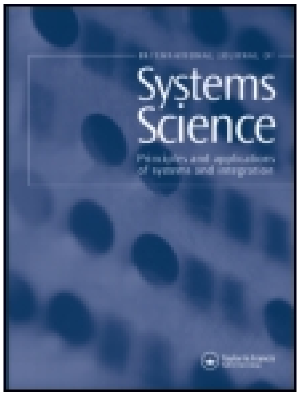

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Feedback stabilization of nonlinear driftless systems with applications to homogeneous-type systems

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Feedback stabilization of nonlinear driftless systems with applications to homogeneous-type systems

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Issues of asymptotic stabilization of nonlinear driftless systems as given by $\dot{x} = g(x)u$ with applications to homogeneous-type driftless systems are presented. Conditions of the existence of a smooth time-invariant stabilizer for general nonlinear driftless systems are obtained by the construction of quadratic-type Lyapunov functions. The proposed conditions do not contradict Brockett's (1983) result for the existence of a smooth time-invariant stabilizer. These results are then employed to study the stabilization problem of homogeneous-type systems. Sufficient conditions are obtained for the stabilization of planar type homogeneous driftless systems with positive order. It is shown that the single input control driftless systems cannot be asymptotically stabilizable by any continuous control if $g(x)$ is a homogeneous function of even order. Moreover, equivalent conditions for the stabilizability of linear driftless systems and the explicit design of stabilizing control laws for bilinear driftless systems are also presented.

1. Introduction

In recent years, the study of the stabilization of nonlinear driftless systems as given by $\dot{x} = g(x)u$ has attracted considerable attention. These studies include the existence conditions of time-invariant smooth stabilizers (Brockett 1983, Liaw and Liang 1993), the design of time-varying stabilizers (Coron 1992, Pomet 1992), the design of time-invariant piecewise smooth stabilizers (Canudas and Sordalen 1992), and applications to the study of a satellite's orbital motion (Ahmed and Sen 1980, 1981) and car-like robot systems (Walsh *et al.* 1994). In the study of homogeneous-type nonlinear systems, substantial literature has been published for system stabilization. Among these results, both Andreini *et al.* (1988) and Tsinias (1990) proposed sufficient Lyapunov conditions for system stabilization, Luesink and Nijmeijer (1989) proposed a constant control law for bilinear systems, while Bacciotti and Boieri (1991) obtained results for the local asymptotic smooth stabilizability of single-input planar bilinear systems. Moreover, Celikovsky (1993) attained the stabilizability of single-input homogeneous bilinear systems of which the matrix of the linear term is semi-stable and the matrix of the bilinear term is skew-symmetric.

There are two major goals of this paper. One is to relax the assumption of system stabilizability on the full rank of $g(0)$ as proposed by Brockett (1983). Quadratic type Lyapunov functions will be proposed to determine the asymptotic stabilizability of driftless systems without the assumptions on $g(0)$. The other goal is to study the stabilizability of homogeneous-type driftless systems. We will show that the single input control driftless systems cannot be asymptotically stabilizable by any continuous control if $g(x)$ is a homogeneous function of even order. The stabilization problem of both linear driftless systems and bilinear driftless systems will also be discussed. Specifically, the equivalent conditions for the stabilizability of linear driftless systems and explicit design of the stabilizing control laws for bilinear driftless systems will be presented.

The paper is organized as follows. In § 2, a quadratic-type Lyapunov function is employed to construct asymptotic stabilizability conditions for general driftless systems. It is followed by the recall of the definition of homogeneous system and the basic properties of a homogeneous function. The asymptotic stabilizability of homogeneous-type systems, specifically for planar systems, linear systems and bilinear systems is discussed.

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Examples are given in § 4 to illustrate the applications. Finally, § 5 summarizes the main results.

2. Stabilization of driftless systems

Consider a class of nonlinear control systems as given by

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $g(x) = (g_1(x), \ldots, g_m(x))$ and $u = (u_1, \ldots, u_m)^T$. Here, $x \in R^n$, $u_i \in R$ for $i = 1, \ldots, m$, and $f(x)$ and $g_i(x)$ for $i = 1, \ldots, m$ are all assumed to be smooth functions. To study the stabilization problems of (1), in general, we need to solve for the equilibrium points of the uncontrolled version of the system. That is, to solve for $x_e \in R^n$-such that $f(x_e) = 0$. Then all the stability analysis and control design will play around the point of x_e. In fact, the autonomous term (or the so-called 'drift-term') $f(x)$ plays a very important role in determining the stability as well as the stabilizability of system (1). For instance, suppose all the eigenvalues of the jacobian matrix $Df(x_e)$ lie in the open left-half of the complex plane. Then the equilibrium point x_e will be asymptotically stable even without control. If some eigenvalues do not have negative real part, the control input u will then play a role in forcing these eigenvalues to lie in the open left-half of the complex plane or in providing nonlinear efforts to guarantee the stability of x_e.

Results have been presented regarding the asymptotic stabilization of the origin of (1) in critical cases, that is, the jacobian matrix $Df(x_e)$ possesses eigenvalues lying on the imaginary axis of the complex plane (Aeyels 1985, Behtash and Sastry 1988, Liaw and Abed 1991, Fu and Abed 1993). The study of the most degenerate critical case of system (1) of which $f(x) = 0$ for all $x \in R^n$ has also attracted attention (Coron 1992, Pomet 1992, Sontag 1993, Canudas and Sordalen 1992, Walsh *et al.* 1994). Such a class of systems is the so-called 'driftless system,' which is represented as

$$\dot{x} = g(x)u = \sum_{k=1}^{m} u_k g_k(x). \tag{2}$$

In system (2), $x \in R^n$ and $u_k \in R$ for all $k = 1, \ldots, m$. Moreover, $g_k(x)$ is assumed to be smooth for all $k \geq 1$.

