國 立 交 通 大 學

統 計 學 研 究 所

博 士 論 文

相依正隨機變數之合其分佈函數之最佳化超立方體估計

An optimal hypercube approximation of the distribution function of sum of positive dependent random variables

研 究 生：盧信銘

指導教授：彭南夫 博士

中華民國九十六年十二月

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摘　　　要

在給定聯合分佈函數之下，對於計算多個相依的正隨機變數合之分佈函數以及機率函數，我們提供一個最佳化的幾何數值方法。這個方法包含對超立方體的積分並估計”半”超立方體的體積。同時我們也提供了數值分析的結果。

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Student: Hsin Ming Lu Advisor: Dr. Nan Fu Peng

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Abstract

We present optimal geometric numerical methods for computing the distribution function and the density function of sum of several positive dependent random variables with known joint distribution function. This method involves integration on high dimensional hypercubesm, and estimating the volume of "half" hypercubes. Numerical results are also presented.

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盧信銘　　　謹誌于

國立交通大學統計研究所

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An optimal hypercube approximation of the distribution function of sum of positive dependent random variables

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Abstract

We present optimal geometric numerical methods for computing the distribution function and the density function of sum of several positive dependent random variables with known joint distribution function. This method involves integration on high dimensional hypercubes, and estimating the volume of "half" hypercubes. Numerical results are also presented.

1. INTRODUCTION

The aim of this paper is to present some geometric ideas on numerically computing the distribution function and the density function of sum of k positive dependent continuous random variables. This problem is originated from acquiring the distribution function and the density function of a renewal in a renewal process in which a renewal contains several dependent steps, each is Weibull distributed. Unlike independent r.v.'s, there have been few papers concentrated on sum of dependent r.v.'s. The result of Serfozo (1986) need strong restrictions on dependent r.v.'s. We offer an innovative idea to express the approximations of distribution function and density function of sum of positive dependent random variables in a closed form. Our numerical method deal with sum of arbitrary k positive dependent r.v.'s as long as their joint distribution function or joint survival function are known and their joint density function is uniformly continuous.

Assume $X_1, X_2, ..., X_k$ are r.v.'s with joint distribution function $G(x_1, x_2, ..., x_k)$ or joint survival function $S(x_1, x_2, ..., x_k)$. Let

$$R_+^k = \left\{ \boldsymbol{x} \in R^k : \ x_i \geq 0, \ \text{for } i = 1, 2, ..., k \right\}.$$

For $\boldsymbol{a} \in R_+^k$ and $b > 0$, we define a hypercube in R_+^k that

$$B_{b,\ \boldsymbol{a}} = \left\{ \boldsymbol{x} \in R_+^k : \ x_i \in [a_i, a_i + b], \ \text{for } i = 1, 2, ..., k \right\}, \tag{1}$$

and define a "half" hypercube for $s \in (0, k)$

$$B_{b,\ \boldsymbol{a}}^s = \left\{ \boldsymbol{x} \in R_+^k : \ \boldsymbol{x} \in B_{b,\ \boldsymbol{a}} \text{ and } \mathbf{1}'\boldsymbol{a} \leq \mathbf{1}'\boldsymbol{x} \leq \mathbf{1}'\boldsymbol{a} + bs \right\} \tag{2}$$

where $\mathbf{1} \in R_+^k$ is a column vector with each element being 1. The hyperplane that is tangent to $B_{b,\ \boldsymbol{a}}^s$ is $\left\{ \boldsymbol{x} \in R^k : \ \mathbf{1}'\boldsymbol{x} = \mathbf{1}'\boldsymbol{a} + bs \right\}$. We call the "half" hypercube $B_{b,\ \boldsymbol{a}}^s$ a type-s set with size b and tail $\boldsymbol{a}$. Without loss of generality, let $k = 3$ and we present type-1 set and type-2 set with size 1 and tail $\mathbf{0}$ in figure 1.

To approximate the distribution function $F(t) = \int_{B_{t,\ \mathbf{0}}^1} g(\boldsymbol{x}) d\boldsymbol{x}$ where g is the joint density function, we in section 2 decompose $B_{t,\ \mathbf{0}}^1$ into union of subsets of the form (1) and (2) and then estimate $B_{t,\ \mathbf{0}}^1$ by some unions and exclusions of hypercubes.

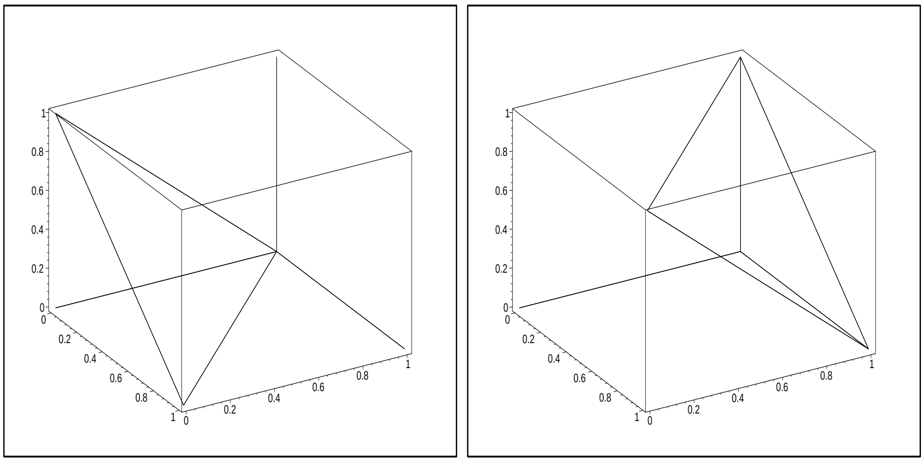

Figure 1: The left is type-1 set and the right is type-2 set.

Integration of g on a hypercube can be obtained easily in terms of G, see Durrett (1994) p120 and p121 as the following formula.

$$P(a_i \leq X_i \leq b_i,\ \text{for } i = 1, 2, ..., k) = \sum_{i=0}^{k} (-1)^i \sum_{\boldsymbol{c}_i \in D_i} G(c_{1_i}, c_{2_i}, ..., c_{k_i}),$$

where $\boldsymbol{c}_i = (c_{1_i}, c_{2_i}, ..., c_{k_i}) \in D_i$ and

$$D_i = \{(d_1, d_2, ..., d_k) : \text{ exact } i\ d's \text{ are } a's \text{ and } (k-i)\ d's \text{ are } b's\}.$$

The key of our method is to find a suitable hypercube, or a few hypercubes, to replace a $B^s_{b,\ \boldsymbol{a}}$ contained in $B^1_{t,\ \boldsymbol{0}}$. To do so, the volume of that $B^s_{b,\ \boldsymbol{a}}$ has to be known. Unlike Albert (2002), Barrow and Smith (1979) and Mitra (1971) in which are presented the distribution function of sum of uniformly distributed r.v.'s, we propose a recursive formula which is easier to obtain. In section 3, we present an optimal approximation and some compared approximations of $F(t)$. In section 4, we present an optimal approximation of $f(t) = \frac{dF(t)}{dt}$. Numerical results are attached in section 5. The first example is of sum of independent Gamma r.v.'s, and it's shown that the excellent performance to the hypercube approximation. The second example involves sum of i.i.d. Weibull r.v.'s, and it's well known that the Weibull model plays an important role in many fields such as reliability applications and so on, see Pham and Lai (2007). Santos Filho and Yacoub(2006) deal with the approximation of probability density function and distribution function of sum of i.i.d. Weibull r.v.'s in a simple and closed form. In our method, not only gives a simple and closed form also offers a good performance in precision of Weibull sums. The third example is about the dependent r.v.'s and some cases can be found in Nelson (1999). Some remaining discussions are listed in section 6.

