國立交通大學

資訊科學系

碩 士 論 文

超立方體中互相獨立線性配置之嵌入研究

Mutually Independent Linear Array Embeddedings in Hypercubes

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中 華 民 國 九 十 四 年 六 月

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國 立 交 通 大 學
資 訊 科 學 系
碩 士 論 文

A Thesis
Submitted to Institute of Computer and Information Science
College of Electrical Engineering and Computer Science
National Chiao Tung University
in partial Fulfillment of the Requirements
for the Degree of
Master
in

Computer and Information Science

June 2005

Hsinchu, Taiwan, Republic of China

中華民國九十四年六月

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摘　　要

在訊息傳遞中，在每個接收點是要避免碰撞的事件發生，因此訊息傳送路徑中互相獨立的特性是相當重要的。我們說兩條相同長度的路徑是獨立的，就代表著除了起始點與終點之外，其餘的時間點中，在同一個時間所經過的目標是不會相同的；在這篇論文中，我們探討研究了在 n 維超立方體中，任意的兩點中可以存在著（n−1）條任意長度之互相獨立的路徑，其長度由兩點間最短(漢明距離)到最長(漢米爾頓距離)都有。

Mutually Independent Linear Array Embeddings in Hypercubes

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Abstract

We say that two paths $P_0 = <u_0,u_1,...u_{k-1}>$ and $P_1 = <v_0,v_1,...,v_{k-1}>$ are independent if $u_0 = v_0$, $l(P_0) = l(P_1)$ and $P_0(i) \neq P_1(i)$ fro every $1 < i < k-1$. The set of paths $\{P_0,P_1,...,P_s\}$ of G are mutually independent if any two different paths in the set are independent. In this paper, we prove that there exist $(n-1)$ mutually independent paths of length l joining any vertices u and v such that $h(u,v)+2 \leq l \leq 2^n-1$ and $n \geq 4$.

誌　　　謝

首先最感謝的是我的指導教授譚建民老師，在這兩年中用心的教導，時時給予鼓勵以及共同討論的幫助，以及共同指導教授徐力行老師給予的教導，同時也高欣欣老師在口試時對這篇論文的指教。在這篇論文形成的階段，堅哥也是給予我可以順利研究下去不可或缺的人物，感謝你的經驗分享，讓我可以更加順利完成論文。

此外，感謝我的學長們，在我剛進來的第一年，很受到你們的照顧，在實驗室的第二年，晃哥和元翔學長繼續就讀博班，而小中中、Panda、史都以及老哲都畢業了，但是和你們時常的聯係讓我也獲益不少，尤其是小中中，在心思成長及人生規劃裡，最要感謝你。當然，寶蓮學姐是一定不可以忘卻的大人物，在妳身旁學習也讓我成長不少，還有妳和弘駿學長的情史，隱蔽得真的讓我們永生難忘。同時，感謝我的學弟們，聖凱、小廖、尚融、銘皇及宙斯，在你們身上，也學到不少做人的道理，也提醒了自己真的不可以怠惰下去，也希望你們順利畢業

最後，感謝我的父母以及所有家人，有你們在後面的支持，才能讓我在求學的旅程上心無旁騖；還有雅梅，是妳讓我踏上這條成功之路，沒有妳，就沒有現在的我。

在此獻給許多幫忙我的人、指導我的人最真誠的感謝，謝謝你們。

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圖　　　目　　　錄

Chapter 1

Introduction

An *interconnection* network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. A network connects the processors of the parallel computer. There are a lot of mutually conflicting requirements in designing the topology of computer networks. It is almost important to design an interconnection network that is the parallel system. A number of *mutually independent path* for specific multiprocessor architectures have been discussed.

The architecture of an *interconnection network* is usually represented as a *graph*. The nodes and edges in a graph correspond to processors and communication links in an interconnection network, respectively. In the design and implementation of local area networks, the ring topology has been used frequently for its good properties such as simplicity , extensibility, regularity and easiness of implementation. To study the *graph embedding problem*, which maps a guest graph into a host graph, is an important issue in evaluating a network. The problem is mapping each node of the guest graph into a node of the host graph, and mapping each edge of the guest graph into an edge of the host graph.

Many interconnection network topologies have been proposed in the literature for the purpose of connecting hundreds or thousands of processing elements. Among these topologies, the *binary n-cube* (abbreviated as *hypercube*) [2], denoted by Q_n is one of the most popular topologies. *Linear arrays* and *rings*, which are two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Some efficient algorithms designed on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in previous works [3, 1].

Chapter 2

Preliminaries

2.1 Notations and Definitions

For the graph theoretical definitions and notations, we follow [1], a *graph* $G = (V, E)$ consists of a finite set V and a subset E of $\{(u, v) \mid u \neq v,\ (u, v)$ is an unordered pair of elements of $V\}$. We call $V = V(G)$ the *vertex set* of G and $E = E(G)$ the *edge set* of G. A graph $G = (V_0 \bigcup V_1, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets V_0 and V_1, such that every edge joins V_0 with V_1. Two vertices ,u and v, have the same color if and only if u and v are in the same partite set. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m \rangle$, in which all the vertices $v_0, v_1, v_2, \ldots, v_m$ are distinct except that possible $v_0 = v_m$. We also write the path $\langle v_0, P, v_m \rangle$, where $P = \langle v_1, v_2, ..., v_{m-1} \rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *Hamming distance* $h(u, v)$ between u and v is the number of different bits in the corresponding strings of both vertices.

An n-dimensional hypercube can be modeled as a graph Q_n, with the vertex set $V(Q_n)$ and the edge set $E(Q_n)$. Each vertex u of Q_n can be distinctly labeled by binary n-bit strings, $u_{n-1}u_{n-2}...u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. If $u(i)$ is the neighbor vertex across dimension i of the vertex u, then the edge between them is said to be on dimension i.

We say that two paths $P_0 = \langle u_0, u_1, \ldots, u_{k-1}\rangle$ and $P_1 = \langle v_0, v_1, \ldots, v_{k-1}\rangle$ are *independent* if $u_0 = v_0$, $l(P_0) = l(P_1)$ and $P_0(i) \neq P_1(i)$ for every $1 < i < k-1$. The set of paths $\{P_0, P_1, \ldots, P_s\}$ of G are *mutually independent* if any two different paths in the set are independent. In this paper, we prove that there exist $(n-1)$ mutually independent paths of length l joining any vertices u and v such that $h(u,v)+2 \leq l \leq 2^n - 1$ and $n \geq 4$.

2.2 Basic properties of Q_n

This paper is aimed at embedding linear arrays and all possible length of paths into the hypercubes. We use induction to prove our main results. Lemmas 1 contribute to the induction basis for inductive proof of our main results.

Lemma 1 *[4] The hypercube Q_n is bipanconnected if $n \geq 2$.*

Chapter 3

Mutually independent linear array embeddings

Lemma 2 *Assume $n \geqslant 3$. Let x be any node of Q_n and u and v be any two nodes that are different color as x of Q_n. Then, there exists a path of length l of $Q_n - \{x\}$ joining u to v for $h(u,v) \leq l \leq 2^n - 2$ and l is even.*

Proof. We proof this lemma by induction on n. It is easy to construct the path of length $h(u,v)$, and we claim to prove the length is $h(u,v)+2 \leq l \leq 2^k - 2$ and l is even. Since Q_3 is node transitive, we can assume that $x = 000$. All of the paths with $n = 3$ are listed below:

chose $u = 001$, $v = 010$ $h(u,v) = 2$
$(001, 101, 111, 011, 010)$
$(001, 101, 100, 110, 111, 011, 010)$
chose $u = 001$, $v = 111$ $h(u,v) = 2$
$(001, 101, 100, 110, 111)$
$(001, 101, 100, 110, 010, 011, 111)$

The lemma hold for $n = 3$ above list. As the inductive hypothesis, we assume that the lemma is true for every integer $n < k$, for all $k \geq 3$. Let $x = x_{k-1}x_{k-2}...x_1x_0$, $u = u_{k-1}u_{k-2}...u_1u_0$ and $v = v_{k-1}v_{k-2}...v_1v_0$. Either $x_i = u_i$ or $x_i = u_i$ will satisfy for some i. Accordingly, Q_k can be decomposed into two subcube Q_{k-1}^0 and Q_{k-1}^1 by dimension i and either u or v is in the same subcube as x. Without loss of generality, we may assume that u is in the same subcube as x and $\{u, x\} \in V(Q^0_{k-1})$. The proof of this lemma is classified in three cases.

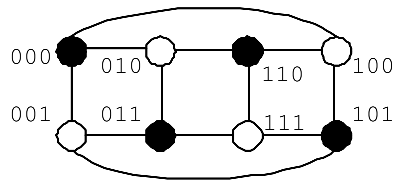


Figure 3.1: The Hypercube Q_3.

Case 1. $v \in V(Q^0_{k-1})$. (see Fig. 3.2(a)).