From the viewpoint of system stabilization, there are two major differences between systems (1) and (2). One, is that system (2) has an infinite number of equilibrium points, while (1) generally has finite ones. Secondly, the system (2) always needs some control force to damp out the disturbances, while system (1) might not need a control force to damp out the disturbance if it appears in the stable manifold of the uncontrolled model. Due to these two major differences, it will be much harder to stabilize an operating point of driftless system (2) and more conditions will be required on $g(x)$ to fulfil the tasks.

In the following, for simplicity and without loss of generality, the origin is supposed to be the operating point of interest. In addition, the deleted neighbourhood of the origin is defined as a neighbourhood around the origin without containing the origin itself.

We now recall a necessary and sufficient condition obtained by Brockett (1983) for the existence of a smooth time-invariant stabilizing control law for system (2) as given below.

Lemma 1: (*Brockett* 1983): *Suppose all the vectors* $g_k(0)$ *in* (2) *are linearly independent. Then there exists a smooth time-invariant asymptotic stabilizer for the origin of system* (2) *if and only if* $m = n$.

Lemma 1 above provides a necessary and sufficient condition for the existence of a smooth stabilizing controller for system (2) while all the vectors $g_k(0)$ are assumed to be linearly independent. However, the linear independency of $g_k(0)$ is not a necessary condition to identify the feedback stabilizability of system (2). In the next example, we show that there does exist a smooth time-invariant stabilizer for system (2) while the assumption of linear independency (as required in Lemma 1) does not hold.

Example 1: Consider a planar system as given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2^3 \end{pmatrix} u, \tag{3}$$

where $a > 0$, $b > 0$ and $u \in R$. In this case, $g(x_1, x_2)^T = (ax_1, bx_2^3)^T$ and $x = (x_1, x_2)^T$. It is easy to check that $g(0, 0) = 0$ and the number of inputs is less than the number of states. Thus, the linear independency of columns of $g(0, 0)$ as required in Lemma 1 does not hold. Choose the control input $u = -(a^2x_1^2 + b^2x_2^4)$ and let $V = ax_1^2 + bx_2^2$ be a Lyapunov function candidate for system (3). We obtain $\dot{V} = -2(a^2x_1^2 + b^2x_2^4)^2 < 0$ for all $x \neq 0$. According to Lyapunov stability criteria, we can hence conclude that the origin of system (3) is globally asymptotically stable. □

In the following, we will try to relax the stabilizing condition given in Lemma 1. First, by the definition of the equilibrium point, we have the next result to provide a necessary condition of the asymptotic stabilization of system (2).

Lemma 2: *Suppose the origin of system* (2) *is locally asymptotically stabilizable. Then, for any asymptotic stabilizer,* $u = u(x)$ *for the origin,* $g(x)u(x) \neq 0$ *around some deleted neighbourhood of the origin.*

From Lemma 2, if the origin of (2) is asymptotically stabilizable, then $g(x)$ and the control u cannot vanish around some deleted neighbourhood Ω of the origin, as well as the control u cannot be orthogonal to the row space of $g(x)$ on Ω. However, $g(x)u$ must vanish at the origin since the origin is an equilibrium point of (2).

Next, by choosing $V(x) = x^T Px$ as a Lyapunov function candidate, we have the following result for guaranteeing the asymptotic stability of the origin of system (2).

Theorem 1: *Let P be a symmetric positive definite matrix. Then, the origin of system (2) is asymptotically stable if either of the following two conditions holds:*

(i) $x^T Pg(x) \neq 0$ *around a deleted neighbourhood of the origin with control input given by*

$$u = -\sum_{i=1}^{m} \alpha_i (x^T \mathrm{P} g_i(x)) e_i, \quad (4)$$

where e_i *denotes the ith cartesian unit column vector, and* $\alpha_i > 0$ *for all* $i = 1, \ldots, m$.

(ii) *For some i,* $x^T P g_i(x) \neq 0$ *around a deleted neighbourhood of the origin with the asymptotic stabilizer chosen in the form as given in (5) or (6) below*

$$u = -\beta(x^T P g_i(x)) e_i \quad (5)$$

$$u = -\beta \operatorname{sgn}(x^T P g_i(x)) e_i, \quad (6)$$

where $\beta > 0$ *and* sgn(·) *denotes the sign function.*

By employing the same quadratic type Lyapunov function candidate as that for Theorem 1, it is obvious to have the next negative result for the stabilization of system (2).

Theorem 2: *If* $x^T P g(x) = 0$ *on a neighbourhood of the origin, then the origin of (2) is not asymptotically stabilizable.*

Remark 1: It is easy to check that condition (i) of Theorem 1 holds if $g(0)$ of system (2) is a non-singular matrix. Thus, Theorem 1 does not contradict the result of Lemma 1. However, the assumptions of Theorem 1 do not require $g(0)$ being of full rank. □

Remark 2: Although the stabilizability of Example 1 cannot be determined by Lemma 1, Theorem 1, however, can provide help by choosing P as an identity matrix of R^n. We then have $x^T P g(x) = a x_1^2 + b x_2^4 \neq 0$ for all $x_1 \neq 0$ or $x_2 \neq 0$. Thus, the origin of (3) is concluded by Theorem 1 to be asymptotically stabilizable. □

3. Applications to homogeneous-type driftless systems

In this section, we first recall the definition of a homogeneous system. A property of a homogeneous function is also presented. We will then apply the results of Theorem 1 to study the stabilization problem of homogeneous-type systems for both single input and multiple input control cases.

