|\tilde{\mathbf{x}}_{k+i}(0)\| + \alpha$ for all $i = 1, \ldots, n-k$. Hence,

$$\|\tilde{\mathbf{x}}(t)\| \le ce^{-\frac{b}{2}t}\|\tilde{\mathbf{x}}(0)\| + n\alpha$$

for some constant c. Thus, system (2.15) is bounded dissipative with respect to $((n+1)\alpha, ((c+1)n+c)\alpha)$. □

3.3.3 Coupled Duffing Oscillators

Another formulation not considered in [4,5] is Duffing oscillators. Specifically, the individual system considered is defined by

$$\dot{x}_1 = -\alpha x_1 - x_2^3 + a\cos wt \tag{3.27a}$$

$$\dot{x}_2 = x_1, \tag{3.27b}$$

where α and a are positive constants. Letting $\mathbf{x} = (x_1, x_2)^T$, we have

$$\mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), f_2(\mathbf{x})) = (-\alpha x_1 - x_2^3 + a\cos wt, x_1). \tag{3.28a}$$

Assume the coupling matrices $\mathbf{D}$ and $\mathbf{G}$ are, respectively,

$$\mathbf{D}(c) = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} \tag{3.28b}$$

and

$$\mathbf{G}(\epsilon, r) = \begin{pmatrix} -2\epsilon & \epsilon - r & 0 & \cdots & 0 & \epsilon + r \\ \epsilon + r & -2\epsilon & \epsilon - r & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -2\epsilon & \epsilon - r \\ \epsilon - r & 0 & \cdots & 0 & \epsilon + r & -2\epsilon \end{pmatrix}, \tag{3.28c}$$

where $\epsilon > 0$ and r are scalar diffusive and gradient coupling parameters, respectively. First, we prove the bounded dissipation of systems (3.27). Setting $\tilde{\mathbf{x}}_2^3 = (x_{1,2}^3, \ldots, x_{m,2}^3)^T$, and $\mathbf{g}(t) = a\cos(wt)\,(1, \cdots, 1)^T$. We see that (3.5) becomes

$$\dot{\tilde{\mathbf{x}}}_1 = -\alpha\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2^3 + \mathbf{g}(t) + dc\mathbf{G}(\epsilon, r)\tilde{\mathbf{x}}_2 + d\mathbf{G}(\epsilon, r)\tilde{\mathbf{x}}_1 \tag{3.29a}$$

$$\dot{\tilde{\mathbf{x}}}_2 = \tilde{\mathbf{x}}_1. \tag{3.29b}$$

We consider the following scalar-valued function as the Lyapunov function of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$

$$U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) = \frac{1}{2} < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > + \sum_{i=1}^{m} \frac{x_{i,2}^4}{4} + c < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1 >, \tag{3.30}$$

Taking the time derivative of U along solutions of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$, we have

$$\begin{aligned}
\frac{dU}{dt} &=< \tilde{\mathbf{x}}_1, \dot{\tilde{\mathbf{x}}}_1 > + \sum_{i=1}^{m} x_{i,2}^3 x_{i,1} + c < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > + c < \tilde{\mathbf{x}}_2, \dot{\tilde{\mathbf{x}}}_1 > \\
&= (c - \alpha) < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > - c\alpha < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1 > - c < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2^3 > + < \tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2, \mathbf{g}(t) > \\
&\quad + d < \tilde{\mathbf{x}}_1, \mathbf{G}(\epsilon, r)\tilde{\mathbf{x}}_1 > + 2dc < \tilde{\mathbf{x}}_1, \mathbf{G}(\epsilon, r)\tilde{\mathbf{x}}_2 > + dc^2 < \tilde{\mathbf{x}}_2, \mathbf{G}(\epsilon, r)\tilde{\mathbf{x}}_2 > \\
&= (c - \alpha) < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > - c\alpha < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1 > - c < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2^3 > + < \tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2, \mathbf{g}(t) > \\
&\quad + d\,(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) \left(\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes \mathbf{G}(\epsilon, r) \right) \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{pmatrix} \\
&\leq (c - \alpha) < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > - c\alpha < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1 > - c < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2^3 > + < \tilde{\mathbf{x}}_1 + c\tilde{\mathbf{x}}_2, \mathbf{g}(t) >
\end{aligned}$$

Note that the last inequality holds true since

$$\begin{aligned}
&\left(\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes \mathbf{G}(\epsilon, r) \right) + \left(\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes \mathbf{G}(\epsilon, r) \right)^T \\
= &\begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix} \otimes (\mathbf{G}(\epsilon, r) + \mathbf{G}(\epsilon, r)^T),
\end{aligned}$$

and $\mathbf{G}(\epsilon, r) + \mathbf{G}(\epsilon, r)^T$ is a nonpositive definite matrix. On the other hand, since

$$< \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2^3 >= \sum_{i=1}^{m} x_{2,i}^4 \geq \frac{1}{m} \left(\sum_{i=1}^{m} x_{i,2}^2 \right)^2 \geq \frac{1}{m} \|\tilde{\mathbf{x}}_2\|_2^4,$$

we have

$$\begin{aligned}
\frac{dU}{dt} &\leq (c - \alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + c\alpha\|\tilde{\mathbf{x}}_2\|_2\|\tilde{\mathbf{x}}_1\|_2 - \frac{c}{m}\|\tilde{\mathbf{x}}_2\|_2^4 + \sqrt{m}a(\|\tilde{\mathbf{x}}_1\|_2 + c\|\tilde{\mathbf{x}}_2\|_2) \\
&=: u(\|\tilde{\mathbf{x}}_2\|_1, \|\tilde{\mathbf{x}}_2\|_2).
\end{aligned}$$

We are now in a position to show bounded dissipation of the coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$.

Proposition 3.3.2.

(i) If c satisfies the inequality

$$0 < c < \min\{\frac{4\alpha}{4+\alpha^2 m}, \alpha\} = \frac{4\alpha}{4+\alpha^2 m}. \tag{3.31}$$

Then there exists a constant c_0 so that $\frac{dU}{dt} < 0$ for $\|\tilde{\mathbf{x}}_2\|_1^2 + \|\tilde{\mathbf{x}}_2\|_2^2 \geq c_0$.

(ii) If $c = 0$, then the first assertion of the proposition still holds true.

Proof. Suppose $\|\tilde{\mathbf{x}}_2\|_2 \geq 1$. Then

$$\begin{aligned} u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2) &\leq (c-\alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + c\alpha\|\tilde{\mathbf{x}}_2\|_2\|\tilde{\mathbf{x}}_1\|_2 - \frac{c}{m}\|\tilde{\mathbf{x}}_2\|_2^2 + \sqrt{m}a(\|\tilde{\mathbf{x}}_1\|_2 + c\|\tilde{\mathbf{x}}_2\|_2) \\ &=: \bar{u}(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2). \end{aligned}$$

It then follows from (3.31) that the the level curve of $\bar{u}$ is a bounded closed curve. We shall call such curve ellipse-like is an elliptic in the plane. Thus, there exists a c_1 so that $\frac{dU}{dt} < 0$ whenever $\|\tilde{\mathbf{x}}_2\|_1^2 + \|\tilde{\mathbf{x}}_2\|_2^2 \geq c_1$ and $\|\tilde{\mathbf{x}}_2\|_2 \geq 1$. Let $\|\tilde{\mathbf{x}}_2\|_2 < 1$ and $\|\tilde{\mathbf{x}}_2\|_1^2 + \|\tilde{\mathbf{x}}_2\|_2^2 \geq c_2$. Here c_2 is a constant to be determined. Then

$$u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2) \leq (c-\alpha)\|\tilde{\mathbf{x}}_1\|_2^2 + (c\alpha + \sqrt{m}a)\|\tilde{\mathbf{x}}_1\|_2 + \sqrt{m}ac =: h(\|\tilde{\mathbf{x}}_1\|_2).$$

Since $h(\|\tilde{\mathbf{x}}_1\|_2)$ is a parabola-like curve which is open downward, there exists a $c_3 > 1$ such that $h(\|\tilde{\mathbf{x}}_1\|_2) < 0$ whenever $\|\tilde{\mathbf{x}}_1\|_2 \geq c_3$. Thus, if $c_2 \geq c_3^2+1$, then $u(\|\tilde{\mathbf{x}}_1\|_2, \|\tilde{\mathbf{x}}_2\|_2) < 0$ whenever $\|\tilde{\mathbf{x}}_2\|_2 < 1$ and $\|\tilde{\mathbf{x}}_1\|_2^2 + \|\tilde{\mathbf{x}}_2\|_2^2 \geq c_2$. Picking $c_0 = \max\{c_1, c_2\}$, we have that the assertion of the proposition holds true. □

Proposition 3.3.3. *Assume (3.31) holds true. Then $\lim_{r\to\infty} U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) = \infty$, where $r = \sqrt{\|\tilde{\mathbf{x}}_1\|^2 + \|\tilde{\mathbf{x}}_2\|^2}$.*

Proof. From (3.30), we have that

$$U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) = \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + \sum_{i=1}^{m} \frac{x_{i,2}^4}{4} + c < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_1 >$$
$$\geq \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + \frac{1}{4m}\|\tilde{\mathbf{x}}_2\|^4 - c\|\tilde{\mathbf{x}}_2\| \cdot \|\tilde{\mathbf{x}}_1\|,$$

Let $\frac{1}{4m}b_1^2 > c^2$. Then suppose $\|\tilde{\mathbf{x}}_2\| > b_1$, we have

$$U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) \geq \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 + c^2\|\tilde{\mathbf{x}}_2\|^2 - c\|\tilde{\mathbf{x}}_2\|\|\tilde{\mathbf{x}}_1\| =: h_1(\|\tilde{\mathbf{x}}_1\|, \|\tilde{\mathbf{x}}_2\|).$$

Since the level curve of $h_1(\|\tilde{\mathbf{x}}_1\|, \|\tilde{\mathbf{x}}_2\|)$ is elliptic-like in the plane. Thus, for any given $M > 0$, there exists a $d_1 > 0$ such that $U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) > M$ whenever $\|\tilde{\mathbf{x}}_1\|^2 + \|\tilde{\mathbf{x}}_2\|^2 \geq d_1^2$ and $\|\tilde{\mathbf{x}}_2\| > b_1$.
Let $\|\tilde{\mathbf{x}}_2\| \leq b_1$. Then

$$U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) \geq \frac{1}{2}\|\tilde{\mathbf{x}}_1\|^2 - cb_1\|\tilde{\mathbf{x}}_1\| =: h_2(\|\tilde{\mathbf{x}}_1\|, \|\tilde{\mathbf{x}}_2\|),$$

since $h_2(\|\tilde{\mathbf{x}}_1\|, \|\tilde{\mathbf{x}}_2\|)$ is a parabola-like curve which is open upward in the plane. Thus, for any given $M > 0$, there exists a $d_2 > 0$ such that $U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) > M$ whenever $\|\tilde{\mathbf{x}}_1\|^2 + \|\tilde{\mathbf{x}}_2\|^2 \geq d_2^2$ and $\|\tilde{\mathbf{x}}_2\| \leq b_1$. Picking $\delta = \max\{d_1, d_2\}$, we have that $U(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) > M$ for all $\|\tilde{\mathbf{x}}_1\|^2 + \|\tilde{\mathbf{x}}_2\|^2 \geq \delta^2$. Thus, the assertion of the proposition holds true. ☐

Theorem 3.3.4. *The coupled system* $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$ *is bounded dissipative if condition (3.31) holds true.*

Proof. The proof is direct consequences of Propositions 3.3.2 and 3.3.3. ☐

Note that

$$f_2(\mathbf{u}) - f_2(\mathbf{v}) = 0(u_2 - v_2) + (u_1 - v_1) \tag{3.32}$$

and so the matrix measure of the corresponding $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}$ is zero. To apply our theorem, we need to make the following coordinate change.

Letting $y_2 = x_2$ and $y_1 = qx_1 + px_2$, we see that (3.27a,b) becomes

$$\dot{y}_1 = (\frac{p}{q} - \alpha)y_1 + p(\alpha - \frac{p}{q})y_2 - qy_2^3 + qa\cos wt =: \bar{\mathbf{f}}_1(\mathbf{y}) \tag{3.33a}$$

$$\dot{y}_2 = \frac{-p}{q}y_2 + \frac{1}{q}y_1 =: \bar{\mathbf{f}}_2(\mathbf{y}), \tag{3.33b}$$

and the corresponding coupled system (3.7) becomes

$$\begin{aligned}\dot{\tilde{\mathbf{y}}}_1 = (\frac{p}{q} - \alpha)\tilde{\mathbf{y}}_1 + p(\alpha - \frac{p}{q})\tilde{\mathbf{y}}_2 - q\tilde{\mathbf{y}}_2^3 + \mathbf{g}(t)\\ + d(qc - p)\mathbf{G}(\epsilon, r)\tilde{\mathbf{y}}_2 + d\mathbf{G}(\epsilon, r)\tilde{\mathbf{y}}_1\end{aligned} \tag{3.34a}$$

$$\dot{\tilde{\mathbf{y}}}_2 = -\frac{q}{p}\tilde{\mathbf{y}}_2 + \frac{1}{q}\tilde{\mathbf{y}}_1, \tag{3.34b}$$

where $\tilde{\mathbf{y}}_2^3 = (y_{1,2}^3, \ldots, y_{m,2}^3)^T$ and $\mathbf{g}(t) = a\cos(wt)\,(1, \cdots, 1)^T$.. In the following, we choose (p, q) to be $(1, c - \frac{1}{d})$ as $c > 0$, and to be $(-1, -\frac{1}{d})$ as $c = 0$, respectively. Then in the case of $c > 0$, (3.34) becomes

$$\begin{aligned}\dot{\tilde{\mathbf{y}}}_1 &= d\mathbf{G}(\epsilon, r)\tilde{\mathbf{y}}_1 + (c - \alpha - \frac{1}{d})\tilde{\mathbf{y}}_1 + (\alpha - c + \frac{1}{d})\tilde{\mathbf{y}}_2 - \tilde{\mathbf{y}}_2^3 + \mathbf{g}(t) + \mathbf{G}(\epsilon, r)\tilde{\mathbf{y}}_2\\ &=: d\mathbf{G}(\epsilon, r)\tilde{\mathbf{y}}_1 + \tilde{\mathbf{F}}_c(\tilde{\mathbf{y}}, t)\\ \dot{\tilde{\mathbf{y}}}_2 &= -\frac{1}{c - \frac{1}{d}}\tilde{\mathbf{y}}_2 + \tilde{\mathbf{y}}_1.\end{aligned}$$

The purpose of the coordinate transformation is two-fold. First, to make the dynamics of the linear part on the uncoupled space stable. In this case, the coefficient of $\tilde{\mathbf{y}}_2$ becomes negative when $d > \frac{2}{c}$. Second, to make sure the parameters in the nonlinear part of coupled space contain no bad influence of d, coupling strength. Otherwise, we may not be able to control its corresponding dynamics by choosing d large.