2. DECOMPOSITION OF $B^1_{t,\ \mathbf{0}}$ AND THE VOLUME OF $B^s_{b,\ \boldsymbol{a}}$

In this section, we present a decomposition of $B^1_{t,\ \mathbf{0}}$ and furthermore this decomposition can be used to obtain the volume of $B^s_{b,\ \boldsymbol{a}}$ contained in $B^1_{t,\ \mathbf{0}}$. For a positive integer j, defining

$$\omega^{(j)} = \left\{ \boldsymbol{x} \in R^k_+ :\ \mathbf{1}'\boldsymbol{x} = j \text{ and } x_i \in Z^+ \cup \{0\}\,,\ \text{for } i = 1, 2, ..., k \right\},$$

we have the following decomposition of $B^1_{t,\ \mathbf{0}}$.

Theorem 1. *Let $c \in (0,1)$. If $\left[\frac{1}{c}\right] < k$, then*

$$B^1_{t,\ \mathbf{0}} = \bigcup_{j=0}^{\left[\frac{1}{c}\right]} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}}\ , \tag{3}$$

and if $\left[\frac{1}{c}\right] \geq k$, then

$$B^1_{t,\ \mathbf{0}} = \left(\bigcup_{j=0}^{\left[\frac{1}{c}\right]-k} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{ct,ct\boldsymbol{y}} \right) \bigcup \left(\bigcup_{j=\left[\frac{1}{c}\right]-k+1}^{\left[\frac{1}{c}\right]} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}} \right) . \tag{4}$$

Furthermore, the intersection of any two sets in (3) *and* (4) *has Lebesque measure zero.*

Proof. We prove (4) only, for (3) can be obtained from (4). Suppose $\left[\frac{1}{c}\right] \geq k$ and let

$$\bar{B}^1_{t,\ \mathbf{0}} = \left(\bigcup_{j=0}^{\left[\frac{1}{c}\right]-k} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{ct,ct\boldsymbol{y}} \right) \bigcup \left(\bigcup_{j=\left[\frac{1}{c}\right]-k+1}^{\left[\frac{1}{c}\right]} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{ct,ct\boldsymbol{y}} \right) .$$

By the structure of $\bar{B}^1_{t,\ \mathbf{0}}$, it is obvious that the intersection of any two sets in $\bar{B}^1_{t,\ \mathbf{0}}$ has Lebesque measure zero. For $\boldsymbol{x} \in B^1_{t,\ \mathbf{0}}$, we observe that there exists k nonnegative integers $d_1, d_2, ..., d_k$ such that $x_i \in [ctd_i, ct(d_i+1)]$. Since $t \geq \sum_{i=1}^k x_i \geq \sum_{i=1}^k ctd_i$, we have $\sum_{i=1}^k d_i \leq \left[\frac{1}{c}\right]$. Hence $\boldsymbol{x} \in B_{ct,ct\boldsymbol{d}} \subset \bar{B}^1_{t,\ \mathbf{0}}$. Therefore, $B^1_{t,\ \mathbf{0}} \subset \bar{B}^1_{t,\ \mathbf{0}}$.

For $0 \leq j \leq \left[\frac{1}{c}\right] - k$, we have $\frac{1}{c} \geq j + k$. Therefore, if $\boldsymbol{y} \in \omega^{(j)}$ and $\boldsymbol{x} \in B_{ct,ct\boldsymbol{y}}$, then $x_i \in [cty_i, ct(y_i+1)] \subset [0,t]$ and

$$0 \leq ctj = ct\mathbf{1}'\boldsymbol{y} \leq \mathbf{1}'\boldsymbol{x} \leq ct\mathbf{1}'(\boldsymbol{y}+\mathbf{1}) = ct(j+k) \leq ct \cdot \frac{1}{c} = t.$$

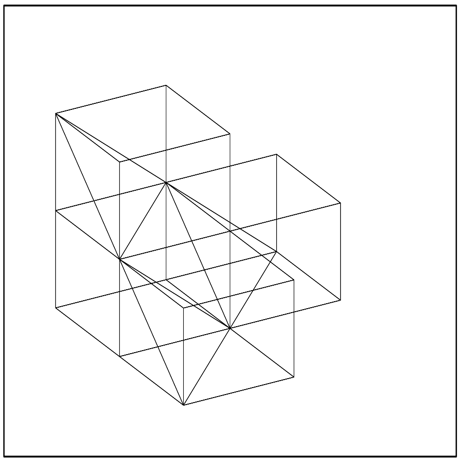

Figure 2: The unions of type-1 set and type-2 set in three-dimension.

Hence $\boldsymbol{x} \in B^1_{t,\,\mathbf{0}}$. We have $B_{ct,ct\boldsymbol{y}} \subset B^1_{t,\,\mathbf{0}}$ or $B_{ct,ct\boldsymbol{y}} \bigcap B^1_{t,\,\mathbf{0}} = B_{ct,ct\boldsymbol{y}}$.

For $\left[\frac{1}{c}\right] - k + 1 \le j \le \left[\frac{1}{c}\right]$ and $\boldsymbol{y} \in \omega^{(j)}$, if $\boldsymbol{x} \in B_{ct,ct\boldsymbol{y}} \bigcap B^1_{t,\,\mathbf{0}}$, then $x_i \in [cty_i, ct(y_i+1)]$ and

$$ct\mathbf{1}'\boldsymbol{y} \le \mathbf{1}'\boldsymbol{x} \le t = ctj + ct\left(\frac{1}{c} - j\right) = ct\mathbf{1}'\boldsymbol{y} + ct\left(\frac{1}{c} - j\right).$$

Hence $\boldsymbol{x} \in B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}}$ and we have $B_{ct,ct\boldsymbol{y}} \bigcap B^1_{t,\,\mathbf{0}} \subset B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}}$. Conversely, if $\boldsymbol{x} \in B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}}$ then $x_i \in [cty_i, ct(y_i+1)] \subset [0,t]$ and

$$0 \le ctj = ct\mathbf{1}'\boldsymbol{y} \le \mathbf{1}'\boldsymbol{x} \le ct\mathbf{1}'\boldsymbol{y} + ct\left(\frac{1}{c} - j\right) = t.$$

Hence $\boldsymbol{x} \in B^1_{t,\,\mathbf{0}}$ and so $B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}} \subset B_{ct,ct\boldsymbol{y}} \bigcap B^1_{t,\,\mathbf{0}}$. We have $B^{\frac{1}{c}-j}_{ct,ct\boldsymbol{y}} = B_{ct,ct\boldsymbol{y}} \bigcap B^1_{t,\,\mathbf{0}}$, then (4) holds. $\square$