It is known (e.g. Hahn 1967) that a function h is said to be 'homogeneous of order p' if $h(\lambda x) = \lambda^p h(x)$ for all $\lambda > 0$. A nonlinear time-invariant system is said to be a *homogeneous system* if its system dynamics are a homogeneous function. Based on the definition of a homogeneous function, it is not difficult to obtain the following result.

Lemma 3: *Suppose* $h(x)$ *is a continuous homogeneous function in* $x \in R^n$ *of order p. Then the following three conditions hold:*

(i) $h(x)$ *is a constant function for all* $x \in R^n$ *if* $p = 0$;

(ii) $h(0) = 0$ *if* $p > 0$;

(iii) *if* $h(x)$ *does not vanish on a closed sphere of* R^n *centred at the origin, then* $h(x) \neq 0$ *for all* $x \neq 0$. *Otherwise, there exists at least one-dimensional solutions of* $h(x) = 0$ *in* R^n *passing through the origin.*

It is known that the origin of a homogeneous vector field of even integer order cannot be asymptotically stabilizable (Corollary 2.1 of Koditschek and Narendra 1982). Moreover, it is noted that if system (2) possesses an asymptotic homogeneous stabilizer, then local stabilizability is equivalent to global stabilizability (Hahn 1967). Thus if $g(x)$ is a homogeneous matrix function of integer order p, and u is supposed to be a homogeneous asymptotic stabilizer of integer order q, then q must be a number such that $p + q$ is an odd number. One of the main goals of this paper can hence be to construct a feasible order of a homogeneous-type state feedback stabilizer for the case of which $g(x)$ is of integer order.

A direct application of Theorem 1 to the case for which $g(x)$ is a homogeneous matrix function may conclude the asymptotic stabilizability of the origin of system (2). Also, it is easy to check that the control law as given in (4) will result in an odd degree homogeneous matrix function for the closed loop system dynamics if condition (i) of Theorem 1 holds. However, in general, it is hard to check condition (i) of Theorem 1 around a deleted neighbourhood of the origin. In the following, we will try to relax the checking condition for specific homogeneous-type driftless systems. First, we will prove that, for single control input, the homogeneous-type driftless system of even order will not be asymptotically stabilizable by time-invariant feedback control. It is followed by the stabilization design for a planar homogeneous-type driftless system. We then focus on the study of the stabilization for both linear and bilinear systems, these two types of systems are a special class of homogeneous-type driftless systems. To facilitate the study, in the remainder of this section, we assume $g(x)$ is a homogeneous function in system (2).

3.1. *Single-input system with even order*

We now consider the case in which $g(x)$ is an even order homogeneous vector function. For the case in which the number of system states equals one (i.e. $n = 1$), the design is trivial since any homogeneous polynomial of odd degree with sign opposite to the sign of system dynamics can play an asymptotic stabilizer. However, for a general case in which $n > 1$, the conditions of Theorem 1 cannot hold since $x^T Pg(x)$ is a homogeneous polynomial of odd degree. However, we cannot conclude directly that the origin of such a system is not asymptotically stabilizable because Theorem 1 only provides sufficient conditions for determining the stabilizability of system (2). An example given by Artstein (1983) and Sontag (1989), as in (7) below, has been shown to be not asymptotically stabilizable by any continuous state feedback from a topological point of view

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} u. \tag{7}$$

In the theorem below, we will show that the origin of a general single-input system with even order can never be asymptotically stabilizable by any continuous state feedback.

Theorem 3: *The origin of system (2) is not asymptotically stabilizable by any continuous state feedback for the case of which $m = 1$ and $n > 1$ if the system dynamics $g(x)$ are a homogeneous vector field of even order.*

Proof: Prove by contradiction to assume the origin is asymptotically stable by some continuous state feedback control law $u = u(x)$. Then, by Lemma 2, u cannot vanish and change sign around a deleted neighbourhood of the origin. Otherwise, by the Intermediate Value Theorem, u will vanish at some point in any arc that joins those two points if it has a sign change. This leads to the origin as an accumulation point of system equilibrium points, which contradicts the results of Lemma 2. Thus, without loss of generality, we can assume

$$u(x) > 0 \quad \text{for all} \quad x \neq 0. \tag{8}$$

For any given $x_0 \neq 0$, let $x^*(s)$ denote the unique solution of the system

$$\dot{x} = g(x) \quad \text{with} \quad x(0) = x_0. \tag{9}$$

Consider the differential equation

$$\frac{ds}{dt} = u(x^*(s)) \quad \text{with} \quad s(0) = 0. \tag{10}$$

It is noted that solution of system (10) exists and $s(t)$ is a strictly increasing function since $u(x) > 0$ for all $x \neq 0$. Define

$$y^*(t) = x^*(s(t)), \tag{11}$$

we then have

$$\begin{aligned} \frac{d}{dt} y^*(t) &= \frac{d}{ds} x^*(s(t)) \frac{ds}{dt} \\ &= g(x^*(s(t)))u(x^*(s(t))) \\ &= g(y^*(t))u(y^*(t)) \end{aligned}$$

with

$$y^*(0) = x^*(s(0)) = x^*(0) = x_0.$$

This results in $y^*(t)$ as the solution of system (2) with initial state $y^*(0) = x_0$. If $s(t)$ has a finite limit as t approaches ∞, say $\lim_{t\to\infty} s(t) = s_0$, then

$$\lim_{t\to\infty} y^*(t) = \lim_{t\to\infty} x^*(s(t)) = x^*(s_0) \neq 0. \tag{12}$$

This implies that $x^*(s_0)$ is an equilibrium point of system (2). We can hence conclude that the origin of system (2) is not asymptotically stable. Next, if $\lim_{t\to\infty} s(t) = \infty$, then the two systems (2) and (9) have common trajectory $y^*(t) = x^*(s(t))$ with different time scales. According to Corollary 2.1 of Koditschek and Narendra (1982), there exists some trajectory $x^*(s)$ of system (9) to show that the origin of system (9) is not asymptotically stable. Thus, $y^*(t) = x^*(s(t))$ is also a solution of system (2) and can be used to provide the non-asymptotic stability of the origin of (2). The conclusion of Theorem 3 is hence implied. □

Note that the results of Theorem 3 above cannot be extended to a general multi-input case. An example is given below to demonstrate the statement.