It is then easy to check that assumptions for Theorem 3.2.1 are all satisfied. Finally, we will show that if $\frac{4\alpha}{4+\alpha m^2} > c \geq 0$, $\epsilon > 0$ and $r \in \mathbb{R}$, then coupled system $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$ is bounded dissipative. Thus, we can summarize the results as follows

Theorem 3.3.5. *Let* $\mathbf{f}$, $\mathbf{D}(c)$ *and* $\mathbf{G}(\epsilon, r)$ *be given as in (3.27a), (3.27b) and (3.27c), respectively. Let* $0 \leq c < \frac{4\alpha}{4+\alpha m^2}$. *Then the coupled system* $(\mathbf{D}(c), \mathbf{G}(\epsilon, r), \mathbf{F}(\mathbf{x}, t))$ *is globally synchronized provided that* d *is chosen sufficiently large.*

Proof. It remains only to verify that $\mathbf{G}(\epsilon, r)$ satisfies assumptions (3.3a,b). Indeed $\mathbf{G}(\epsilon, r)$ is a circulant matrix (see e.g., [15]), the eigenvalues λ_k of $\mathbf{G}(\epsilon, r)$ are

$$\lambda_k = -2\epsilon(1 - \cos\frac{2k\pi}{n}) - i\,2r\sin\frac{2k\pi}{n}, \quad k = 0, \ldots, m-1.$$

□

Remark 3.3.6. *(i) It was shown in [21] that there are positive constants d_0 and c_0 such that, for $d \geq d_0$, $c \geq c_0$, the system $(\mathbf{D}(c), \mathbf{G}(\epsilon, 0), \mathbf{F})$ given in (3.33) is synchronized. Our results also work for the case that $c_0 = 0$ or $\mathbf{G}(\epsilon, r)$, $r \neq 0$. (ii) It was also shown in [1] that there are positive constants d_0 and c_0 such that for $d \geq d_0$, $c \geq c_0$, the system $(\mathbf{D}(c), \mathbf{G}, \mathbf{F})$ is synchronized. Here $-\mathbf{G}$ is a positive definite matrix.*

3.3.4 Coupled Lorenz-Like System

Finally, we also explore the example in [57]. Specifically, the individual system we obtained that is related to the Lorenz system was given by

$$\begin{aligned}
\frac{dx_1}{dt} &= -\sigma(x_1 - x_2) + x_5, \\
\frac{dx_2}{dt} &= \rho x_1 - x_2 - x_1 x_3, \\
\frac{dx_3}{dt} &= x_1 x_2 - \beta x_3, \\
\frac{dx_4}{dt} &= -x_4^3 + x_5, \\
\frac{dx_5}{dt} &= -x_1 - x_4 - 8x_5,
\end{aligned} \tag{3.35}$$

where σ, β are positive constants, and ρ is a real number. Assume the coupling matrices $\mathbf{D}$ and $\mathbf{G}$ are, respectively,

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 \\ 0 & 0 & 0 & 0 & d_5 \end{pmatrix}, \tag{3.36}$$

and

$$\mathbf{G}(\beta) = \begin{pmatrix} -1-\beta & 1 & 0 & \cdots & 0 & \beta \\ 1 & -2 & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -2 & 1 \\ \beta & 0 & \cdots & 0 & 1 & -1-\beta \end{pmatrix}, \tag{3.37}$$

where $d_i = 0$ or 1, $1 \leq i \leq 5$ and $0 \leq \beta \leq 1$. Specifically, $\mathbf{G}(\beta)$ is diffusively coupled with mixed boundary conditions. To prove that bounded dissipation of coupled Lorenz-Like system (3.35), we first show that bounded dissipation of the individual system. Let the scalar-valued function defined as follows,

$$U(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}[\frac{12}{7}x_1^2 + \sigma x_2^2 + \sigma(x_3 - \rho)^2 + \frac{12}{7}x_4^2 + \frac{12}{7}x_5^2] \tag{3.38}$$

Taking the time derivative of U along solutions of the individual system, we have

$$\begin{aligned} \frac{dU}{dt} &= \frac{12}{7}x_1\dot{x}_1 + \sigma x_2\dot{x}_2 + \sigma(x_3 - \rho)\dot{x}_3 + \frac{12}{7}x_4\dot{x}_4 + \frac{12}{7}x_5\dot{x}_5 \\ &= -\frac{12}{7}\sigma x_1^2 + \frac{12}{7}\sigma x_1 x_2 - \sigma x_2^2 - \sigma\beta x_3^2 + \sigma\beta\rho x_3 - \frac{12}{7}x_4^4 - \frac{96}{7}x_5^2 \end{aligned}$$

Note that $x^4 > x^2 - 1$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned} \frac{dU}{dt} &< -\frac{12}{7}\sigma x_1^2 + \frac{12}{7}\sigma x_1 x_2 - \sigma x_2^2 - \sigma\beta x_3^2 + \sigma\beta\rho x_3 - \frac{12}{7}x_4^2 - \frac{96}{7}x_5^2 \\ &= -\frac{3}{7}\sigma(\frac{2}{\sqrt{13}}x_1 + \frac{3}{\sqrt{13}}x_2)^2 - \frac{16}{7}\sigma(\frac{3}{\sqrt{13}}x_1 - \frac{2}{\sqrt{13}}x_2)^2 - \sigma\beta(x_3 - \frac{\rho}{2})^2 \\ &\quad - \frac{12}{7}x_4^2 - \frac{96}{7}x_5^2 + \frac{\sigma\beta\rho^2}{4}. \end{aligned}$$

Let the set $\mathbf{S}$ be defined by

$$\begin{aligned} \mathbf{S} = \{(x_1, x_2, x_3, x_4, x_5) \quad | \quad & \frac{3}{7}\sigma(\frac{2}{\sqrt{13}}x_1 + \frac{3}{\sqrt{13}}x_2)^2 + \frac{16}{7}\sigma(\frac{3}{\sqrt{13}}x_1 - \frac{2}{\sqrt{13}}x_2)^2 \\ + \sigma\beta(x_3 - \frac{\rho}{2})^2 + \frac{12}{7}x_4^2 + \frac{96}{7}x_5^2 < r^2, \ & \text{where } r^2 > \frac{\sigma\beta\rho^2}{4}, \ \sigma, \ \beta > 0, \ \text{and } \rho \in \mathbb{R}\} \end{aligned}$$

Immediately, we have the set $\mathbf{S}$ is an solid ellipse provided that $r^2 > \frac{\sigma\beta\rho^2}{4}$. Moreover, $\mathbf{S}$ is a bounded dissipation region of the individual system with $\sigma,\ \beta > 0,\ \rho \in \mathbb{R}$.

Next, we prove that the bounded dissipation of the coupled Lorenz-like system. Setting the scalar-valued function $V : \mathbb{R}^{nm} \to \mathbb{R}$ which is defined by

$$\begin{aligned} V(\tilde{\mathbf{x}}) =& \frac{1}{2}[\frac{12}{7} < \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_1 > + \sigma < \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_2 > \\ & + \sigma < (\tilde{\mathbf{x}}_3 - \rho\tilde{\mathbf{e}}), (\tilde{\mathbf{x}}_3 - \rho\tilde{\mathbf{e}}) > + \frac{12}{7} < \tilde{\mathbf{x}}_4, \tilde{\mathbf{x}}_4 > + \frac{12}{7} < \tilde{\mathbf{x}}_5, \tilde{\mathbf{x}}_5 >] \end{aligned}$$

where $\tilde{\mathbf{e}} = (1, \cdots, 1)^T$, $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}_i$, $1 \le i \le 5$, are defined in (3.4). Then, taking the time derivative of V along solutions of the coupled Lorenz-like system of the form in (3.35) and combining the scalar-valued function U, we have

$$\begin{aligned} \frac{dV}{dt}(\tilde{\mathbf{x}}) =& \frac{12}{7} < \tilde{\mathbf{x}}_1, \frac{d\tilde{\mathbf{x}}_1}{dt} > + \sigma < \tilde{\mathbf{x}}_2, \frac{d\tilde{\mathbf{x}}_2}{dt} > \\ & + \sigma < (\tilde{\mathbf{x}}_3 - \rho\tilde{\mathbf{e}}), \frac{d\tilde{\mathbf{x}}_3}{dt} > + \frac{12}{7} < \tilde{\mathbf{x}}_4, \frac{d\tilde{\mathbf{x}}_4}{dt} > + \frac{12}{7} < \tilde{\mathbf{x}}_5, \frac{d\tilde{\mathbf{x}}_5}{dt} > \\ =& \sum_{i=1}^{m} \frac{U(\mathbf{x}_i)}{dt} + \sum_{j=\{1,4,5\}} \frac{12}{7} d_j < \mathbf{x}_j, \mathbf{G}(\beta)\mathbf{x}_j > \\ & + \sum_{j=\{2,3\}} \sigma d_j < \mathbf{x}_j, \mathbf{G}(\beta)\mathbf{x}_j > - d_3 < \rho\tilde{\mathbf{e}}, \mathbf{G}(\beta)\mathbf{x}_3 > \end{aligned}$$

Note that $\mathbf{G}(\beta)^T\tilde{\mathbf{e}} = \vec{\mathbf{0}}$, we have $d_3 < \rho\tilde{\mathbf{e}}, \mathbf{G}(\beta)\mathbf{x} >_3 = 0$, for all $1 \le \beta \le 1$. Furthermore, $\mathbf{G}(\beta)$ is also a semi-negative definite matrix, then the following inequality holds true,

$$\frac{dV(\tilde{\mathbf{x}})}{dt} \le \sum_{i=1}^{m} \frac{dU(\mathbf{x}_i)}{dt} \tag{3.39}$$

Similarly, let the set $\bar{\mathbf{S}}$ be defined by

$$\begin{aligned} \bar{\mathbf{S}} = \{\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m) \quad | \quad & \sum_{i=1}^{m} [\frac{3}{7}\sigma(\frac{2}{\sqrt{13}}\mathbf{x}_{i1} + \frac{3}{\sqrt{13}}\mathbf{x}_{i2})^2 + \frac{16}{7}\sigma(\frac{3}{\sqrt{13}}\mathbf{x}_{i1} - \frac{2}{\sqrt{13}}\mathbf{x}_{i2})^2 \\ & + \sigma\beta(\mathbf{x}_{i3} - \frac{\rho}{2})^2 + \frac{12}{7}\mathbf{x}_{i4}^2 + \frac{96}{7}\mathbf{x}_{i5}^2] < R^2, \text{ where } R^2 > \frac{m\sigma\beta\rho^2}{4}, \ \sigma, \ \beta > 0, \text{ and } \rho \in \mathbb{R}\} \end{aligned}$$

Immediately, we have the set $\bar{\mathbf{S}}$ is also an solid ellipse provided that $R^2 > \frac{m\sigma\beta\rho^2}{4}$. Moreover, $\bar{\mathbf{S}}$ is a bounded dissipation region of the coupled Lorenz-like system with σ,

$\beta > 0$, $\rho \in \mathbb{R}$.

The coupling matrix $\mathbf{D}$ in the following is assumed to be $diag([1,0,0,1,0])$. In the case, the "coupled" nonlinearities f_1 and f_4 are clearly Lipschitz on the absorbing domain, and the differences of the "uncoupled" nonlinearities f_2, f_3, and f_5 are

$$\begin{aligned}
f_2(\mathbf{u}) - f_2(\mathbf{v}) &= -(u_2 - v_2) - u_1(u_3 - v_3) + (\rho - v_3)(u_1 - v_1),\\
f_3(\mathbf{u}) - f_3(\mathbf{v}) &= u_1(u_2 - v_2) - \beta(u_3 - v_3) + v_2(u_1 - v_1),\\
f_5(\mathbf{u}) - f_5(\mathbf{v}) &= -8(u_5 - v_5) - (u_1 - v_1) - (u_4 - v_4).
\end{aligned}$$

Clearly, to apply the given main theorem directly to claim the synchronization reaches, the best strategy to deal with "uncoupled " nonlinearities is to split them into two parts, one is f_2 and f_3, and the other is f_5. Then

$$\begin{aligned}
\mathbf{Q}_{\mathbf{u},\mathbf{v},1} &= \begin{pmatrix} -1 & -u_1 \\ u_1 & -\beta \end{pmatrix},\\
\mathbf{Q}_{\mathbf{u},\mathbf{v},2} &= -8.
\end{aligned}$$

In this way, the matrix measure of $\mathbf{Q}_{\mathbf{u},\mathbf{v},1}$ and $\mathbf{Q}_{\mathbf{u},\mathbf{v},2}$ is $\max\{-1,-\beta\}$ and -8, respectively. Thus, condition (3.19b) is satisfied. The remainder parts $\mathbf{r}_1 = \begin{pmatrix} (\rho - v_3)(u_1 - v_1) \\ v_2(u_1 - v_1) \end{pmatrix}$, and $\mathbf{r}_2 = -(u_1 - v_1) - (u_4 - v_4)$ also satisfy condition (3.19c). Thus, *the coupled system* $(\mathbf{D}, \mathbf{G}, \mathbf{F}(\mathbf{x}))$ *is uniformly globally synchronized provided the coupling strength d is large enough.*

Our theorems can be applied to quite many cases of the coupling matrix $\mathbf{D}$, which can be checked easily and similarly as above arguments. However, there exists some cases that uniformly global synchronization can be seen from the simulation of computer, but our theorems can not be applied directly. Here, we give a comparison with theoretical and numerical results as follows.

Location	Appli.	Simul.	Location	Appli.	Simul.
x_1	N	T	x_1, x_2	N	T
x_2	N	T	x_1, x_3	N	T
x_3	N	F	x_1, x_4	T	T
x_4	N	F	x_1, x_5	N	T
x_5	N	F	x_2, x_3	N	T
			x_2, x_4	N	T
			x_2, x_5	N	T
			x_3, x_4	N	F
			x_3, x_5	N	F
			x_4, x_5	N	F

Location	Appli.	Simul.	Location	Appli.	Simul.
x_1, x_2, x_3	N	T	x_1, x_2, x_3, x_4	T	T
x_1, x_2, x_4	T	T	x_1, x_2, x_3, x_5	N	T
x_1, x_2, x_5	N	T	x_1, x_2, x_4, x_5	T	T
x_1, x_3, x_4	T	T	x_1, x_3, x_4, x_5	T	T
x_1, x_3, x_5	N	T	x_2, x_3, x_4, x_5	T	T
x_1, x_4, x_5	T	T	x_1, x_2, x_3, x_4, x_5	T	T
x_2, x_3, x_4	N	T			
x_2, x_3, x_5	T	T			
x_2, x_4, x_5	T	T			
x_3, x_4, x_5	N	F			

Chapter 4

Wavelet Method for Chaotic Control

The main results of the second part of the thesis are contained in this chapter. Controlling chaos via wavelet transform was proposed by Wei, Zhan and Lai [48]. It was reported there that by modifying a tiny fraction of the wavelet subspace of a coupling matrix, the transverse stability of the synchronous manifold $\mathfrak{M}$ of a coupled chaotic system could be dramatically enhanced. Such phenomena are analytically verified when the coupling matrix is diffusively coupled with periodic and Neumann boundary conditions. The results in this part are reorganized from papers in [25,26].

4.1 Wavelet Method for the Diffusively Coupled with Mix Boundary Conditions

Let

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}_{n\times n,} \tag{4.1a}$$

be a matrix with the dimension of each block matrix A_{kl} being $2^i \times 2^i$. By an i-scale wavelet operator W [14,48], the matrix A is transformed into $W(A)$ of the form

$$W(A) = \begin{pmatrix} \widetilde{A}_{11} & \cdots & \widetilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \widetilde{A}_{n1} & \cdots & \widetilde{A}_{nn} \end{pmatrix}_{n\times n,} \tag{4.1b}$$

where each entry of $\widetilde{A}_{kl}$ is the average of entries of A_{kl}, $1 \le k, l \le n$. That is, for any matrix B of dimension $2^i \times 2^i$, the kl entry $(\tilde{B})_{kl}$ of $\tilde{B}$ is defined to be

$$(\tilde{B})_{kl} = \frac{\alpha}{2^{2i}} \sum_{l=1}^{2^i} \sum_{k=1}^{2^i} (B)_{kl}.$$

Here α is a scaler factor.