By induction hypothesis, there exists a path of length l of Q^0_{k-1} joining u and v for any $h(u,v)+2 \leq l \leq 2^{k-1}-2$ and l is even. Suppose that $2^{k-1} \leq l \leq 2^k - 2$ and l is even. Let R be one of the longest path of $Q^0_{k-1} - \{x\}$ joining u and v. Let (w, z) be any edge on R. We can write R as $\langle u, R_0, w, z, R_1, v\rangle$. By definition, $w^{(1)}$ and $z^{(1)}$ are vertices in Q^1_{k-1}. By Lemma 1, there exists a path P in Q^1_{k-1} joining $w^{(1)}$ and $z^{(1)}$ for $1 \leq l(P) \leq 2^{k-1}-1$ and $l(P)$ is odd. Thus, $\langle u, R_0, w, w^{(1)}, P, z^{(1)}, z, R_1, v\rangle$ is a path of length l in Q_n connecting u and v.

Case 2. $v \in V(Q^1_{k-1})$. (see Fig. 3.2(b)).

Let $y^{(1)}$ be one neighbor of v such that $y \neq u$. Thus, $h(u, y) = h(u, v)$. Suppose that $h(u,v)+2 \leq l \leq 2^k - 2$ for l is even. By induction hypothesis, there exist a path R joining u and y for any $h(u, y) \leq l(R) \leq 2^{k-1}-2$ and $l(R)$ is even. Let $l_1 = l - l(R) - 1$. Then l_1 is odd and $1 \leq l_1 \leq 2^{k-1}$. By Lemma 1, there exists a path P of length l_1 in Q^1_{k-1} joining $y^{(1)}$ and v. Thus, $\langle u, R, y, y^{(1)}, P, v\rangle$ is a path of length l in Q_n joining u and v.

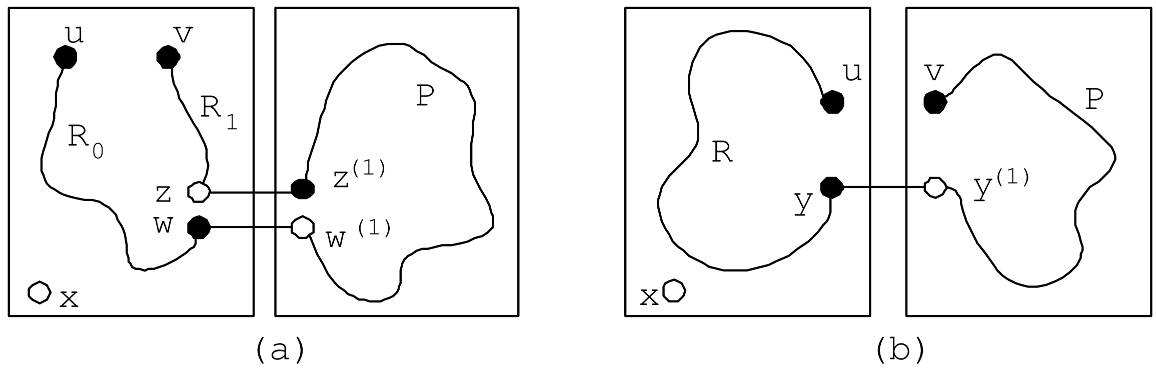


Figure 3.2: Illustration for the Lemma 2.

□

Lemma 3 *Let x and y be any two nodes from different partite set of Q_4, and let u and v be any two vertices from different partire set of $Q_4 - \{x, y\}$. Then, there exists a path P of $Q_n - \{x, y\}$ joining u and v such that $h(u, v) \leq l(P) \leq 13$ and $l(P)$ is odd.*

Proof. Since Q_4 is node transitive, we can assume that $x = 0000$. Moreover, we suppose that $y = 0001$ or 0111 such that the distance between x and y is either 1 or 3. Q_4 can be decomposed into two subcubes Q_3^0 and Q_3^1 by dimension 0 or 3 such that x and y are in the same subcase. Without loss of generality, we may assume that $x, y \in V(Q_3^0)$. The proof of this lemma is classified in two cases.

Case I. y=0001.

There exists a hamiltonian cycle $C = \langle 0100, 0101, 0111, 0011, 0010, 0110, 0100 \rangle$ of $Q_3^0 - \{x, y\}$. We can write the cycle C as $\langle a_0, a_1, a_2, a_3, a_4, a_5, a_0 \rangle$. In the other hand, there exist a path of length 5 joining a_i and a_j of $Q_3^0 - \{x, y\}$ if (a_i, a_j) is lying on C for $i \neq j$. The proof of this situation is classified in three cases.

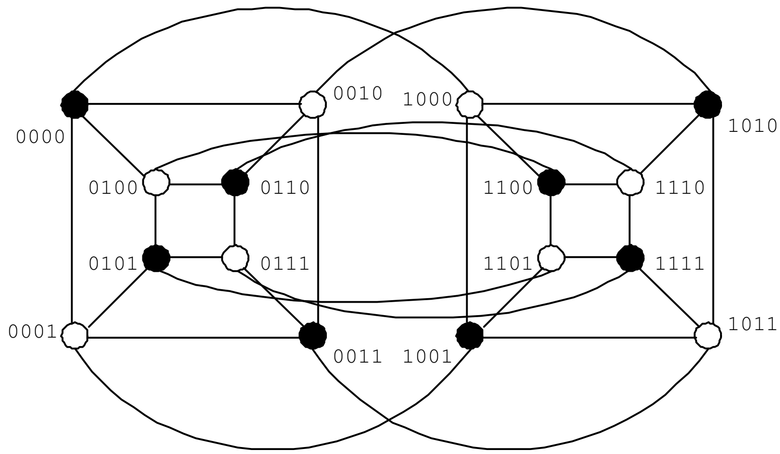


Figure 3.3: The Hypercube Q_4.

Subcase I.1. $u, v \in V(Q^0_{k-1})$.

(a) $h(u, v) = 1$. Suppose that (u, v) is lying on C. By definition, $u^{(1)}$ and $v^{(1)}$ are vertices in Q^1_3 and $u^{(1)}$ is adjacent to $v^{(1)}$. Suppose that $l = 3$. We can construct P as $\langle u, u^{(1)}, v^{(1)}, v\rangle$. By above discuss, there exists one path of length 5 joining any edge that on the C. Suppose that $7 \leq l \leq 13$ and l is odd. There exist one path R of length 5 joining u and v. Let (w, z) be any edge on R and we can write R as $\langle u, R_0, w, z, R_1, v\rangle$. By definition, $(w^{(1)}, z^{(1)})$ is in Q^1_3. By Lemma 1, there exist a path S of length l_1 joining $w^{(1)}$ and $z^{(1)}$ of Q^1_3 for any $1 \leq l_1 \leq 7$ and l_1 is odd. Thus, $\langle u, R_0, w, w^{(1)}, S, z^{(1)}, z, R_1, v\rangle$ is one path of length l joining u and v.

Suppose that (u, v) is not lying on C. In this situation, we only discuss one case about $(u, v) = (0110, 0111)$. We can find a path of length 3 as $\langle a_0, a_1, a_2, a_3\rangle$ like $\langle 0110, 0100, 0101, 0111\rangle$. By definition, $a^{(1)}_1$ and $a^{(1)}_2$ are in the Q^1_3. By Lemma 1, there exist a path S of length l_1 joining $a_1^{(1)}$ and $a_2^{(1)}$ of Q_3^1 for any $1 \leq l_1 \leq 7$ and l_1 is odd. Thus, $u, a_1, a_1^{(1)}, S, a_2^{(1)}, a_2, v$ is a path of length l joining u and v for $5 \leq l \leq 11$ and l is odd. Assume that $l = 13$. $P = \langle 0110, 0100, 0101, 1101, 1100, 1000, 1001, 1011, 1111, 1110, 1010, 0010, 0011, 0111\rangle$ is the path of length l joining u and v.

(b) $h(u, v) = 3$. In this situation, we only discuss one case about $(u, v) = (0100, 0101)$. There exists a path R of length 5 joining u and v of $Q_3^0 - \{x, y\}$ as $R = \langle 0100, 0101, 0111, 0110, 0010, 0011\rangle$. Let w be $R(4)$ on R. We can write the path R as $\langle u, R_0, w, v\rangle$. By definition, $w^{(1)}, v^{(1)}$ are both in Q_3^1. By Lemma 1, there exist a path S of length l_1 joining $w^{(1)}$ and $v^{(1)}$ of Q_3^1 for any $1 \leq l_1 \leq 7$ and l_1 is odd. Thus, $\langle u, R_0, w, w^{(1)}, S, v^{(1)}, v\rangle$ is the path of length l joining u and v for any $7 \leq l \leq 13$.

Subcase I.2. $u \in V(Q_{k-1}^0)$ and $v \in V(Q_{k-1}^1)$ or $u \in V(Q_{k-1}^1)$ and $v \in V(Q_{k-1}^0)$.

Without loss of generality, we assume that $u \in V(Q_{k-1}^0)$ and $v \in V(Q_{k-1}^1)$.

(a) $h(u, v) = 1$. Let w be the neighbor of u for $w \in V(Q_3^0)$ and $w \neq \{x, y\}$ and (u, w) is lying on C. In addition, let z be the neighbor of v for $z \in V(Q_3^0)$ and z is adjacent to w. By Lemma 1, there exist a path S of length l_1 joining z and v of Q_3^1 for any $1 \leq l_1 \leq 7$ and l_1 is odd. Therefor, $\langle u, w, z, S, v\rangle$ is the path of length l joining u and v for any $3 \leq l \leq 9$ and l is odd. Suppose that $11 \leq l \leq 13$ and l is odd. By above discuss, there exists a path R of length 5 joining u and w of $Q_3^0 - \{x, y\}$. Thus, $\langle u, R, w, z, S, v\rangle$ is the path of length l joining u and v.