Example 2: Consider a planar driftless system $\dot{x} = g(x)u$ with $x = (x_1, x_2)^T \in R^2$ and

$$g(x) = \begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix}. \tag{13}$$

It is easy to check that $x^T g(x) \neq 0$ for all $x \neq 0$. Thus, by Theorem 1, the origin is concluded to be asymptotically stabilizable. □

3.2. *Planar systems of positive order*

In this subsection, we consider the case in which $g(x)$ in system (2) is a single-input planar homogeneous function of positive order. By the application of Theorem 1 to provide the existence of an asymptotic stabilizer for system (2), we need to have a non-zero value for $x^T Pg(x)$ for all x around a deleted neighbourhood of the origin. Instead of checking the values of $x^T Pg(x)$ for all x around the deleted neighbourhood of the origin, for the planar system we may only need to check the sign change of $x^T Pg(x)$ on some closed contour enclosing the origin while providing the existence of some $x_0 \neq 0$ with $x_0^T P g(x_0) \neq 0$.

We have the next theorem.

Theorem 4: *Suppose $g(x)$ in system (2) is a continuous homogeneous vector field of order $p > 0$ with $n = 2$ and $m = 1$. Then the origin of system (2) is globally asymptotically stabilizable if the following two conditions hold.*

(i) *There exists a symmetric positive definite matrix P such that $x^T P g(x)$ does not change sign on some closed contour Γ enclosing the origin. Also, $x_0^T P g(x_0) \neq 0$ for some $x_0 \neq 0$.*

(ii) *$g(x) \neq 0$ on the closed contour Γ.*

Moreover, the asymptotic stabilizer u can be chosen by

$$u = -\alpha(x)\,\mathrm{sgn}\,(x_0^T P g(x_0)), \tag{14}$$

where $\alpha(x)$ is a locally positive definite function or positive constant.

Proof: Let $V = \frac{1}{2}x^T P x$ be the Lyapunov function candidate for system (2). We then have

$$\dot{V} = x^T P \dot{x} = x^T P g(x) u. \tag{15}$$

From condition (i), for a homogeneous system we have $|x^T P g(x)| \geq 0$ for all $x \in R^2$. By choosing control u as given in (14), we hence have that $g(x)u$ is a continuous function and

$$\dot{V} = -|\alpha(x) x^T P g(x)| \leq 0. \tag{16}$$

Since $g(x) \neq 0$ on the contour Γ enclosing the origin, from property (iii) of Lemma 3 we then have $g(x) \neq 0$ for all $x \neq 0$. Thus, the invariant set in the set $R := \{x \mid \dot{V}(x) = 0\}$ only contains the point $x = 0$. Referring to the invariant set theorem (see e.g. Hahn 1967), we can conclude that the origin of (2) is globally asymptotically stable while u is chosen as in (14). □

Note that it is not difficult to extend the result of Theorem 4 above to the general multi-input case of which each column of $g(x)$ may have a different order. Details are omitted.

Remark 3: Suppose the homogeneous vector field $g(x)$ satisfies the so-called 'Lyapunov condition' (see e.g. Tsinias 1990), that is, $x^T P g(x) \neq 0$ for all non-zero x in the driftless system case. Then the results of Theorem 4 agree with those obtained by Tsinias (1990). However, in Theorem 4 above, we allow $x^T P g(x) = 0$ for some $x \neq 0$. Thus, Theorem 4 provides more relaxing conditions for determining the stabilizability of planar-type homogeneous driftless systems. □

Remark 4: In general, the results of Theorem 4 cannot be extended to the case of which $n > 2$. A counter-example can be given by $g(x, y, z) = (x, -z, y)^T \in R^3$. It is observed that $g(x, y, z)$ is a continuous homogeneous vector field of order 1, $(x, y, z)g(x, y, z) = x^2$ does not change sign on the whole R^3 and $e_1^T g(e_1) = 1 \neq 0$ for $e_1 = (1, 0, 0)^T$. However, the origin is not asymptotically stabilizable since every trajectory with initial states lying on the YZ-plane is a periodic trajectory of the YZ-plane regardless of the value of control input u. □

3.3. *Linear systems and bilinear systems*

In the following, we consider two special classes of homogeneous type driftless systems, namely, linear systems and bilinear systems. Stabilizability conditions for these two classes of systems will be obtained to link with existing known results.

First, consider the system dynamics of $g(x)$ in (2) as a constant matrix, say, $g(x) = B$. We have the next obvious result.