For a given matrix, the above wavelet transform allows a perfect reconstruction (inverse wavelet transform), by which there is nothing to gain: $A = W^{-1}(W(A))$. In [48], a simple operator O_{k} is introduced to attain a desirable coupling matrix. That is,

$$C = W^{-1}(O_{\mathrm{k}}(W(A))) = A + (k-1)W(A) =: A + \alpha W(A), \tag{4.1c}$$

where O_{k} be the multiplication of a scalar factor α on each block matrix $\widetilde{A}_{kl}$. To verify this phenomenon mathematically, we first consider the coupling matrix $A = \mathbf{G}(\beta)$, as given in (3.37). Let $l = \frac{m}{2^j} \in \mathbb{N}$, where i is a fixed positive integer. We then write A into an $l \times l$ block matrix of the form.

$$A = \mathbf{G}(\beta) = \begin{pmatrix} A_1(\beta) & A_2(1) & 0 & . & . & 0 & A_2^T(\beta) \\ A_2^T(1) & A_1(1) & A_2(1) & . & . & 0 & 0 \\ 0 & . & . & . & & & . \\ . & & . & . & . & & . \\ . & & & . & . & . & 0 \\ 0 & 0 & . & . & A_2^T(1) & A_1(1) & A_2(1) \\ A_2(\beta) & 0 & . & . & 0 & A_2^T(1) & \overline{A}_1(\beta) \end{pmatrix}_{l\times l,} \tag{4.2a}$$

where

$$A_1(\beta) = \begin{pmatrix} -1-\beta & 1 & & & & \\ 1 & -2 & 1 & & \mathbf{0} & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & & 1 & -2 \end{pmatrix}_{2^i \times 2^i},$$

$$\bar{A}_1(\beta) = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & \mathbf{0} & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & 1 & -1-\beta \end{pmatrix}_{2^i \times 2^i}, \tag{4.2b}$$

and

$$A_2(\beta) = \begin{pmatrix} 0 & . & . & . & 0 \\ . & & & & . \\ . & & & & . \\ 0 & & & & 0 \\ \beta & 0 & . & . & 0 \end{pmatrix}_{2^i \times 2^i} . \tag{4.2c}$$

Then the newly transformed coupling matrix $\mathbf{G} = \mathbf{G}(\alpha, \beta)$ is an $l \times l$ block matrix of the following form.

$$\mathbf{G}(\alpha,\beta) = \begin{pmatrix} \mathbf{G}_1(\alpha,\beta) & \mathbf{G}_2(\alpha,1) & 0 & \cdots & 0 & \mathbf{G}_2^T(\alpha,\beta) \\ \mathbf{G}_2^T(\alpha,1) & \mathbf{G}_1(\alpha,1) & \mathbf{G}_2(\alpha,1) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{G}_2^T(\alpha,1) & \mathbf{G}_1(\alpha,1) & \mathbf{G}_2(\alpha,1) \\ \mathbf{G}_2(\alpha,\beta) & 0 & \cdots & 0 & \mathbf{G}_2^T(\alpha,1) & \hat{\mathbf{I}}\mathbf{G}_1(\alpha,\beta)\hat{\mathbf{I}} \end{pmatrix}_{l \times l.} \tag{4.3a}$$

Here

$$\mathbf{G}_1(\alpha,\beta) = \begin{pmatrix} -1-\beta & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{2^j \times 2^j} - \frac{\alpha(1+\beta)}{2^{2j}} ee^T$$

$$=: A_1(\beta, 2^j) - \frac{\alpha(1+\beta)}{2^{2j}} ee^T, \tag{4.3b}$$

where $e = (1, 1, ..., 1)^T$, j is a positive integer, $\alpha > 0$ is a (wavelet) scalar factor and $\beta \in \mathbb{R}$ represents a mixed boundary constant. Moreover,

$$\mathbf{G}_2(\alpha,\beta) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \beta & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}} ee^T$$

$$=: A_2(\beta, 2^j) + \frac{\alpha\beta}{2^{2j}} ee^T, \tag{4.3c}$$

$$\hat{\mathbf{I}} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \cdot & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{4.3d}$$

The dimension of $\mathbf{G}(\alpha,\beta)$ is $l2^j \times l2^j$. From here on, we shall call l and j the block and the wavelet dimensions of $\mathbf{G}(\alpha,\beta)$, respectively.

The matrix $\mathbf{G}(\alpha,\beta)$ carries a new relationship among the coupled oscillators, which might not be as simple as the original matrix A.

4.2 Perturbed Block Circulant Matrix and Their Eigenvalue Problems

Of concern here is the eigencurve problem for a class of "perturbed" block circulant matrices.

$$\mathbf{G}(\alpha,\beta)\mathbf{b} = \lambda(\alpha,\beta)\mathbf{b}. \tag{4.4}$$

Here $\mathbf{G}(\alpha,\beta)$ is a block circulant matrix (see e.g., [15]) only if $\beta = 1$. It is well-known, see e.g., Theorem 5.6.4 of [15], that for each α the eigenvalues of $\mathbf{G}(\alpha,1)$ consists of eigenvalues of a certain linear combinations of its block matrices. Such results are called the reduced eigenvalue problem for $\mathbf{G}(\alpha,1)$.
Writing the eigenvalue problem $\mathbf{G}(\alpha,\beta)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_l)^T$ and $\mathbf{b}_i \in \mathbb{C}^{2^j}$, in block component form, we get

$$\mathbf{G}_2^T(\alpha,1)\mathbf{b}_{i-1} + \mathbf{G}_1(\alpha,1)\mathbf{b}_i + \mathbf{G}_2(\alpha,1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \;\; 1 \le i \le l. \tag{4.5a}$$

Mixed boundary conditions would yield that

$$\mathbf{G}_2^T(\alpha,1)\mathbf{b}_0 + \mathbf{G}_1(\alpha,1)\mathbf{b}_1 + \mathbf{G}_2(\alpha,1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = \mathbf{G}_1(\alpha,\beta)\mathbf{b}_1 + \mathbf{G}_2(\alpha,1)\mathbf{b}_2 + \mathbf{G}_2^T(\alpha,\beta)\mathbf{b}_l,$$

and

$$\begin{aligned}\mathbf{G}_2^T(\alpha,1)\mathbf{b}_{l-1} + \mathbf{G}_1(\alpha,1)\mathbf{b}_l + \mathbf{G}_2(\alpha,1)\mathbf{b}_{l+1} &= \lambda\mathbf{b}_l \\ &= \mathbf{G}_2(\alpha,\beta)\mathbf{b}_1 + \mathbf{G}_2^T(\alpha,1)\mathbf{b}_{l-1} + \hat{\mathbf{I}}\mathbf{G}_1(\alpha,\beta)\hat{\mathbf{I}}\mathbf{b}_l,\end{aligned}$$

or, equivalently,

$$\mathbf{G}_2^T(\alpha,1)\mathbf{b}_0 = (\mathbf{G}_1(\alpha,\beta) - \mathbf{G}_1(\alpha,1))\mathbf{b}_1 + \mathbf{G}_2^T(\alpha,\beta)\mathbf{b}_l$$

$$= [\begin{pmatrix} 1-\beta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \frac{\alpha(1-\beta)}{2^{2j}} ee^T]\mathbf{b}_1 + [\begin{pmatrix} 0 & \cdots & 0 & \beta \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \frac{\alpha\beta}{2^{2j}} ee^T]\mathbf{b}_l$$

$$= (1-\beta)\mathbf{G}_2^T(\alpha, 1)\hat{\mathbf{I}}\mathbf{b}_1 + \beta\mathbf{G}_2(\alpha, 1)\mathbf{b}_l, \tag{4.5b}$$

and

$$\mathbf{G}_2(\alpha, 1)\mathbf{b}_{l+1} = (\hat{\mathbf{I}}\mathbf{G}_1(\alpha, \beta)\hat{\mathbf{I}} - \mathbf{G}_1(\alpha, 1))\mathbf{b}_l + \mathbf{G}_2(\alpha, \beta)\mathbf{b}_1$$

$$= (1-\beta)\mathbf{G}_2^T(\alpha, 1)\hat{\mathbf{I}}\mathbf{b}_l + \beta\mathbf{G}_2(\alpha, 1)\mathbf{b}_1. \tag{4.5c}$$

To study the block difference equation (4.5), we set

$$\mathbf{b}_j = \delta^j \boldsymbol{v}, \tag{4.6}$$

where $\boldsymbol{v} \in \mathbb{C}^{2^j}$ and $\delta \in \mathbb{C}$.

Substituting (4.6) into (4.5a), we have

$$[\mathbf{G}_2^T(\alpha, 1) + \delta(\mathbf{G}_1(\alpha, 1) - \lambda\mathbf{I}) + \delta^2\mathbf{G}_2(\alpha, 1)]\boldsymbol{v} = 0. \tag{4.7}$$

To have a nontrivial solution $\boldsymbol{v}$ satisfying (4.7), we need to have

$$det[\mathbf{G}_2^T(\alpha, 1) + \delta(\mathbf{G}_1(\alpha, 1) - \lambda I) + \delta^2\mathbf{G}_2(\alpha, 1)] = 0. \tag{4.8}$$

Definition 4.2.1. *Equation (4.8) is to be called the characteristic equation of the block difference equation (4.5a). Let* $\delta_k = \delta_k(\lambda) \neq 0$ *and* $\mathbf{v}_k = \mathbf{v}_k(\lambda) \neq 0$ *be complex numbers and vectors, respectively, satisfying (4.7). Here* $k = 1, 2, ..., m$ *and* $m \leq 2^j$. *Assume that there exists a* $\lambda \in \mathbb{C}$, *such that* $\boldsymbol{b}_j = \Sigma_{k=1}^m c_k \delta_k^j(\lambda)\mathbf{v}_k(\lambda)$, *j=0,1,...,l+1, satisfy equation (4.5b,c), where* $c_k \in \mathbb{C}$. *If, in addition,* $\boldsymbol{b}_j$, $j = 1, 2, ..., l$, *are not all zero vectors, then such* $\delta_k(\lambda)$ *is called a characteristic value of equation (4.5) or (4.4) with respect to* λ *and* $\mathbf{v}_k(\lambda)$ *its corresponding characteristic vector.*

Remark 4.2.2. *Clearly, for each* α *and* β, λ *in the Definition of 4.2.1 is an eigenvalue of* $\mathbf{G}(\alpha, \beta)$.

Should no ambiguity arises, we will write $\mathbf{G}_2^T(\alpha, 1) = \mathbf{G}_2^T$, $\mathbf{G}_1(\alpha, 1) = \mathbf{G}_1$ and $\mathbf{G}_2(\alpha, 1) = \mathbf{G}_2$. Likewise, we will write $A_2(\beta, 2^j) = A_2(\beta)$ and $A_1(\beta, 2^j) = A_1(\beta)$.

Proposition 4.2.3. *Let* $\rho(\lambda) = \{\delta_i(\lambda) : \delta_i(\lambda)$ *is a root of equation (4.8)*$\}$, *and let* $\overline{\rho}(\lambda) = \{\frac{1}{\delta_i(\lambda)} : \delta_i(\lambda)$ *is a root of equation (4.8)*$\}$. *Then* $\rho(\lambda) = \overline{\rho}(\lambda)$. *Let* δ_i *and* δ_k *be in* $\rho(\lambda)$. *We further assume that* δ_i *and* $\boldsymbol{v}_i = (v_{i1}, \cdots, v_{i2^j})^T$ *satisfy (4.7). Suppose* $\delta_i \cdot \delta_k = 1$. *Then* δ_k *and* $\boldsymbol{v}_k = (v_{i2^j}, v_{i2^j-1}, \cdots, v_{i2}, v_{i1})^T =: \boldsymbol{v}_i^s$ *also satisfy (4.7). Conversely, if* $\delta_i \cdot \delta_k \neq 1$, *then* $\boldsymbol{v}_k \neq \boldsymbol{v}_i^s$.

Proof. To proof $\rho(\lambda) = \overline{\rho}(\lambda)$, we see that

$$det[\mathbf{G}_2^T + \delta(\mathbf{G}_1 - \lambda I) + \delta^2\mathbf{G}_2] = \delta^2 det[\frac{1}{\delta^2}\mathbf{G}_2^T + \frac{1}{\delta}(\mathbf{G}_1 - \lambda I) + \mathbf{G}_2]$$
$$= \delta^2 det[\frac{1}{\delta^2}\mathbf{G}_2^T + \frac{1}{\delta}(\mathbf{G}_1 - \lambda I) + \mathbf{G}_2]^T = \delta^2 det[\mathbf{G}_2^T + \frac{1}{\delta}(\mathbf{G}_1 - \lambda I) + \frac{1}{\delta^2}\mathbf{G}_2].$$

Thus, if δ is a root of equation (4.8), then so is $\frac{1}{\delta}$. To see the last assertion of the proposition, we write equation (4.7) with $\delta = \delta_i$ and $\boldsymbol{v} = \boldsymbol{v}_i$ in component form.

$$\sum_{m=1}^{2^j}[(\mathbf{G}_2^T)_{lm}v_{im} + \delta_i(\bar{\mathbf{G}}_1)_{lm}v_{im} + \delta_i^2(\mathbf{G}_2)_{lm}v_{im}] = 0, l = 1, 2, ..., 2^j. \quad (4.9)$$

Here $\bar{\mathbf{G}}_1 = \mathbf{G}_1 - \lambda I$. Now the right hand side of (4.9) becomes

$$(\frac{1}{\delta_k})^2\{\sum_{m=1}^{2^j}[(\mathbf{G}_2)_{l(2^j+1-m)}v_{i(2^j+1-m)} + \delta_k(\bar{\mathbf{G}}_1)_{l(2^j+1-m)}v_{i(2^j+1-m)}$$
$$+\delta_k^2(\mathbf{G}_2^T)_{l(2^j+1-m)}v_{i(2^j+1-m)}]\}$$
$$= (\frac{1}{\delta_k})^2\{\sum_{m=1}^{2^j}[(\mathbf{G}_2^T)_{(2^j+1-l)m}v_{i(2^j+1-m)} + \delta_k(\bar{\mathbf{G}}_1)_{(2^j+1-l)m}v_{i(2^j+1-m)}$$
$$+\delta_k^2(\mathbf{G}_2)_{(2^j+1-l)m)}v_{i(2^j+1-m)}]\}, l = 1, 2, ..., 2^j. \tag{4.10}$$

We have used the fact that

$$(A)_{(2^j+1-l)m} = (A^T)_{l(2^j+1-m)}, \tag{4.11}$$

where $A = \mathbf{G}_2^T$ or $\bar{\mathbf{G}}_1$ or $\mathbf{G}_2$ to justify the equality in (4.10). However, (4.11) follows from (4.4c) and (4.4d). Letting $v_{i(2^j+1-m)} = v_{km}$, we have that the pair $(\delta_k, \boldsymbol{v}_k)$ satisfies (4.7). Suppose $\boldsymbol{v}_k = \boldsymbol{v}_i^s$, we see, similarly, that the pair $(\frac{1}{\delta_i}, \boldsymbol{v}_k)$ also satisfy (4.7). Thus $\frac{1}{\delta_i} = \delta_k$.

□

Remark 4.2.4. *Equation (4.8) is a palindromic equation. That is for each* λ, δ *and* δ^{-1} *are both the roots of (4.8). However, eigenvalue problem discussed here is not a palindromic eigenvalue problem [23].*

Definition 4.2.5. *We shall call* $\mathbf{v}^s$ *and* $-\mathbf{v}^s$*, the symmetric vector and antisymmetric vector of* $\mathbf{v}$*, respectively. A vector* $\mathbf{v}$ *is symmetric (resp., antisymmetric) if* $\mathbf{v} = \mathbf{v}^s$ *(resp.,* $\mathbf{v} = -\mathbf{v}^s$*).*

Theorem 4.2.6. *Let* $\delta_k = e^{\frac{\pi k}{l}i}$*, k is an integer and* $i = \sqrt{-1}$*, then* δ_{2k}*, k=0,1,...,l-1, are characteristic values of equation (4.5) with* $\beta = 1$*. For each* α*, if* $\lambda \in \mathbb{C}$ *satisfies*

$$det[\mathbf{G}_2^T + \delta_{2k}(\mathbf{G}_1 - \lambda I) + \delta_{2k}^2\mathbf{G}_2] = 0,$$

for some $k \in \mathbb{Z}$*,* $0 \leq k \leq l-1$*, then* λ *is an eigenvalue of* $\mathbf{G}(\alpha, 1)$*.*

Proof. Let λ be as assumed. Then there exists a $\boldsymbol{v} \in \mathbb{C}^{2^j}$, $\boldsymbol{v} \neq \mathbf{0}$ such that

$$[\mathbf{G}_2^T + \delta_{2k}(\mathbf{G}_1 - \lambda I) + \delta_{2k}^2 \mathbf{G}_2]\boldsymbol{v} = \mathbf{0}.$$

Let $\mathbf{b}_j = \delta_{2k}^j \boldsymbol{v}$, $0 \leq j \leq l+1$. Then such $\mathbf{b}_j' s$ satisfy (4.5a), (4.5b), and (4.5c). We just proved the assertion of the theorem. □

Corollary 4.2.7. *Set*

$$\Gamma_k = \mathbf{G}_1 + \delta_{2l-k}\mathbf{G}_2^T + \delta_k \mathbf{G}_2. \tag{4.12}$$

Then the eigenvalues of $\mathbf{G}(\alpha, 1)$*, for each* α*, consists of eigenvalues of* Γ_k*,* $k = 0, 2, 4, ..., 2(l-1)$*. That is* $\rho(\mathbf{G}(\alpha, 1)) = \bigcup_{k=0}^{l-1} \rho(\Gamma_{2k})$*. Here* $\rho(A) =$ *the spectrum of the matrix* A*.*

Remark 4.2.8. $\mathbf{G}(\alpha, 1)$ *is a block circulant matrix. The assertion of Corollary 4.2.7 is not new (see e.g., Theorem 5.6.4 of [15]). Here we merely gave a different proof.*

To study the eigenvalue of $\mathbf{G}(\alpha, 0)$ for each α, we begin with considering the eigenvalues and eigenvectors of $\mathbf{G}_2^T + \mathbf{G}_1 + \mathbf{G}_2$ and $\mathbf{G}_2^T - \mathbf{G}_1 + \mathbf{G}_2$.