Corollary 1. *For a positive integer $n \ge k$,*

$$\begin{aligned} B^1_{t,\,\mathbf{0}} &= \left(\bigcup_{j=0}^{n-k} \bigcup_{\boldsymbol{y} \in \omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \right) \bigcup \left(\bigcup_{j=n-k+1}^{n} \bigcup_{\boldsymbol{y} \in \omega^{(j)}} B^{n-j}_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \right) \\ &= \left(\bigcup_{j=0}^{n-k} \bigcup_{\boldsymbol{y} \in \omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \right) \bigcup \left(\bigcup_{j=0}^{k-1} \bigcup_{\boldsymbol{y} \in \omega^{(n-j)}} B^{j}_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \right). \end{aligned} \tag{5}$$

According to corollary 1, for any "section size" $n \in \mathcal{N}$ and $n \geq k$, we can decompose $B^1_{t,\ \mathbf{0}}$ into unions of hypercubes and unions of type-j set, for $j = 0, 1, 2, ..., k-1$. In figure 2, we show the unions of type-1 set and type-2 set in three dimension. Note that type-0 set has Lebesque measure zero. Denote m_k the Lebesque measure in R^k. It is obvious that

$$m_k(B^s_{b,\ \boldsymbol{a}}) = b^k m_k(B^s_{1,\ \boldsymbol{a}}) = b^k m_k(B^s_{1,\ \mathbf{0}}).$$

The following theorem is an easy consequence of theorem 1.

Theorem 2. *For* $s \in (0, 1]$

$$m_k(B^s_{1,\ \mathbf{0}}) = \frac{s^k}{k!}, \tag{6}$$

and for $s \in (1, k)$

$$m_k(B^s_{1,\ \mathbf{0}}) = \frac{s^k}{k!} - \sum_{x=1}^{[s]} C^{k+x-1}_x m_k(B^{s-x}_{1,\ \mathbf{0}}). \tag{7}$$

PROOF. (6) is trivial and we prove (7) only. For $s \in (1, k)$, we let $c = \frac{1}{s}$ and apply (3) to obtain

$$\begin{aligned} m_k(B^1_{s,\ \mathbf{0}}) &= \sum_{x=0}^{[s]} \sum_{\boldsymbol{y} \in \omega^{(x)}} m_k(B^{s-x}_{1,\ \boldsymbol{y}}) \\ &= \sum_{x=0}^{[s]} \sum_{\boldsymbol{y} \in \omega^{(x)}} m_k(B^{s-x}_{1,\ \mathbf{0}}) \\ &= \sum_{x=0}^{[s]} C^{k+x-1}_x m_k(B^{s-x}_{1,\ \mathbf{0}}). \end{aligned}$$

The last equality holds since the number of $\boldsymbol{y} \in \omega^{(x)}$ is C^{k+x-1}_x, and $m_k(B^1_{s,\ \mathbf{0}}) = \frac{s^k}{k!}$, we obtain (7). □

Note that the volume of $B^s_{1,\ \mathbf{0}}$ also represents the distribution function of sum of uniformly distributed r.v.'s that have been shown in Albert (2002), Barrow and Smith (1979) and Mitra (1971). In a geometric viewpoint, we present the volume of $B^s_{1,\ \mathbf{0}}$ through a recursive formula in theorem 2 above. Therefore, we have the volume of type-s set with size b and tail $\boldsymbol{a}$, that is $m_k(B^s_{b,\ \boldsymbol{a}})$.

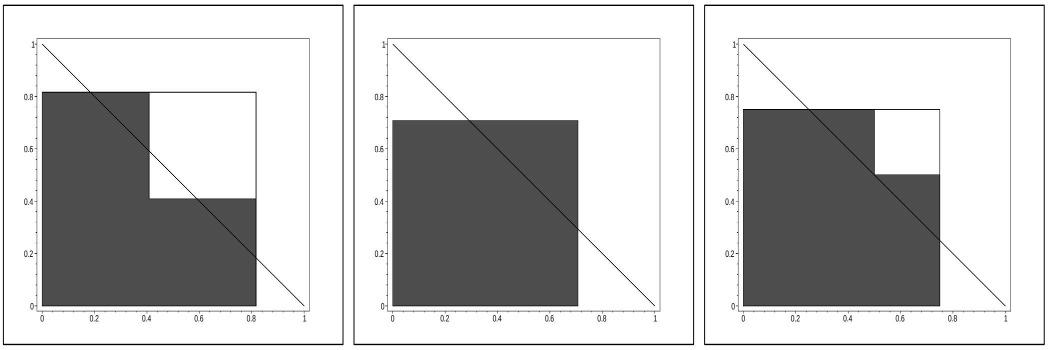

Figure 3: The left is optimal case, the middle is case 1 and the right is case 2

3. AN OPTIMAL APPROXIMATION OF $F(t)$

We in this section provide an optimal volume-invariant approximation to $F(t)$. That is, we present an optimal set $\hat{B}^1_{t,\ \mathbf{0}}$ such that $m_k\left(\hat{B}^1_{t,\ \mathbf{0}}\right) = m_k\left(B^1_{t,\ \mathbf{0}}\right)$ and approximate $F(t) = \int_{B^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x})$ by $\int_{\hat{B}^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x})$. We first find an optimal estimate of $B^j_{\frac{t}{n},\ \frac{t}{n}\boldsymbol{y}}$ the type-j set with tail $\frac{t}{n}\boldsymbol{y}$ and size $\frac{t}{n}$ and, without loss of generality, manipulate this method in $B^j_{1,\ \mathbf{0}}$ the type-1 set with tail $\mathbf{0}$ and size 1, for $1 \leq j \leq k-1$ and $j \in Z^+$. Let

$$
\begin{aligned}
A_j = \Big\{ B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}} :\ & 0 < d < l \leq 1 \\
& \text{and } m_k\left(B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}}\right) = m_k\left(B^j_{1,\ \mathbf{0}}\right) \Big\}
\end{aligned} \tag{8}
$$

be a class of sets, each of which is the difference of two hypercubes and volume-invariant from $B^j_{1,\ \mathbf{0}}$. Given l, the d in (8) is $d = \left(l^k - m_k\left(B^j_{1,\ \mathbf{0}}\right)\right)^{\frac{1}{k}}$. First, We present two different volume-invariant sets as the following.

Case 1: $d_{1j} = 0$ and $l_{1j} = \left(m_k\left(B^j_{1,\ \mathbf{0}}\right)\right)^{\frac{1}{k}}$;

Case 2: d_{2j} and l_{2j} satisfy $l_{2j} - d_{2j} = \frac{j}{k}$ and $l^k_{2j} - d^k_{2j} = m_k\left(B^j_{1,\ \mathbf{0}}\right)$, where $B_{l_{ij},\ \mathbf{0}} \setminus B_{d_{ij},\ (l_{ij}-d_{ij})\mathbf{1}} \in\ A_j$, $i = 1, 2$.

