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Conditional diagnosability of hypercubes under the comparison diagnosis model ☆,☆☆

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ABSTRACT

Processor fault diagnosis plays an important role in multiprocessor systems for reliable computing, and the diagnosability of many well-known networks has been explored. Lai et al. proposed a novel measure of diagnosability, called conditional diagnosability, by adding an additional condition that any faulty set cannot contain all the neighbors of any vertex in a system. We make a contribution to the evaluation of diagnosability for hypercube networks under the comparison model and prove that the conditional diagnosability of n-dimensional Hypercube Q_n is $3(n-2)+1$ for $n \geqslant 5$. The conditional diagnosability of Q_n is about three times larger than the classical diagnosability of Q_n.

1. Introduction

With the continuous increase in the size of a multiprocessor system, the complexity of the system can adversely affect its reliability. In order to maintain reliability, the system should be able to identify faulty vertices and replace them with fault-free ones. The process of identifying faulty vertices is called the *diagnosis* of the system, and the *diagnosability* of the system refers to the maximum number of faulty vertices that can be identified by the system.

Several different problems and models of fault diagnosis have been studied [6,7,12,13,20,22]. There are two fundamental approaches to system-level diagnosis: *tested-based diagnosis* and *comparison-based diagnosis*. In 1967, the Preparata, Metze, and Chien (PMC) model was proposed for system-level diagnosis in multiprocessor systems [17]. The PMC model uses tested-based diagnosis approach, under which a processor performs the diagnosis by testing on neighboring processors via the communication links between them. The PMC model was also used [2,4,9,10,12]. Comparison-based diagnosis is an attractive alternative to investigate the problem of fault diagnosis. In 1980, Malek and Maeng introduced the comparison model using Comparison-based diagnosis approach, also known as the MM model [14,15]. In this model, the number of faulty vertices is limited and all faults are permanent. The MM model deals with the faulty diagnosis by sending the same input (or task) from a vertex w to each pair of distinct neighbors, u and v, and then comparing their responses. The vertex w is called the *comparator* of vertices u and v. Different comparators may examine the same pair of vertices. The result of the comparison is either the two responses agreed or two responses disagreed. Based on the results of all the comparisons, one need to decide the faulty or fault-free status of the processors in the system. Using a comparison diagnosis model, Sengupta and Dahbura described a diagnosable system and presented a polynomial algorithm to determine the set of all faulty vertices [19].

The hypercube structure [18] is a well-known interconnection topology for multiprocessor systems. An n-dimensional hypercube can be modeled as a graph Q_n with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. There are 2^n vertices in Q_n, and each vertex has degree n. Each vertex v in Q_n can be distinctly labeled by a binary n-bit string, $v = v_{n-1}v_{n-2}\cdots v_1v_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. There is also a recursive definition of the Q_n. The hypercube Q_1 is a complete graph K_2 with two vertices $\{0,1\}$. For $n \geqslant 2$, Q_n is constructed from two copies of Q_{n-1} by adding a perfect matching between them.

Reviewing some previous papers [1,3,6–11,13,18,20], the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n, all have diagnosability n under the comparison model or the PMC model. The diagnosability of the Star S_n is shown to be $n-1$ under the comparison model [22]. In classical measures of system-level diagnosability for multiprocessor systems, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is

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fault-free or faulty. As a consequence, the diagnosability of a system is limited by its minimum degree. Therefore, Lai et al. introduced a restricted diagnosability of multiprocessor systems called conditional diagnosability in [12]. Lai et al. considered a measure by restricting that, for each processor v in a system, not all the processors which are directly connected to v fail at the same time. Under this condition, Lai et al. showed that the conditional diagnosability of n-dimensional Hypercube Q_n is $4(n-2)+1$ under the PMC model. In this paper, we study the diagnosability of the hypercube networks under the comparison model, and prove that the conditional diagnosability of n-dimensional Hypercube Q_n is $3(n-2)+1$ for $n \geqslant 5$. The conditional diagnosability of Q_n is about three times larger than that of the classical diagnosability of Q_n. We also make some comments in the conclusion section to explain why the increase in diagnosability under the comparison model is lower than that under the PMC model.

The rest of this paper is organized as follows: Section 2 provides preliminaries and previous results for diagnosing a system. In Section 3, we study the conditional diagnosability of the hypercube Q_n under the comparison model. Finally, our conclusions are given in Section 4.

2. Preliminaries and previous results

A multiprocessor system can be represented by a graph $G=(V,E)$, where the set of vertices $V(G)$ represents processors and the set of edges $E(G)$ represents communication links between processors. Throughout this paper, we focus on undirected graphs without loops and follow [21] for graph theoretical definitions and notations.

Let $G=(V,E)$ be a graph and $v \in V(G)$ be a vertex. The neighborhood $N(v)$ of vertex v is the set of all vertices that are adjacent to v. The cardinality $|N(v)|$ is called the degree of v, denoted by $deg_G(v)$ or simply $deg(v)$. For a subset of vertices $V' \subset V(G)$, the neighborhood set of the vertex set V' is defined as $N(V') = \bigcup_{v\in V'} N(v) \setminus V'$. For a set of vertices (respectively, edges) S, we use the notation $G \setminus S$ to denote the graph obtained from G by removing all the vertices (respectively, edges) in S. The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial. The connectivity $\kappa(G)$ of a graph $G=(V,E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph.