Proposition 4.2.9. *Let* $T_1(\mathbf{G})$ *(resp.,* $T_2(\mathbf{G})$*) be the set of linearly independent eigenvectors of the matrix* C *that are symmetric (resp., antisymmetric). Then* $|T_1(\mathbf{G}_2^T + \mathbf{G}_1 + \mathbf{G}_2)| = |T_2(\mathbf{G}_2^T + \mathbf{G}_1 + \mathbf{G}_2)| = |T_1(\mathbf{G}_2^T - \mathbf{G}_1 + \mathbf{G}_2)| = |T_2(\mathbf{G}_2^T - \mathbf{G}_1 + \mathbf{G}_2)| = 2^{j-1}$*. Here* $|A|$ *denote the cardinality of the set* A*.*

Proof. We will only illustrate the case for $\mathbf{G}_2^T - \mathbf{G}_1 + \mathbf{G}_2 =: \mathbf{G}$. We first observe that $|T_1(\mathbf{G})|$ is less than or equal to 2^{j-1}. So is $|T_2(\mathbf{G})|$. We also remark the cardinality of the set of all linearly independent eigenvectors of $\mathbf{G}$ is 2^j. If $0 < |T_1(\mathbf{G})| < 2^{j-1}$, there must exist an eigenvector $\boldsymbol{v}$ for which $\boldsymbol{v} \neq \boldsymbol{v}^s$, $\boldsymbol{v} \neq -\boldsymbol{v}^s$ and $\boldsymbol{v} \notin span\{T_1(\mathbf{G}), T_2(\mathbf{G})\}$, the span of the vectors in $T_1(\mathbf{G})$ and $T_2(\mathbf{G})$. It then follows from Proposition 2.1 that $\boldsymbol{v} + \boldsymbol{v}^s$, a symmetric vector, is in the $span\{T_1(\mathbf{G})\}$. Moreover, $\boldsymbol{v} - \boldsymbol{v}^s$ is in $span\{T_2(\mathbf{G})\}$. Hence $\boldsymbol{v} \in span\{T_1(\mathbf{G}), T_2(\mathbf{G})\}$, a contradiction. Hence, $|T_1(\mathbf{G})| = 2^{j-1}$. Similarly, we conclude that $|T_2(\mathbf{G})| = 2^{j-1}$. □

Theorem 4.2.10. *Let* $\delta_k = e^{\frac{\pi k}{l} i}$*, k is an integer,* $i = \sqrt{-1}$*. For each* α*, if* $\lambda \in \mathbb{C}$ *satisfies*

$$det[\mathbf{G}_2^T + \delta_k(\mathbf{G}_1 - \lambda I) + \delta_k^2 \mathbf{G}_2] = 0,$$

for some $k \in \mathbb{Z}$*,* $1 \leq k \leq l-1$*, then* λ *is an eigenvalue of* $\mathbf{G}(\alpha, 0)$*. Let* λ *be the eigenvalue of* $\mathbf{G}_2^T + \mathbf{G}_1 + \mathbf{G}_2$ *(resp.,* $-\mathbf{G}_2^T + \mathbf{G}_1 - \mathbf{G}_2$*) for which its associated eigenvector* $\boldsymbol{v}$ *satisfies* $\hat{\mathbf{I}}\boldsymbol{v} = \boldsymbol{v}$ *(resp.,* $\hat{\mathbf{I}}\boldsymbol{v} = -\boldsymbol{v}$*), then* λ *is also an eigenvalue of* $\mathbf{G}(\alpha, 0)$*.*

Proof. For any $1 \leq k \leq l-1$, let δ_k be as assumed. Let λ_k and ν_k be a number and a nonzero vector, respectively, satisfying

$$[\mathbf{G}_2^T + \delta_k(\mathbf{G}_1 - \lambda_k I) + \delta_k^2 \mathbf{G}_2]\boldsymbol{v}_k = \mathbf{0}. \tag{4.13}$$

Using Proposition 4.2.3, we see that λ_k satisfies

$$det[\mathbf{G}_2^T + \delta_{2l-k}(\mathbf{G}_1 - \lambda_k I) + \delta_{2l-k}^2 \mathbf{G}_2] = 0. \tag{4.14}$$

Let $\boldsymbol{v}_{2l-k}$ be a nonzero vector satisfying $[\mathbf{G}_2^T + \delta_{2l-k}(\mathbf{G}_1 - \lambda_k I) + \delta_{2l-k}^2 \mathbf{G}_2]\boldsymbol{v}_{2l-k} = \mathbf{0}$. Letting

$$\mathbf{b}_i = \delta_k^i \boldsymbol{v}_k + \delta_k \delta_{2l-k}^i \boldsymbol{v}_{2l-k}, i = 0, 1, ..., l+1,$$

we conclude, via (4.13) and (4.14), that $\mathbf{b}_i$ satisfy (4.5a) with $\lambda = \lambda_k$. Moreover,

$$\hat{\mathbf{I}}\mathbf{b}_1 = \delta_k \hat{\mathbf{I}}\boldsymbol{v}_k + \hat{\mathbf{I}}\boldsymbol{v}_{2n-k} = \delta_k \boldsymbol{v}_{2l-k} + \boldsymbol{v}_k = \mathbf{b}_0.$$

We have used Proposition 4.2.3 to justify the second equality above. Similarly, $\mathbf{b}_{l+1} = \hat{\mathbf{I}}\mathbf{b}_l$. To see $\lambda = \lambda_k$, $1 \leq k \leq l-1$, is indeed an eigenvalue of $\mathbf{G}(\alpha, 0)$ for each α, it remains to show that $\mathbf{b}_i \neq \mathbf{0}$ for some i. Using Proposition 4.2.3, we have that there exists an m, $1 \leq m \leq 2^j$ such that $v_{km} = v_{(2l-k)(2^j-m+1)} \neq 0$. We first show that $\mathbf{b}_0 \neq \mathbf{0}$. Let m be the index for which $v_{km} \neq 0$. Suppose $\mathbf{b}_0 = \mathbf{0}$. Then

$$v_{km} + \delta_k v_{(2l-k)m} = 0$$

and

$$v_{k(2^j-m+1)} + \delta_k v_{(2l-k)(2^j-m+1)} = v_{(2l-k)m} + \delta_k v_{km} = 0.$$

And so, $v_{km} = \delta_k^2 v_{km}$, a contradiction. Let λ and $\boldsymbol{v}$ be as assumed in the last assertion of theorem. Letting $\mathbf{b}_i = \boldsymbol{v}$ (resp., $\mathbf{b}_i = (-1)^i \boldsymbol{v}$), we conclude that λ is an eigenvalue of $\mathbf{G}(\alpha, 0)$ with corresponding eigenvector $(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l)^T$. Thus, λ_k is an eigenvalue of $\mathbf{G}(\alpha, 0)$ for each α. □

Corollary 4.2.11. *Let $\delta_k = e^{\frac{\pi k}{l} i}$, k is an integer, $i = \sqrt{-1}$. Then, for each α, $\rho(C(\alpha, 0)) = \bigcup_{k=1}^{l-1} \rho(\Gamma_k) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_l)$, where $\rho^S(A)$ (resp., $\rho^{AS}(A)$) the set of eigenvalues of A for which their corresponding eigenvectors are symmetric (resp., antisymmetric).*

We next consider the eigenvalues of $\mathbf{G}(\alpha, \beta)$.

Theorem 4.2.12. *Let $\delta_k = e^{\frac{\pi k}{l} i}$, k is an integer, $i = \sqrt{-1}$. Then, for each α,*

$$\rho(\mathbf{G}(\alpha, \beta)) \supset \begin{cases} \bigcup_{k=1}^{[\frac{l}{2}]} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0), & l \text{ is odd}, \\ \bigcup_{k=1}^{\frac{l}{2}-1} \rho(\Gamma_{2k}) \bigcup \rho^S(\Gamma_0) \bigcup \rho^{AS}(\Gamma_l), & l \text{ is even}. \end{cases}$$

Here $[\frac{l}{2}]$ is the greatest integer that is less than or equal to $\frac{l}{2}$.

Proof. We illustrate only the case that l is even. Assume that k is such that $1 \leq k \leq \frac{l}{2} - 1$. Let $\mathbf{b}_i = \delta_{2k}^i \boldsymbol{v}_{2k} + \delta_{2k} \delta_{2l-2k}^i \boldsymbol{v}_{2l-2k}$, we see clearly that such $\mathbf{b}_i$, $i = 0, 1, l, l+1$, satisfy both Neumann and periodic boundary conditions, respectively. And so

$$\mathbf{b}_0 = (1-\beta)\mathbf{b}_0 + \beta \mathbf{b}_0 = (1-\beta)\hat{I}\mathbf{b}_1 + \beta \mathbf{b}_l,$$

and

$$\mathbf{b}_{l+1} = (1-\beta)\mathbf{b}_{l+1} + \beta\mathbf{b}_{l+1} = (1-\beta)\hat{I}\mathbf{b}_l + \beta\mathbf{b}_1.$$

Here, δ_{2k}, $1 \leq k \leq \frac{l}{2} - 1$, are characteristic values of equation of (4.5). Thus, if $\lambda \in \rho(\Gamma_{2k})$, then λ is an eigenvalue of $\mathbf{G}(\alpha, \beta)$. The assertions for Γ_0 and Γ_n can be done similarly. □

Remark 4.2.13. *If n is an even number, for each α and β, half of the eigenvalues of $\mathbf{G}(\alpha, \beta)$ are independent of the choice of β. The other characteristic values of (4.5) seem to depend on β. It is of interest to find them.*

4.3 The Chaotic Control for Periodic and Neumann Boundary Conditions

We begin with considering the eigencurves of Γ_k, as given in (4.12). Clearly,

$$\Gamma_k = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \delta_{2l-k} \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \delta_k & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}_{m \times m} - \frac{\alpha(2 - 2\cos\frac{\pi k}{l})}{m} ee^T$$

$$=: D_1(k) - \alpha(k)ee^T, \tag{4.15}$$

where $m = 2^j$. We next find a unitary matrix to diagonalize $D_1(k)$.

Remark 4.3.1. *Let $(\lambda(k), \mathbf{v}(k))$ be the eigenpair of $D_1(k)$. If $e^T\mathbf{v}(k) = 0$, then $\lambda(k)$ is also an eigenvalue of Γ_k.*

Proposition 4.3.2. *Let*

$$\theta_{l,k} = \frac{2l\pi}{m} + \frac{k\pi}{nm}, l = 0, 1, ..., m-1, \tag{4.16a}$$

$$\boldsymbol{p}_l(k) = \left(e^{i\theta_{l,k}}, e^{i2\theta_{l,k}}, \cdots, e^{im\theta_{l,k}}\right)^T \tag{4.16b}$$

and

$$P(k) = \left(\frac{\boldsymbol{p}_0(k)}{\sqrt{m}}, \cdots, \frac{\boldsymbol{p}_{m-1}(k)}{\sqrt{m}} \right). \tag{4.16c}$$

(i) Then $P(k)$ is a unitary matrix and $P^H(k)D_1(k)P(k) = Diag(\lambda_{0,k} \cdots \lambda_{m-1,k})$, where P^H is the conjugate transpose of P, and

$$\lambda_{l,k} = 2\cos\theta_{l,k} - 2, l = 0, 1, .., m-1. \tag{4.16d}$$

(ii) Moreover, for $0 \leq k \leq 2l$, the eigenvalues of $D_1(k)$ are distinct if and only if $k \neq 0, l$ or $2l$.

Proof. Let $\mathbf{b} = (b_1, ..., b_m)^T$. Writing the eigenvalue problem $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ in component form, we get

$$b_{j-1} - (2+\lambda)b_j + b_{j+1} = 0, j = 2, 3, ..., m-1, \tag{4.17a}$$

$$-(2+\lambda)b_1 + b_2 + \delta_{2n-k}b_m = 0, \tag{4.17b}$$

$$\delta_k b_1 + b_{m-1} - (2+\lambda)b_m = 0. \tag{4.17c}$$

Set $b_j = \delta^j$, where δ satisfies the characteristic equation $1 - (2+\lambda)\delta + \delta^2 = 0$ of the system $D_1(k)\mathbf{b} = \lambda\mathbf{b}$. Then the boundary conditions (4.17b) and (4.17c) are reduced to

$$\delta^m = \delta_k. \tag{4.18}$$

Thus, the solutions $e^{i\theta_{l,k}}$, $l = 0, 1, ..., m-1$, of (4.18) are the candidates for the characteristic values of (4.17). Substituting $e^{i\theta_{l,k}}$ into (4.17a) and solving for λ, we see that $\lambda = \lambda_{l,k}$ are the candidates for the eigenvalues of $D_1(k)$. Clearly, $(\lambda, \mathbf{b}) = (\lambda_{l,k}, \boldsymbol{p}_l(k))$ satisfies $D_1(k)\mathbf{b} = \lambda\mathbf{b}$ and $\mathbf{b} = \boldsymbol{p}_l(k) \neq 0$. Thus, $\lambda = \lambda_{l,k}$ are, indeed, the eigenvalues of $D_1(k)$. To complete the proof of the proposition, it suffices to show that $P(k)$ is unitary. To this end, we need to compute $\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k)$. Clearly, $\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_l(k) = m$.

Now, let $l \neq l'$, we have that

$$\boldsymbol{p}_l^H(k) \cdot \boldsymbol{p}_{l'}(k) = \sum_{j=1}^{m} e^{ij(\theta_{l,k} - \theta_{l',k})} = \sum_{j=1}^{m} e^{ij(\frac{2(l-l')}{m}\pi)} = \frac{r(1-r^m)}{1-r} = 0,$$

where $r = e^{i(\frac{2(l-l')}{m}\pi)}$. Hence, $P(k)$ is unitary. The last assertion of the proposition is obvious. □

To prove the main results in this section, we also need the following proposition. Some of assertions of the proposition are from Theorem 8.6.2 of [2].

