Anisotropically inflating universes in a scalar-tensor theory

W. F. Kao[1,*] and Ing-Chen Lin[2]

[1]*Institute of Physics, Chiao Tung University, Hsin Chu, Taiwan*
[2]*Department of Physics, Tsing-Hua University, Hsin Chu, Taiwan*
(Received 23 September 2008; published 3 February 2009)

We show that a Brans-Dicke model admits some anisotropically inflating solutions which are identical to the solutions found in a higher derivative pure gravity theory. These inflating solutions were shown to break the cosmic no-hair theorem such that they do not approach the de Sitter universe at large times. The stability conditions of these solutions in this scalar-tensor theory are shown explicitly in this paper. It is shown that there exist unstable modes of the anisotropic perturbations. Therefore the inflating solutions are unstable in the scalar-tensor theory.

DOI: 10.1103/PhysRevD.79.043001 PACS numbers: 95.30.Sf, 04.50.Kd, 98.80.Bp, 98.80.Jk

I. INTRODUCTION

The inflationary universe is central to the astronomical observations aim to test and understand how the universe might have evolved from a general initial condition into its present state of large-scale isotropy and homogeneity together with an almost flat spectrum of near-Gaussian fluctuations. There is a period of accelerated expansion during the epoch of the early universe [1]. It has been shown that the simplest physically motivated inflationary scenario is induced by the acceleration driven by a scalar field with a constant potential. It is therefore important to find out whether universal acceleration and an asymptotic approach to the de Sitter metric always occurs. In fact, a series of cosmic no-hair theorems of varying strengths and degrees of applicability have been proved in support of certain constraints on the field parameters for its occurrence [2–8]. Note that from the point of view of an effective theory of gravity, a system with quadratic curvature terms should be understood as some perturbative correction to Einstein gravity suitable in some energy scale. Therefore, higher derivative gravity theories are supposed to play an important role in the physics of the early universe. Moreover, the conformal equivalence between these higher-order theories in vacuum and general relativity in the presence of a scalar field has also been shown in Refs. [9–11]. For example, a conformal transformation has been used to prove that a general theory with the action $S = \int d^Dx\sqrt{g}[F(\phi, R) - (\epsilon/2)(\nabla\phi)^2]$, where $F(\phi, R)$ is an arbitrary function of a scalar ϕ and scalar curvature R, is equivalent to a system described by the Einstein-Hilbert action plus scalar fields. Therefore it is interesting to study the cross relations between the higher-order theories and general relativity in the presence of a scalar field.

In particular, higher-order gravity theories give field equations which are higher order than two in time derivatives and generally have runaway solutions. The runaway solutions are supposed to be unphysical because they grow with time scales which are beyond the limits of validity of the theory. Thus, in this context, probably not all solutions have physical significance [12]. For instance, it is interesting to learn that the expanding solution found in the higher-order gravity theory studied in Ref. [13] does not have a limit in general relativity (i.e. it is not defined for $\beta \rightarrow 0$). An isotropic example of this phenomena is the Starobinsky inflation [14]. Indeed, it was shown that when quadratic terms are added to the Lagrangian of general relativity then new types of a cosmological solution arise when $\Lambda > 0$ which have no counterparts in general relativity in the Bianchi type II and type VI_h spaces [13]. Theses solutions inflate anisotropically and do not approach the de Sitter spacetime at large times. They hence provide counterexamples to the expectation that a cosmic no-hair theorem will continue to hold in simple higher-order extensions of general relativity. Other consequences of these higher-order theories have also been studied in [15–19].

A pure gravity theory which is quadratic in the scalar curvature and the Ricci tensor was considered in Ref. [13] for the model consists of the four-dimensional gravitational action

$$S_{\mathrm{BH}} = \frac{1}{2}\int d^4x\sqrt{g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda). \quad (1)$$

The Einstein equations can be shown to be [13]

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (2)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu}/2$ and

$$\begin{aligned}\Phi_{\mu\nu} \equiv {}& 2\alpha R(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}) + (2\alpha + \beta) \\ & \times (g_{\mu\nu}D^2 - D_\mu D_\nu)R + \beta D^2(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) \\ & + 2\beta(R_{\mu\sigma\nu\rho} - \tfrac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}. \end{aligned} \quad (3)$$

Here the tensor $\Phi_{\mu\nu}$ incorporates the deviation from regular Einstein gravity related to the coupling constants α and β.