Let $S_1, S_2 \subseteq V(G)$ be two distinct sets. The symmetric difference of the two sets S_1 and S_2 is defined as the set $S_1 \Delta S_2 = (S_1 \setminus S_2) \bigcup (S_2 \setminus S_1)$.

The comparison diagnosis model [14,15] is proposed by Malek and Maeng. In this model, a self-diagnosable system is often represented by a multigraph $M(V,C)$, where V is the same vertex set defined in G and C is the labeled edge set. Let $(u,v)_w$ be a labeled edge. If (u,v) is an edge labeled by w, then $(u,v)_w$ is said to belong to C, which implies that the vertex u and v are being compared by vertex w. The same pair of vertices may be compared by different comparators, so M is a multigraph. For $(u,v)_w \in C$, we use $r((u,v)_w)$ to denote the result of comparing vertices u and v by w such that $r((u,v)_w)=0$ if the outputs of u and v agree, and $r((u,v)_w)=1$ if the outputs disagree. In this model, if $r((u,v)_w)=0$ and w is fault-free, then both u and v are fault-free. If $r((u,v)_w)=1$, then at least one of the three vertices u, v, w must be faulty. If the comparator w is faulty, then the result of the comparison is unreliable that means both $r((u,v)_w)=0$ and $r((u,v)_w)=1$ are possible outputs, and it outputs only one of these two possibilities. In this paper, we consider a complete diagnosis that means each vertex diagnoses all pairs of distinct neighbors. For an n-dimensional Hypercube Q_n, each vertex has degree n, and therefore, there are $\binom{n}{2}$ comparisons for each vertex acting as a comparator. Furthermore, there are 2^n vertices in Q_n so the total number of comparisons is $\binom{n}{2}2^n = O(n^2 2^n)$.

The collection of all comparison results, defined as a function σ: $C \to \{0,1\}$, is called the *syndrome* of the diagnosis. A subset $F \subset V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V \setminus F$ are fault-free. A system is said to be *diagnosable* if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. In [19], a system is called a t-diagnosable system if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. A faulty comparator can lead to unreliable results. So, a set of faulty vertices may produce different syndromes. Let $\sigma_F = \{\sigma | \sigma$ is compatible with $F\}$. Two distinct sets $F_1, F_2 \subset V$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \bigcap \sigma_{F_2} \neq \emptyset$; otherwise, F_1, F_2 are said to be *distinguishable*. There are several different ways to verify a system to be t-diagnosable under the comparison approach. The following theorem given by Sengupta and Dahbura [19] is a necessary and sufficient condition for ensuring distinguishability.

Theorem 1. *[19] Let $G=(V,E)$ be a graph. For any two distinct sets $F_1, F_2 \subset V$, (F_1, F_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied (see Fig. 1):*

1. $\exists u,w \in V \setminus \{F_1 \bigcup F_2\}$ *and* $\exists v \in F_1 \triangle F_2$ *such that* $(u,v)_w \in C$,
2. $\exists u,v \in F_1 \setminus F_2$ *and* $\exists w \in V \setminus \{F_1 \bigcup F_2\}$ *such that* $(u,v)_w \in C$, *or*
3. $\exists u,v \in F_2 \setminus F_1$ *and* $\exists w \in V \setminus \{F_1 \bigcup F_2\}$ *such that* $(u,v)_w \in C$.

Before studying the conditional diagnosability of the hypercube, we need some definitions for further discussion. Let $G=(V,E)$ be a graph. For any set of vertices $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by the vertex subset U. Let H be a subgraph of G and v be a vertex in H. We use $V(H;3) = \{v \in V(H) | \, deg_H(v) \geqslant 3\}$ to represent the set of vertices which has degree 3 or more in H. Let $F_1, F_2 \subseteq V(G)$ be two distinct sets and $S = F_1 \bigcap F_2$. We use $C_{F_1 \Delta F_2, S}$ to denote the subgraph induced by the vertex subset $(F_1 \Delta F_2) \bigcup \{u |$ there exists a vertex $v \in F_1 \Delta F_2$ such that u and v are connected in $G \setminus S\}$. The following result is a useful sufficient condition for checking whether (F_1, F_2) is a distinguishable pair.

Theorem 2. *Let $G=(V,E)$ be a graph. For any two distinct sets $F_1, F_2 \subset V$ with $|F_i| \leqslant t$, $i=1,2$, and $S = F_1 \bigcap F_2$. (F_1, F_2) is distinguishable if, the subgraph $C_{F_1 \Delta F_2, S}$ of $G \setminus S$ contains at least $2(t-|S|)+1$ vertices having degree 3 or more.*

Proof. Given any pair of distinct sets of vertices $F_1, F_2 \subset V$ with $|F_i| \leqslant t$, $i=1,2$. Let $S = F_1 \bigcap F_2$, then $0 \leqslant |S| \leqslant t-1$, and $|F_1 \Delta F_2| \leqslant 2(t-|S|)$. Consider the subgraph $C_{F_1 \Delta F_2, S}$, the number of vertices having degree 3 or more is at least $2(t-|S|)+1$ in $C_{F_1 \Delta F_2, S}$, the subgraph $C_{F_1 \Delta F_2, S}$ contains at least $2(t-|S|)+1$ vertices. There is at least one vertex with degree 3 or more lying in $C_{F_1 \Delta F_2, S} \setminus F_1 \Delta F_2$. Let u be one of such vertices with degree 3 or more. Let i, j, and k be three distinct vertices linked to u. If one of i, j, and k lies in

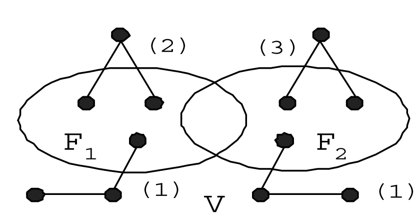

Fig. 1. Description of distinguishability for Theorem 1.