Proposition 4.3.3. *Suppose* $D = diag(d_1, ..., d_m) \in \mathbb{R}^{m \times m}$ *and that the diagonal entries satisfy* $d_1 > \cdots > d_m$*. Let* $\gamma \neq 0$ *and* $\boldsymbol{z} = (z_1, ..., z_m)^T \in \mathbb{R}^n$*. Assume that* $(\lambda_i(\gamma), \boldsymbol{v}_i(\gamma))$ *are the eigenpairs of* $D + \gamma \boldsymbol{z}\boldsymbol{z}^T$ *with* $\lambda_1(\gamma) \geq \lambda(\gamma) \geq ... \geq \lambda_m(\gamma)$*. (i) Let* $A = \{k : 1 \leq k \leq m, z_k = 0\}$*,* $A^c = \{1, ..., m\} - A$*. If* $k \in A$*, then* $d_k = \lambda_k$*. (ii) Assume* $\alpha > 0$*. Then the following interlacing relations hold* $\lambda_1(\gamma) \geq d_1 \geq \lambda_2(\gamma) \geq d_2 \geq ... \geq \lambda_m(\gamma) \geq d_m$*. Moreover, the strict inequality holds for these indexes* $i \in A^c$*. (iii) Let* $i \in A^c$*,* $\lambda_i(\gamma)$ *are strictly increasing in* γ *and* $\lim_{\alpha \to \infty} \lambda_i(\gamma) = \bar{\lambda}_i$ *for all* i*, where* $\bar{\lambda}_i$ *are the roots of* $g(\lambda) = \sum_{k \in A^c} \frac{z_i^2}{d_k - \lambda}$ *with* $\bar{\lambda}_i \in (d_i, d_{i-1})$*. In case that* $1 \in A^c$*,* $d_0 = \infty$*.*

Proof. The proof of interlacing relations in (ii) and the assertion in (i) can be found in Theorem 8.6.2 of [2]. We only prove the remaining assertions of the proposition. Rearranging $\mathbf{z}$ so that $\mathbf{z}^T = (0, 0, ..., 0, z_{i_1}, ..., z_{i_k}) =: (0, ..., 0, \bar{\boldsymbol{z}}^T)$, where $i_1 < i_2 < ... < i_k$ and $i_j \in A^c$, $j = 1, ..., k$. The diagonal matrix D is rearranged accordingly. Let $D = diag(D_1, D_2)$, where $D_2 = diag(d_{i_1}, ..., d_{i_k})$. Following Theorem 8.6.2 of [2], we see that $\lambda_{i_j}(\gamma)$ are the roots of the scalar equation $f_\gamma(\lambda)$, where

$$f_\gamma(\lambda_{i_j}(\gamma)) = 1 + \gamma \sum_{j=1}^{k} \frac{z_j^2}{d_{i_j} - \lambda_{i_j}(\gamma)} = 0. \tag{4.19}$$

Differentiate the equation above with respect to γ, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} + (\gamma \sum_{j=1}^{k} \frac{z_{i_j}^2}{(d_{i_j} - \lambda_{i_k}(\gamma))^2}) \frac{d\lambda_{i_j}(\gamma)}{d\gamma} = 0.$$

Thus,

$$\frac{d\lambda_{i_j}(\gamma)}{d\gamma} = \frac{1}{\gamma^2} \sum_{j=1}^{k} \frac{z_{i_j}^2}{(d_{i_j} - \lambda_{i_j}(\gamma))^2} > 0.$$

Clearly, for each i_j, the limit of $\lambda_{i_j}(\gamma)$ as $\gamma \to \infty$ exists, say $\bar{\lambda}_{i_j}$. Since, for $d_{i_j} < \lambda < d_{i_j - 1}$,

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \lambda_{i_j}(\gamma)} = \frac{1}{\gamma}.$$

Taking the limit as $\alpha \to \infty$ on both side of the equation above, we get

$$\sum_{j=1}^{k} \frac{z_{i_j}^2}{d_{i_j} - \bar{\lambda}_{i_j}} = 0 \tag{4.20}$$

as desired. □

We are now in the position to state the following theorems.

Theorem 4.3.4. *let l and $m = 2^j$ be given positive integers. For each k, $k = 1, 2, \cdots, l-1$, and α, we denote by $\lambda_{l,k}(\alpha)$, $l = 0, 1, \cdots, 2^j - 1$, the eigenvalues of Γ_k. For $k = 1, 2, \cdots, l-1$, we let $(\lambda_{l,k}, u_{l,k})$, $l = 0, 1, \cdots, 2^j - 1$, be the eigenpairs of $D_1(k)$, as defined in (4.15). Then the following hold true.*
(i) $\lambda_{l,k}(\alpha)$ is strictly decreasing in α, $l = 0, 1, \cdots, 2^j - 1$ and $k = 1, 2, \cdots, l-1$.
(ii)There exist $\lambda^*_{l,k}$ such that $\lim_{\alpha \to \infty} \lambda_{l,k}(\alpha) = \lambda^*_{l,k}$. Moreover, $g_k(\lambda^*_{l,k}) = 0$, where

$$g_k(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k} + \lambda)}. \tag{4.21}$$

Proof. The first assertion of the theorem follows from proposition 4.3.3-(iii). Let k be as assumed. Set, for $l = 0, 1, ..., m-1$,

$$z_{l+1} = \boldsymbol{p}_l^H(k)e = \sum_{j=1}^{m} e^{ij\theta_{l,k}} = \frac{e^{-\theta_{l,k}}(1 - e^{-im\theta_{l,k}})}{1 - e^{-\theta_{l,k}}} = \frac{e^{-\theta_{l,k}}(1 - e^{-ik\frac{\pi}{n}})}{1 - e^{-\theta_{l,k}}}.$$

Then

$$\bar{\boldsymbol{z}}_{l+1}\boldsymbol{z}_{l+1} = \frac{2 - 2\cos m\theta_{l,k}}{2 - 2\cos\theta_{l,k}} = \frac{2\cos\frac{k\pi}{n} - 2}{\lambda_{l,k}} \neq 0. \tag{4.22}$$

Let $P(k)$ be as given in (4.16c). Then

$$-P^H(k) \cdot \Gamma_k \cdot P(k) = Diag(-\lambda_{0,k}, ..., -\lambda_{m-1,k}) + \alpha(k)P_l^H(k)e(P_l^H(k)e)^H.$$

Note that if k is as assumed, it follows from Proposition 4.3.1-(ii) that $\lambda_{l,k}$, $l = 0, ..., m-1$, are distinct. Thus, we are in the position to apply Proposition 4.3.3. Specifically, by noting $A^c = \phi$, we see that $\lambda_{0,k}^*$ satisfies $g(\lambda) = 0$, where

$$g(\lambda) = \sum_{l=1}^{m} \frac{1}{(\lambda_{l-1,k})(\lambda_{l-1,k} + \lambda)}.$$

We have used (4.16d), (4.20) and (4.22) to find $g(\lambda)$. □

We next give an upper bound for $\lambda_{0,k}^*$, $k = 1, 2, \cdots, n-1$.

Theorem 4.3.5. *The following inequalities hold true.*

$$\lambda_{0,k}^* < \lambda_{0,l}, \qquad k = 1, 2, \cdots, l-1. \tag{4.23}$$

Proof. To complete the proof of (4.23), it suffices to show that $g_k(-\lambda_{0,l}) < 0$. Now,

$$g_k(-\lambda_{0,l}) = \sum_{l=1}^{m} \frac{1}{[2cos(\frac{2(l-1)\pi}{m} + \frac{k\pi}{lm}) - 2][2cos(\frac{2(l-1)\pi}{m} + \frac{k\pi}{lm}) - 2cos\frac{\pi}{m}]}$$

$$=: h(m,l,k) = h(2^j,l,k). \tag{4.24}$$

We shall prove that $h(2^j,l,k) < 0$ by the induction on j. For $j=1$, $h(2,l,k) = \frac{1}{2}\left[\frac{1}{cos^2(\frac{k\pi}{2l})-1}\right] < 0$, $k = 1,2,\cdots,l-1$. Assume $h(2^j,l,k) < 0$. Here, $l \in \mathbb{N}$ and $k = 1,2,\cdots,l-1$. We first note that

$$cos\left(\frac{2(2^j+i-1)\pi}{2^{j+1}} + \frac{k\pi}{2^{j+1}l}\right) = -cos\left(\frac{2(i-1)\pi}{2^{j+1}} + \frac{k\pi}{2^{j+1}l}\right)$$

$$=: -cos\theta_{i-1,k,j+1}, \quad i = 1,2,\cdots,2^j. \tag{4.25}$$

Moreover, upon using (4.25), we get that

$$\frac{1}{(cos\theta_{i-1,k,j+1}-1)(cos\theta_{i-1,k,j+1}-cos\theta_{0,l,j+1})} + \frac{1}{(cos\theta_{2^j+i-1,k,j+1}-1)(cos\theta_{2^j+i-1,k,j+1}-cos\theta_{0,l,j+1})}$$

$$= \frac{1}{(cos\theta_{i-1,k,j+1}-1)(cos\theta_{i-1,k,j+1}-cos\theta_{0,l,j+1})} + \frac{1}{(cos\theta_{i-1,k,j+1}+1)(cos\theta_{i-1,k,j+1}+cos\theta_{0,l,j+1})}$$

$$= \frac{2cos^2\theta_{i-1,k,j+1} + 2cos\theta_{0,l,j+1}}{(cos^2\theta_{i-1,k,j+1}-1)(cos^2\theta_{i-1,k,j+1}-cos^2\theta_{0,l,j+1})}$$

$$= \frac{8(cos^2\theta_{i-1,k,j+1} + cos\theta_{0,l,j+1})}{(cos2\theta_{i-1,k,j+1}-1)(cos2\theta_{i-1,k,j+1}-cos2\theta_{0,l,j+1})}$$

$$= \frac{2(cos^2\theta_{i-1,k,j+1} + cos\theta_{0,l,j+1})}{(cos\theta_{i-1,k,j}-1)(cos\theta_{i-1,k,j}-cos\theta_{0,l,j})}. \tag{4.26}$$

We are now in a position to compute $h(2^{j+1}, l, k)$. Using (4.26), we get that

$$h(2^{j+1}, l, k) = \sum_{l=1}^{2^{j+1}} \frac{1}{4(cos\theta_{l-1,k,j+1} - 1)(cos\theta_{l-1,k,j+1} - cos\theta_{0,l,j+1})}$$

$$= \sum_{l=1}^{2^j} \frac{2(cos^2\theta_{l-1,k,j+1} + cos\theta_{0,l,j+1})}{(cos\theta_{l-1,k,j} - 1)(cos\theta_{i-1,k,j} - cos\theta_{0,l,j})}$$

$$\leq 8(cos^2\theta_{0,k,j+1} + cos\theta_{0,l,j+1})h(2^j, l, k). \tag{4.27}$$

We have used the facts that $cos^2\theta_{0,k,j+1} > cos^2\theta_{i-1,k,j+1}$, $i = 2, \cdots, 2^j$, and that the first term (i=1) of the summation in (4.27) is negative while all the others are positive to justify the inequality in (4.27). It then follows from (4.27) that $h(2^{j+1}, l, k) < 0$. We just complete the proof of the theorem. □

Theorem 4.3.6. *Let l and j be the block and wavelet dimensions of $\mathbf{G}(\alpha, 1)$, respectively. Assume l and j are any positive integers. Let $\lambda_2(\alpha)$ be the second eigencurve of $\mathbf{G}(\alpha, 1)$. Then the following hold.*
(i) $\lambda_2(\alpha)$ is a nonincreasing function of α.
(ii) If l is an even number, then $\lambda_2(\alpha) = \lambda_{0,n}$ whenever $\alpha \geq \alpha^*$ for some $\alpha^* > 0$.
(iii) If l is an odd number, then $\lambda_2(\alpha) < \lambda_{0,n}$ whenever $\alpha \geq \overline{\alpha}$ for some $\overline{\alpha} > 0$.

Proof. We first remark that in the case of $\beta = 1$, the set of the indexes $k's$ in (4.15) is $\{0, 2, 4, ..., 2(l-1)\} := I_l$. Suppose l is an even number. Then $l \in I_l$. Thus, $\delta_l = -1$, $\theta_{0,l} = \frac{\pi}{m}$, and $\boldsymbol{p}_0(l) = \left(e^{i\frac{\pi}{m}}, e^{i\frac{2\pi}{m}}, \cdots, e^{i\pi}\right)^T$. Applying Proposition 4.3.2, we see that $\boldsymbol{p}_0(l) - \boldsymbol{p}_0^s(l)$, an antisymmetric vector, is also an eigenvector of $D_1(l)$. And so $e^T(\boldsymbol{p}_0(l) - \boldsymbol{p}_0^s(l)) = 0$. It then follows from Remark 4.3.1 that $\lambda_{0,l}$ is an eigenvalue of $\Gamma_l = D_1(l) - \rho(l)ee^T$ for all α. The first and second assertions of the theorem now follow from Theorems 4.3.4 and 4.3.5. Let l be an odd number. Then $\delta_i \cdot \delta_i \neq 1$ for any $i \in I_n$. Thus, if the pair $(\delta_i, \boldsymbol{v}_i)$ satisfy (4.7), then $\boldsymbol{v}_i \neq -\boldsymbol{v}_i^s$. Otherwise, the pair $(\delta_i, \boldsymbol{v}_i - (-\boldsymbol{v}_i)^s) = (\delta_i, \boldsymbol{v}_i + \boldsymbol{v}_i^s)$ also satisfy (4.7). This is a contradiction to the last assertion in Proposition 4.2.3. Thus, $\boldsymbol{v}_i^H \cdot e \neq 0$ for any $i \in I_n$. We then conclude, via Proposition 4.3.3-(iii) and Theorem 4.3.5, that the last assertion of the theorem holds. □

Remark 4.3.7. *(i)Let the number of uncoupled (chaotic) oscillators be $N = l2^j$. If l is an odd number, then the wavelet method for controlling the coupling chaotic oscillators work even better in the sense that the critical coupling strength ϵ can be made even smaller. (ii)For l being a multiple of 4 and $j \in \mathbb{N}$, the assertions in Theorem 3.3 was first proved in [6] by a different method.*

Theorem 4.3.8. *Let l and j be the block and wavelet dimensions of $\mathbf{G}(\alpha, 0)$, respectively. Assume l and j are any positive integers. Let $\lambda_2(\alpha)$ be the second eigencurve of $\mathbf{G}(\alpha, 0)$. Then for any l, there exists a $\widetilde{\alpha}$ such that $\lambda_2(\alpha) = \lambda_{0,n}$ whenever $\alpha \geq \widetilde{\alpha}$.*

Remark 4.3.9. *For $l \in \mathbb{N}$ and $j = 1$, the explicit formulas for the eigenvalues of $\mathbf{G}(\alpha, 0)$ was obtained in [4]. Such results are possible due to the fact that the dimension of the matrices in (4.8) is 2×2.*

4.4 Numerical Illustrations for Periodic and Neumann Boundary Conditions

To illustrate how such wavelet transform affects the critical coupling strength, we consider $\mathbf{G}$ to be diffusively coupled with Periodic and Neumann Boundary Conditions.

4.4.1 Periodic Boundary Conditions

In this section, we consider the nearest neighbor coupling with periodic boundary conditions. The resulting coupling matrix $\mathbf{G}(1)$ is given as in (3.37). Let the dimension of $A_1(1)$, $A_2(1)$ and $\bar{A}_1(1)$ be 2×2. Then

$$A_1(1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = \bar{A}_1(1), A_2(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.28a}$$

$$\tilde{A}_1(1) = \alpha \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \tilde{\bar{A}}_1, \tilde{A}_2(1) = \alpha \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \tag{4.28b}$$

Then $\mathbf{G}_i(1) = A_i(1) + \tilde{A}_i(1)$, i=1, 2, $\bar{\mathbf{G}}_1(1) = \bar{A}_1(1) + \tilde{\bar{A}}_1(1)$. Thus,

$$\mathbf{G}_1(1) = \begin{pmatrix} -\frac{1}{2}(4+\alpha) & \frac{1}{2}(2-\alpha) \\ \frac{1}{2}(2-\alpha) & -\frac{1}{2}(4+\alpha) \end{pmatrix} = \bar{\mathbf{G}}_1(1), \mathbf{G}_2(1) = \begin{pmatrix} \frac{\alpha}{4} & \frac{\alpha}{4} \\ \frac{1}{4}(4+\alpha) & \frac{\alpha}{4} \end{pmatrix}. \tag{4.28c}$$

We begin with identifying some trivial eigenvalues of $\mathbf{G}(\alpha, 1)$.