Next, finding the optimal α_j and β_j as the following. Let $0 < \alpha_j < \beta_j \leq 1$ satisfy

$$
\begin{aligned}
& m_k\left(B^j_{1,\ \mathbf{0}} \setminus \left(B_{\beta_j,\ \mathbf{0}} \setminus B_{\alpha_j,\ (\beta_j-\alpha_j)\mathbf{1}}\right)\right) \\
=& m_k\left(\left(B_{\beta_j,\ \mathbf{0}} \setminus B_{\alpha_j,\ (\beta_j-\alpha_j)\mathbf{1}}\right) \setminus B^j_{1,\ \mathbf{0}}\right) \\
=& \inf_{B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}} \in A_j} m_k\left(\left(B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}}\right) \setminus B^j_{1,\ \mathbf{0}}\right).
\end{aligned}
$$

The first equality holds since $m_k\left(B_{\beta_j,\ \mathbf{0}} \setminus B_{\alpha_j,\ (\beta_j-\alpha_j)\mathbf{1}}\right) = m_k\left(B^j_{1,\ \mathbf{0}}\right)$. It is easy to see that the α_j and β_j can be minimized the nonoverlapping part of $B^j_{1,\ \mathbf{0}}$ and $B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}} \in A_j$. In figure 3, we present the optimal volume-invariant set and two different cases as above in two-dimension. The dark parts in figure 3 represents the estimate of $B^j_{1,\ \mathbf{0}}$. To obtain the optimal α_j and β_j, we need the following theorem.

Theorem 3. *For $B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}} \in A_j$, we have*

$$
\begin{aligned}
& m_k\left(\left(B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}}\right) \setminus B^j_{1,\ \mathbf{0}}\right) \\
=& m_k\left(B^j_{1,\ \mathbf{0}}\right) - l^k m_k\left(B^{\frac{j}{l}}_{1,\ \mathbf{0}}\right) + I\left(l-d < \frac{j}{k}\right) d^k m_k\left(B^{\frac{j-k(l-d)}{d}}_{1,\ \mathbf{0}}\right),
\end{aligned}
$$

where I is an indicator function.

PROOF. If $l-d \geq \frac{j}{k}$, then $m_k\left(B_{d,\ (l-d)\mathbf{1}} \bigcap B^j_{1,\ \mathbf{0}}\right) = 0$. We have

$$
\begin{aligned}
& m_k\left(\left(B_{l,\ \mathbf{0}} \setminus B_{d,\ (l-d)\mathbf{1}}\right) \setminus B^j_{1,\ \mathbf{0}}\right) \\
=& m_k\left(\left(B_{l,\ \mathbf{0}} \setminus B^j_{1,\ \mathbf{0}}\right) \setminus B_{d,\ (l-d)\mathbf{1}}\right) \\
=& m_k\left(B_{l,\ \mathbf{0}} \setminus B^j_{1,\ \mathbf{0}}\right) - m_k\left(\left(B_{l,\ \mathbf{0}} \setminus B^j_{1,\ \mathbf{0}}\right) \bigcap B_{d,\ (l-d)\mathbf{1}}\right) \\
=& m_k\left(B_{l,\ \mathbf{0}}\right) - m_k\left(B_{l,\ \mathbf{0}} \bigcap B^j_{1,\ \mathbf{0}}\right) - m_k\left(B_{d,\ (l-d)\mathbf{1}}\right) \\
=& l^k - l^k m_k\left(B_{1,\ \mathbf{0}} \bigcap B^j_{\frac{1}{l},\ \mathbf{0}}\right) - d^k \\
=& l^k - d^k - l^k m_k\left(B^{\frac{j}{l}}_{1,\ \mathbf{0}}\right) \\
=& m_k\left(B^j_{1,\ \mathbf{0}}\right) - l^k m_k\left(B^{\frac{j}{l}}_{1,\ \mathbf{0}}\right).
\end{aligned}
$$

The second equality from the bottom holds since $\frac{1}{l} > 1$. If $l-d < \frac{j}{k}$, then

$$
m_k\left(B_{d,\ (l-d)\mathbf{1}} \bigcap B^j_{1,\ \mathbf{0}}\right) = m_k\left(B^{\frac{j-k(l-d)}{d}}_{d,\ (l-d)\mathbf{1}}\right) = d^k m_k\left(B^{\frac{j-k(l-d)}{d}}_{1,\ \mathbf{0}}\right).
$$

Hence we have theorem 3. □

Some numerical results of α_j and β_j are listed in table 1, for $j = 0, 1, ..., k-1$. Note that for $j = 0$, we let $\alpha_0 = \beta_0 = 0$.

Let

$$
\hat{B}^1_{t,\ \mathbf{0}} = \left(\bigcup_{j=0}^{n-k} \bigcup_{\boldsymbol{y} \in \omega^{(j)}} B_{\frac{t}{n}, \frac{t}{n}\boldsymbol{y}}\right) \bigcup \left(\bigcup_{j=0}^{k-1} \bigcup_{\boldsymbol{y} \in \omega^{(n-j)}} \left(B_{\frac{t}{n}\beta_j, \frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\alpha_j, \frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right)\right)
$$

Table 1: The optimal α_j and β_j

k	j	β_j	α_j	k	j	β_j	α_j
2	1	0.816497	0.408249	6	1	0.349801	0.276119
					2	0.687728	0.541633
3	1	0.602942	0.374505		3	0.927982	0.719393
	2	0.999999	0.550319		4	0.999999	0.657172
					5	1.000000	0.334008
4	1	0.483559	0.337726				
	2	0.895692	0.615614	7	1	0.307856	0.251513
	3	1.000000	0.451797		2	0.610611	0.498411
					3	0.856245	0.693457
5	1	0.405585	0.305058		4	0.988259	0.783328
	2	0.782639	0.585206		5	0.999999	0.586961
	3	0.992610	0.716314		6	1.000000	0.295597
	4	1.000000	0.383834				

be our estimate of $B^1_{t,\,\mathbf{0}}$. The second part of the right hand side of $\hat{B}^1_{t,\,\mathbf{0}}$ represents the estimate of the unions of $B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}$, for $\boldsymbol{y} \in \omega^{(n-j)}$ and $j = 0, 1, ..., k-1$. In each j, we have

$$
\begin{aligned}
m_k\left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right) &= \left(\frac{t}{n}\right)^k m_k\left(B_{\beta_j,\,\boldsymbol{y}} \setminus B_{\alpha_j,\,\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}}\right) \\
&= \left(\frac{t}{n}\right)^k m_k\left(B_{\beta_j,\,\mathbf{0}} \setminus B_{\alpha_j,\,(\beta_j-\alpha_j)\mathbf{1}}\right) \\
&= \left(\frac{t}{n}\right)^k m_k\left(B^j_{1,\,\mathbf{0}}\right) = m_k\left(B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}\right).
\end{aligned}
$$

Therefore, we obtain $m_k\left(B^1_{t,\,\mathbf{0}}\right) = m_k\left(\hat{B}^1_{t,\,\mathbf{0}}\right)$. In the other hand, the nonoverlapping parts of $B^1_{t,\,\mathbf{0}}$ and $\hat{B}^1_{t,\,\mathbf{0}}$ are minimized, thus $\hat{B}^1_{t,\,\mathbf{0}}$ would be a reasonable and optimal estimate of $B^1_{t,\,\mathbf{0}}$ of (5).