(b) $h(u, v) = 3$. The same case (a). Let w be the neighbor of u for $w \in V(Q_3^0)$ and $w \neq \{x, y\}$ and (u, w) is lying on C. In addition, let z be the neighbor of v for $z \in V(Q_3^0)$ and $z^{(0)} = w$. This proof is similar to that of above (a) and hence the detailed proof is omitted.

Subcase I.3. $u, v \in V(Q^1_{k-1})$. In this situation, we only discuss one case about $(u, v) = (1110, 1111)$. By Lemma 1, there exist a path S of length l_1 joining u and v of Q^1_3 for any $3 \leq l_1 \leq 7$ and l_1 is odd. Let w be the node $S(l_1 - 1)$ on S. The path S can be wrote as $\langle u, S_0, w, v\rangle$. By definition, $w^{(0)}$ is in Q^0_3. It is easy to check that $w^{(0)}$ is adjacent to $v^{(0)}$ and $(w^{(0)}, v^{(0)})$ is lying on C. By above discuss, there exist a path R of length 5 joining $w^{(0)}$ and $v^{(0)}$. Thus, $\langle u, S_0, w, w^{(0)}, R, v^{(0)}, v\rangle$ is the path of length l joining u and v.

Case II. y=0111.

There exists a hamiltonian cycle $C = \langle 0100, 0101, 0001, 0011, 0010, 0110, 0100\rangle$ of $Q^0_3 - \{x, y\}$. We can write the cycle C as $\langle a_0, a_1, a_2, a_3, a_4, a_5, a_0\rangle$. In the other hand, there exist a path of length 5 joining a_i and a_j of $Q^0_3 - \{x, y\}$ if (a_i, a_j) is lying on C for $i \neq j$. The proof of this case is similar to than Case I and hence the detailed proof is omitted.

□

Lemma 4 *Assume* $n = 3, 4$. *Let* $\{e_i \mid e_i = (w_i, b_i) \in E(Q_n), b_i$ *is black node and* w_i *is white node,* $1 \leq i \leq n-1\}$ *be any n-1 disjoint edges in* Q_n. *Then, there exist n-1 independent paths* $P_1, ..., P_{n-1}$ *of length* l *in* Q_n *joining* w_i *and* b_i *for* $1 \leq l \leq 2^n - 1$.

Proof. It is easy to construct the path of length 1, and the path of length $3 \leq l \leq 7$ such that l is even are listed below:

chose (000, 001) and (101, 100) $l = 3$ (000, 010, 011, 001) (101, 111, 110, 100) $l = 5$ (000, 010, 110, 111, 011, 001) (101, 111, 011, 010, 110, 100) $l = 7$ (000, 010, 110, 100, 101, 111, 011, 011) (101, 111, 011, 001, 000, 010, 110, 100)
chose (000, 001) and (110, 111) $l = 3$ (000, 010, 011, 001) (110, 010, 101, 111) $l = 5$ (000, 010, 110, 111, 011, 001) (110, 100, 000, 001, 101, 111) $l = 7$ (000, 010, 110, 100, 101, 111, 011, 001) (110, 100, 000, 010, 011, 001, 101, 111)
chose (000, 001) and (110, 100) $l = 3$ (000, 010, 011, 001) (110, 111, 101, 100) $l = 5$ (000, 010, 110, 111, 011, 001) (110, 111, 011, 001, 101, 100) $l = 7$ (000, 010, 110, 100, 101, 111, 011, 001) (110, 111, 101, 001, 011, 010, 000, 100)
chose (000, 001) and (101, 111) $l = 3$ (000, 010, 011, 001) (101, 100, 110, 111) $l = 5$ (000, 010, 110, 111, 011, 001) (101, 100, 000, 010, 110, 111) $l = 7$ (000, 010, 110, 100, 101, 111, 011, 001) (101, 100, 000, 010, 011, 001, 101, 111)

By above list, the lemma holds for $n = 3$.

Suppose that $n = 4$. There are 4 dimensions in Q_4, so Q_4 can be decomposed into Q_3^0 and Q_3^1 two subcubes by dimension j such that e_i are not cross edges for all $1 \le i \le 3$. Then, the number of the black nodes is equal to the white nodes in Q_3^i, $i = 0, 1$. Therefor, the proof is divided into two major cases.

Case 1. Not all of the disjoint edges are in the same subcube.

Without loss of generality, we may assume that e_1,e_2 $\in$ $E(Q_3^0)$ and e_3 $\in$ $E(Q_3^1)$. Assume that $i = 1, 2$. Suppose that $1 \le l \le 7$ and l is odd. By above discuss, there exist 2 mutually independent path of length l joining w_i and b_i. By Lemma 1, there exists one path of length l joining w_3 and b_3 of Q_3^1. Suppose that $9 \le l \le 13$ and l is odd. Let R_i be the longest paths of Q_3^0 joining w_i and b_i and R_3 be the longest path of Q_3^1

joining w_3 and b_3. Obviously, $l(R_1) = l(R_2) = l(R_3) = 7$. In addition, let x_j be the node $R_j(6)$ and we can write R_j as $\langle w_j, R_j^0, x_j, b_j \rangle$ for all $1 \leq j \leq 3$. By definition, $x_i^{(1)}$ and $b_i^{(1)}$ are the vertices in Q_3^1. By above discuss, there exist 2 mutually independent path S_i of length l_1 joining $w_i^{(1)}$ and $b_i^{(1)}$ for any $1 \leq l_1 \leq 7$. By definition, $x_3^{(0)}$ and $b_3^{(0)}$ are the vertices in Q_3^0. By Lemma 1, there exist one path S_3 of length l_1 joining $w_3^{(0)}$ and $b_3^{(0)}$ for any $1 \leq l_1 \leq 7$. Thus, $\langle w_i, R_i^0, x_i, x_i^{(1)}, S_i, b_i^{(1)}, b_i \rangle$ and $\langle w_3, R_3^0, x_3, x_3^{(0)}, S_3, b_3^{(0)}, b_3 \rangle$ are 3 mutually independent path of length l joining w_j and b_j for any $1 \leq l \leq 13$ and $1 \leq j \leq 3$.

Case 2. All of the disjoint edges are in the same subcube.

Without loss of generality, we may assume that e_1,e_2,$e_3 \in E(Q_3^0)$. In this situation, we only discuss one case about any two edges are lying on different dimensions. Since Q_4 is vertex transitive, we can assume that $e_1 = (000, 001)$, $e_2 = (100, 110)$ and $e_3 = (111, 011)$. Suppose that $3 \leq l \leq 7$ and l is odd. With above discuss, there exist 2 mutually independent path of length l joining w_i and b_i. In addition, , $w_3^{(1)}$ and $b_3^{(1)}$ are the vertices in Q_3^1. By Lemma 1, there exists one path R of length $l-2$ joining $w_3^{(1)}$ and $b_3^{(1)}$ of Q_3^1. Thus, $\langle w_3, w_3^{(1)}, R, b_3^{(1)}, b_3 \rangle$ is the path of length l joining w_3 and b_3. Suppose that $9 \leq l \leq 15$. The paths are listed below:

$l = 9$
(0000, 0010, 0110, 0100, 0101, 0111, 1111, 1011, 0011, 0011)
(0101, 0111, 0011, 0001, 0000, 0010, 1010, 1110, 0110, 0100)
(0011, 1011, 1010, 1110, 1100, 1000, 1001, 1101, 1111, 0111)
$l = 11$
(0000, 0010, 0110, 0100, 0101, 0111, 1111, 1011, 1001, 1011, 0011, 0011)
(0101, 0111, 0011, 0001, 0000, 0010, 1010, 1000, 1100, 1110, 0110, 0100)
(0011, 1011, 1010, 1110, 1100, 1000, 1001, 0001, 0101, 1101, 1111, 0111)
$l = 13$
(0000, 0010, 0110, 0100, 0101, 0111, 1111, 1110, 1100, 1011, 1001, 1011, 0011, 0011)
(0101, 0111, 0011, 0001, 0000, 0010, 1010, 1000, 1001, 1101, 1100, 1110, 0110, 0100)
(0011, 1011, 1010, 1110, 1100, 1000, 1001, 0001, 0101, 0000, 0100, 1101, 1111, 0111)
$l = 15$
(0000, 0010, 0110, 0100, 0101, 0111, 1111, 1110, 1010, 1000, 1100, 1011, 1001, 1011, 0011, 0011)
(0101, 0111, 0011, 0001, 0000, 0010, 1010, 1000, 1001, 1011, 1111, 1101, 1100, 1110, 0110, 0100)
(0011, 1011, 1010, 1110, 1100, 1000, 1001, 0001, 0101, 0000, 0010, 0110, 0100, 1101, 1111, 0111)

□

Lemma 5 *Assume $n \geq 4$. Let x and y be any two nodes from different partite set of Q_n, and let u and v be any two vertices from different partite set of $Q_n - \{x, y\}$. Then, there exists a path P joining u and v of $Q_n - \{x, y\}$ for $h(u, v) \leq l(P) \leq 2^n - 3$ and $l(P)$ is odd.*

Proof. We prove this lemma by induction on n. By Lemma 3, we observe that the lemma holds for $n = 4$. For $k \geq 4$, we assume that the lemma is true for every integer $n < k$. Let $x = x_{k-1}x_{k-2}...x_1x_0$ and $y = y_{k-1}y_{k-2}...y_1y_0$. Hence $x_i = y_i$ for some i. Accordingly, Q_k can be decomposed into two subcube Q^0_{k-1} and Q^1_{k-1} by dimension i and x and y are in the same subcube. Without loss of generality, we may assume that $x, y \in V(Q^0_{k-1})$. Therefor, the proof is divided into three major cases.