Proposition 4.4.1. *For each α, 0 and -4 are eigenvalues of $\mathbf{G}(\alpha, 1)$. If, in addition, $\frac{l}{2}(> 1)$ is a positive integer, then -2 is also an eigenvalue of $\mathbf{G}(\alpha, 1)$ for any α.*

Proof. Let $\mathbf{G}(\alpha, 1) + 4I = (\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_m)$, where $\mathbf{c}_i$, $1 \leq i \leq m$, are column vectors. Then $\sum_{j=1}^{m}(-1)^{j+1}\mathbf{c}_j = 0$. Thus -4 is an eigenvalue of $\mathbf{G}(\alpha, 1)$ for each $\alpha > 0$. Let $\mathbf{G}(\alpha, 1) + 2I = (\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_m)$. If $m = 2l(> 4)$ is a multiple of four, then $\sum_{j=1}^{m} \delta(j)\mathbf{c}_j = 0$, where

$$\delta(j) = \begin{cases} 1 & if\ j = 4k\ or\ 4k+1\ for\ some\ k, \\ -1 & if\ j = 4k+2\ or\ 4k+3\ for\ some\ k. \end{cases}$$

Thus, -2 is an eigenvalue of $\mathbf{G}(\alpha, 1)$ for each α with such N. □

Writing the corresponding eigenvalue problem $\mathbf{G}(\alpha, 1)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l)^T$ and $\mathbf{b}_i \in \mathbb{C}^2$, in block component form, we have

$$\mathbf{G}_2^T(1)\mathbf{b}_{i-1} + \mathbf{G}_1(1)\mathbf{b}_i + \mathbf{G}_2(1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, \ \ 1 \leq i \leq l. \tag{4.29a}$$

Periodic boundary conditions would yield that

$$\mathbf{G}_2^T(1)\mathbf{b}_0 + \mathbf{G}_1(1)\mathbf{b}_1 + \mathbf{G}_2(1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = \mathbf{G}_1(1)\mathbf{b}_1 + \mathbf{G}_2(1)\mathbf{b}_2 + \mathbf{G}_2^T(1)\mathbf{b}_l$$

and

$$\mathbf{G}_2^T(1)\mathbf{b}_{l-1} + \mathbf{G}_1(1)\mathbf{b}_l + \mathbf{G}_2(1)\mathbf{b}_{l+1} = \lambda\mathbf{b}_n = \mathbf{G}_2(1)\mathbf{b}_1 + \mathbf{G}_2^T(1)\mathbf{b}_{l-1} + \bar{\mathbf{G}}_1(1)\mathbf{b}_l,$$

or, equivalently,

$$\mathbf{b}_0 = \mathbf{b}_l, \mathbf{b}_1 = \mathbf{b}_{l+1}. \tag{4.29b,c}$$

To study the block difference equation (4.29), we first seek to find the solution $\mathbf{b}_i$ of the form.

$$\mathbf{b}_i = \delta^i \begin{pmatrix} 1 \\ \nu \end{pmatrix}. \tag{4.30}$$

Substituting (4.30) into (4.29a), we get

$$[\mathbf{G}_2^T(1) + \delta(\mathbf{G}_1(1) - \lambda I) + \delta^2\mathbf{G}_2(1)] \begin{pmatrix} 1 \\ \nu \end{pmatrix} = 0. \tag{4.31}$$

To have a nontrivial solution $\begin{pmatrix} 1 \\ \nu \end{pmatrix}$ to equation (4.31), we need to have

$$det[\mathbf{G}_2^T(1) + \delta(\mathbf{G}_1(1) - \lambda I) + \delta^2\mathbf{G}_2(1)] = 0, \tag{4.32a}$$

or, equivalently,

$$\alpha\delta^4 + (4\alpha+4+2\alpha\lambda)\delta^3 - (8+10\alpha+16\lambda+4\alpha\lambda+4\lambda^2)\delta^2 + (4\alpha+4+2\alpha\lambda)\delta + \alpha = 0. \tag{4.32b}$$

Equation (4.32b) is to be called the characteristic equation of the block difference equation(4.29a). To study the property of equation (4.32b), we need the following proposition.

Proposition 4.4.2. *Let D_1, D_2 and D_3 be 2×2 matrices. Suppose $D_1 = D_3^T$ and $D_2 = D_2^T$. Let x_1, x_2, x_3 and x_4 be roots of $det[D_1 + xD_2 + x^2 D_3] = 0$, where $x \in \mathbb{C}$. Then we may renumber the subscripts if necessary so that*

$$x_1 x_2 = 1 = x_3 x_4. \tag{4.33a}$$

If, in addition, diagonal elements of D_1 and D_2, respectively, are both equal, then

$$y_1 y_2 = 1 = y_3 y_4. \tag{4.33b}$$

Here $\begin{pmatrix} 1 \\ y_i \end{pmatrix}$, i=1, 2, 3, 4, are vectors satisfying

$$[D_1 + x_i D_2 + x_i^2 D_3] \begin{pmatrix} 1 \\ y_i \end{pmatrix} = 0. \tag{4.33c}$$

Proof. If D_1, D_2 and D_3 are as assumed, then

$$det[D_1 + xD_2 + x^2 D_3] = ax^4 + bx^3 + cx^2 + bx + a \tag{4.34}$$

for some constants $a \neq 0$, b, and c. Letting $y = x + \frac{1}{x}$, then (4.34) can be written as $\alpha y^2 + \beta y + \gamma$, where α, β and γ depend on the constants a, b, and c. Thus $det[D_1 + xD_2 + x^2 D_3] = 0$ is equivalent to $x^2 - \lambda_\pm x + 1 = 0$, where $\lambda_\pm$ are the roots $a_1 y^2 + b_1 y + c_1 = 0$. Consequently, $x_1 x_2 = 1 = x_3 x_4$.
Letting $D_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix} = D_3^T$ and $D_2 = \begin{pmatrix} a_2 & b_2 \\ b_2 & a_2 \end{pmatrix}$, we write (4.33c) in component form.

$$(a_1 + y_i b_1) + (a_2 + y_i b_2) x_i + (a_1 + y_i c_1) x_i^2 = 0, i = 1, 2, 3, 4, \tag{4.35a}$$

$$(c_1 + y_i a_1) + (b_2 + y_i a_2) x_i + (b_1 + y_i a_1) x_i^2 = 0, i = 1, 2, 3, 4, \tag{4.35b}$$

For $i = 1$, (4.35a) is equal to

$$(a_1 + y_1 b_1) + (a_2 + y_1 b_2)\frac{1}{x_2} + (a_1 + y_1 c_1)\frac{1}{x_2^2} = 0$$

or

$$(a_1 + y_1 c_1) + (a_2 + y_1 b_2)x_2 + (a_1 + y_1 b_1)x_2^2 = 0$$

or

$$(c_1 + \frac{1}{y_1}a_1) + (b_2 + \frac{1}{y_1}a_2)x_2 + (b_1 + \frac{1}{y_1}a_1)x_2^2 = 0. \tag{4.35c}$$

Using equations (4.35c), (4.35b) with $i = 2$, and the uniqueness of y_i, $i = 1, 2, 3, 4$, we conclude that $y_1 y_2 = 1$. Similarly, $y_3 y_4 = 1$. We just complete the proof of the proposition. □

We are now in a position to further study equation (4.32). We assume, momentarily, that equation (4.32) has four distinct roots $\delta_1, \delta_2, \delta_3$ and δ_4. The general solutions to (4.29a) can then be written as

$$\mathbf{b}_i = c_1\delta_1^i \begin{pmatrix} 1 \\ \nu_1 \end{pmatrix} + c_2\delta_2^i \begin{pmatrix} 1 \\ \nu_2 \end{pmatrix} + c_3\delta_3^i \begin{pmatrix} 1 \\ \nu_3 \end{pmatrix} + c_4\delta_4^i \begin{pmatrix} 1 \\ \nu_4 \end{pmatrix}. \tag{4.36}$$

Here ν_i, $i = 1, 2, 3, 4$, are some constants depending on δ_i.
Applying (4.36) to boundary conditions (4.29b,c), we get

$$c_1(\delta_1^l - 1)\begin{pmatrix} 1 \\ \nu_1 \end{pmatrix} + c_2(\delta_2^l - 1)\begin{pmatrix} 1 \\ \nu_2 \end{pmatrix} + c_3(\delta_3^l - 1)\begin{pmatrix} 1 \\ \nu_3 \end{pmatrix} + c_4(\delta_4^l - 1)\begin{pmatrix} 1 \\ \nu_4 \end{pmatrix} = 0 \tag{4.37a}$$

and

$$c_1\delta_1(\delta_1^l-1)\begin{pmatrix}1\\ \nu_1\end{pmatrix}+c_2\delta_2(\delta_2^l-1)\begin{pmatrix}1\\ \nu_2\end{pmatrix}+c_3\delta_3(\delta_3^l-1)\begin{pmatrix}1\\ \nu_3\end{pmatrix}$$
$$+c_4\delta_4(\delta_4^l-1)\begin{pmatrix}1\\ \nu_4\end{pmatrix}=0. \tag{4.37b}$$

Writing (4.37) in matrix form, we have

$$\begin{pmatrix}1 & 1 & 1 & 1\\ \nu_1 & \nu_2 & \nu_3 & \nu_4\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\\ \nu_1\delta_1 & \nu_2\delta_2 & \nu_3\delta_3 & \nu_4\delta_4\end{pmatrix} diag(\delta_1^l-1,\delta_2^l-1,\delta_3^l-1,\delta_4^l-1)\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}=0. \tag{4.38}$$

Now if, $diag(\delta_1^l-1,\delta_2^l-1,\delta_3^l-1,\delta_4^l-1)$ is singular, then equation (4.36) has nontrivial solutions c_i , i=1,2,3,4. Note that $diag(\delta_1^l-1,\delta_2^l-1,\delta_3^l-1,\delta_4^l-1)$ is singular if and only if δ_i, i=1,2,3,4, satisfy

$$\delta^l=1 \tag{4.39}$$

and (4.32b). To solve system of equations (4.39) and (4.32b), we first note that

$$\delta_m=e^{i\frac{2m\pi}{l}}, 0\le m\le n-1, \tag{4.40}$$

are roots of equation (4.39). Substituting (4.40) into (4.32b), we get that the imaginary part of the resulting equation is

$$[-4\sin\frac{4m\pi}{l}]\lambda^2+[2\alpha\sin\frac{6m\pi}{l}-(4\alpha+16)\sin\frac{4m\pi}{l}+2\alpha\sin\frac{2m\pi}{l}]\lambda$$
$$+[\alpha\sin\frac{8m\pi}{l}+4(1+\alpha)\sin\frac{6m\pi}{l}-(8+10\alpha)\sin\frac{4m\pi}{l}+4(1+\alpha)\sin\frac{2m\pi}{l}]=0. \tag{4.41}$$

Before we proceed to compute the real part of the resulting equation, we need the following lemma.

Lemma 4.4.3. *Let a, b, and c be any complex number, then*

$$\cos 2\theta(\sin 4\theta + a\sin 3\theta + b\sin 2\theta + a\sin\theta)$$

$$= \sin 2\theta(\cos 4\theta + a\cos 3\theta + b\cos 2\theta + a\cos\theta + 1). \tag{4.42}$$

Since the proof of the lemma is straighforward, we will skip it.

Using (4.41) and (4.42), we see immediately that the real part of (4.32b) with $\delta = e^{i\frac{2m\pi}{l}}$ is a constant multiple $\frac{\sin\frac{4m\pi}{l}}{\cos\frac{4m\pi}{l}}$ of its imaginary part. We next show that (4.41) is indeed the characteristic equation of the matrix $\mathbf{G}(\alpha, 1)$.

Theorem 4.4.4. *: Let $m\times m$, $m = 2k$, $k \in \mathbb{N}$, be the dimension of the matrix $\mathbf{G}(\alpha, 1)$. Let dimension of each block matrix in $\mathbf{G}(\alpha, 1)$ be 2×2. Then the eigenvalues $\lambda^{\pm}_m(\alpha, 1)$ of $C(\alpha, 1)$ are of the following form.*

$$\begin{aligned}\lambda^{\pm}_m(\alpha,1) = \frac{1}{2}(\alpha\cos\frac{2m\pi}{l} - \alpha - 4) &\pm \frac{1}{2}[(\alpha\cos\frac{2m\pi}{l} - \alpha - 4)^2 \\ &+ 4(\alpha\cos^2\frac{2m\pi}{l} + 2(\alpha+1)\cos\frac{2m\pi}{l} - 2 - 3\alpha)]^{\frac{1}{2}} \\ &=: \check{\lambda}_m(\alpha,1) \pm \hat{\lambda}_m(\alpha,1), m = 0, 1, ..., l-1.\end{aligned} \tag{4.43}$$

Proof. Solving (4.41), we get (4.43). Using Proposition 4.4.2, we see that if $\delta = 1$ or -1 is a root of equation (4.32b), then the multiplicity of $\delta = 1$ or -1 is both two. Thus, we have only proved the following. (i)If $\frac{l}{2}$ is not a positive integer, then for each α, $\lambda^{\pm}_m(\alpha, 1)$, $m = 1, 2, ..., l-1$, are eigenvalues of $\mathbf{G}(\alpha, 1)$. (ii)If $\frac{l}{2}$ is a positive integer, then for each α, $\lambda^{\pm}_m(\alpha, 1)$, $m = 1, 2, ..., \frac{l}{2} - 1, \frac{l}{2} + 1, ..., l-1$, are eigenvalues of $\mathbf{G}(\alpha, 1)$. To complete the proof of the theorem, it remains to show that for each α, $\lambda^{\pm}_0(\alpha, 1)(= 0, -4)$ are eigenvalues of $\mathbf{G}(\alpha, 1)$ for each α and that if, additionally, $\frac{l}{2} > 1$ is a positive integer, then for each α, $\lambda^{\pm}_{\frac{l}{2}}(\alpha, 1)(= -2, -2-2\alpha)$ are also eigenvalues of

$\mathbf{G}(\alpha, 1)$. Using Proposition 4.4.1, we only need to show that $-2-2\alpha = (\lambda^-_{\frac{l}{2}}(\alpha, 1))$ is an eigenvalue of $\mathbf{G}(\alpha, 1)$ for fixed α. To this end, we see that

$$trace\ of\ \mathbf{G}(\alpha, 1) = -l(\alpha+4). \tag{4.44}$$

Let $m = 2l > 4$ be a multiple of four, then

$$\lambda^+_{\frac{l}{2}}(\alpha, 1) + (\sum_{j=1, j\neq\frac{l}{2}}^{l} \lambda^{\pm}_j(\alpha, 1)) + \lambda^{\pm}_0(\alpha, 1) = -2-(l-2)(\alpha+4)-4. \tag{4.45}$$

Using (4.44) and (4.45), we have that the remaining eigenvalue of $\mathbf{G}(\alpha, 1)$ for each α is $-2-2\alpha$, which is equal to $\lambda^-_{\frac{l}{2}}(\alpha, 1)$. We thus complete the proof of the theorem. □

Proposition 4.4.5. *For all $\alpha > 0$, we have that $\hat{\lambda}_m(\alpha, 1) > 0$, $\check{\lambda}_m(\alpha, 1) < 0$ and $\lambda^{\pm}_m(\alpha, 1) \leq 0$.*

Proof. Obviously, $\check{\lambda}_m(\alpha, 1) < 0$. Now, letting $t = \cos\frac{2m\pi}{l}$, we have that

$$\begin{aligned}4(\hat{\lambda}_m(\alpha, 1))^2 &= (t-1)^2\alpha^2 + 4(t^2-1)\alpha + 8(1+t)\\ &= ((t-1)\alpha + 2(t+1))^2 + 4(1-t^2) > 0\end{aligned}$$

for any $\alpha > 0$. Thus $\hat{\lambda}_m(\alpha, 1) > 0$. To prove the last assertion of the proposition, we note, via (4.43), that

$$0 > 4(\alpha\cos^2\frac{2m\pi}{l} + 2(\alpha+1)\cos\frac{2m\pi}{l} - 2 - 3\alpha) =: l.$$

Thus,

$$2\lambda^{\pm}_m(\alpha, 1) = 2\check{\lambda}_m(\alpha, 1) \pm (4\check{\lambda}^2_m(\alpha, 1) + l)^{\frac{1}{2}} \leq 0.$$

We just complete the proof of the proposition. □

Proposition 4.4.6. *If $\frac{l}{2}$ is not a positive integer, then the eigencurves $\lambda_m^{\pm}(\alpha, 1)$, $m = 1, 2, ..., l-1$, are strictly decreasing in $\alpha \in (0, \infty)$. If $\frac{l}{2}(> 1)$ is a positive integer, then $\lambda_m^{\pm}(\alpha, 1)$, $m = 1, 2, ..., \frac{l}{2}-1, \frac{l}{2}+1, ..., l-1$, and $\lambda_{\frac{l}{2}}^{-}(\alpha, 1)$ are strictly decreasing in $\alpha \in (0, \infty)$.*

Proof. Letting $t = \cos\frac{2m\pi}{l}$, we write (4.43) as

$$\begin{aligned}\lambda_m^{\pm}(\alpha, 1) &= \frac{1}{2}\{\alpha(t-1) - 4 \pm [(t-1)^2\alpha^2 + 4(t^2-1)\alpha + 8(1+t)]^{\frac{1}{2}}\} \\ &=: \frac{1}{2}\{\alpha(t-1) - 4 \pm (t_\alpha)^{\frac{1}{2}}\} =: \lambda_t^{\pm}(\alpha).\end{aligned} \tag{4.46}$$

Then

$$2\frac{d\lambda_m^{\pm}(\alpha, 1)}{d\alpha} = (t-1)(1 \pm \frac{(t-1)\alpha + 2(t+1)}{\sqrt{t_\alpha}}).$$

A direct computation would yield that

$$t_\alpha \geq ((t-1)\alpha + 2(t+1))^2.$$

Thus, $\frac{d\lambda_m^{\pm}(\alpha,1)}{d\alpha} \leq 0$. The equality holds only if t=1 or t= -1 for λ_m^{+}. ☐

Proposition 4.4.7. *(i) In the $\alpha - \lambda$ plane, $\lambda_t^{+}(\alpha, 1)$ intersect with $\lambda = -2 + k$ at $\alpha_{t,k}$, where*

$$\alpha_{t,k} = \frac{2(1+t) - k^2}{(1-t)(1+t+k)}. \tag{4.47}$$

(ii) For $-1 \leq t < 1$, $\lim\limits_{\alpha\to\infty} \lambda_t^{+}(\alpha, 1) = -(t+3)$.