$C_{F_1\Delta F_2,S} \setminus F_1\Delta F_2$, condition 1 of Theorem 1 holds obviously. Suppose all these three vertices belong to $F_1\Delta F_2$. Without loss of generality, assume i lies in $F_1 \setminus F_2$, one of the two cases will happen: (1) if j lies in $F_1 \setminus F_2$, condition 2 of Theorem 1 holds; or, (2) if j lies in $F_2 \setminus F_1$, wherever k lies in $F_1 \setminus F_2$ or $F_2 \setminus F_1$, condition 2 or 3 of Theorem 1 holds. So (F_1,F_2) is a distinguishable pair and the proof is complete. □

By Theorem 2, we now propose a sufficient condition to verify whether a system is t-diagnosable under the comparison diagnosis model.

Corollary 1. *Let $G=(V,E)$ be a graph. G is t-diagnosable if, for each set of vertices $S \subset V$ with $|S| = p$, $0 \leqslant p \leqslant t-1$, every connected component C of $G \setminus S$ contains at least $2(t-p)+1$ vertices having degree at least three. More precisely, $|V(C;3)| \geqslant 2(t-p)+1$.*

3. Conditional diagnosability of Q_n

In classical measures of diagnosability for multiprocessor systems under the comparison model, if all the neighbors of some processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. For example, consider an n-dimensional Hypercube Q_n and two faulty sets $F_1,F_2 \subset V(Q_n)$ as shown in Fig. 2. As we observe the all neighbors of vertex v are included in F_1 and F_2. Let $F_1 = N(v) \bigcup \{v\}$ and $F_2 = N(v)$, then $|F_1| = n+1$ and $|F_2| = n$. By Theorem 1, F_1 and F_2 are indistinguishable under the comparison model. So the diagnosability of a system is limited by its minimum vertex degree.

In an n-dimensional Hypercube Q_n, Q_n has $\binom{2^n}{n}$ vertex subsets of size n, among which there are only 2^n vertex subsets which contains all the neighbors of some vertex. Since the ratio $2^n \Big/ \binom{2^n}{n}$ is very small for large n, the probability of a faulty set containing all the neighbors of any vertex is very low. For this reason, Lai et al. introduced a new restricted diagnosability of multiprocessor systems called conditional diagnosability in [12]. They consider the situation that any faulty set cannot contain all the neighbors of any vertex in a system. In the following, we need some terms to define the conditional diagnosability formally. A faulty set $F \subset V$ is called a *conditional faulty set* if $N(v) \not\subseteq F$ for every vertex $v \in V$. A system $G=(V,E)$ is said to be *conditionally t-diagnosable* if F_1 and F_2 are distinguishable, for each pair of conditional faulty sets $F_1,F_2 \subset V$, and $F_1 \neq F_2$, with $|F_1| \leqslant t$ and $|F_2| \leqslant t$. The maximum value of t such that G is conditionally t-diagnosable is called the *conditional diagnosability* of G, written as $t_c(G)$. It is trivial that $t_c(G) \geqslant t(G)$.

Lemma 1. *Let G be a multiprocessor system. Then, $t_c(G) \geqslant t(G)$.*

Let $G=(V,E)$ be a graph and $F_1,F_2 \subset V$, $F_1 \neq F_2$. We say (F_1,F_2) is a distinguishable conditional-pair (an indistinguishable conditional-pair, respectively) if F_1 and F_2 are conditional faulty sets and are distinguishable (indistinguishable, respectively). Before discussing the conditional diagnosability, we have some observations as follows: Let $F_1,F_2 \subset V$ be an indistinguishable conditional-pair. Let $X = V \backslash (F_1 \bigcup F_2)$. Since F_1 and F_2 are an indistinguishable conditional-pair, none of the three conditions of Theorem 1 holds and every vertex has at least one fault-free neighbor. Let vertex $u \in X$. If $N(u) \bigcap X \neq \emptyset$, then $N(u) \bigcap (F_1\Delta F_2) = \emptyset$ (see Fig. 3a); otherwise $|N(u) \bigcap (F_1 \setminus F_2)| = 1$ and $|N(u) \bigcap (F_2 \setminus F_1)| = 1$ (see Fig. 3b). Let vertex $v \in F_1\Delta F_2$. If $N(v) \bigcap X = \emptyset$, then $|N(v) \bigcap (F_1 \setminus F_2)| \geqslant 1$ and $|N(v) \bigcap (F_2 \setminus F_1)| \geqslant 1$ (see Fig. 3c). We state this fact in the following lemma.

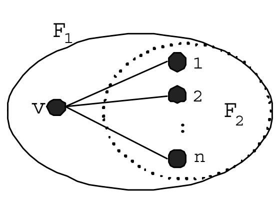

Fig. 2. An indistinguishable pair (F_1,F_2).