Furthermore, our approximation $\hat{F}(t)$ to $F(t)$ is

$$
\begin{aligned}
\hat{F}(t) &= \int_{\hat{B}^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}\in\omega^{(j)}} \int_{B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}\in\omega^{(n-j)}} \left[\int_{B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \right. \\
&\qquad \left. - \int_{B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \right] \qquad (9) \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}\in\omega^{(j)}} P\left(\frac{t}{n} y_i \le X_i \le \frac{t}{n}(y_i+1),\ i=1,2,...,k \right) \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}\in\omega^{(n-j)}} \left[P\left(\frac{t}{n} y_i \le X_i \le \frac{t}{n}(y_i+\beta_j),\ i=1,2,...,k \right) \right. \\
&\left. -P\left(\frac{t}{n}[y_i+(\beta_j-\alpha_j)] \le X_i \le \frac{t}{n}(y_i+\beta_j),\ i=1,2,...,k \right) \right].
\end{aligned}
$$

The following theorem shows that $\hat{F}(t)$ converges to $F(t)$ when section size n tends to infinity under the condition that the joint density function g is uniformly continuous. Before the theorem, we need the following lemma which is elementary.

Lemma 1. *Given n and t, suppose that g is uniformly continuous in R^k_+, and for all $\boldsymbol{y} \in R^k_+$ and $\boldsymbol{z},\ \boldsymbol{w} \in B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}$, then we have*

$$\mid g(\boldsymbol{z}) - g(\boldsymbol{w}) \mid \le \frac{M}{n},\ \textit{for some } M > 0.$$

Theorem 4. *Suppose that g is uniformly continuous in $[0,t]^k$, then*

$$\mid \hat{F}(t) - F(t) \mid \longrightarrow 0 \ \textit{as } n \longrightarrow +\infty.$$

Furthermore, the convergence rate of $\hat{F}(t)$ is n^{-2}.

Proof. Note that

$$
\begin{aligned}
& B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \setminus \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]} \right) \subset B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \\
\text{and } & \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]} \right) \setminus B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \subset B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}.
\end{aligned}
$$

For each $\boldsymbol{y} \in \ \omega^{(n-j)}$, and fixing a point $\boldsymbol{z} \in B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}$, we have by lemma 1 that

$$
\begin{aligned}
& \mid \hat{F}(t) - F(t) \mid \\
=& \mid \int_{\hat{B}^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) - \int_{B^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \mid \\
=& \mid \int_{\hat{B}^1_{t,\ \mathbf{0}} \backslash B^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) - \int_{B^1_{t,\ \mathbf{0}} \backslash \hat{B}^1_{t,\ \mathbf{0}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \mid \\
=& \mid \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} \left(\int_{\left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right) \backslash B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \right. \\
& \qquad \left. - \int_{B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \backslash \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right)} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \right) \mid \\
\leq& \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} \mid \int_{\left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right) \backslash B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
& \qquad - \int_{B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \backslash \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right)} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \mid \\
\leq& \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} \mid \int_{\left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right) \backslash B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}} \left(g(\boldsymbol{z}) + \frac{M}{n} \right) m_k(d\boldsymbol{x}) \\
& \qquad - \int_{B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \backslash \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \backslash B_{\frac{t}{n}\alpha_j,\frac{t}{n}[\boldsymbol{y}+(\beta_j-\alpha_j)\mathbf{1}]}\right)} \left(g(\boldsymbol{z}) - \frac{M}{n} \right) m_k(d\boldsymbol{x}) \mid \\
=& \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} \frac{2M}{n} m_k \left(B^j_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}} \setminus \left(B_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\alpha_j,\frac{t}{n}(\boldsymbol{y}+\beta_j-\alpha_j)\mathbf{1}} \right) \right) \\
=& \frac{2M}{n} \left(\frac{t}{n} \right)^k \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} m_k \left(B^j_{1,\ \mathbf{0}} \setminus \left(B_{\beta_j,\ \mathbf{0}} \setminus B_{\alpha_j,\ (\beta_j-\alpha_j)\mathbf{1}} \right) \right) \\
=& \frac{2M}{n} \left(\frac{t}{n} \right)^k \sum_{j=0}^{k-1} \sum_{\boldsymbol{y} \in \omega^{(n-j)}} \left[m_k \left(B^j_{1,\ \mathbf{0}} \right) - \beta_j^k m_k \left(B^{\frac{j}{\beta_j}}_{1,\ \mathbf{0}} \right) \right. \\
& \qquad \left. + I\left(\beta_j - \alpha_j < \frac{j}{k} \right) \alpha_j^k m_k \left(B^{\frac{j-k\left(\beta_j-\alpha_j\right)}{\alpha_j}}_{1,\ \mathbf{0}} \right) \right] \\
\leq& \frac{2M}{n} \left(\frac{t}{n} \right)^k k c_1 n^{k-1} \longrightarrow 0 \text{ as } n \longrightarrow +\infty,
\end{aligned}
$$

for some $c_1 > 0$. The last inequality holds due to the fact that the number of elements in $\omega^{(n-j)}$ is $C_{n-j}^{k+n-j-1}$ when n is suitably larger than k. $\Box$

In order to show the priority of the optimal approximation of $F(t)$, we also present two compared volume-invariant estimates of $B^1_{t,\ \mathbf{0}}$. For $j = 0, 1, ..., k-1$, the first estimate $\hat{B}^1_{t,\ \mathbf{0}(1)}$, is to let $d_{1j} = 0$ and $l_{1j} = \left(m_k\left(B^j_{1,\ \mathbf{0}}\right)\right)^{\frac{1}{k}}$, and the second estimate $\hat{B}^1_{t,\ \mathbf{0}(2)}$, is to let d_{2j} and l_{2j} satisfy $l^k_{2j} - d^k_{2j} = m_k\left(B^j_{1,\ \mathbf{0}}\right)$ and $l_{2j} - d_{2j} = \frac{j}{k}$, where $B_{l_{ij},\ \mathbf{0}} \setminus B_{d_{ij},\ (l_{ij}-d_{ij})\mathbf{1}} \in A_j$, $i = 1, 2$. That is

$$\hat{B}^1_{t,\ \mathbf{0}(1)} = \left(\bigcup_{j=0}^{n-k} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}\right) \bigcup \left(\bigcup_{j=0}^{k-1} \bigcup_{\boldsymbol{y}\in\omega^{(n-j)}} B_{\frac{t}{n}l_{1j},\frac{t}{n}\boldsymbol{y}}\right)$$

and

$$\hat{B}^1_{t,\ \mathbf{0}(2)} = \left(\bigcup_{j=0}^{n-k} \bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}\right) \bigcup \left(\bigcup_{j=0}^{k-1} \bigcup_{\boldsymbol{y}\in\omega^{(n-j)}} \left(B_{\frac{t}{n}l_{2j},\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}d_{2j},\frac{t}{n}[\boldsymbol{y}+(l_{2j}-d_{2j})\mathbf{1}]}\right)\right).$$

Therefore, we have

$$\begin{aligned}
\hat{F}_{(1)}(t) &= \int_{\hat{B}^1_{t,\ \mathbf{0}(1)}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}\in\omega^{(j)}} P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+1),\ i = 1, 2, ..., k\right) \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}\in\omega^{(n-j)}} P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+l_{1j}),\ i = 1, 2, ..., k\right)
\end{aligned}$$

and

$$\begin{aligned}
\hat{F}_{(2)}(t) &= \int_{\hat{B}^1_{t,\ \mathbf{0}(2)}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}\in\omega^{(j)}} P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+1),\ i = 1, 2, ..., k\right) \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}\in\omega^{(n-j)}} \left[P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+1_{2j}),\ i = 1, 2, ..., k\right)\right. \\
&\left. -P\left(\frac{t}{n}[y_i+(l_{2j}-d_{2j})] \le X_i \le \frac{t}{n}(y_i+l_{2j}),\ i = 1, 2, ..., k\right)\right]
\end{aligned}$$

to be the compared approximations with respect to $\hat{F}(t)$. Some numerical results of $\hat{F}(t)$, $\hat{F}_{(1)}(t)$ and $\hat{F}_{(2)}(t)$ are shown in section 5.