Proof. Solving equation $-2 + k = \lambda_t^{+}(\alpha, 1)$, we easily get that $\alpha_{t,k}$ are as asserted. Rewriting $\lambda_t^{+}(\alpha, 1)$ as

$$\lambda_t^+(\alpha, 1) = \frac{-2\alpha(t-1)(t+3) + 4(1-t)}{\alpha(t-1) - 4 - \sqrt{t_\alpha}},$$

we see that $\lim_{\alpha \to \infty} \lambda_t^+(\alpha, 1) = -(t+3)$ for $-1 \le t < 1$. $\square$

Theorem 4.4.8. *Let m be any positive even integer. The dimension of each block matrix in $\mathbf{G}(\alpha, 1)$ is 2×2. Then (i) Suppose m is a multiple of four and $m > 4$. For each $\alpha > 0$, let $\lambda_2(\alpha, 1)$ be the second largest eigenvalue of $\mathbf{G}(\alpha, 1)$. Then $\lambda_2(\alpha, 1) = \lambda_1^+(\alpha, 1)$, for $0 \le \alpha \le \frac{1}{\sin^2 \frac{\pi}{l}} =: \alpha_1$; and $\lambda_2(\alpha, 1) = \lambda_{\frac{n}{2}}^+(\alpha, 1) = -2$ for all $\alpha \in [\alpha_1, \infty)$. See Figure 4.1.*
(ii) Suppose m is not a multiple of four. Then there exists a $\tilde{\alpha}_c$ such that $\lambda_2(\alpha, 1) = \lambda_{[\frac{l}{2}]}^+(\alpha)$ for all $\alpha \ge \tilde{\alpha}_c$. Here $[\frac{l}{2}]$ = the largest positive integer that is less than or equal to $\frac{l}{2}$. Moreover, $\lambda_2(\alpha, 1) < -2$ whenever $\alpha > \alpha_1$. See Figure 4.2.

Proof. For $\alpha_{t,k}$ to be positive, we must have

$$2(1+t) > k^2. \tag{4.48}$$

Now,

$$\begin{aligned} (1-t)^2(1+t+k)^2 \frac{d\alpha_{t,k}}{dt} &= 2(t+1)^2 - k^3 + 4k - 2tk^2 \\ &> (1+t)k^2 - k^3 + 4k - 2tk^2 \\ &= -k(k^2 + (t-1)k - 4) \\ &= -k(k - t_+)(k - t_-), \end{aligned}$$

where $t_\pm = \frac{1-t \pm \sqrt{16+(1-t)^2}}{2}$. Note that we have used (4.48) to justify the above inequality. Moreover $t_- < 0$ and $t_+ \ge 2$. Thus, $\frac{d\alpha_{t,k}}{dt} > 0$ whenever $\lambda = -2 + k$, $0 \le k < 2$, and $\lambda = \lambda_t^+(\alpha, 1)$ have the intersections intersect at the positive $\alpha_{t,k}$. Upon using Proposition 4.4.6, we conclude that for $0 \le m \le l-1$, the portion of the graphs of $\lambda_m^+(\alpha, 1)$ lying above the line $\lambda = -2$ do not intersect each other. Thus, $\lambda_2(\alpha, 1)$ is as asserted.

By Proposition 4.4.7-(ii), we have that

$$\lim_{\alpha\to\infty} \lambda_m^+(\alpha,1) = -(\cos\frac{2m\pi}{l} + 3) =: \lambda_m^\infty = \lambda_t^\infty.$$

Then λ_m^∞, $0 < m \le l-1$, have a maximum at $m = [\frac{l}{2}]$. Thus, there exists a $\tilde{\alpha}_c$ such that $\lambda_2(\alpha,1) = \lambda^+_{[\frac{l}{2}]}(\alpha,1)$ for all $\alpha \ge \tilde{\alpha}_c$. The last assertion of the theorem follows from Proposition 4.4.7-(i) and Proposition 4.4.1. □

Remark 4.4.9. *(i) Since $\lambda_t^+(\alpha,1)$ is increasing in t and λ_t^∞ is decreasing in t. The eigencurves $\lambda_m^+(\alpha,1)$, $0 < m \le [\frac{l}{2}]$, must be crossing each other.*
(ii) The first columnin Table 4.1 contains the values of $\lambda_m^\pm(1,1)$, m=0,1,...,5, while the second column contains the eigenvalues of $\mathbf{G}(1,1)$ obtained by using Mathematica. As indicated, the $\mathbf{G}(1,1)$ and $\mathbf{G}(5,1)$ obtained by both methods are identical. The values $\lambda_m^\pm(3,1)$, m=0,1,...,8, in the first and third columns of Table 4.2 are computed by Maple, while those in the second and forth columns are computed by Matlab. Some discrepancies between the values in the respective columns occur due to the round-off errors.
(iii) Figure 4.1 illustrates the graph of $\lambda_m^\pm(\alpha,1)$, m=0,1,...,5, with l=6. The dotted part of the curve is $\lambda_2(\alpha,1)$. Figure 4.2 gives the same information with l=9.
(iv) We conclude, via the last assertion of Theorem 4.4.8, that the wavelet approach works even better when m is an even number but not a multiple of four. Indeed, in such case, it synchronizes faster when α is chosen to be the critical value $\tilde{\alpha}_c$.

l=6			
$\lambda_m^{\pm}(1,1)$	eigenvalues of $\mathbf{G}(1,1)$	$\lambda_m^{\pm}(5,1)$	eigenvalues of $\mathbf{G}(5,1)$
$\lambda_0^+(1,1) = 0$	0	$\lambda_0^+(5,1) = 0$	0
$\lambda_1^+(1,1) = -\frac{9}{4} + \frac{1}{4}\sqrt{37}$	$-\frac{9}{4} + \frac{1}{4}\sqrt{37}$	$\lambda_1^+(5,1) = -\frac{13}{4} + \frac{1}{4}\sqrt{13}$	$-\frac{13}{4} + \frac{1}{4}\sqrt{13}$
$\lambda_2^+(1,1) = -\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$\lambda_2^+(5,1) = -\frac{23}{4} + \frac{1}{4}\sqrt{181}$	$-\frac{23}{4} + \frac{1}{4}\sqrt{181}$
$\lambda_3^+(1,1) = -2$	-2	$\lambda_3^+(5,1) = -2$	-2
$\lambda_2^+(1,1) = -\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$\lambda_4^+(5,1) = -\frac{23}{4} + \frac{1}{4}\sqrt{181}$	$-\frac{23}{4} + \frac{1}{4}\sqrt{181}$
$\lambda_2^+(1,1) = -\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} + \frac{1}{4}\sqrt{13}$	$\lambda_5^+(5,1) = -\frac{13}{4} + \frac{1}{4}\sqrt{13}$	$-\frac{13}{4} + \frac{1}{4}\sqrt{13}$
$\lambda_0^-(1,1) = -4$	-4	$\lambda_0^-(5,1) = -4$	-4
$\lambda_1^-(1,1) = -\frac{9}{4} - \frac{1}{4}\sqrt{37}$	$-\frac{9}{4} - \frac{1}{4}\sqrt{37}$	$\lambda_1^-(5,1) = -\frac{13}{4} - \frac{1}{4}\sqrt{13}$	$-\frac{13}{4} - \frac{1}{4}\sqrt{13}$
$\lambda_2^-(1,1) = -\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$\lambda_2^-(5,1) = -\frac{23}{4} - \frac{1}{4}\sqrt{181}$	$-\frac{23}{4} - \frac{1}{4}\sqrt{181}$
$\lambda_3^-(1,1) = -4$	-4	$\lambda_3^-(5,1) = -12$	-12
$\lambda_4^-(1,1) = -\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$\lambda_4^-(5,1) = -\frac{23}{4} - \frac{1}{4}\sqrt{181}$	$-\frac{23}{4} - \frac{1}{4}\sqrt{181}$
$\lambda_5^-(1,1) = -\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$-\frac{11}{4} - \frac{1}{4}\sqrt{13}$	$\lambda_5^-(5,1) = -\frac{13}{4} - \frac{1}{4}\sqrt{13}$	$-\frac{13}{4} - \frac{1}{4}\sqrt{13}$

Table 4.1

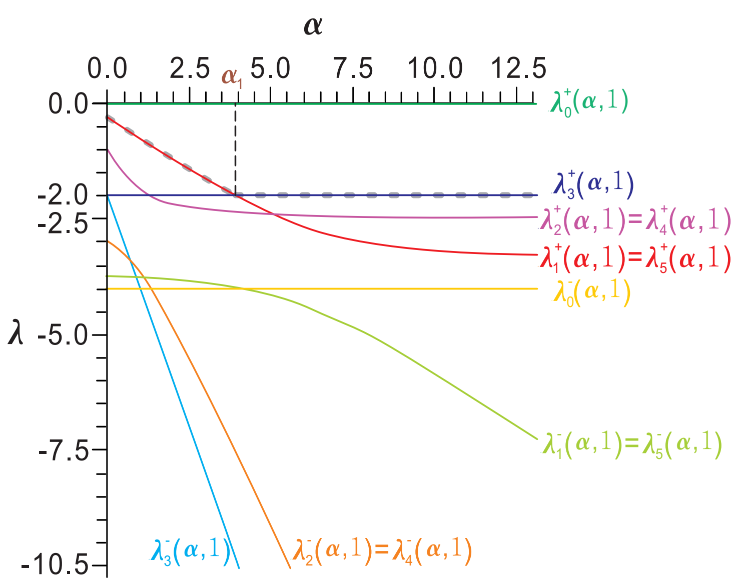

Figure 4.1

l=9			
$\lambda_m^{\pm}(3,1)$	eigenvalues of $\mathbf{G}(3,1)$	$\lambda_m^{\pm}(10,1)$	eigenvalues of $\mathbf{G}(10,1)$
$\lambda_0^+(3,1) = 0$	0	$\lambda_0^+(10,1) = 0$	0
$\lambda_1^+(3,1) \approx -0.7967$	-0.7967	$\lambda_1^+(10,1) \approx -2.2938$	-2.2930
$\lambda_2^+(3,1) \approx -2.2524$	-2.2525	$\lambda_2^+(10,1) \approx -3.0135$	-3.0140
$\lambda_3^+(3,1) \approx -2.2975$	-2.2974	$\lambda_3^+(10,1) \approx -2.4465$	-2.4466
$\lambda_4^+(3,1) \approx -2.0399$	-2.0399	$\lambda_4^+(10,1) \approx -2.0535$	-2.0542
$\lambda_5^+(3,1) \approx -2.0399$	-2.0399	$\lambda_5^+(10,1) \approx -2.0535$	-2.0542
$\lambda_6^+(3,1) \approx -2.2975$	-2.2974	$\lambda_6^+(10,1) \approx -2.4465$	-2.4466
$\lambda_7^+(3,1) \approx -2.2524$	-2.2525	$\lambda_7^+(10,1) \approx -3.0135$	-3.0140
$\lambda_8^+(3,1) \approx -0.7967$	-0.7967	$\lambda_8^+(10,1) \approx -2.2938$	-2.2930
$\lambda_0^-(3,1) = -4$	-4	$\lambda_0^-(10,1) = -4$	-4
$\lambda_1^-(3,1) \approx -3.9051$	-3.9052	$\lambda_1^-(10,1) \approx -4.0458$	-4.0465
$\lambda_2^-(3,1) \approx -4.2268$	-4.2265	$\lambda_2^-(10,1) \approx -9.2505$	-9.2495
$\lambda_3^-(3,1) \approx -6.2025$	-6.2026	$\lambda_3^-(10,1) \approx -16.5534$	-16.5534
$\lambda_4^-(3,1) \approx -7.7791$	-7.7792	$\lambda_4^-(10,1) \approx -21.3427$	-21.3427
$\lambda_5^-(3,1) \approx -7.7791$	-7.7792	$\lambda_5^-(10,1) \approx -21.3427$	-21.3427
$\lambda_6^-(3,1) \approx -6.2025$	-6.2026	$\lambda_6^-(10,1) \approx -16.5534$	-16.5534
$\lambda_7^-(3,1) \approx -4.2268$	-4.2265	$\lambda_7^-(10,1) \approx -9.2505$	-9.2495
$\lambda_8^-(3,1) \approx -3.9051$	-3.9052	$\lambda_8^-(10,1) \approx -4.0458$	-4.0465

Table 4.2

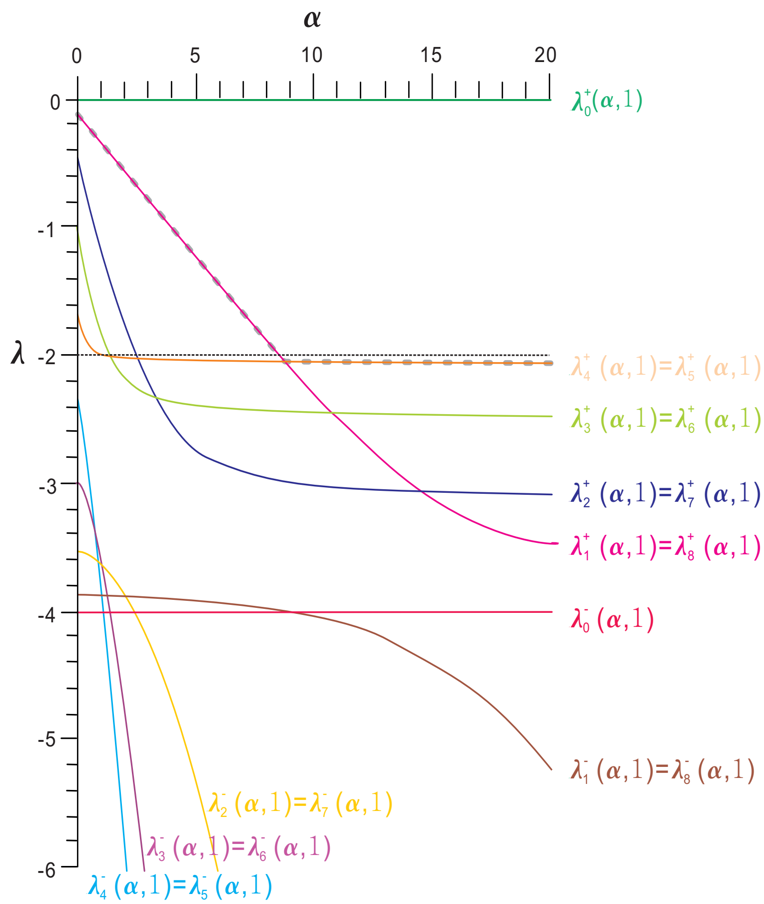

Figure 4.2

4.4.2 Neumann Boundary Conditions

In this section, we consider the nearest neighbor coupling with Neumann boundary conditions. The resulting coupling matrix $\mathbf{G}$ is then $\mathbf{G}(0)$, given as in (3.37).
With $i = 1$, we have

$$A_1(0) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \bar{A}_1(0) = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, A_2(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_1(1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, A_2(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{A}_2(0) = \alpha \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\tilde{A}_1(0) = \alpha \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} = \tilde{\tilde{A}}_1(0), \tilde{A}_2(1) = \alpha \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

and

$$\tilde{A}_1(1) = \alpha \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{4.49}$$