Lemma 2. *Let $G=(V,E)$ be a graph and $F_1,F_2 \subset V$ be an indistinguishable conditional-pair. Let $X = V \backslash (F_1 \bigcup F_2)$. The following three conditions holds:*

1. $|N(u) \bigcap (F_1\Delta F_2)| = 0$ *for* $u \in X$ *and* $N(u) \bigcap X \neq \emptyset$,
2. $|N(u) \bigcap (F_1 \setminus F_2)| = 1$ *and* $|N(u) \bigcap (F_2 \setminus F_1)| = 1$ *for* $u \in X$ *and* $N(u) \bigcap X = \emptyset$, *and*
3. $|N(v) \bigcap (F_1 \setminus F_2)| \geqslant 1$ *and* $|N(v) \bigcap (F_2 \setminus F_1)| \geqslant 1$ *for* $v \in F_1\Delta F_2$ *and* $N(v) \bigcap X = \emptyset$.

Now, we give an example to show that the conditional diagnosability of the hypercube Q_n is no greater than $3(n-2)+2$, $n \geqslant 5$. As shown in Fig. 4, we take a cycle of length four in Q_n. Let $\{v_1,v_2,v_3,v_4\}$ be the four consecutive vertices on this cycle, and let $F_1 = N(\{v_1,v_3,v_4\}) \bigcup \{v_1\}$ and $F_2 = N(\{v_1,v_3,v_4\}) \bigcup \{v_3\}$, then $|F_1| = |F_2| = 3(n-2)+2$. It is straightforward to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable by Theorem 1. Note that the hypercube Q_n has no cycle of length 3 and any two vertices have at most two common neighbors. As we can see, $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$ and $|F_1 \bigcap F_2| = 3(n-2)+1$. Therefore, Q_n is not conditionally $(3(n-2)+2)$-diagnosable and $t_c(Q_n) \leqslant 3(n-2)+1$, $n \geqslant 3$. Then, we shall show that Q_n is conditionally t-diagnosable, where $t = 3(n-2)+1$.

Lemma 3. *$t_c(Q_n) \leqslant 3(n-2)+1$ for $n \geqslant 3$.*

Let F be a set of vertices $F \subset V(Q_n)$ and C be a connected component of $Q_n \setminus F$. We need some results on the cardinalities of F and $V(C)$ under some restricted conditions. The results are listed in Lemmas 4 and 8. In Lemma 4, Lai et al. proved that deleting at most $2(n-1)-1$ vertices from Q_n, the incomplete hypercube Q_n has one connected component containing at least $2^n-|F|-1$ vertices. We expand this result further. In Lemma 8, we show that deleting at most $3n-6$ vertices from Q_n, the incomplete hypercube Q_n has one connected component containing at least $2^n-|F|-2$ vertices.

Lemma 4 [12]. *Let Q_n be an n-dimensional hypercube, $n \geqslant 3$, and let F be a set of vertices $F \subset V(Q_n)$ with $n \leqslant |F| \leqslant 2(n-1)-1$. Suppose that $Q_n \setminus F$ is disconnected. Then $Q_n \setminus F$ has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of $Q_n \setminus F$ contains $2^n-|F|-1$ vertices.*

In order to prove Lemma 8, we need some preliminary results as follows.

Lemma 5 [18]. *Let Q_n be an n-dimensional hypercube. The connectivity of Q_n is $\kappa(Q_n) = n$.*

Lemma 6. *For any three vertices x, y, z in Q_4, $|N(\{x,y,z\})| \geqslant 7$.*

Proof. A four-dimensional hypercube Q_4 can be divided into two Q_3's, denoted by Q_3^L and Q_3^R. Any two vertices in the Q_n have at most two common neighbors. If these three vertices x, y, z all fall in Q_3^L, then x, y, z have at least four neighboring vertices, all in Q_3^L. Besides, x, y, z have three more neighboring vertices in Q_3^R. Therefore, $|N(\{x,y,z\})| \geqslant 4+3 = 7$. Suppose now x, y fall in Q_3^L, z falls in Q_3^R. Vertex x and y have at least four neighboring vertices, all in Q_3^L. Vertex z will bring in at least three neighboring vertices in Q_3^R. Therefore, $|N(\{x,y,z\})| \geqslant 4+3 = 7$. □

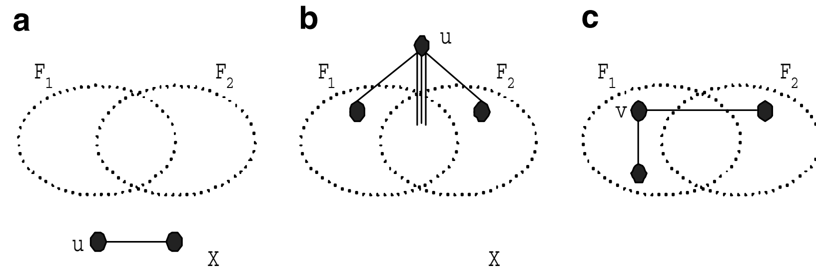

Fig. 3. An indistinguishable conditional-pair (F_1,F_2).

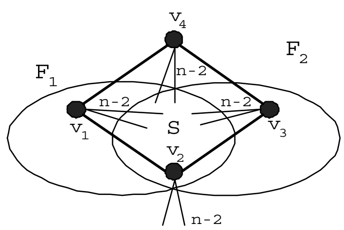

Fig. 4. An indistinguishable conditional-pair (F_1,F_2), where $|F_1| = |F_2| = 3(n-2)+2$.

We are going to prove Lemma 8 by induction on n, and we need a base case to start with. As we observed, for $n = 4$, we found a counter example that the result of Lemma 8 does not hold. So we have to start with $n = 5$.