4. AN OPTIMAL APPROXIMATION OF $f(t)$

Our optimal estimate $\tilde{f}(t)$ of $f(t)$ is the derivative of $\tilde{F}(t)$ where $\tilde{F}(t)$ is a slight modification of $\hat{F}(t)$. Therefore, points of R_+^{k-1} show up throughout this section. In order to be consistent with the notations in the previous sections, we denote without alteration a vector in R^k by $\boldsymbol{a}$ or $\boldsymbol{x}$, etc, and denote a vector in R^{k-1} by $\boldsymbol{a}^{(k-1)}$ or $\boldsymbol{x}^{(k-1)}$, etc. Furthermore, we let $\mathbf{1}_i \in R^k$ and $\mathbf{1}_i = (0, ..., 0, 1, 0, ..., 0)$, the ith component of which is one and the others are zero, for $i = 1, 2, ..., k$. For any fixed component index $i \in \{1, 2, ..., k\}$, let

$$D^i_{b,\ \boldsymbol{a}+(s-a_i)\mathbf{1}_i} = \left\{\boldsymbol{x} \in R^k_+ :\ a_j \le x_j \le a_j + b,\ j \ne i,\ x_i = s\right\} \tag{10}$$

be a "hyperdisc" with ith component equals to s and the others fall in the interval $[a_j, a_j + b]$, for $j \ne i$. The hyperdisc $D^i_{b,\ \boldsymbol{a}+(s-a_i)\mathbf{1}_i}$ in (10) is in fact $(k-1)$-dimensional and we denote m_{k-1} the Lebesque measure in R^{k-1}. The next lemma is elementary.

Lemma 2.

$$\begin{aligned} &\frac{d}{dt}\int_{B_{tb,ta}} g(\boldsymbol{x})m_k(d\boldsymbol{x}) \\ &= \sum_{i=1}^{k}\left[(a_i + b)\int_{D^i_{tb,t(\boldsymbol{a}+b\mathbf{1}_i)}} g(\boldsymbol{z})m_{k-1}(d\boldsymbol{z}) - a_i\int_{D^i_{tb,t\boldsymbol{a}}} g(\boldsymbol{z})m_{k-1}(d\boldsymbol{z})\right]. \end{aligned} \tag{11}$$

Let $B^1_{t,\ \mathbf{0}^{(k-1)}} \subset R^{k-1}_+$ be similarly defined as (2), we have

$$f(t) = \int_{B^1_{t,\ \mathbf{0}^{(k-1)}}} g(\boldsymbol{x}^{(k-1)}, t - \mathbf{1}^{(k-1)\prime}\boldsymbol{x}^{(k-1)})m_{k-1}\left(d\boldsymbol{x}^{(k-1)}\right). \tag{12}$$

Note that the following notation $\boldsymbol{y}_{(j)} = \left(y_{(j),1}, y_{(j),2}, ..., y_{(j),k}\right) \in\ \omega^{(j)}$ that coincides with $\boldsymbol{y} = (y_1, y_2, ..., y_k) \in\ \omega^{(j)}$ and it is not difficult to see from (9) and (11) that

$$
\begin{aligned}
\hat{f}(t) &= \frac{d\hat{F}(t)}{dt} \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}_{(j)} \in \omega^{(j)}} \sum_{i=1}^{k} \left[\frac{y_{(j),i}+1}{n} \int_{D^i_{\frac{t}{n},\frac{t}{n}(\boldsymbol{y}_{(j)}+\mathbf{1}_i)}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right. \\
&\qquad \left. - \frac{y_{(j),i}}{n} \int_{D^i_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}_{(j)}}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right] \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}_{(n-j)} \in \omega^{(n-j)}} \sum_{i=1}^{k} \left\{ \left[\frac{y_{(n-j),i}+\beta_j}{n} \int_{D^i_{\frac{t}{n}\beta_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+\beta_j \mathbf{1}_i)}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right. \right. \\
&\qquad \left. - \frac{y_{(n-j),i}}{n} \int_{D^i_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}_{(n-j)}}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right] \\
&\qquad - \left[\frac{y_{(n-j),i}+\beta_j}{n} \int_{D^i_{\frac{t}{n}\alpha_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+(\beta_j-\alpha_j)\mathbf{1}+\alpha_j \mathbf{1}_i)}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right. \\
&\qquad \left. \left. - \frac{y_{(n-j),i}+(\beta_j-\alpha_j)}{n} \int_{D^i_{\frac{t}{n}\alpha_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+(\beta_j-\alpha_j)\mathbf{1})}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right] \right\}.
\end{aligned}
$$

Since $y_{(j),i}+1 = y_{(j+1),i}$ and $\boldsymbol{y}_{(j)}+\mathbf{1}_i \in \omega^{(j+1)}$, we can reduce the first mathematical expression of the right hand side of the above formula to obtain

$$
\begin{aligned}
\hat{f}(t) = \sum_{i=1}^{k} \sum_{\boldsymbol{y}_{(n-k+1)} \in \omega^{(n-k+1)}} & \frac{y_{(n-k+1),i}}{n} \int_{D^i_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}_{(n-k+1)}}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \\
+ \sum_{i=1}^{k} \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}_{(n-j)} \in \omega^{(n-j)}} & \left\{ \left[\frac{y_{(n-j),i}+\beta_j}{n} \int_{D^i_{\frac{t}{n}\beta_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+\beta_j \mathbf{1}_i)}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right. \right. \\
- \frac{y_{(n-j),i}}{n} & \left. \int_{D^i_{\frac{t}{n}\beta_j,\frac{t}{n}\boldsymbol{y}_{(n-j)}}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right] \\
- & \left[\frac{y_{(n-j),i}+\beta_j}{n} \int_{D^i_{\frac{t}{n}\alpha_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+(\beta_j-\alpha_j)\mathbf{1}+\alpha_j \mathbf{1}_i)}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right. \\
- & \left. \left. \frac{y_{(n-j),i}+(\beta_j-\alpha_j)}{n} \int_{D^i_{\frac{t}{n}\alpha_j,\frac{t}{n}(\boldsymbol{y}_{(n-j)}+(\beta_j-\alpha_j)\mathbf{1})}} g(\boldsymbol{z}) m_{k-1}(d\boldsymbol{z}) \right] \right\}.
\end{aligned} \tag{13}
$$