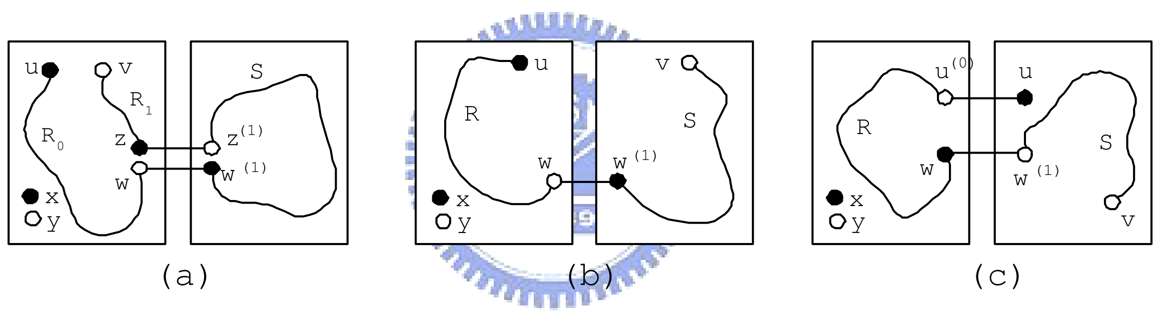


Figure 3.4: Illustration for the Lemma 5.

Case 1. $u, v \in V(Q^0_{k-1})$. By inductive hypothesis, there exists a path of length l_0 connecting u and v of $Q_k - \{x, y\}$ for any $h(u, v) \leq l_0 \leq 2^{k-1} - 3$ such that l_0 is odd. Suppose that $2^{k-1} - 1 \leq l \leq 2^k - 3$ with l is odd. Let R be one of the longest path of Q^0_{k-1} joining u and v. Since $l(R) = 2^{k-1} - 3 \geq 5$ if $k \geq 4$, there exists an edge (w, z) in R. We can write the path R as $\langle u, R_0, w, z, R_1, v\rangle$. In subcube Q^1_{k-1}, let $w^{(1)}$ and $z^{(1)}$ be the neighbors of w and z. By Lemma 1, we can find a path S joining $w^{(1)}$ and $z^{(1)}$ for

$1 \leq l(S) \leq 2^{k-1} - 1$ with $l(S)$ is odd. Therefor, $P = \langle u, R_0, w, w^{(1)}, S, z^{(1)}, z, R_1, v \rangle$ is a path of length l joining u and v of $Q_k - \{x, y\}$.

Case 2. $u \in V(Q^0_{k-1})$ and $v \in V(Q^1_{k-1})$ or $u \in V(Q^1_{k-1})$ and $v \in V(Q^0_{k-1})$. Without loss of generality, we may assume that $u \in V(Q^0_{k-1})$ and $v \in V(Q^1_{k-1})$. Let $w^{(1)}$ be one neighbor of v and $w^{(1)} \in V(Q^1_{k-1})$. By definition, w is the neighbor of $w^{(1)}$ and $w \in V(Q^0_{k-1})$. Obviously, $h(u, w) = h(w^{(1)}, v) = 1$. By inductive hypothesis, there exists a path R of length l_0 connecting u and w of $Q_k - \{x, y\}$ for $1 \leq l_0 \leq 2^{k-1} - 3$ and l_0 is odd. Let $l_1 = l - l_0 - 1$. By Lemma 1, there exists a path S of length l_1 joining $w^{(1)}$ and v. Thus, $P = \langle u, R, w, w^{(1)}, S, v \rangle$ is a path joining u and v of $Q_k - \{x, y\}$ for $h(u, v) \leq l(P) \leq 2^k - 3$.

Case 3. $u, v \in V(Q^1_{k-1})$. In this subcase discussion, we assume that at most one vertex in $\{u, v\}$ is adjacent to $\{x, y\}$. Otherwise, Q_k can be decomposed into another two subcubes by another dimension j for x and y in the same subcube and the proof is the same as Case 1. Without loss of generality, we may assume that u is not adjacent to $\{x, y\}$. By Lemma 1, there exists a path of length l joining u and v of Q^1_{k-1} for any $h(u, v) \leq l \leq 2^{k-1} - 1$ such that l is odd. Suppose that $2^{k-1} + 1 \leq l \leq 2^k - 3$ and l is odd. By definition, $u^{(0)}$ is vertex in Q^0_{k-1}. Let w be any vertex that are different color as $u^{(0)}$ of Q^0_{k-1} and $w \neq \{x, y\}$. Let $w^{(1)}$ be the neighbor of w and $w^{(1)} \in V(Q^1_{k-1})$. By Lemma 2, there exist a path S joining $w^{(1)}$ and v of $Q^1_{k-1} - \{u\}$ for any $h(w^{(1)}, v) \leq l(S) \leq 2^{k-1} - 2$ and $l(S)$ is even. Let $l_0 = l - l(S) - 2$. Then l_0 is odd and $1 \leq l_0 \leq 2^{k-1} - 3$. By induction hypothesis, there exists a path R of length l_0 joining $u^{(0)}$ and w. Thus, $P = \langle u, u^{(0)}, R, w, w^{(1)}, S, v \rangle$ is a path joining u and v of $Q_k - \{x, y\}$ for $h(u, v) \leq l(P) \leq 2^k - 3$.

□

Lemma 6 *Assume $n \geq 3$. Let $\{e_i \mid e_i = (w_i, b_i) \in E(Q_n), b_i$ is black node and w_i is white node, $1 \leq i \leq n-1\}$ be any n-1 disjoint edges in Q_n. Then, there exist n-1 independent paths $P_1, ..., P_{n-1}$ of length l in Q_n joining w_i and b_i for $1 \leq l \leq 2^n - 1$.*

Proof. We prove this lemma by induction on n. By Lemma 4, we observe that the lemma holds for $n = 3, 4$. As the inductive hypothesis, we assume that the lemma is true for $3 \leq k < n$. There are k dimensions in Q_k, so Q_k can be decomposed into Q^0_{k-1} and Q^1_{k-1} two subcubes by dimension j such that e_i are not cross edges for all $1 \leq i \leq k-1$. Then the number of the black nodes is equal to the white nodes in Q^i_{k-1}, $i = 1, 2$. The proof is divided into two major cases.

Case 1. Not all of the disjoint edges are in the same subcube.

Without loss of generality, we may assume that $e_i \in E(Q^0_{k-1})$ and $e_j \in E(Q^1_{k-1})$ for $1 \leq i \leq j \leq k-1$ and $|e_i| + |e_j| = k-1$. Suppose that $1 \leq l \leq 2^{k-1} - 1$ and l is odd. By above discuss, there exist i mutually independent path of length l of Q^0_{k-1} joining w_i and b_i and j mutually independent path of length l of Q^1_{k-1}. joining w_j and b_j. Suppose that $2^{k-1} + 1 \leq l \leq 2^k - 1$ and l is odd. Let R_i be the longest paths of Q^0_{k-1} joining w_i and b_i and R_j be the longest path of Q^1_{k-1} joining w_j and b_j. Obviously, $l(R_i) = l(R_j) = 2^{k-1} - 1$. In addition, let x_i be the node $R_i(2^{k-1} - 2)$ and we can write R_i as $\langle w_i, R^0_i, x_i, b_i \rangle$. By definition, $x^{(1)}_i$ and $b^{(1)}_i$ are the vertices in Q^1_{k-1}. By above discuss, there exist i mutually independent path S_i of length l_1 joining $w^{(1)}_i$ and $b^{(1)}_i$ for any $1 \leq l_1 \leq 2^{k-1} - 1$. The same as above proof. Let x_j be the node $R_j(2^{k-1} - 2)$ on R_j and we can write R_j as $\langle w_j, R^0_j, x_j, b_j \rangle$. By definition, $x^{(0)}_j$ and $b^{(0)}_j$ are the vertices in Q^0_{k-1}. By above discuss, there exist j mutually independent path S_j of length l_1 joining $w_j^{(0)}$ and $b_j^{(0)}$ for any $1 \leq l_1 \leq 2^{k-1} - 1$. Thus, $\langle w_i, R_i^0, x_i, x_i^{(1)}, S_i, b_i^{(1)}, b_i \rangle$ and $\langle w_j, R_j^0, x_j, x_j^{(0)}, S_j, b_j^{(0)}, b_j \rangle$ are $k-1$ mutually independent path of length l for any $1 \leq l \leq 2^k - 1$.