Theorem 5: *Suppose $g(x) = B$ in system (2). Then the following three statements are equivalent:*

(i) *the origin of system (2) is globally asymptotically stabilizable;*

(ii) *there exists some symmetric positive definite matrix P such that $x^T P B \neq 0$ for all $x \neq 0$;*

(iii) *the matrix B has rank n.* □

The stabilization problem of the bilinear systems has recently received considerable attention owing to a great number of remarkable applications (e.g. see Luesink and Nijmeijer 1989, Tsinias 1990, Bacciotti and Boieri 1991, Celikovsky 1993). In the following, we will focus on the study of the asymptotic stabilization problem of bilinear driftless systems as given by

$$\dot{x} = \sum_{i=1}^{m} u_i B_i x, \tag{17}$$

where $x \in R^n$, $u_i \in R$ and $B_i \in R^{n \times n}$ for each i.

For the case of single-input planar bilinear driftless systems. Bacciotti and Boieri (1991) have shown that the origin possesses a smooth asymptotic stabilizer if and only if B is a definite matrix. For the general bilinear system, the existence of an asymptotic stabilizer does not require the definiteness of the matrix B, however, it does require some properties of the autonomous term (or the so-called 'drift term') of the system. Details can be referred to the results of Bacciotti and Boieri (1991). Obviously, the requirement of definiteness of matrix B for the stabilization of bilinear driftless systems agrees with the one stated in Theorem 1 of this paper. In the next theorem, we extend the study of Bacciotti and Boieri (1991) to general n-dimensional systems as given by

$$\dot{x} = uBx, \tag{18}$$

where $x \in R^n$ and u is a scalar.

Theorem 6: *For single-input bilinear driftless systems as in* (18), *the following three statements are equivalent.*

(i) *B is definite.*
(ii) $x^{\mathrm{T}}Bx \neq 0$ *for all* $x \neq 0$ (*note that this is condition* (*i*) *of Theorem* 1 *with* $P = I$).
(iii) *The origin of system* (18) *is globally asymptotically stabilizable by a continuous time-invariant asymptotic stabilizer.*

Proof It is clear that (i) $\Rightarrow$ (ii). Moreover, the result of (ii) $\Rightarrow$ (iii) follows from Theorem 1. It remains to show that (iii) $\Rightarrow$ (i). Prove by contradiction to assume that B is not a definite matrix but system (18) possesses a continuous asymptotic stabilizer u. Since the coordinate transformation does not affect the stability property of a dynamical system, we consider system (18) is transformed into a block-diagonal form as given in (19) below by some coordinate transformation,

$$\dot{y} = u\begin{pmatrix} B_+ & 0 & 0 \\ 0 & B_- & 0 \\ 0 & 0 & B_0 \end{pmatrix} y. \tag{19}$$

Here, $\lambda(B_+) \subseteq \mathbb{C}^{\circ +}$, $\lambda(B_-) \subseteq \mathbb{C}^{\circ -}$ and $\lambda(B_0) \subseteq j\omega$-axis. In addition, $B_+ \in R^{n_+ \times n_+}$, $B_0 \in R^{n_0 \times n_0}$, $B_- \in R^{n_- \times n_-}$, $n_+ \geq 0, n_0 \geq 0, n_- \geq 0$, with $n_+ + n_0 + n_- = n$, and $\lambda(\cdot)$ denotes the eigenvalues of the corresponding matrix. It is noted that any continuous asymptotic stabilizer u cannot change sign around some deleted neighbourhood of the origin. Otherwise, there exists some extra equilibrium points within any neighbourhood of the origin, which implies that the origin is not asymptotically stabilizable. Without loss of generality, we can assume $u(y) > 0$ on a deleted neighbourhood Ω of the origin. Then, for the proof of the theorem, we only need to consider the cases of which $n_+ > 0$ and/or $n_0 > 0$.

(i) Case of $n_0 > 0$

If B contains zero eigenvalues, then the origin is not asymptotically stabilizable since every point lying on the eigenspace of B is an equilibrium point no matter what u is chosen (see Lemma 2). Next, if B contains non-zero pure imaginary eigenvalues. Then B_0 contains a block matrix in the form of

$$J = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \tag{20}$$

for some $\beta \neq 0$. Consider the block matrix J associated with the state variables y_i and y_{i+1}. Define $V = y_i^2 + y_{i+1}^2$, we have $\dot{V} = u(y)[2y_i(-\beta y_{i+1}) + 2y_{i+1}(\beta y_i)] = 0$ for all $y = (y_1, \ldots, y_n)^{\mathrm{T}} \in R^n$. This implies that the motion of the system states always lies on the level set of V. The analysis is obviously effective for the case in which B_0 has Jordan block of J. Details are omitted.

(ii) Case of $n_+ > 0$

If $n_+ > 0$, that is, B has eigenvalues with a positive real part, then there exists a symmetric positive definite matrix $P_+ \in R^{n_+ \times n_+}$ which solves $B_+^{\mathrm{T}}P_+ + P_+B_+ = I_{n_+}$, where I_{n_+} is the $n_+ \times n_+$ identity matrix. Choose $P = \operatorname{diag}\{P_+, 0, 0\} \in R^{n \times n}$ and $V = y^{\mathrm{T}}Py$, we have $V > 0$ on $M :=$ the complement of $\{0_{n_+} \times R^{n_0} \times R^{n_-}\}$ with the origin lying on the boundary of M. The time derivative of V gives

$$\dot{V} = u(y)y^{\mathrm{T}}\begin{pmatrix} P_+B_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y = \tfrac{1}{2}u(y)y^{\mathrm{T}}\begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y > 0 \text{ on } M \tag{21}$$

since $u(y) > 0$ for all $y \neq 0$. Thus, by employing Chetaev's Instability Theorem (see e.g. Hahn 1967), the origin is concluded to be unstable.