Recall that we called $\hat{F}(t)$ in (9) volume-invariant, and $\hat{f}(t)$ has similar property as we derive as the following. Set $g = 1$ and $m_k\left(B^1_{t,\,\mathbf{0}}\right) = m_k\left(\hat{B}^1_{t,\,\mathbf{0}}\right)$, we have

$$
\begin{aligned}
& m_{k-1}\left(B^1_{t,\,\mathbf{0}^{(k-1)}}\right) \\
& = \frac{dm_k\left(B^1_{t,\,\mathbf{0}}\right)}{dt} = \frac{dm_k\left(\hat{B}^1_{t,\,\mathbf{0}}\right)}{dt} \\
& = \sum_{i=1}^{k} \sum_{\boldsymbol{y}_{(n-k+1)}\in\omega^{(n-k+1)}} \frac{y_{(n-k+1),i}}{n} m_{k-1}\left(D^i_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}_{(n-k+1)}}\right) \\
& + \sum_{i=1}^{k} \sum_{j=n-k+1}^{n} \sum_{\boldsymbol{y}_{(j)}\in\omega^{(j)}} \left\{\left[\frac{y_{(j),i}+\beta_{n-j}}{n} m_{k-1}\left(D^i_{\frac{t}{n}\beta_{n-j},\frac{t}{n}(\boldsymbol{y}_{(j)}+\beta_{n-j}\mathbf{1}_i)}\right)\right.\right. \\
& - \left.\frac{y_{(j),i}}{n} m_{k-1}\left(D^i_{\frac{t}{n}\beta_{n-j},\frac{t}{n}\boldsymbol{y}_{(j)}}\right)\right] \\
& - \left[\frac{y_{(j),i}+\beta_{n-j}}{n} m_{k-1}\left(D^i_{\frac{t}{n}\alpha_{n-j},\frac{t}{n}(\boldsymbol{y}_{(j)}+(\beta_{n-j}-\alpha_{n-j})\mathbf{1}+\alpha_{n-j}\mathbf{1}_i)}\right)\right. \\
& - \left.\left.\frac{y_{(j),i}+(\beta_{n-j}-\alpha_{n-j})}{n} m_{k-1}\left(D^i_{\frac{t}{n}\alpha_{n-j},\frac{t}{n}(\boldsymbol{y}_{(j)}+(\beta_{n-j}-\alpha_{n-j})\mathbf{1})}\right)\right]\right\}.
\end{aligned} \tag{14}
$$

Noting that in (14) the D's with negative sign are not necessarily subsets of the D's with positive sign, we thus from (14) call $\hat{f}(t) = \frac{d\hat{F}(t)}{dt}$ weighted-signed-measure invariant. The derivative of any other volume-invariant approximation of $F(t)$ of the form (9) should also be weighted-signed-measure invariant. The next theorem shows that $\hat{f}(t)$ is a candidate of approximation of $f(t)$. The proof of theorem 5 below is similar to that of theorem 4, and is hence omitted it here.

Theorem 5. *Suppose that g is uniformly continuous, then*

$$\mid \hat{f}(t) - f(t) \mid \longrightarrow 0 \ \textit{as} \ n \longrightarrow +\infty.$$

Furthermore, the convergence rate of $\hat{f}(t)$ is n^{-1}.

Now we proceed acquiring an optimal approximation of $f(t)$. Let $D^{(j)}_k = \left\{\boldsymbol{x} \in R^k_+ : \mathbf{1}'\boldsymbol{x} = j\right\}$ and observe that for $\boldsymbol{y} \in \omega^{(n-j)}$, $D^{(t)}_k$ mounting on $B^j_{\frac{t}{n},\,\frac{t}{n}\boldsymbol{y}}$ is equivalent to $D^{(j)}_k$ mounting on $B^j_{1,\,\mathbf{0}}$. Denote $dis\left(\boldsymbol{x}, D^{(j)}_k\right) = \frac{|j-\mathbf{1}'\boldsymbol{x}|}{\sqrt{k}}$ the distance between the point $\boldsymbol{x}$ and the hyperplane $D^{(j)}_k$. Recall that a set $B_{l,\,\mathbf{0}} \setminus B_{d,\,(l-d)\mathbf{1}}$ in A_j of (8) plays the major role in estimating $B^j_{1,\,\mathbf{0}}$, where $d = \left(l^k - m_k\left(B^j_{1,\,\mathbf{0}}\right)\right)^{\frac{1}{k}}$. Let the set of vertices of $B_{l,\,\mathbf{0}} \setminus B_{d,\,(l-d)\mathbf{1}}$ is

$$Z_l^{(j)} = \left\{\mathbf{0}, l\sum_{b=1}^{m}\mathbf{1}_{i_b}, (l-d)\mathbf{1}, d\sum_{b=1}^{m}\mathbf{1}_{i_b} + (l-d)\mathbf{1}\right.$$
$$\left.\text{with } \{i_1, i_2, ..., i_m\} \subset \{1, 2, ..., k\}\,,\ m = 1, 2, ..., k-1\right\}.$$

Next theorem makes our criterion for choosing optimal $\tilde{f}(t)$ possible.

Theorem 6. *For each j, $1 \le j \le k-1$ and $j \in Z^+$, we have*

$$\sum_{\boldsymbol{x}\in Z_l^{(j)}} dis\left(\boldsymbol{x}, D_k^{(j)}\right) = \frac{1}{\sqrt{k}}\left(\sum_{m=0}^{k-1} C_m^k|j-ml| + \sum_{m=0}^{k-1} C_m^k|j-kl+(k-m)d|\right).$$

PROOF. Since

$$dis\left(l\sum_{b=1}^{m}\mathbf{1}_{i_b}, D_k^{(j)}\right) = \frac{|\,j-ml\,|}{\sqrt{k}}$$

and

$$dis\left(d\sum_{b=1}^{m}\mathbf{1}_{i_b} + (l-d)\mathbf{1}, D_k^{(j)}\right) = \frac{|\,j-[kl-(k-m)d]\,|}{\sqrt{k}}$$

for all $(i_1, i_2, ..., i_m)$, we have the theorem. $\square$

Our optimal approximation $\tilde{f}(t)$ of $f(t)$ must satisfy $\tilde{f}(t) = \frac{d\tilde{F}(t)}{dt}$ and $\tilde{F}(t) = \int_{\tilde{B}^1_{t,\,\mathbf{0}}} g(\boldsymbol{x})m_k(d\boldsymbol{x})$ where

$$\tilde{B}^1_{t,\,\mathbf{0}} = \left(\bigcup_{j=0}^{n-k}\bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}\right)\bigcup\left(\bigcup_{j=0}^{k-1}\bigcup_{\boldsymbol{y}\in\omega^{(n-j)}}\left(B_{\frac{t}{n}\gamma_j,\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}\delta_j,\frac{t}{n}[\boldsymbol{y}+(\gamma_j-\delta_j)\mathbf{1}]}\right)\right)$$

with γ_j and δ_j satisfying $\delta_j = \left(\gamma_j^k - m_k\left(B^j_{1,\,\mathbf{0}}\right)\right)^{\frac{1}{k}}$ and

$$\sum_{\boldsymbol{x}\in Z_{\gamma_j}^{(j)}} dis\left(\boldsymbol{x}, D_k^{(j)}\right) = \inf_l \sum_{\boldsymbol{x}\in Z_l^{(j)}} dis\left(\boldsymbol{x}, D_k^{(j)}\right).$$

In figure 4, given the optimal γ_1 and δ_1, we show the distance between the point $\boldsymbol{x}$ and the hyperplane in two-dimension. The optimal γ_j and δ_j are listed in table 2, for $j = 0, 1, ..., k-1$. Note that $\gamma_0 = \delta_0 = 0$. Numerical results of $\tilde{f}(t)$ are shown in section 5.