Case 2. All of the disjoint edges are in the same subcube.

Without loss of generality, we may assume that all edges are in Q_{k-1}^0. For convenience, we assume that $1 \leq i \leq k-2$. It is trivial to construct the path of length 1 connecting w_i and b_i. Suppose that $3 \leq l \leq 2^{k-1} - 1$. By induction hypothesis, there exist $k-2$ paths P_i of length l joining w_i and b_i in Q_{k-1}^0. By definition, $w_{k-1}^{(1)}$ and $b_{k-1}^{(1)}$ are the neighbors of w_{k-1} and b_{k-1} for $w_{k-1}^{(1)} \in Q_{k-1}^1$ and $b_{k-1}^{(1)} \in Q_{k-1}^1$. By Lemma 1, we can find a path R with length $l-2$ joining $w_{k-1}^{(1)}$ and $b_{k-1}^{(1)}$. Thus, $\langle w_{k-1}, w_{k-1}^{(1)}, R, b_{k-1}^{(1)}, v_{k-1} \rangle$ is the path P_{k-1} of length l in Q_k joining w_{k-1} and b_{k-1}.

Suppose that $2^{k-1} + 1 \leq l \leq 2^k - 1$. Assume that $2^{k-1} - 3 \leq l_0 \leq 2^{k-1} - 1$. With above discussion, let R_i be $k-2$ mutually independent paths with length l_0 joining w_i and b_i. Let x_i and y_i be the nodes $R_i(l_0 - 2)$ and $R_i(l_0 - 1)$ on R_i. We can write R_i as $\langle w_i, R_i^0, x_i, y_i, b_i \rangle$. Let $x_i^{(1)}$ and $y_i^{(1)}$ be the neighbors of x_i and y_i in Q_{n-1}^1. By Lemma 6, there exist $k-2$ path S_i joining $x_i^{(1)}$ and $y_i^{(1)}$ in Q_{k-1}^1 for $3 \leq l(S_i) \leq 2^{k-1} - 1$. Thus, $P_i = \langle w_i, R_i^0, x_i, x_i^{(1)}, S_i, y_i^{(1)}, y_i, b_i \rangle$ are $k-2$ paths with length l in Q_k joining w_i and b_i. By definition, $w_{k-1}^{(1)}$ and $b_{k-1}^{(1)}$ are the neighbors of w_{k-1} and b_{k-1} for $w_{k-1}^{(1)} \in Q_{k-1}^1$ and $b_{k-1}^{(1)} \in Q_{k-1}^1$. Let z be one neighbor of b_{k-1} and $z \neq y_i$. Otherwise, b_{k-1} is adjacent to y_i and we can construct P_j as $\langle w_i, R_i^0, x_i, x_i^{(1)}, S_i, y_i^{(1)}, y_i, b_{k-1} \rangle$ such that one neighbor of b_j is not equal all y_i for $j \in [1, k-2]$. In addition, let a be any node that are different color as w_{k-1} and $a \neq x_i$. By definition, $z^{(1)}$ and $a^{(1)}$ are the vertices in $V(Q_{k-1}^1)$ and $z^{(1)} \neq y_i^{(1)}$

and $a^{(1)} \neq x_i^{(1)}$. By Lemma 5, there exist a path R_{k-1} with length $l_0 - 2$ connecting $w_{k-1}^{(1)}$ and $a^{(1)}$ of $Q_{k-1}^1 - \{z^{(1)}, b_{k-1}^{(1)}\}$. By Lemma 3, there exist a path S_{t-1} with length $|S_i| - 2$ joining a and z. Thus, $P_{k-1} = \langle w_{k-1}, w_{k-1}^{(1)}, R_{k-1}, a^{(1)}, a, S_{k-1}, z, z^{(1)}, b_{k-1}^{(1)}, b_{k-1}\rangle$ is the $k-1$ mutually independent path with length l joining w_{k-1} and b_{k-1}.

□

Lemma 7 *Assume that $n \geq 3$. Let v be any vertex of Q_n. There exist n-1 independent path $P_1, ..., P_{n-1}$ of length l in Q_n from v to v_i such that v_i is the neighbor of v for $1 \leq i \leq n-1$ and $1 \leq l \leq 2^n - 1$.*

Proof. We prove this lemma by induction on n. Since Q_3 is node transitive, we can assume that $v = 000$ and $v_1 = 001$ and $v_2 = 010$. The required path of $n = 3$ are listed below:

chose $v = 000$, $v_1 = 001$ $v_2 = 010$
$l = 1$
$(000, 001)$
$(000, 010)$
$l = 3$
$(000, 100, 101, 001)$
$(000, 001, 011, 010)$
$l = 5$
$(000, 100, 110, 111, 101, 001)$
$(000, 001, 101, 100, 110, 010)$
$l = 7$
$(000, 100, 110, 010, 011, 111, 101, 001)$
$(000, 001, 011, 111, 101, 100, 110, 010)$

The lemma hold for $n = 3$ above list. As the inductive hypothesis, we assume that the lemma is true for $3 \leq k < n$.

Without loss of generality, we may assume the subcube is Q_{t-1}^0. The proof of this subcase is classified in three parts.

For convenience, we assume that $1 \leq i \leq k-2$. It is trivial to construct the path of length 1 connecting v and v_i. Suppose that $3 \leq l \leq 2^{k-1} - 1$. By induction hypothesis,

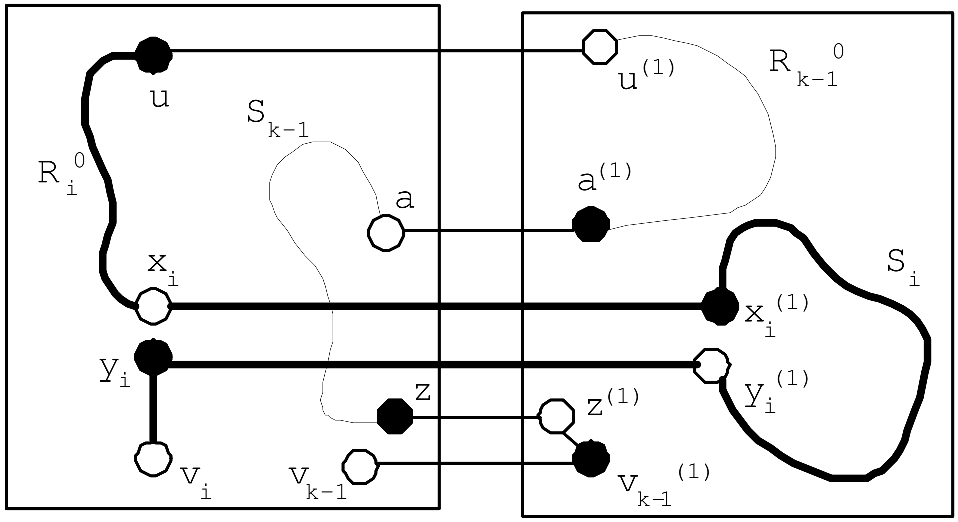


Figure 3.5: Illustration for the Lemma 7.

there exist $k-2$ paths P_i of length l joining v and v_i in Q^0_{k-1}. By definition, $v^{(1)}$ and $v^{(1)}_{k-1}$ are the neighbors of v and v_{k-1} for $v^{(1)} \in Q^1_{k-1}$ and $v^{(1)}_{k-1} \in Q^1_{k-1}$. By Lemma 1, we can find a path R with length $l-2$ joining $v^{(1)}$ and $v^{(1)}_{k-1}$. Thus, $\langle v, v^{(1)}, R, v^{(1)}_{k-1}, v_{k-1}\rangle$ is the path P_{k-1} of length l in Q_k joining v and v_{k-1}.

Suppose that $2^{k-1}+1 \le l \le 2^k - 1$. Assume that $2^{k-1}-3 \le l_0 \le 2^{k-1}-1$. With above discussion, let R_i be $k-2$ mutually independent paths with length l_0 joining v and v_i. Let x_i and y_i be the nodes $R_i(l_0-2)$ and $R_i(l_0-1)$ on R_i. We can write R_i as $\langle v, R^0_i, x_i, y_i, v_i\rangle$. Let $x^{(1)}_i$ and $y^{(1)}_i$ be the neighbors of x_i and y_i in Q^1_{n-1}. By Lemma 6, there exist $k-2$ path S_i joining $x^{(1)}_i$ and $y^{(1)}_i$ in Q^1_{k-1} for $3 \le l(S_i) \le 2^{k-1}-1$. Thus, $P_i = \langle v, R^0_i, x_i, x^{(1)}_i, S_i, y^{(1)}_i, y_i, v_i\rangle$ are $k-2$ paths with length l in Q_k joining v and v_i. By definition, $v^{(1)}$ and $v^{(1)}_{k-1}$ are the neighbors of v and v_{k-1} for $v^{(1)} \in Q^1_{k-1}$ and $v^{(1)}_{k-1} \in Q^1_{k-1}$. Let z be one neighbor of v_{k-1} and $z \ne y_i$. Otherwise, v_{k-1} is adjacent to y_i and we can construct P_j as $\langle v, R^0_i, x_i, x^{(1)}_i, S_i, y^{(1)}_i, y_i, v_{k-1}\rangle$ such that one neighbor of v_j is not equal all y_i for $j \in [1, k-2]$. In addition, let a be any node that are different color as v_{k-1} and $a \neq x_i$. By definition, $z^{(1)}$ and $a^{(1)}$ are the vertices in $V(Q^1_{k-1})$ and $z^{(1)} \neq y^{(1)}_i$ and $a^{(1)} \neq x^{(1)}_i$. By Lemma 5, there exist a path R_{k-1} with length $l_0 - 2$ connecting $u^{(1)}$ and $a^{(1)}$ of $Q^1_{k-1} - \{z^{(1)}, v^{(1)}_{k-1}\}$. By Lemma 3, there exist a path S_{k-1} with length $|S_i| - 2$ joining a and z. Thus, $P_{k-1} = \langle v, v^{(1)}, R_{k-1}, a^{(1)}, a, S_{k-1}, z, z^{(1)}, v^{(1)}_{k-1}, v_{k-1}\rangle$ is the $k-1$ mutually independent path with length l joining v and v_{k-1}.