Lemma 7. *Let Q_5 be a five-dimensional hypercube, and let F be a set of vertices $F \subset V(Q_5)$ with $|F| \leqslant 3n-6 = 9$. Then $Q_5 \setminus F$ has a connected component containing at least $2^n - |F| - 2 = 30 - |F|$ vertices.*

Proof. A five-dimensional hypercube Q_5 can be divided into two Q_4's, denoted by Q_4^L and Q_4^R. Let $F_L = F \bigcap V(Q_4^L), 0 \leqslant |F_L| \leqslant 9$ and $F_R = F \bigcap V(Q_4^R), 0 \leqslant |F_R| \leqslant 9$. Then $|F| = |F_L| + |F_R|$. Without loss of generality, we may assume that $|F_L| \geqslant |F_R|$. In the following proof, we consider three cases by the size of F_R: (1) $0 \leqslant |F_R| \leqslant 2$, (2) $|F_R| = 3$, and (3) $|F_R| = 4$.

Case 1: $0 \leqslant |F_R| \leqslant 2$.

Since $\kappa(Q_4) = 4$, $Q_4^R \setminus F_R$ is connected and $|V(Q_4^R \setminus F_R)| = 2^4 - |F_R|$. Let $F_R^{(L)} \subset V(Q_4^L)$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in Q_4^L \setminus F_L \setminus F_R^{(L)}$, there is exactly one vertex $v^{(R)}$ in $Q_4^R \setminus F_R$, such that $(v,v^{(R)}) \in E(Q_5)$. Besides, $|V(Q_4^L \setminus F_L \setminus F_R^{(L)})| \geqslant 2^4 - |F_L| - |F_R|$. Hence $Q_5 \setminus F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - |F_R|] = 32 - |F| - |F_R| \geqslant 30 - |F|$ vertices.

Case 2: $|F_R| = 3$.

Since $\kappa(Q_4) = 4$, $Q_4^R \setminus F_R$ is connected and $|V(Q_4^R \setminus F_R)| = 2^4 - |F_R|$. Let $F_R = \{x,y,z\}$ and $F_R^{(L)} = \{x^{(L)}, y^{(L)}, z^{(L)}\} \subset V(Q_4^L)$, where $(x,x^{(L)})$, $(y,y^{(L)})$, $(z,z^{(L)}) \in E(Q_5)$. For each vertex $v \in Q_4^L \setminus F_L \setminus F_R^{(L)}$, there is exactly one vertex $v^{(R)}$ in $Q_4^R \setminus F_R$, such that $(v,v^{(R)}) \in E(Q_5)$. If at least one of the three vertices $x^{(L)}$, $y^{(L)}$, $z^{(L)}$ belongs to F_L, then $|V(Q_4^L \setminus F_L \setminus F_R^{(L)})| \geqslant 2^4 - |F_L| - 2$. Hence $Q_5 \setminus F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - 2] = 30 - |F|$ vertices; otherwise, $|V(Q_4^L \setminus F_L \setminus F_R^{(L)})| \geqslant 2^4 - |F_L| - 3$. Since $|F_L| \leqslant 6$, by Lemma 6, $x^{(L)}$, $y^{(L)}$, $z^{(L)}$ have at least one neighboring vertex in $Q_4^L \setminus F_L \setminus F_R^{(L)}$. Hence $Q_5 \setminus F$ has a connected component that contains at least $[2^4 - |F_R|] + [2^4 - |F_L| - 3] + 1 = 30 - |F|$ vertices.

Case 3: $|F_R| = 4$.

Since $|F_R| = 4$ and $|F_L| \leqslant 5$, by Lemma 4, $Q_4^L \setminus F_L$ ($Q_4^R \setminus F_R$, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^4 - |F_L| - 1$ ($2^4 - |F_R| - 1$, respectively) vertices. Since $|V(C_L)| \geqslant |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u,v) \in E(Q_5)$. Hence $Q_5 \setminus F$ has a connected component that contains at least $[2^4 - |F_L| - 1] + [2^4 - |F_R| - 1] = 30 - |F|$ vertices.

Consequently, the lemma holds. □

We now prove Lemma 8.

Lemma 8. *Let Q_n be an n-dimensional hypercube, $n \geqslant 5$, and let F be a set of vertices $F \subset V(Q_n)$ with $|F| \leqslant 3n-6$. Then $Q_n \setminus F$ has a connected component containing at least $2^n - |F| - 2$ vertices.*

Proof. We prove the lemma by induction on n. By Lemma 7, the lemma holds for $n = 5$. As the inductive hypothesis, we assume that the result is true for Q_{n-1}, for $|F| \leqslant 3(n-1) - 6$, and for some $n \geqslant 6$. Now we consider Q_n, $|F| \leqslant 3n-6$. An n-dimensional hypercube Q_n can be divided into two Q_{n-1}'s, denoted by Q_{n-1}^L and Q_{n-1}^R. Let $F_L = F \bigcap V(Q_{n-1}^L), 0 \leqslant |F_L| \leqslant 3n-6$ and $F_R = F \bigcap V(Q_{n-1}^R), 0 \leqslant |F_R| \leqslant 3n-6$. Then $|F| = |F_L| + |F_R|$. Without loss of generality, we may assume that $|F_L| \geqslant |F_R|$. In the following proof, we consider two cases by the size of F_R: 1) $0 \leqslant |F_R| \leqslant 2$ and 2) $|F_R| \geqslant 3$.

Case 1: $0 \leqslant |F_R| \leqslant 2$.