From the discussions above, we can conclude that the origin of (18) is not asymptotically stabilizable by any continuous time-invariant asymptotic stabilizer if B is indefinite. This results in (iii) $\Rightarrow$ (i). □

In the following, we will apply Theorem 6 above to study the asymptotic stabilization problems of the general bilinear driftless systems as defined in (17) in which $m \geq 1$.

Next result follows readily from Theorem 6.

Theorem 7: *Suppose the r matrices of* B_i *in system* (17) *are definite with* $r \geq 1$. *Then the origin of system* (17) *is asymptotically stabilizable by the control in the form of* (4) *or* (5).

For the case of which none of the B_i in (17) is a definite matrix, coordinate transformation may provide a means of determining the stabilizability of system (17). Suppose there exists a non-singular matrix S by which more than two B_i can be simultaneously transformed into either upper triangular matrices or lower triangular matrices. We may then apply Theorem 7 to determine the asymptotic stabilizability of system (17) as in the next theorem.

Denote $\operatorname{Re}\{\cdot\}$ the real part, $(\cdot)_{kk}$ the kth diagonal element of a matrix and e_i the ith cartesian unit column vector, respectively, which are used throughout the remainder of this paper.

We have the following result.

Theorem 8: *Suppose there exists a non-singular matrix* S *such that* $S^{-1}B_iS$ *are upper triangular matrices for* $i = 1, \ldots, p$ *with* $p \leq m$. *If there is a set of real numbers* $c_1, \ldots, c_p$ *such that* $\operatorname{Re}\{\sum_{j=1}^{p} c_j(S^{-1}B_jS)_{kk}\} < 0$ *for* $k = 1, \ldots, n$, *then the origin of* (17) *is globally asymptotically stabilizable by constant control* $u = \sum_{j=1}^{p} c_je_j$ *or by state feedback control* $u = -\sum_{j=1}^{p} c_j^2(x^{\mathrm{T}}B_j)e_j$.

The next result follows readily from Theorem 8.

Corollary 1: *Suppose two matrices B_i and B_j in (17) can be simultaneously triangularizable by S with*

$$\mathrm{Re}\,(S^{-1}B_iS)_{kk}\} > \mathrm{Re}\,\{(S^{-1}B_jS)_{kk}\} \text{ for all } k = 1, \ldots, n.$$

Then the origin of system (17) is globally asymptotically stabilizable by constant control $u = c(e_i - e_j)$ for $c < 0$ or state feedback control $u = -c^2[(x^TB_ix)e_i - (x^TB_jx)e_j]$ for $c \neq 0$.

In general, it may not be easy to check the existence of matrix S as defined in Theorem 8. A sufficient condition for guaranteeing the existence of such a linear transformation S is that matrices $B_1, \ldots, B_p$ form a commuting family. Here, a family of $n \times n$ matrices $F \subseteq \mathbb{C}^{n\times n}$ is called a commuting family if $AB = BA$ for any $A, B \in F$.

We recall the next result.

Lemma 4 (e.g. Horn and Johnson 1985): *Let $F \subseteq \mathbb{C}^{n\times n}$ be a commuting family. Then there exists a unitary matrix $S \in \mathbb{C}^{n\times n}$ such that $S^{-1}AS$ is an upper triangular matrix for all $A \in F$.*

A proof of Lemma 4 can be found in Horn and Johnson (1985). For completeness and easier implementation, an algorithm is proposed in the Appendix to construct such a unitary matrix S of Lemma 4.

Denote by F_S a commuting family of which all the matrices in F_S can be simultaneously triangularizable by matrix S. The next result follows readily from Lemma 4 and Theorem 8.

Corollary 2: *Suppose in system (17) there are p matrices, say $B_1, \ldots, B_p$ which belong to a commuting family F_S. If there is a set of real numbers $c_1, \ldots, c_p$ such that*

$$\mathrm{Re}\left\{\sum_{j=1}^{p} c_j(S^{-1}B_jS)_{kk}\right\} < 0 \text{ for all } k = 1, \ldots, n, \tag{22}$$

then the origin of system (17) is globally asymptotically stabilizable by constant control $u = \sum_{j=1}^{p} c_je_j$ or by state feedback control $u = \sum_{j=1}^{p} c_j^2(x^TB_j)e_j$.

4. Illustrative examples

In this section, we consider three examples to demonstrate the main results of the paper.

Example 3: Consider the example of attitude stabilization of a rigid body as adopted from Meyer (1971) and Canudas de Wit and Sordalen (1992),

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta\tan\phi & 1 & -\cos\theta\tan\phi \\ -\sin\theta\sec\phi & 0 & \cos\theta\sec\phi \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \tag{23}$$

where $(\phi, \theta, \psi)^T$ denotes the Euler angles describing the orientation and $(\omega_1, \omega_2, \omega_3)^T$ denotes the rotational control velocity, respectively. The model is only valid locally for $-\pi/2 < \theta < \pi/2$ due to the fact that the original system possesses a singularity at which $\cos\theta = 0$.

System (23) is an example of a driftless system with $x = (\phi, \theta, \psi)^T$, $u = (\omega_1, \omega_2, \omega_3)^T$ and

$$g(x) = \begin{pmatrix} \cos x_2 & 0 & \sin x_2 \\ \sin x_2\tan x_1 & 1 & -\cos x_2\tan x_1 \\ -\sin x_2\sec x_1 & 0 & \cos x_2\sec x_1 \end{pmatrix}. \tag{24}$$

It is observed that $g(0)$ has rank 3. This implies that $x^Tg(x) \neq 0$ around some deleted neighbourhood of the origin. Thus, the origin of the system (23) is concluded to be asymptotically stabilizable by Lemma 1 (or by Theorem 1). Moreover, we have that $x^Tg(x) = 0$ holds only at the origin. This leads to the global-like stabilization of the origin while the control input is applied in the form of (4). □

The following example illustrates the application of Theorem 4.