□

Theorem 1 *Assume $n \geq 4$. Given any two vertices u, v in Q_n and the distance $d(u,v) = d$. There exist n-1 mutually independent path $P_1, ..., P_{n-1}$ of length l joining u and v in Q_n for $l = d+2, d+4, ..., 2^n - 1 - \lceil \frac{(-1)^d+1}{2} \rceil$.*

Proof. We prove this lemma by induction on n. The required path of $n = 4$ are listed below:

chose $u = 0000,\ v = 0001,\ h(u,v) = 1$
(0000, 0100, 0101, 0001)
(0000, 0010, 0011, 0001)
(0000, 1000, 1001, 0001)
(0000, 0100, 0110, 0111, 0101, 0001)
(0000, 0010, 1010, 1011, 0011, 0001)
(0000, 1000, 1100, 1101, 1001, 0001)
(0000, 0100, 0110, 0010, 0011, 0111, 0101, 0001)
(0000, 0010, 1010, 1110, 1111, 1011, 1001, 0001)
(0000, 1000, 1001, 1011, 1010, 0010, 0011, 0001)
(0000, 0100, 0110, 0010, 1010, 1011, 0011, 0111, 0101, 0001)
(0000, 0010, 0011, 0111, 0110, 0100, 0101, 1101, 1001, 0001)
(0000, 1000, 1010, 1110, 1100, 1101, 1111, 1011, 0011, 0001)
(0000, 0100, 0110, 0010, 1010, 1110, 1111, 1011, 0011, 0111, 0101, 0001)
(0000, 0010, 0011, 0111, 0110, 0100, 0101, 1101, 1100, 1000, 1001, 0001)
(0000, 1000, 1100, 1101, 1111, 0111, 0110, 1110, 1010, 1011, 0011, 0001)
(0000, 0100, 0101, 0111, 0011, 1011, 1010, 1000, 1100, 1101, 1001, 1011, 0011, 0001)
(0000, 0010, 0110, 0100, 0101, 0111, 0011, 1011, 1010, 1000, 1100, 1101, 1001, 0001)
(0000, 1000, 1010, 1011, 1001, 1101, 1100, 1110, 1111, 0111, 0110, 0100, 0101, 0001)
(0000, 0100, 0101, 0111, 0110, 0010, 1010, 1000, 1100, 1110, 1111, 1011, 1001, 1011, 0011, 0001)
(0000, 0010, 0110, 0100, 0101, 0111, 0011, 1011, 1010, 1000, 1100, 1110, 1111, 1101, 1001, 0001)
(0000, 1000, 1010, 1011, 1001, 1101, 1100, 1110, 1111, 0111, 0011, 0010, 0110, 0100, 0101, 0001)

chose $u = 0000,\ v = 0111,\ h(u,v) = 3$
(0000, 0100, 0110, 0010, 0011, 0111)
(0000, 0010, 0011, 0001, 0101, 0111)
(0000, 1000, 1010, 1011, 1111, 0111)
(0000, 0100, 0110, 0010, 0011, 0001, 0101, 0111)
(0000, 0010, 0011, 0001, 0101, 0100, 0110, 0111)
(0000, 1000, 1100, 1110, 1010, 1011, 1111, 0111)
(0000, 0100, 0110, 0010, 0011, 0001, 1001, 1011, 0011, 0111)
(0000, 0010, 0011, 0001, 0101, 0100, 1100, 1110, 0110, 0111)
(0000, 1000, 1010, 1110, 0110, 0010, 0011, 1011, 1111, 0111)
(0000, 0100, 0110, 0010, 0011, 0001, 1001, 1000, 1100, 1101, 1111, 0111)
(0000, 0010, 0011, 0001, 0101, 0100, 1100, 1101, 1111, 1110, 0110, 0111)
(0000, 1000, 1010, 1110, 0110, 0010, 0011, 1011, 1001, 0001, 0101, 0111)
(0000, 0001, 0101, 0100, 0110, 0010, 0011, 1011, 1001, 1000, 1010, 1110, 1111, 0111)
(0000, 0010, 0011, 0001, 0101, 1101, 1001, 1000, 1010, 1110, 1100, 0100, 0110, 0111)
(0000, 1000, 1001, 1011, 1010, 1110, 1100, 0100, 0110, 0010, 0011, 0001, 0101, 0111)
(0000, 0001, 0101, 0100, 0110, 0010, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 1110, 1111, 0111)
(0000, 0010, 0011, 0001, 0101, 1101, 1001, 1000, 1010, 1011, 1111, 1110, 1100, 0100, 0110, 0111)
(0000, 1000, 1001, 1011, 1010, 1110, 1111, 1101, 1100, 0100, 0110, 0010, 0011, 0001, 0101, 0111)

chose $u = 0000,\ v = 0110,\ h(u,v) = 2$
(0000, 0001, 0011, 0010, 0110)
(0000, 0010, 1010, 1110, 0110)
(0000, 0100, 0101, 0111, 0110)
(0000, 0001, 0011, 0111, 0101, 0100, 0110)
(0000, 0010, 1010, 1011, 1111, 1110, 0110)
(0000, 0100, 0101, 0001, 0011, 0111, 0110)
(0000, 0001, 0011, 0111, 0101, 1101, 1100, 0100, 0110)
(0000, 0010, 1010, 1011, 1111, 1011, 1010, 1110, 0110)
(0000, 0100, 0101, 0001, 0011, 1011, 1111, 0111, 0110)
(0000, 0001, 0011, 0111, 0101, 1101, 1001, 1000, 1100, 0100, 0110)
(0000, 0010, 1010, 1011, 1111, 1110, 1100, 0100, 0101, 0111, 0110)
(0000, 0100, 0101, 0001, 0011, 0111, 1111, 1011, 1010, 1110, 0110)
(0000, 0001, 0011, 0111, 0101, 1101, 1001, 1000, 1010, 1011, 1111, 1110, 0110)
(0000, 0010, 1010, 1011, 1111, 0111, 0011, 0001, 1001, 1101, 1100, 0100, 0110)
(0000, 1000, 1001, 1101, 1100, 1110, 1010, 1011, 1111, 0111, 0011, 0010, 00110)
(0000, 0001, 0011, 0111, 0101, 0100, 1100, 1000, 1001, 1101, 1111, 1011, 1010, 0010, 0110)
(0000, 0010, 1010, 1110, 1100, 1000, 1001, 1101, 1111, 1011, 0011, 0111, 0101, 0100, 0110)
(0000, 1000, 1001, 1011, 1010, 0010, 0011, 0001, 0101, 0100, 1100, 1101, 1111, 1110, 0110)

chose $u = 0000,\ v = 1111,\ h(u,v) = 4$
(0000, 0001, 0011, 0010, 0110, 0111, 1111)
(0000, 0010, 0110, 0100, 0101, 1101, 1111)
(0000, 1000, 1100, 1101, 1001, 1011, 1111)
(0000, 0001, 0011, 0010, 0110, 0111, 0101, 1101, 1111)
(0000, 0010, 0110, 0111, 0101, 1101, 1100, 1110, 11111)
(0000, 1000, 1100, 1101, 1001, 0001, 0101, 0111, 1111)
(0000, 0001, 0011, 0111, 0101, 0100, 0110, 0010, 1010, 1011, 1111)
(0000, 0010, 0110, 0100, 1100, 1101, 1001, 0001, 0101, 0111, 1111)
(0000, 1000, 1100, 1101, 1001, 0001, 0101, 0111, 0110, 1110, 1111)
(0000, 0001, 0011, 0111, 0101, 0100, 0110, 0010, 1010, 1000, 1001, 1011, 1111)
(0000, 0010, 0110, 0100, 1100, 1000, 1010, 1011, 1001, 0001, 0101, 0111, 1111)
(0000, 1000, 1100, 1101, 1001, 0001, 0011, 0111, 0101, 0100, 0110, 1110, 1111)
(0000, 0001, 0011, 0111, 0101, 0100, 0110, 0010, 1010, 1110, 1100, 1000, 1001, 1101, 1111)
(0000, 0010, 0110, 0100, 1100, 1110, 1010, 1000, 1001, 1101, 0101, 0001, 0011, 0111, 1111)
(0000, 0100, 0101, 0001, 0011, 1011, 1001, 1101, 1100, 1000, 1010, 0010, 0110, 1110, 1111)

The lemma holds for $n = 4$ above list. As the inductive hypothesis, we assume that the lemma is true for every integer $n < k$, for all $k \geq 4$. Therefor, the proof is divided into two major cases.