Since $0 \leqslant |F_R| \leqslant 2$, $Q_{n-1}^R \setminus F_R$ is connected and $|V(Q_{n-1}^R \setminus F_R)| = 2^{n-1} - |F_R|$. Let $F_R^{(L)} \subset V(Q_{n-1}^L)$ be the set of vertices which has neighboring vertices in F_R. For each vertex $v \in Q_{n-1}^L \setminus F_L \setminus F_R^{(L)}$, there is exactly one vertex $v^{(R)}$ in $Q_{n-1}^R \setminus F_R$, such that $(v,v^{(R)}) \in E(Q_n)$. Besides, $|V(Q_{n-1}^L \setminus F_L \setminus F_R^{(L)})| \geqslant 2^{n-1} - |F_L| - |F_R|$. Hence $Q_n \setminus F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - |F_R|] = 2^n - |F| - |F_R| \geqslant 2^n - |F| - 2$ vertices.

Case 2: $|F_R| \geqslant 3$.

Since $|F_R| \geqslant 3$, $3 \leqslant |F_L| \leqslant 3(n-1) - 6$, and $3 \leqslant |F_R| \leqslant 3(n-1) - 6$. By the inductive hypothesis, $Q_{n-1}^L \setminus F_L$ ($Q_{n-1}^R \setminus F_R$, respectively) has a connected component C_L (C_R, respectively) that contains at least $2^{n-1} - |F_L| - 2$ ($2^{n-1} - |F_R| - 2$, respectively) vertices. Next, we divide the case into three subcases: (2.1) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q_{n-1}^R \setminus F_R$ is disconnected, (2.2) $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q_{n-1}^R \setminus F_R$ is connected, and (2.3) $|V(C_L)| \geqslant 2^{n-1} - |F_L| - 1$ and $|V(C_R)| \geqslant 2^{n-1} - |F_R| - 1$.

Case 2.1: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q_{n-1}^R \setminus F_R$ **is disconnected.**

This is an impossible case. Since $\kappa(Q_{n-1}) = n-1$, $|F_R| \geqslant n-1$. By Lemma 4, $|F_L| \geqslant 2((n-1)-1)$. Then the total number of faulty vertices is at least $(n-1) + 2((n-1)-1) = 3n-5$ which is greater than $3n-6$, a contradiction.

Case 2.2: $|V(C_L)| = 2^{n-1} - |F_L| - 2$ and $Q_{n-1}^R \setminus F_R$ **is connected.**

Since $Q_{n-1}^R \setminus F_R$ is connected, $|V(Q_{n-1}^R \setminus F_R)| = 2^{n-1} - |F_R|$. Since $|V(C_L)| \geqslant |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u,v) \in E(Q_n)$. Hence $Q_n \setminus F$ has a connected component that contains at least $[2^{n-1} - |F_R|] + [2^{n-1} - |F_L| - 2] = 2^n - |F| - 2$ vertices.

Case 2.3: $|V(C_L)| \geqslant 2^{n-1} - |F_L| - 1$ and $|V(C_R)| \geqslant 2^{n-1} - |F_R| - 1$.

Since $|V(C_L)| \geqslant |F_R| + 1$, there exists a vertex $u \in C_L$ and a vertex $v \in C_R$ such that $(u,v) \in E(Q_n)$. Hence $Q_n \setminus F$ has a connected component that contains at least $[2^{n-1} - |F_L| - 1] + [2^{n-1} - |F_R| - 1] = 2^n - |F| - 2$ vertices.

This completes the proof of the lemma. □

By Lemma 8, we have the following corollary.

Corollary 2. *Let Q_n be an n-dimensional hypercube, $n \geqslant 5$, and let F be a set of vertices $F \subset V(Q_n)$ with $|F| \leqslant 3n-6$. Then $Q_n \setminus F$ satisfies one of the following conditions:*

1. *$Q_n \setminus F$ is connected.*
2. *$Q_n \setminus F$ has two components, one of which is K_1, and the other one has $2^n - |F| - 1$ vertices.*
3. *$Q_n \setminus F$ has two components, one of which is K_2, and the other one has $2^n - |F| - 2$ vertices.*
4. *$Q_n \setminus F$ has three components, two of which are K_1, and the third one has $2^n - |F| - 2$ vertices*

Let $G = (V,E)$ be a graph. A subset M of $E(G)$ is called a matching in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be matched under M. A vertex cover of G is a subset $\mathscr{K}$ of $V(G)$ such that every edge of G has at least one end in $\mathscr{K}$. A subset I of $V(G)$ is called an independent set of G if no two vertices of I are adjacent in G. To prove the conditional diagnosability of the hypercube, we need the following classical results.

Theorem 3 [21]. *Let $G = (V,E)$ be a bipartite graph. The maximum size of a matching in G equals the minimum size of a vertex cover of G.*

Proposition 1 [21]. *Let $G = (V,E)$ be a bipartite graph. The set $I \subset V(G)$ is a maximum independent set of G if and only if $V \setminus I$ is a minimum vertex cover of G.*

The hypercube can be described as follows: Let Q_n denote an n-dimensional hypercube. Q_1 is a complete graph with two vertices labeled with 0 and 1, respectively. For $n \geqslant 2$, each Q_n consists of two Q_{n-1}'s, denoted by Q^0_{n-1} and Q^1_{n-1}, with a perfect matching M between them. That is, M is a set of edges connecting the vertices of Q^0_{n-1} and the vertices of Q^1_{n-1} in a one-to-one manner. It is easy to see that there are 2^{n-1} edges between Q^0_{n-1} and Q^1_{n-1}. The hypercube is a bipartite graph with 2^n vertices. Hence, we have the following Lemma.