New classes of exact solutions are found in a spatially homogeneous universe of the Bianchi type II space given

*gore@mail.nctu.edu.tw

by the metric

$$ds^2_{\rm II} = -dt^2 + e^{2bt}\left[dx + \frac{a}{2}(zdy - ydz)\right]^2 + e^{bt}(dy^2 + dz^2), \tag{4}$$

where

$$a^2 = \frac{11 + 8\Lambda(11\alpha + 3\beta)}{30\beta}, \qquad b^2 = \frac{8\Lambda(\alpha + 3\beta) + 1}{30\beta}. \tag{5}$$

Here a and b are some constant functions of α and β. These solutions are spacetime homogeneous with a five-dimensional isotropy group. They have a one-parameter family of four-dimensional Lie groups [20–22]. Interesting discussions related to these solutions can be found in Ref. [13].

Note that the no-hair theorem for Einstein gravity states that the presence of a positive cosmological constant drives the late-time evolution towards the de Sitter spacetime for Bianchi type I–VIII spaces [6]. The matter sources are required to obey the strong-energy condition. It has also been shown that the cosmic no-hair theorem cannot be proved and counterexamples exist if this condition does not hold exactly [7,23–26].

The Bianchi type solutions given above inflate in the presence of a positive cosmological constant Λ. They are, however, neither de Sitter, nor asymptotically de Sitter. Because of the complexity of the equations of motion it is difficult to extract information about the stability of these nonperturbative solutions in general. We found, however, that this solution is also a solution to a Brans-Dicke type scalar-tensor theory in the Bianchi type II space. We will show that this is indeed true for a scalar-tensor theory. The stability conditions of anisotropic perturbations will also be presented shortly in this paper. We will show, however, that these anisotropically inflating solutions are not stable under field perturbations. The critical role of the scalar field in this theory may hopefully shed more light on the stability problem of the anisotropic universes. Similar solutions can also be found in a type VI Bianchi space which will be discussed elsewhere.

II. THE SCALAR-TENSOR THEORY

The Barrow-Hervik (BH) solution (4) can also be written as, in a different form,

$$ds^2 = -dt^2 + a_1^2(t)dr^2 + g_{mn}dx^m dx^n \tag{6}$$

with $(x^0, x^1, x^2, x^3) = (t, r, z, \phi)$, $a_1^2(t) = \exp[bt]/a^2$ and $a_2^2(t) = \exp[2bt]/a^2$ where a and b denote some constant functions of α and β given by Eq. (5). Here

$$g_{mn} = \begin{pmatrix} a_2^2(t) & ra_2^2(t) \\ ra_2^2(t) & a_1^2(t) + r^2 a_2^2(t) \end{pmatrix}. \tag{7}$$

We will show that the BH solution (6) is also a solution to the Brans-Dicke model given by

$$S = \int d^4x\sqrt{g}\left[\frac{\psi}{2}R - \frac{w}{2\psi}D^\mu\psi D_\mu\psi - \lambda\psi\right]. \tag{8}$$

Indeed, the variation equation of the metric field gives

$$\begin{aligned} G^\mu{}_\nu &= t^\mu_\nu \\ &= \frac{1}{\psi}\left\{\frac{w}{\psi}\partial^\mu\psi\partial_\nu\psi + D^\mu D_\nu\psi \right. \\ &\quad \left. - g^\mu{}_\nu\left[\frac{w}{2\psi}\partial^\sigma\psi\partial_\sigma\psi + \lambda\psi\right]\right\} \end{aligned} \tag{9}$$

with t^μ_ν as the energy-momentum tensor associated with the scalar field ψ. In addition, the trace equation implies that

$$(2\omega + 3)D^2\psi = -2\lambda\psi. \tag{10}$$

Assuming that ψ is a function of t only, we can show that

$$t^2{}_3 = r(H_1 - H_2)\frac{\dot\psi}{\psi}. \tag{11}$$

It is straightforward to show that the metric solution found in Ref. [13] is also a solution to the scalar-tensor theory if

$$\begin{aligned} \psi &= \psi_0\exp\left[-2\frac{a^2 + b^2}{b}t\right], \\ w &= -\frac{3}{2} + \delta_1 = -\frac{3}{2} + \frac{4a^2 + b^2}{8(a^2 + b^2)}, \\ \lambda &= \frac{a^2(4a^2 + b^2)}{2b^2} > 0. \end{aligned} \tag{12}$$

Therefore, we can show that $-11/8 \le w \le -1$ for all possible combinations of a and b. The scalar-tensor theory with the BH solution has a negative kinetic energy term. The negative kinetic energy term can be removed from the action by a proper choice of conformal coordinate via the conformal transformation $\tilde g_{\mu\nu} = \psi g_{\mu\nu}$. As a result, the scalar-tensor action becomes