Case I. u and v are the same colored vertices.

In this case, let $u = u_{k-1}u_{k-2}...u_1u_0$ and $v = v_{k-1}v_{k-2}...v_1v_0$. Hence $u_i \neq v_i$ for some i. Accordingly, Q_k can be decomposed into two subcube Q^0_{k-1} and Q^1_{k-1} by dimension i. Therefor, u and v are in the different subcube. Without loss of generality, we may assume that $u \in V(Q^0_{k-1})$ and $v \in V(Q^1_{k-1})$.

For convenience, we assume that $1 \leq i \leq k-2$.

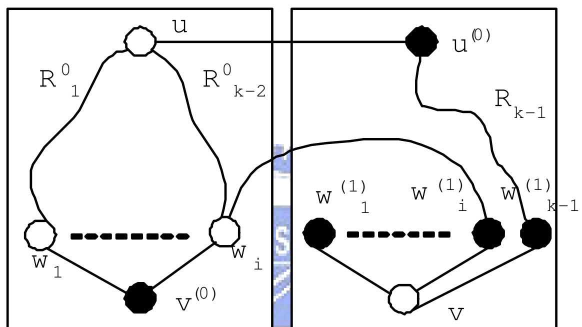


Figure 3.6: Illustration for the Case I.

Suppose that $h(u, v) + 2 \leq l \leq 2^{k-1}$ and l is even. Let $v^{(0)}$ be the neighbor of v and $v^{(0)} \in V(Q^0_{k-1})$. Thus, $h(u, v^{(0)}) = h(u, v) - 1$. Assume that $h(u, v^{(0)}) + 2 \leq l_0 \leq 2^{k-1} - 1$ for l_0 is odd. By induction hypothesis, there are $k-2$ mutually independent path R_i of length l_0 connecting u and $v^{(0)}$. Let w_i be the nodes $R_i(l_0-1)$ on R_i. We can write the path R_i as $\langle u, R^0_i, w_i, v^{(0)}\rangle$. Let $w^{(1)}_i$ be the neighbors of w_i for $w^{(1)}_i \in V(Q^1_{k-1})$. Obviously, $w^{(1)}_i$ are the neighbors of v. Therefor, $P_i = \langle u, R^0_i, w_i, w^{(1)}_i, v\rangle$ are $k-2$ mutually independent path of length l joining u and v. By definition, $u^{(1)}$ is the neighbor of u and $u^{(1)} \in V(Q^1_{k-1})$. Let $w^{(1)}_{k-1}$ be the neighbor of v and $w^{(1)}_{k-1} \in V(Q^1_{k-1})$ and $w^{(1)}_{k-1} \neq w^{(1)}_i$. Obviously, $u^{(1)}$ and

$w^{(1)}_{k-1}$ are the same colored vertices. By Lemma 2, there is a path R_{k-1} with length $l-2$ of $Q^1_{k-1}-\{v\}$ joining $u^{(1)}$ and $w^{(1)}_{k-1}$. Thus, $P_{k-1}=\langle u,u^{(1)},R_{k-1},w^{(1)}_{k-1},v\rangle$ is the $k-1$ mutually independent path of length l joining u and v.

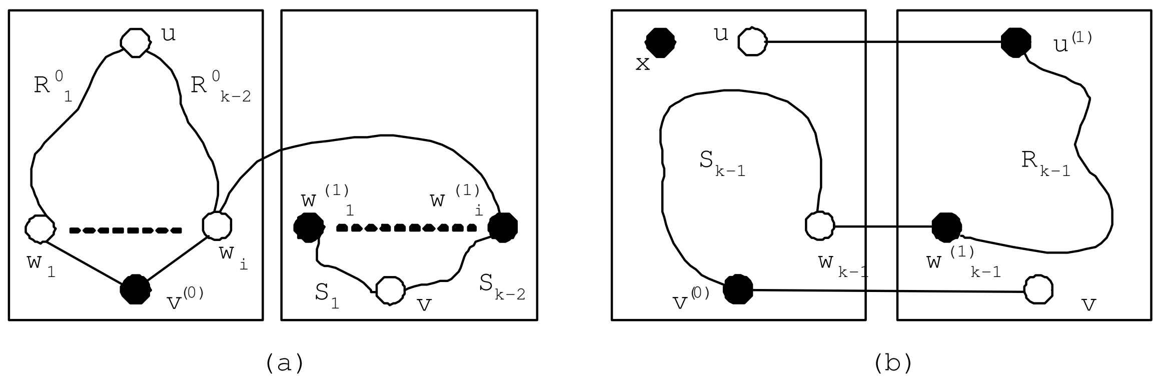


Figure 3.7: Illustration for the Case I.

Suppose that $2^{k-1}+2\leq l\leq 2^k-2$ and l is even. By above discussion, there exist $k-2$ path R^0_i of length $2^{k-1}-2$ joining u and w_i in Q^0_{k-1} and one path R_{k-1} of length $2^{k-1}-2$ joining $u^{(1)}$ and $w^{(1)}_{k-1}$ in Q^1_{k-1}. By Lemma 7, there exist $k-2$ mutually independent path S_i in Q^1_{k-1} from v to $w^{(1)}_i$ for $3\leq l(S_i)\leq 2^{k-1}-1$. Thus, $P_i=\langle u,R^0_i,w_i,w^{(1)}_i,S_i,v\rangle$ are $k-2$ mutually independent path with length l joining u and v. Let x be any vertex that are different color as u of Q^0_{k-1} and $x\neq v^{(0)}$. By Lemma 5, there exists a path S_{k-1} joining w_{k-1} and $v^{(0)}$ of $Q^0_{k-1}-\{u,x\}$ for $1\leq l(S_{k-1})\leq 2^{k-1}-3$. Therefore, $P_{k-1}=\langle u,u^{(1)},R_{k-1},w^{(1)}_{k-1},w_{k-1},S_{k-1},v^{(0)},v\rangle$ is the $k-1$ mutually independent path with length l joining u and v.

Case II. u and v are different colored vertices.

For convenience, we assume that $1\leq i\leq k-2$.

Subcase II-1. $h(u, v) < k$.

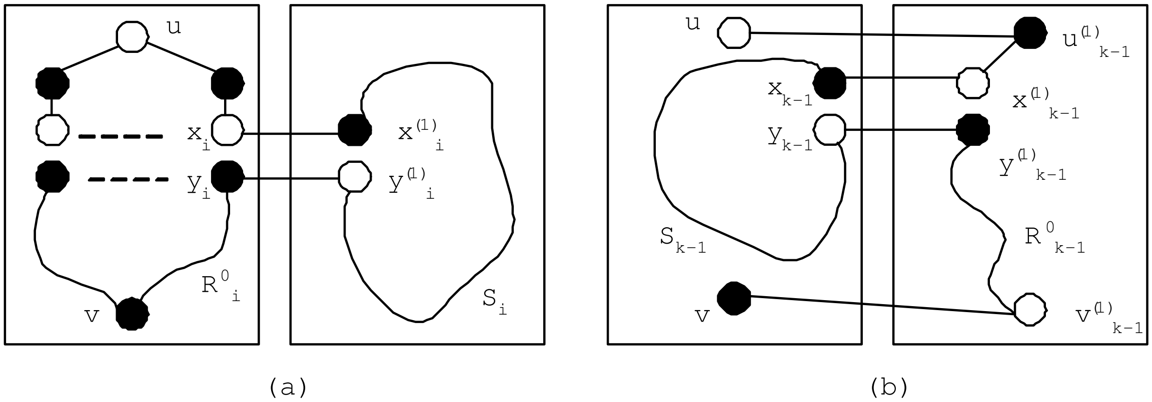


Figure 3.8: Illustration for the Case II-1.

In this case, let $u = u_{k-1}u_{k-2}...u_1u_0$ and $v = v_{k-1}v_{k-2}...v_1v_0$. Hence $u_i = v_i$ for some i. Accordingly, Q_k can be decomposed into two subcube Q^0_{k-1} and Q^1_{k-1} by dimension i. Therefor, u and v are in the same subcube. Without loss of generality, we may assume that u and v are both in Q^0_{k-1}.