Lemma 9. *Let Q_n be an n-dimensional hypercube. In hypercube Q_n, the maximum size of a matching, the minimum size of a vertex cover and the maximum size of an independent set are all 2^{n-1}.*

We are now ready to show that the conditional diagnosability of Q_n is $3(n-2)+1$ for $n \geqslant 5$. Let $F_1,F_2 \subset V(Q_n)$ be two conditional faulty sets with $F_1 \leqslant 3(n-2)+1$ and $F_2 \leqslant 3(n-2)+1$, $n \geqslant 5$. We shall show our result by proving that (F_1,F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Lemma 10. *Let Q_n be an n-dimensional hypercube with $n \geqslant 5$. For any two conditional faulty sets $F_1,F_2 \subset V(Q_n)$, and $F_1 \neq F_2$, with $F_1 \leqslant 3(n-2)+1$ and $F_2 \leqslant 3(n-2)+1$. Then (F_1,F_2) is a distinguishable conditional-pair under the comparison diagnosis model.*

Proof. We use Theorem 2 to prove this result. Let $S = F_1 \bigcap F_2$, then $0 \leqslant |S| \leqslant 3(n-2)$. We will show that, deleting S from Q_n, the subgraph $C_{F_1\Delta F_2,S}$ containing $F_1\Delta F_2$ has "many" vertices having degree 3 or more. More precisely, we are going to prove that, in the subgraph $C_{F_1\Delta F_2,S}$ the number of vertices having degree 3 or more is at least $2[3(n-2)+1-|S|]+1 = 6n-2|S|-9$. In the following proof, we consider three cases by the size of S: (1) $0 \leqslant |S| \leqslant n-1$, (2) $|S| = n$, and (3) $n+1 \leqslant |S| \leqslant 3(n-2)$.

Case 1: $0 \leqslant |S| \leqslant n-1$.

Since the connectivity of Q_n is n, $Q_n \setminus S$ is connected, the subgraph $C_{F_1\Delta F_2,S}$ is the only component in $Q_n \setminus S$. Since the hypercube Q_n has no cycle of length three and any two vertices have at most two common neighbors, it is straightforward, though tedious, to check that the number of vertices which has degree 2 or 1 is at most 2 in $C_{F_1\Delta F_2,S}$. Hence, the number of vertices having degree 3 or more is at least $2^n - |S| - 2$ which is greater than $6n-2|S|-9$, for $n \geqslant 5$. By Theorem 2, (F_1,F_2) is a distinguishable conditional-pair under the comparison diagnosis model.

Case 2: $|S| = n$.

If $Q_n \setminus S$ is disconnected, by Lemma 4, $Q_n \setminus S$ has one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, this is an impossible case. So $Q_n \setminus S$ is connected, and the subgraph $C_{F_1\Delta F_2,S}$ is the only component in $Q_n \setminus S$. Let $U = Q_n \setminus (F_1 \bigcup F_2)$. If there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then the condition 1 of Theorem 1 holds and therefore (F_1,F_2) is a distinguishable conditional-pair; otherwise $V(U)$ is an independent set. Since $|S| = n$ and $|F_1\Delta F_2| \leqslant 2(2n-5)$, $|V(U)| \geqslant 2^n - 2(2n-5) - n = 2^n - 5n + 10$. By Lemma 9, the maximum size of a independent set is 2^{n-1} in Q_n. Comparing the lower bound $2^n - 5n + 10$ and the upper bound 2^{n-1}, we have $2^n - 5n + 10 > 2^{n-1}$ for $n \geqslant 5$, a contradiction.

Case 3: $n+1 \leqslant |S| \leqslant 3(n-2)$.

By Corollary 2, there are four cases in $Q_n \setminus S$ we need to consider. For case 1 of Corollary 2, $Q_n \setminus S$ is connected, the proof is exactly the same as that of Case 2, and hence the detail is omitted. For case 2 and 4 of Corollary 2, $Q_n \setminus S$ has at least one trivial component $\{v\}$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. Since F_1 and F_2 are two conditional faulty sets, the two cases are disregarded. Therefore, we only need to consider that $Q_n \setminus S$ has two components, one of which is K_2 and the other one has $2^n - |S| - 2$ vertices. Let (x,y) be the component with only one edge. Since $N(\{x,y\}) \subseteq S$ and F_1 and F_2 do not contain all the neighbors of any vertex, vertex x and y cannot belong to $F_1\Delta F_2$. So the subgraph $C_{F_1\Delta F_2,S}$ is the other large connected component of $Q_n \setminus S$. Let $U = Q_n \setminus (F_1 \bigcup F_2) \setminus \{x,y\}$. If no two vertices of $V(U)$ are adjacent, then $V(U)$ is an independent set and $|V(U)| \geqslant 2^n - 6n + |S| + 8$. By Lemma 9, the maximum size of a matching is $2^{n-1} - 1$ in $Q_n \setminus \{x,y\}$. By Theorem 3 and Proposition 1, the maximum size of a independent set is $2^{n-1} - 1$ in $Q_n \setminus \{x,y\}$. Comparing the lower bound $2^n - 6n + |S| + 8$ and the upper bound $2^{n-1} - 1$, we have $2^n - 6n + |S| + 8 > 2^{n-1} - 1$ for $n \geqslant 5$, $n+1 \leqslant |S| \leqslant 3(n-2)$, a contradiction. Hence, there exist two vertices u and v in $V(U)$ such that u is adjacent to v, then condition 1 of Theorem 1 is satisfied and therefore (F_1,F_2) is a distinguishable conditional-pair.