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{1}{2}\psi R - \frac{w}{2\psi}D^\alpha\psi D_\alpha\psi - \lambda\psi\right] \\ &= \int d^4x\sqrt{-\tilde g}\left[\frac{1}{2}\tilde R - \frac{\delta_1}{2\psi^2}\tilde D^\alpha\psi\tilde D_\alpha\psi - \frac{\lambda}{\psi}\right] \end{aligned} \tag{13}$$

in the conformal frame. Writing $\psi = \psi_0\exp[-2\phi]$, the action becomes

$$S = \int d^4x\sqrt{-\tilde g}\left[\frac{1}{2}\tilde R - 2\delta_1\tilde D^\alpha\phi\tilde D_\alpha\phi - \lambda\exp[2\phi]\right]. \tag{14}$$

A new set of solutions in this coordinate,

$$\tilde a_i = a_i\psi_0\exp\left[-2\frac{a^2 + b^2}{b}t\right], \qquad \phi = \frac{a^2 + b^2}{b}t, \tag{15}$$

can therefore be obtained straightforwardly from the BH solution (4).

III. THE STABILITY ANALYSIS

By writing

$$\psi = \psi_0 \exp[-2\phi], \tag{16}$$

$$a_i = \frac{1}{a} \exp[A_i(t)] \tag{17}$$

such that $H_i = \dot{A}_i$, we can show that the field equations (9) become

$$\dot{A}_1^2 + 2\dot{A}_1\dot{A}_2 - \frac{1}{4}a^2 \exp[2A_2 - 4A_1] = 2\omega\dot{\phi}^2 + 2(2\dot{A}_1 + \dot{A}_2)\dot{\phi} + \lambda, \tag{18}$$

$$(2\omega + 3)[\ddot{\phi} + (2\dot{A}_1 + \dot{A}_2)\dot{\phi} - 2\dot{\phi}^2] + \lambda = 0, \tag{19}$$

$$\ddot{A}_1 + 2\dot{A}_1^2 - \ddot{A}_2 - \dot{A}_2^2 - \dot{A}_1\dot{A}_2 - a^2 \exp[2A_2 - 4A_1] = 2(\dot{A}_1 - \dot{A}_2)\dot{\phi}. \tag{20}$$

Here the first equation is the G^t_t equation, the second equation is the trace equation (10), and the last equation is the G^2_3 equation. It can be shown that the BH solution given by the set of solutions $A_1 = bt/2$, $A_2 = bt$, and $\phi = (a^2 + b^2)/b$ does solve the above equations.

The perturbation equations of δA_i and $\delta\phi$ with respect to the BH solution given above can be shown to be

$$D\delta\mathcal{A} \equiv \begin{pmatrix} (4x+1)\nu - x & (2x+1)\nu + x/2 & -(4x+3/2) \\ 2(x+1)\nu & (x+1)\nu & \nu - 4x - 2 \\ \nu^2 - (2x+1)\nu + 4x & -\nu^2 + (2x-1/2)\nu - 2x & 1 \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{pmatrix} = 0. \tag{21}$$

Here we have defined $\delta A_3 \equiv \delta\dot{\phi}/b$ and $x \equiv a^2/b^2$ for convenience. Note that the perturbation equation becomes a dimensionless form in this format. In addition, we have also assumed that the perturbation equations are of the form $\delta A_i = k_i \exp[b\nu t]$ with k_i some constant initial perturbations. The differential forms of the perturbation equations can be restored by replacing the constant eigenvalues ν by ∂_t/b in the above equation. It is clear that a nontrivial solution exists only if $\det\mathcal{D} = 0$. This is equivalent to solving an eigenvalue problem. As a result, all linear combinations of the solutions

$$\delta A_i = \sum_j k_{ij} \exp[b\nu_j t] \tag{22}$$

to the eigenvalue equation (21) are also solutions to the perturbation equation (21). Here k_{ij} are some constant initial perturbations associated with the jth eigenmode solution, and the eigenvalues ν_j are the solutions to the perturbation equation.

This equation can be further simplified to a 2×2 matrix equation by solving one of the equations which gives, e.g.,

$$\delta A_3 = \frac{1}{8x+3}\{[2(4x+1)\nu - 2x]\delta A_1 + [2(2x+1)\nu + x]\delta A_2\}. \tag{23}$$