Example 4: Consider a planar homogeneous system of order 3 as given by

$$\dot{x} = g(x)u, \quad \text{with} \quad g(x) = \begin{pmatrix} x_1^3 - 2x_1x_2^2 \\ x_2^3 \end{pmatrix}, \tag{25}$$

where $x = (x_1, x_2)^T$. It is easy to check that

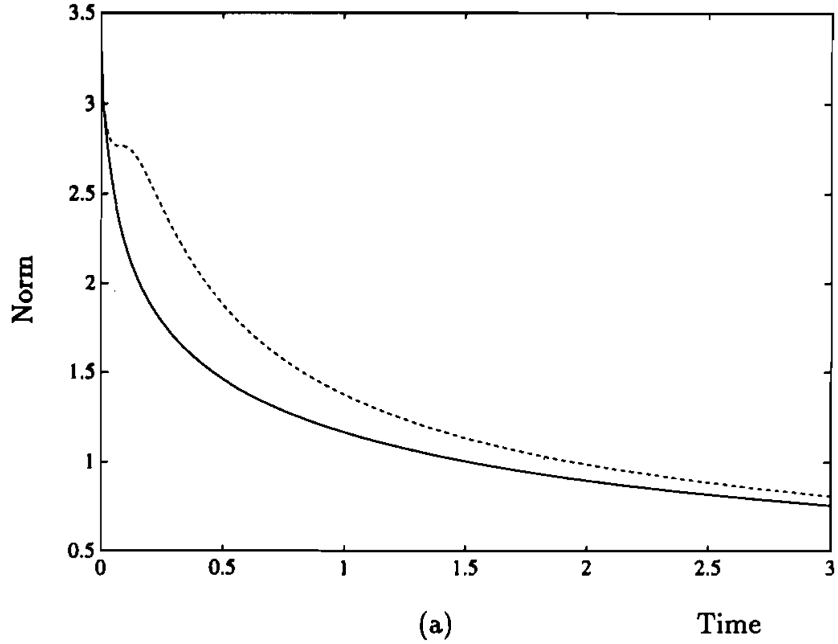
$$x^Tg(x) = (x_1^2 - x_2^2)^2 \tag{26}$$

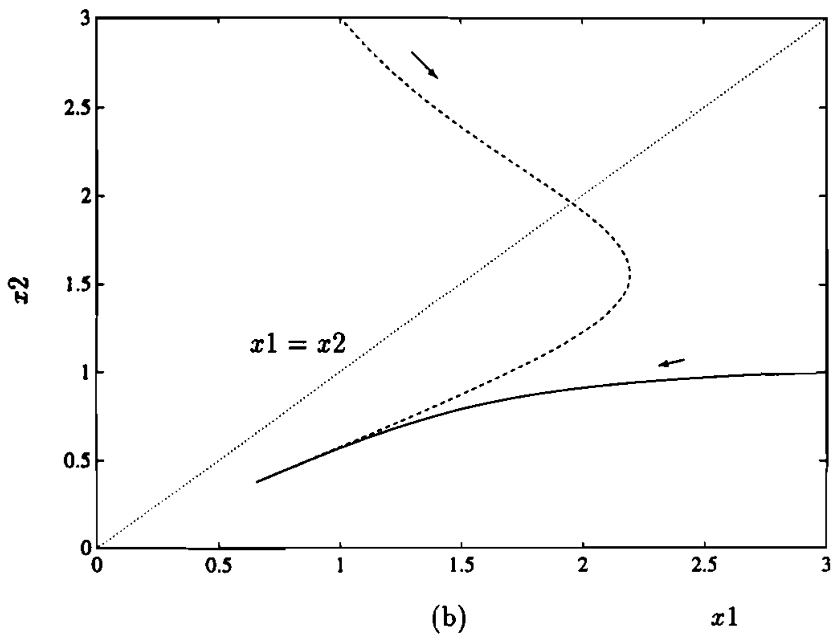
is also a homogeneous function and $x^Tg(x) > 0$ except for all x lying on the lines $x_1 = \pm x_2$. By the application of Theorem 4, we can conclude that the origin of system (25) is globally asymptotically stabilizable. Numerical simulations are given in the Figure to demonstrate the conclusions. In the Figure, the dotted line denotes the timing trajectory corresponding to the initial position at $x_0 = (1, 3)^T$ and the solid line is that for the initial position at $x_0 = (3, 1)^T$. In the simulations, the control input is chosen as $u = -1$ and the vertical varible of part (a) of the Figure denotes the two-norm value of the state variables. □

The next example illustrates the application of Corollary 2.