In Case 1, we prove that at least one of the conditions of Theorem 1 is satisfied in subgraph $C_{F_1\Delta F_2,S}$. In Case 2 and 3, the condition 1 of Theorem 1 holds in subgraph $C_{F_1\Delta F_2,S}$. Therefore, (F_1,F_2) is a distinguishable conditional-pair under the comparison diagnosis model. □

We now present our main result which can be stated as follows.

Theorem 4. *The conditional diagnosability of Q_n is $t_c(Q_n) = 3(n-2)+1$ for $n \geqslant 5$, $t_c(Q_3) = 3$ and $t_c(Q_4) = 5$.*

Proof. By Lemma 3, $t_c(Q_n) \leqslant 3(n-2)+1$, and by Lemma 10, Q_n is conditionally $(3(n-2)+1)$-diagnosable for $n \geqslant 5$. Hence, $t_c(Q_n) = 3(n-2)+1$ for $n \geqslant 5$. For Q_3 and Q_4, we observe that Q_3 is not conditionally four-diagnosable and Q_4 is not conditionally six-diagnosable, as shown in Fig. 5. So, $t_c(Q_3) \leqslant 3$ and $t_c(Q_4) \leqslant 5$. Hence, the conditional diagnosabilities of Q_3 and Q_4 are both strictly less than $3(n-2)+1$.

For the three-dimensional hypercube Q_3, Q_3 is three-diagnosable and it is not conditionally four-diagnosable. It follows from Lemma 1 that $t_c(Q_3) = 3$. For the four-dimensional hypercube Q_4, we can use the similar technique used in proving Lemma 10 to prove that for any two conditional faulty sets $F_1,F_2 \subset V(Q_4)$, and $F_1 \neq F_2$, with $|F_1| \leqslant 5$ and $|F_2| \leqslant 5$, then (F_1,F_2) is a distinguishable conditional-pair under the comparison model. Hence, the conditional diagnosability of Q_4 is 5. □

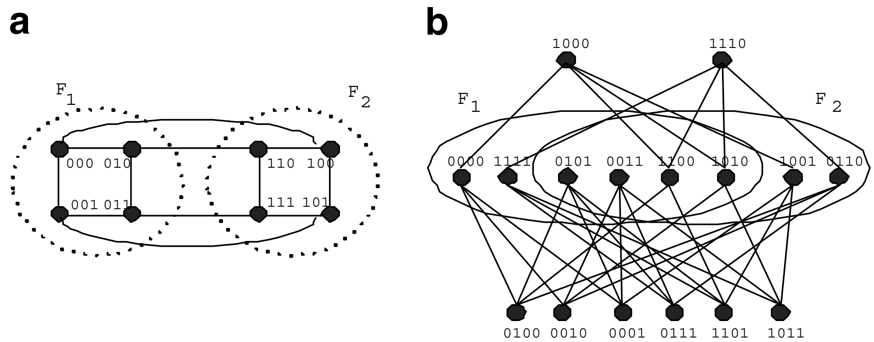

Fig. 5. Two indistinguishable conditional-pairs for Q_3 and Q_4.

4. Conclusions

In the real world, processors fail independently and with different probabilities. The probability that any faulty set contains all the neighbors of some processor is very small [5,16] so we are interested in the study of conditional diagnosability. A new diagnosis measure proposed by Lai et al. [12], it restricts that each processor of a system is incident with at least one fault-free processor. In this paper, we use the hypercube as an example and show that the conditional diagnosability of Q_n is $3(n-2)+1$ under the comparison model. This number $3(n-2)+1$ is about three times as large as the classical diagnosability.

In this paper, we study the conditional diagnosability of Q_n under the comparison model. Under the PMC model, however, the conditional diagnosability of Q_n is shown to be $4(n-2)+1$ by Lai et al. [12]. In order to understand why the increase in diagnosability under the comparison model is lower than that under the PMC model, we take a look at Fig. 4. As shown in Fig. 4, there are two conditional faulty sets F_1 and F_2 with $|F_1| = |F_2| = 3(n-2)+2$. As shown, F_1 and F_2 are indistinguishable, and therefore the conditional diagnosability of Q_n is no greater than $3(n-2)+2$ under the comparison model. We now consider the same conditional faulty sets under the PMC model in Fig. 4, either the edge (v_4, v_1) or the edge (v_4, v_3) provides "effective" test to distinguish the syndrome of F_1 and F_2 under the PMC model, namely v_4 tests v_1 or v_4 tests v_3. Therefore F_1 and F_2 are distinguishable. However, v_4 compares v_1 and v_3 is not an effective comparison to distinguish the syndrome of F_1 and F_2 under the comparison model. On the other hand, see Fig. 1, every effective comparison under the comparison model provides effective test under the PMC model. So the conditional diagnosability of Q_n under the comparison model is intuitively lower than that under the PMC model. In this paper, we give a complete proof to support our intuition and finally obtain the main result.

Several different fault diagnosis models have gained much attention in the study of fault diagnosis. It is worth to investigate the conditional diagnosability of a system under various models. It is also an attractive work to develop more different measures of diagnosability based on network topology and network reliability.

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