A direct calculation would yield that

$$\mathbf{G}_2(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

,

$$\mathbf{G}_1(0) = \begin{pmatrix} -\frac{1}{4}(4+\alpha) & \frac{1}{4}(4-\alpha) \\ \frac{1}{4}(4-\alpha) & -\frac{1}{4}(8+\alpha) \end{pmatrix}, \mathbf{G}_2(1) = \begin{pmatrix} \frac{\alpha}{4} & \frac{\alpha}{4} \\ \frac{1}{4}(\alpha+4) & \frac{\alpha}{4} \end{pmatrix},$$

$$\mathbf{G}_1(1) = \begin{pmatrix} -\frac{1}{2}(4+\alpha) & \frac{1}{2}(2-\alpha) \\ \frac{1}{2}(2-\alpha) & -\frac{1}{2}(4+\alpha) \end{pmatrix}, \bar{\mathbf{G}}_1(0) = \begin{pmatrix} -\frac{1}{4}(8+\alpha) & \frac{1}{4}(4-\alpha) \\ \frac{1}{4}(4-\alpha) & -\frac{1}{4}(4+\alpha) \end{pmatrix}. \tag{4.50}$$

As in the case of periodic boundary conditions, the eigenvalue problem $\mathbf{G}(\alpha, 0)\mathbf{b} = \lambda\mathbf{b}$, where $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_l)^T$, $\mathbf{b}_i \in \mathbb{C}^2$, can be formed as block difference equation

$$\mathbf{G}_2^T(1)\mathbf{b}_{i-1} + \mathbf{G}_1(1)\mathbf{b}_i + \mathbf{G}_2(1)\mathbf{b}_{i+1} = \lambda\mathbf{b}_i, 1 \le i \le l. \tag{4.51}$$

With Neumann boundary conditions, $\mathbf{b}_0$ and $\mathbf{b}_{l+1}$ must satisfy the following equations

$$\mathbf{G}_1(0)\mathbf{b}_1 + \mathbf{G}_2(1)\mathbf{b}_2 = \lambda\mathbf{b}_1 = \mathbf{G}_2^T(1)\mathbf{b}_0 + \mathbf{G}_1(1)\mathbf{b}_1 + \mathbf{G}_2(1)\mathbf{b}_2 \tag{4.52a}$$

and

$$\mathbf{G}_2^T(1)\mathbf{b}_{l-1} + \bar{\mathbf{G}}_1(0)\mathbf{b}_l = \lambda\mathbf{b}_l = \mathbf{G}_2^T(1)\mathbf{b}_{l-1} + \mathbf{G}_1(1)\mathbf{b}_l + \mathbf{G}_2(1)\mathbf{b}_{l+1} \tag{4.52b}$$

Solving (4.52a) and (4.52b), respectively, we get

$$\mathbf{b}_0 = (\mathbf{G}_2^T(1))^{-1}(\mathbf{G}_1(0) - \mathbf{G}_1(1))\mathbf{b}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{b}_1 \tag{4.53a}$$

and

$$\mathbf{b}_{l+1} = \mathbf{G}_2(1)^{-1}(\bar{\mathbf{G}}_1(0) - \mathbf{G}_1(1))\mathbf{b}_l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{b}_l. \tag{4.53b}$$

We then see that the characteristic equation of the block difference equation (4.51) is

$$det[\mathbf{G}_2^T(1) + \delta(\mathbf{G}_1(1) - \lambda I) + \delta^2\mathbf{G}_2] = 0. \tag{3.6a}$$

Here δ is such that $\mathbf{b}_i = \delta^i \begin{pmatrix} 1 \\ \nu \end{pmatrix}$, where ν is a constant depending on δ. Expanding the determinant in (4.54a), we get

$$\alpha\delta^4 + 2(2\alpha + 2 + \lambda\alpha)\delta^3 - 2(4 + 5\alpha + 2(\alpha + 4)\lambda + 2\lambda^2)\delta^2$$
$$+2(2\alpha + 2 + \lambda\alpha)\delta + \alpha = 0. \tag{4.54b}$$

We assume, momentarily, that equation (4.54b) has four distinct roots δ_1, δ_2, δ_3 and δ_4. The general solutions to (4.51) can then be written as

$$\mathbf{b}_i = \sum_{j=1}^{4} c_j \delta_j^i \begin{pmatrix} 1 \\ \nu_j \end{pmatrix}. \tag{4.55}$$

Substituting (4.55) into boundary conditions (4.53), we get

$$\begin{pmatrix} \delta_1\nu_1 - 1 & \delta_2\nu_2 - 1 & \delta_3\nu_3 - 1 & \delta_4\nu_4 - 1 \\ \delta_1 - \nu_1 & \delta_2 - \nu_2 & \delta_3 - \nu_3 & \delta_4 - \nu_4 \\ \delta_1^l(\delta_1\nu_1 - 1) & \delta_2^l(\delta_2\nu_2 - 1) & \delta_3^l(\delta_3\nu_3 - 1) & \delta_4^l(\delta_4\nu_4 - 1) \\ \delta_1^l(\delta_1 - \nu_1) & \delta_2^l(\delta_2 - \nu_2) & \delta_3^l(\delta_3 - \nu_3) & \delta_4^l(\delta_4 - \nu_4) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} =: D\mathbf{c} = 0, \tag{4.56}$$

where $\mathbf{c} = (c_1, c_2, c_3, c_4)^T$. We are now in a position to simplify $detD$.

$$detD = (\delta_2\nu_2)(\delta_4\nu_4)\begin{vmatrix} \delta_1\nu_1 - 1 & 1-\delta_1\nu_1 & \delta_3\nu_3 - 1 & 1-\delta_3\nu_3 \\ \delta_1 - \nu_1 & \nu_1 - \delta_1 & \delta_3 - \nu_3 & \nu_3 - \delta_3 \\ \delta_1^l(\delta_1\nu_1 - 1) & \delta_2^l(1-\delta_1\nu_1) & \delta_3^l(\delta_3\nu_3 - 1) & \delta_4^l(1-\delta_3\nu_3) \\ \delta_1^l(\delta_1 - \nu_1) & \delta_2^l(\nu_1 - \delta_1) & \delta_3^l(\delta_3 - \nu_3) & \delta_4^l(\nu_3 - \delta_3) \end{vmatrix}$$

$$= (\delta_2\nu_2)(\delta_4\nu_4)(\delta_1^l - \delta_2^l)(\delta_3^l - \delta_4^l)\begin{vmatrix} 0 & 1-\delta_1\nu_1 & 0 & 1-\delta_3\nu_3 \\ 0 & \nu_1 - \delta_1 & 0 & \nu_3 - \delta_3 \\ \delta_1\nu_1 - 1 & \delta_2^l(1-\delta_1\nu_1) & \delta_3\nu_3 - 1 & \delta_4^l(1-\delta_3\nu_3) \\ \delta_1 - \nu_1 & \delta_2^l(\nu_1 - \delta_1) & \delta_3 - \nu_3 & \delta_4^l(\nu_3 - \delta_3) \end{vmatrix}$$

$$= (\delta_2\nu_2)(\delta_4\nu_4)(\delta_1^l - \delta_2^l)(\delta_3^l - \delta_4^l)$$
$$\{[(\delta_1\nu_1 - 1)(\nu_3 - \delta_3) + (\delta_1 - \nu_1)(\delta_3\nu_3 - 1)]\begin{vmatrix} 1-\delta_1\nu_1 & 1-\delta_3\nu_3 \\ \nu_1 - \delta_1 & \nu_3 - \delta_3 \end{vmatrix}\}.$$

Therefore, $detD$ being equal to zero amounts to $\delta_i^{2l} = 1$ for $i = 1, 2, 3, 4$.
To get the characteristic equation of $\mathbf{G}(\alpha, 0)$, we need to solve $\delta^{2l} = 1$ and equation (4.54b). This leads to the following theorem.

Theorem 4.4.10. *Let m be any positive even integer. The dimension of each block matrix in $\mathbf{G}(\alpha, 0)$ is 2×2. Let $\lambda_m^{\pm}(\alpha, 0)$ be defined as follows.*

$$\begin{aligned}\lambda_m^{\pm}(\alpha, 0) =& \frac{1}{2}(\alpha\cos\frac{m\pi}{l} - \alpha - 4) \\ & \pm \frac{1}{2}[(\alpha\cos\frac{m\pi}{l} - \alpha - 4)^2 + 4(\alpha\cos^2\frac{m\pi}{l} + 2(\alpha+1)\cos\frac{m\pi}{l} - 2 - 3\alpha)]^{\frac{1}{2}}.\end{aligned} \tag{4.57}$$

Then $\lambda_m^{\pm}(\alpha, 0)$, $m = 1, 2, ..., l-1$, $\lambda_0^+(\alpha, 0) = 0$ and $\lambda_l^+(\alpha, 0) = -2$ are eigenvalues of $\mathbf{G}(\alpha, 0)$ for each $\alpha > 0$.

Proof. Substituting $\delta = e^{i\frac{m\pi}{l}}$, $0 \le m \le l-1$, into (4.54b), we get (4.57). Clearly, if $\delta \neq 1$ or -1, or equivalently, $\cos\frac{m\pi}{l} \neq 1$ or -1, then $\lambda_m^{\pm}(\alpha, 0)$, $m = 1, 2, .., l-1$, are eigencurves of $\mathbf{G}(\alpha, 0)$. Since $0 = \lambda_0^+(\alpha, 0)$ is an eigenvalue of $\mathbf{G}(\alpha, 0)$ for all α, we only need to show that $\lambda_l^+(\alpha,0)$ is, indeed, the eigenvalue of $\mathbf{G}(\alpha,0)$ for each α. To this end, we see that $trace(\mathbf{G}(\alpha,0)) = -(l-2)(\alpha+4)-6-\alpha$. However, $\lambda_0^+(\alpha,0)+\sum_{j=1}^{l-1}\lambda_j^{\pm}(\alpha,0) = -(l-1)(\alpha+4) =: k$. Thus, $trace(\mathbf{G}(\alpha,0)) - k = -2 = \lambda_l^+(\alpha,0)$. We just complete the proof of the theorem. □

Remark 4.4.11. *(i) Letting $t = \cos\frac{m\pi}{l}$, $\lambda_m^{\pm}(\alpha,0) = \lambda_t^{\pm}(\alpha,0)$ and treating t as a real parameter, we see that for fixed $\alpha > 0$, the eigenvalues of $\mathbf{G}$ with periodic boundary conditions and Neumann boundary conditions, respectively, lie on the curve $\lambda_t^{\pm}(\alpha,0)$ in $t-\lambda$ plane.*
(ii) Note that $\lambda_m^{\pm}(\alpha,0) = \lambda_{2l-m}^{\pm}(\alpha,0)$.

Theorem 4.4.12. *: For each α, let $\lambda(\alpha,0)$ be the second largest eigenvalue of $\mathbf{G}(\alpha,0)$. Then $\lambda(\alpha,0) = \lambda_1^+(\alpha,0)$, for $0 \le \alpha \le \frac{1}{\sin^2\frac{\pi}{2l}} =: \bar{\alpha}_1$; and $\lambda(\alpha,0) = \lambda_l^+(\alpha,0) = -2$ for all $\alpha \in [\bar{\alpha}_1,\infty)$.*

We skip the proof of theorem due to its similarity with that of Theorem 4.4.8-(ii).

l=3			
$\lambda_m^{\pm}(2,0)$	eigenvalues of $\mathbf{G}(2,0)$	$\lambda_m^{\pm}(5,0)$	eigenvalues of $\mathbf{G}(5,0)$
$\lambda_0^+(2,0)=0$	0	$\lambda_0^+(5,0)=0$	0
$\lambda_1^+(2,0)=-\frac{5}{2}+\frac{1}{2}\sqrt{7}$	$-\frac{5}{2}+\frac{1}{2}\sqrt{7}$	$\lambda_1^+(5,0)=-\frac{13}{4}+\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}+\frac{1}{4}\sqrt{13}$
$\lambda_2^+(2,0)=-\frac{7}{2}+\frac{1}{2}\sqrt{7}$	$-\frac{7}{2}+\frac{1}{2}\sqrt{7}$	$\lambda_2^+(5,0)=-\frac{23}{4}+\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}+\frac{1}{4}\sqrt{181}$
$\lambda_3^+(2,0)=-2$	-2	$\lambda_3^+(5,0)=-2$	-2
$\lambda_1^-(2,0)=-\frac{5}{2}-\frac{1}{2}\sqrt{7}$	$-\frac{5}{2}-\frac{1}{2}\sqrt{7}$	$\lambda_1^-(5,0)=-\frac{13}{4}-\frac{1}{4}\sqrt{13}$	$-\frac{13}{4}-\frac{1}{4}\sqrt{13}$
$\lambda_2^-(2,0)=-\frac{7}{2}-\frac{1}{2}\sqrt{7}$	$-\frac{7}{2}-\frac{1}{2}\sqrt{7}$	$\lambda_2^-(5,0)=-\frac{23}{4}-\frac{1}{4}\sqrt{181}$	$-\frac{23}{4}-\frac{1}{4}\sqrt{181}$

Table 4.3

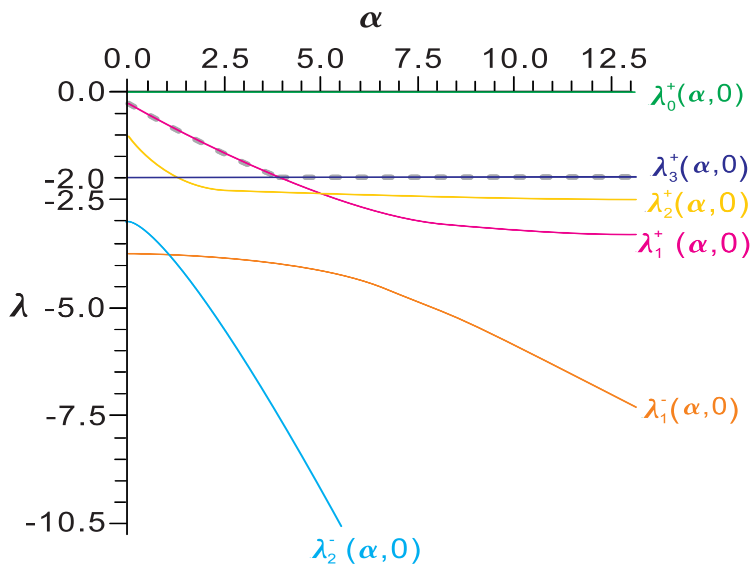

Figure 4.3

Remark 4.4.13. *Table 4.3 and Figure 4.3 illustrate, again, the accuracy of our theorems.*

Chapter 5

Concluding Chapter

We conclude this chapter by mentioning some possible future work.

(i) It is of great interest to extend our method to study the real world topology.

(ii) It is certainly worthwhile to study how bounded dissipation of the coupled system is related to the uncoupled dynamics and its connectivity topology.

(iii) It is interesting to study (global) synchronization of coupled system which lacks bounded dissipation such as the Rössler system.

(iv) It is desirable to solve the reduced problem of $\mathbf{G}(\alpha, \beta)$, $0 < \beta < 1$.

(v) It is also of considerable interest to study the wavelet transform on coupled map lattices.

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