Example 5: Consider a two-input bilinear driftless system given by

$$\dot{x} = u_1B_1x + u_2B_2x \tag{27}$$





(a) Norm and (b) phase trajectories of Example 4.

with

$$B_1 = \begin{pmatrix} -2 & 0 & 0 \\ -10 & 5 & -3 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} -1 & 0 & 0 \\ -6 & 4 & -1 \\ 3 & 0 & 2 \end{pmatrix}. \tag{28}$$

It is not difficult to check that both matrices B_1 and B_2 are indefinite but commutative. From Lemma 4, we know that these two matrices can be simultaneously transformed into upper triangular matrices by some unitary matrix S. Following the algorithm as given in the Appendix, one choice of such matrix S can be obtained as $S = S_1 \cdot S_2$, where

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{29}$$

The matrix S is hence calculated as

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (30)$$

We then have

$$S^{-1}B_1S = \begin{pmatrix} 5 & -3 & 10 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

and

$$S^{-1}B_2S = \begin{pmatrix} 4 & -1 & -6 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}. \quad (31)$$

According to Theorem 8, we can choose the stabilizing control laws of $u_1 = 2$ and $u_2 = -3$. In such a design, the eigenvalues of the closed-loop system are obtained as -2, -8 and -1. This leads to the asymptotic stability of system (27). Note that, by Theorem 8, the state feedback control law of $u = -(4x^TB_1x, 9x^TB_2x)^T$ can be another choice for providing the stability of the origin. □

Another practical example of the driftless system regarding the stabilization of a satellite's orbital motion can be found in Ahmed and Sen (1980, 1981). The stabilizability of that system can be determined by Theorem 1 with the matrix P being an identity matrix.

5. Conclusions

In this paper, we have established existence conditions of the asymptotic stabilizer for nonlinear driftless systems. The asymptotic stabilizers are explicitly obtained either in state feedback form or in constant gain form. These results are then applied to the study of homogeneous-type systems with positive order.

Appendix

In the following, we propose an algorithm to construct a unitary matrix S of Lemma 4. First, we state the following two results.

Lemma 5: *Suppose any two matrices of the set $\{B_1, \ldots, B_p\}$ are commuting. Then there exists a common eigenvector of $B_1, \ldots, B_p$. That is, there exists a non-zero vector v satisfying*

$$B_iv = \lambda_iv \text{ for some } \lambda_i \in \lambda(B_i) \text{ and for all } i = 1, \ldots, p, \quad (A\ 1)$$

where $\lambda(\cdot)$ denotes the eigenvalue of the corresponding matrix.

Proof: The proof of this result can be found in Lemma 1.3.17 of Horn and Johnson (1985). Moreover, a constructive way of finding common eigenvector v can also be found in problem 1.3.11 of Horn and Johnson (1985). □

Lemma 6: *Let $F_p = \{B_1, \ldots, B_p\}$ be a commuting family of matrices of $\mathbb{C}^{n\times n}$. Suppose S is a unitary matrix satisfying*

$$S^{-1}B_iS = \begin{pmatrix} \lambda_i & M_i \\ 0 & \hat{B}_i \end{pmatrix} \text{ for all } i = 1, \ldots, p, \quad (A\ 2)$$

where $\lambda_i \in \mathbb{C}$, $\hat{B}_i \in \mathbb{C}^{(n-1)\times(n-1)}$ and M_i denotes some matrix. Then, $\hat{F}_p = \{\hat{B}_1, \ldots, \hat{B}_p\}$ is also a commuting family of matrices of $\mathbb{C}^{(n-1)\times(n-1)}$.

Proof: Since $B_iB_j = B_jB_i$, by (A 2) we have

$$S\begin{pmatrix} \lambda_i & M_i \\ 0 & \hat{B}_i \end{pmatrix},\begin{pmatrix} \lambda_j & M_j \\ 0 & \hat{B}_j \end{pmatrix}S^{-1} = S\begin{pmatrix} \lambda_j & M_j \\ 0 & \hat{B}_j \end{pmatrix}\begin{pmatrix} \lambda_i & M_i \\ 0 & \hat{B}_i \end{pmatrix}S^{-1}. \quad (A\ 3)$$

This results in

$$\begin{pmatrix} \lambda_i\lambda_j & \lambda_iM_j + M_i\hat{B}_j \\ 0 & \hat{B}_i\hat{B}_j \end{pmatrix} = \begin{pmatrix} \lambda_j\lambda_i & \lambda_jM_i + M_j\hat{B}_i \\ 0 & \hat{B}_j\hat{B}_i \end{pmatrix}. \quad (A\ 4)$$

Thus, we have $\hat{B}_i\hat{B}_j = \hat{B}_j\hat{B}_i$, the conclusion is hence implied. □

Now, we can construct an algorithm to derive matrix S as follows.

Step 1. Let $j = 0$.

Step 2. Given a commuting family of matrices in $\mathbb{C}^{(n-j)\times(n-j)}$, say $\{B_{1j}, \ldots, B_{pj}\}$.

Find a common eigenvector v of $\{B_{1j}, \ldots, B_{pj}\}$. That is, find v such that

$$B_{ij}v = \lambda_iv \quad \text{for some} \quad \lambda_i \in \lambda(B_{ij}) \quad \text{and for all} \quad i = 1, \ldots, p. \quad (A\ 5)$$

Step 3. Construct a unitary matrix $\tilde{S}_j$ by placing v_1, which is obtained in Step 1, in the first column. This results in

$$\tilde{S}_j^{-1}B_{ij}\tilde{S}_j = \begin{pmatrix} \lambda_i & M \\ 0 & B_{i,j+1} \end{pmatrix} \text{ for all } i = 1, \ldots, p, \quad (A\ 6)$$

where $M \in \mathbb{C}^{1\times(n-j-1)}$ and $B_{i,j+1} \in \mathbb{C}^{(n-j-1)\times(n-j-1)}$.

Step 4. If $j = n - 1$, then go to Step 5. Otherwise, let $j = j + 1$ and go to Step 2.

Step 5.

$$S = S_{n-1} \ldots S_1S_0,$$

where

$$S_j = \begin{cases} \begin{pmatrix} I_j & 0 \\ 0 & \tilde{S}_j \end{pmatrix} \in \mathbb{C}^{n \times n} & \text{for } j > 0, \\ \tilde{S}_0 & \text{for } j = 0. \end{cases} \quad \text{(A 7)}$$

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