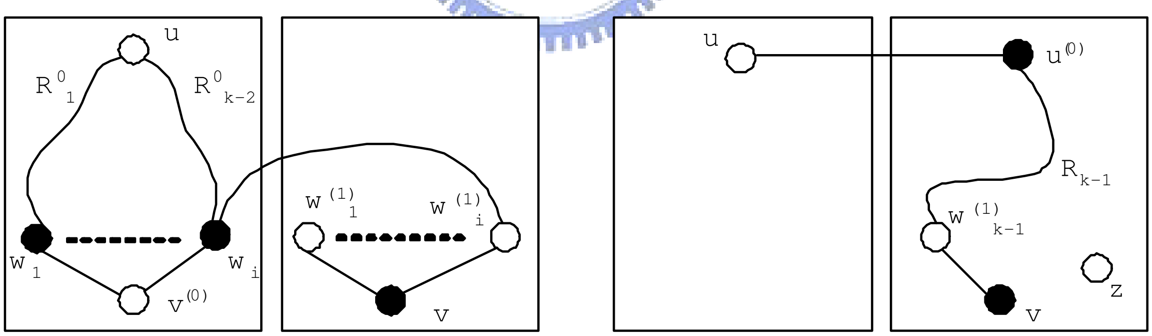
Suppose that $h(u, v) + 2 \leq l \leq 2^{k-1} - 1$ and l is odd. By inductive hypothesis, there are $k - 2$ mutually independent paths of length l joining u and v in Q^0_{k-1}. By definition, $u^{(1)}$ and $v^{(1)}$ are the neighbors of u and v for $u^{(1)}, v^{(1)} \in V(Q^1_{k-1})$. By Lemma 1, we can find a path R with length $l - 2$ joining $u^{(1)}$ and $v^{(1)}$ in Q^1_{k-1}. Thus, $\langle u, u^{(1)}, R, v^{(1)}, v\rangle$ is the path P_{k-1} of length l in Q_k joining u and v.

Suppose that $2^{k-1} + 1 \leq l \leq 2^k - 3$ for l is odd. With above discussion, let R_i be $k-2$ mutually independent paths of length l_0 joining u and v in Q^0_{k-1} for any $l_0 = 2^{k-1} - 1$. Let x_i and y_i be the nodes $R_i(2)$ and $R_i(3)$ on R_i. We can write R_i as $\langle u, R_i(1), x_i, y_i, R^0_i, v\rangle$. By definition, $x^{(1)}_i$ and $y^{(1)}_i$ are the neighbors of x_i and y_i for $\{x^{(1)}_i, y^{(1)}_i\} \in V(Q^1_{k-1})$.

By Lemma 6, there exist $k-2$ independent path S_i joining $x_i^{(1)}$ and $y_i^{(1)}$ in Q_{k-1}^1 for $1 \leq l(S_i) \leq 2^{k-1}-3$. Thus, $P_i = \langle u, R_i(1), x_i, x_i^{(1)}, S_i, y_i^{(1)}, y_i, R_i^0, v\rangle$ is $k-2$ mutually independent paths of length l in Q_k joining u and v.

By definition, $u^{(1)}$ and $v^{(1)}$ are neighbors of u and v for $u^{(1)}, v^{(1)} \in V(Q_{k-1}^1)$. By Lemma 1, there exists a path R_{k-1} of length $|R_i|-2$ joining $u^{(1)}$ and $v^{(1)}$. Let $x_{k-1}^{(1)}$ and $y_{k-1}^{(1)}$ be the nodes $R_{k-1}(1)$ and $R_{k-1}(2)$ on R_{k-1}. We can write R_{k-1} as $\langle u^{(1)}, x_{k-1}^{(1)}, y_{k-1}^{(1)}, R_{k-1}^0, v^{(1)}\rangle$. By definition, x_{k-1} and y_{k-1} are vertices in Q_{k-1}^0. By Lemma 5, there exists a path S_{k-1} joining x_{k-1} and y_{k-1} of $Q_{k-1}^0 - \{u, v\}$ for $1 \leq l(S_{k-1}) \leq 2^{k-1}-2$. Thus, $P_{k-1} = \langle u, u^{(1)}, x_{k-1}^{(1)}, x_{k-1}, R_{k-1}, y_{k-1}, y_{k-1}^{(1)}, R_{k-1}^0, v^{(1)}, v\rangle$ is the $k-1$ mutually independent path with length l joining u and v.

Subcase II-2. $h(u,v) = k$. We may choose a dimension i with the same way of the proof of Case (I) to split Q_k into two subcubes Q_{k-1}^0 and Q_{k-1}^1. Without loss of generality, we assume that $u \in V(Q_{k-1}^0)$ and $v \in V(Q_{k-1}^1)$.



(a) (b)

Figure 3.9: Illustration for the Case II-2.

Suppose that $h(u,v)+2 \leq l \leq 2^{k-1}-1$ and l is odd. Let $v^{(0)}$ be the neighbor of v and $v^{(0)} \in V(Q^0_{k-1})$. Assume that $h(u,v^{(0)})+2 \leq l_0 \leq 2^{k-1}-2$ for l_0 is even. By induction hypothesis, there are $k-2$ mutually independent path R_i of length l_0 connecting u and $v^{(0)}$. Let w_i be the nodes $R_i(l_0-1)$ on R_i. We can write the path R_i as $\langle u, R^0_i, w_i, v^{(0)}\rangle$. Let $w^{(1)}_i$ be the neighbors of w_i for $w^{(1)}_i \in V(Q^1_{k-1})$. Obviously, $w^{(1)}_i$ are the neighbors of v. Therefor, $P_i = \langle u, R^0_i, w_i, w^{(1)}_i, v\rangle$ are $k-2$ mutually independent path with length l joining u and v. By definition, $u^{(1)}$ is the neighbor of u and $u^{(1)} \in V(Q^1_{k-1})$. Let $w^{(1)}_{k-1}$ be the neighbor of v and $w^{(1)}_{k-1} \in V(Q^1_{k-1})$ and $w^{(1)}_{k-1} \neq w^{(1)}_i$. Obviously, $u^{(1)}$ and $w^{(1)}_{k-1}$ are different colored vertices. Let z be any vertex that are different color as v and $v \neq w^{(1)}_{k-1}$. By Lemma 5, there is a path R_{k-1} with length l_0-1 of $Q^1_{k-1}-\{z,v\}$ joining $u^{(1)}$ and $w^{(1)}_{k-1}$. Thus, $P_{k-1} = \langle u, u^{(1)}, R_{k-1}, w^{(1)}_{k-1}, v\rangle$ is the $k-1$ mutually independent path of length l joining u and v.

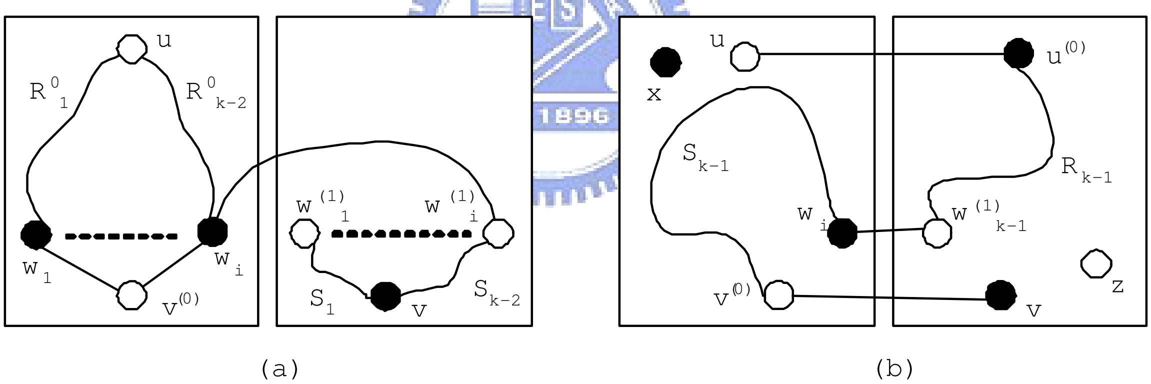


Figure 3.10: Illustration for the Case II-2.

Suppose that $2^{k-1}+1 \leq l \leq 2^k-3$ and l is odd. By above discussion, there exist $k-2$ path R^0_i of length $2^{k-1}-3$ joining u and w_i in Q^0_{k-1} and one path R_{k-1} of length $2^{k-1}-3$ joining $u^{(1)}$ and $w^{(1)}_{k-1}$ in Q^1_{k-1}. By Lemma 7, there exist $k-2$ mutually independent path

S_i in Q^1_{k-1} from v to $w_i^{(1)}$ for $3 \leq l(S_i) \leq 2^{k-1} - 1$. Thus, $P_i = \langle u, R_i^0, w_i, w_i^{(1)}, S_i, v\rangle$ are $k-2$ mutually independent path with length l joining u and v. Let x be any vertex that are different color as u of Q^0_{k-1} and $x \neq v^0$. By Lemma 5, there exists a path S_{k-1} joining w_{k-1} and v^0 of $Q^0_{k-1} - \{u, x\}$ for $1 \leq l(S_{k-1}) \leq 2^{k-1} - 3$. Therefore, $P_{k-1} = \langle u, u^{(1)}, R_{k-1}, w_{k-1}^{(1)}, w_{k-1}, v^{(0)}, v\rangle$ is the $k-1$ mutually independent path with length l joining u and v.

$\square$

Chapter 4

conclusion

Since every component in the interconnection network may have different reliability, it is important to consider properties of a network with mutually independent linear array embeddings. In this paper, the n-dimensional hypercube with $(n-1)$ mutually independent path of any length l joining any vertices u and v for $h(u,v) \leq l \leq 2^n - 1$. It is also impossible to make n mutually independent paths and cycles except one case that u is adjacent to v.

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