This will replace all the effects of the perturbation δA_3 as a linear combination of δA_1 and δA_2. As a result, the 3×3 matrix equation $\mathcal{D}\mathcal{A} = 0$ reduces to a 2×2 matrix equation

$$\mathcal{D}_2\delta\mathcal{A}_2 \equiv \begin{pmatrix} (8x+2)\Delta + 2(\nu - 8x^2 - x) & (4x+2)\Delta - 2(\nu + 16x^2 + 8x) \\ (8x+3)\Delta - (\nu - 8x^2 - x) & -(8x+3)\Delta + (\nu + 16x^2 + 8x)/2 \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta A_2 \end{pmatrix} = 0 \tag{24}$$

with $\Delta \equiv \nu^2 - 2x\nu + 3x$ for convenience. Therefore, $\det\mathcal{D}_2 = 0$ can be calculated to give

$$\Delta(\Delta - 3x) = 0. \tag{25}$$

This gives $\nu = 0$ or $\nu = 2x$ for $\Delta = 3x$ and $\nu = x \pm \sqrt{x^2 - 3x}$ for $\Delta = 0$. In fact, the trivial solution $\nu = 0$ is a degenerated solution to the perturbation equation. It is a degenerated solution in a sense that $\delta A_3 = 0$ (i.e. $\delta\phi =$ constant) is also a solution to the perturbation equations. Therefore, there are always unstable modes in this theory given by $\nu = 2x$ and $\nu = x \pm \sqrt{x^2 - 3x}$. As a result, the anisotropically inflating solution (6) will be unstable after a period of inflation of the order of $1/(b\nu)$. This will bring the anisotropically inflating universe to an end. The resulting universe is then expected to return to the de Sitter space provided proper constraints on the field parameters are obeyed such that the de Sitter space becomes a stable final state.

In summary, new classes of exact anisotropically inflating solutions are found in spatially homogeneous universes of the Bianchi type II space in Ref. [13]. We are motivated to study a scalar-tensor theory that also admits such inflating solutions in the early times. Indeed, we found that these solutions are also solutions to a Brans-Dicke type scalar-tensor theory in the Bianchi type II space. In addition, we also derive the stability conditions of the anisotropic perturbations in this paper. We prove that these anisotropically inflating solutions admit unstable modes and will bring the inflationary universe to an end after a brief period of time of the order $1/(b\nu)$ described by the unstable mode $\delta H_i = k_i \exp[b\nu t]$.

Note that the effective energy-momentum tensor t^μ_ν of the inflating solution in the scalar-tensor theory (8) is exactly the same as the effective energy-momentum tensor of the model studied in Ref. [13]. Therefore, the energy conditions of this model are still the same as the energy conditions obeyed by the higher derivative model. The stability behavior of the scalar-tensor theory is not, however, identical to the higher derivative model. The critical role of the scalar field in the inflationary era therefore deserves more attention.

ACKNOWLEDGMENTS

This research is supported in part by the NSC of Taiwan under Contract No. NSC-95-2112-M-009-032-MY3. We would like to thank the referee for useful comments and suggestions.

[1] A. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. B **129**, 177 (1983).
[2] J. D. Barrow, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983), p. 267.
[3] W. Boucher and G. W. Gibbons, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983), p. 273.
[4] A. A. Starobinskii, Sov. Phys. JETP Lett. **37**, 66 (1983).
[5] L. G. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).
[6] R. Wald, Phys. Rev. D **28**, 2118 (1983).
[7] J. D. Barrow, Phys. Lett. B **187**, 12 (1987).
[8] J. D. Barrow, Phys. Lett. B **235**, 40 (1990); J. D. Barrow and P. Saich, Phys. Lett. B **249**, 406 (1990); J. D. Barrow and A. R. Liddle, Phys. Rev. D **47**, R5219 (1993); A. D. Rendall, Classical Quantum Gravity **22**, 1655 (2005).
[9] J. D. Barrow, Nucl. Phys. **B296**, 697 (1988).
[10] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).
[11] K. I. Maeda, Phys. Rev. D **39**, 3159 (1989).
[12] E. E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).
[13] J. D. Barrow and Sigbjorn Hervik, Phys. Rev. D **73**, 023007 (2006).
[14] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
[15] A. Berkin, Phys. Rev. D **44**, 1020 (1991).
[16] E. Bruning, D. Coule, C. Xu, Gen. Relativ. Gravit. **26**, 1197 (1994).
[17] H.-J. Schmidt, arXiv:gr-qc/0407095.
[18] A. D. Linde, Phys. Lett. B **175**, 395 (1986); A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).
[19] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University, Oxford, 1986).
[20] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. (N.Y.) **17**, 986 (1976).
[21] J. Lauret, Math. Ann. **319**, 715 (2001); Quart. J. Math. **52**, 463 (2001); Math. Z. **241**, 83 (2002); Differential Geometry and its Applications **18**, 177 (2003).
[22] S. Hervik, CMS Summer 2004 meeting, Halifax, NS, Canada (unpublished).
[23] J. D. Barrow, Phys. Lett. B **180**, 335 (1986).
[24] J. D. Barrow, Nucl. Phys. **B310**, 743 (1988).
[25] J. D. Barrow, Phys. Lett. B **183**, 285 (1987).
[26] N. Kaloper, Phys. Rev. D **44**, 2380 (1991).