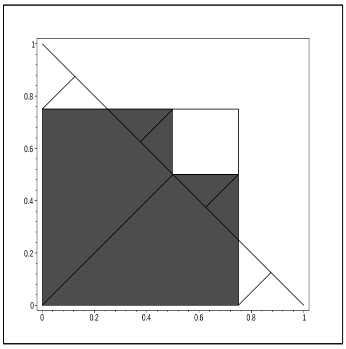

Figure 4: The distance between the point $\boldsymbol{x}$ and the hyperplane in the optimal case

Table 2: The optimal γ_j and δ_j

k	j	γ_j	δ_j	k	j	γ_j	δ_j
2	1	0.750004	0.250011	6	1	0.348916	0.273116
					2	0.682740	0.524109
3	1	0.590962	0.341192		3	0.905298	0.607947
	2	0.977867	0.466793		4	1.000000	0.657177
					5	0.999996	0.333149
4	1	0.470973	0.294631				
	2	0.863099	0.484134	7	1	0.304235	0.237667
	3	0.999999	0.451796		2	0.603599	0.471527
					3	0.848451	0.662802
5	1	0.399080	0.282191		4	0.972706	0.702235
	2	0.771495	0.545529		5	0.997605	0.495816
	3	0.971687	0.619478		6	0.999981	0.251129
	4	0.999138	0.331933				

Table 3: Sum of independent Gamma random variables, each with scale parameter 1 and shape parameter 1, 1, 2, 3 and 4, respectively.

t	n	$F(t)$	$\hat{F}(t)$	$RE(\hat{F}(t))$	$f(t)$	$\tilde{f}(t)$	$RE(\tilde{f}(t))$
3	25	0.000292	0.000293	0.003556	0.000810	0.000813	0.002956
	50		0.000293	0.000894		0.000811	0.000637
6	25	0.042621	0.042621	0.000014	0.041303	0.041232	0.001732
	50		0.042621	0.000002		0.041275	0.000675
9	25	0.294012	0.293557	0.002190	0.118580	0.118156	0.003575
	50		0.293897	0.000552		0.118431	0.001255
12	25	0.652771	0.651690	0.004767	0.104837	0.104576	0.002487
	50		0.652499	0.001196		0.104723	0.001093
15	25	0.881536	0.880589	0.009064	0.048611	0.048691	0.001652
	50		0.881299	0.002264		0.048602	0.000172
18	25	0.969634	0.969163	0.015991	0.014985	0.015120	0.009002
	50		0.969517	0.003974		0.015008	0.001523
21	25	0.993749	0.993588	0.025957	0.003485	0.003554	0.019769
	50		0.993709	0.006411		0.003499	0.004011

5. NUMERICAL RESULTS

We present numerical computation on three types of distributions, each involves sum of five random variables. In each case, we choose the section size $n = 25$ and $n = 50$.

Example 1: (Independent but non-identically distributed r.v.'s) The first includes five independent Gamma r.v.'s, each with scale parameter 1 and shape parameter 1, 1, 2, 3 and 4, respectively. It's well known that the sum of five Gamma r.v.'s is still Gamma distributed with scale parameter 1 and shape parameter 11. The main results and the compared approximations are listed in table 3 and 4, respectively. For $i = 1, 2$, let $RE(\hat{F}(t)) = \frac{|F(t)-\hat{F}(t)|}{F(t)(1-F(t))}$, $RE(\hat{F}_{(i)}(t)) = \frac{|F(t)-\hat{F}_{(i)}(t)|}{F(t)(1-F(t))}$ and $RE(\tilde{f}(t)) = \frac{|f(t)-\tilde{f}(t)|}{f(t)}$ stand for corresponding relative error of $\hat{F}(t)$, $\hat{F}_{(i)}(t)$ and $\tilde{f}(t)$ when $F(t)$ and $f(t)$ are available. Figure 5 compares the exact cumulative distribution function and probability density function with the approximation results (box symbol) in table 3 for section size $n = 50$. It's shown that the hypercube approxi-

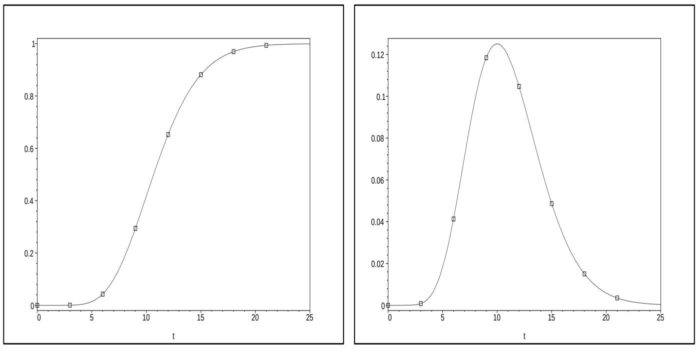

Figure 5: Comparison of the exact cdf and pdf with the hypercube approximations

Table 4: Sum of independent Gamma random variables, each with scale parameter 1 and shape parameter 1, 1, 2, 3 and 4, respectively. (compared approximations)

t	n	$F(t)$	$\hat{F}_{(1)}(t)$	$RE(\hat{F}_{(1)}(t))$	$\hat{F}_{(2)}(t)$	$RE(\hat{F}_{(2)}(t))$
3	25	0.000292	0.000294	0.005904	0.000294	0.005557
	50		0.000293	0.001527	0.000293	0.001438
6	25	0.042621	0.042618	0.000083	0.042618	0.000070
	50		0.042620	0.000011	0.042621	0.000009
9	25	0.294012	0.293211	0.003857	0.293257	0.003636
	50		0.292810	0.000972	0.293822	0.000916
12	25	0.652771	0.650877	0.008353	0.650980	0.007901
	50		0.652295	0.002100	0.652321	0.001982
15	25	0.881536	0.879870	0.015948	0.879956	0.015125
	50		0.881120	0.003981	0.881143	0.003763
18	25	0.969634	0.968799	0.028344	0.968840	0.026941
	50		0.969427	0.007006	0.969438	0.006632
21	25	0.993749	0.993461	0.046448	0.993474	0.044239
	50		0.993679	0.011345	0.993682	0.010752

Table 5: Sum of i.i.d. Weibull random variables, each with the same shape parameter 2 and scale parameter 1.

t	n	$\hat{F}(t)$	$\hat{F}_{(1)}(t)$	$\hat{F}_{(2)}(t)$	$\tilde{f}(t)$
2.814097	25	0.050203	0.050410	0.050384	0.601481
	50	0.050026	0.050075	0.050068	0.601295
3.357615	25	0.150178	0.150540	0.150498	1.249670
	50	0.149882	0.149965	0.149955	1.251369
3.702083	25	0.250187	0.250554	0.250516	1.630926
	50	0.249910	0.249991	0.249982	1.634485
3.987351	25	0.350015	0.350294	0.350271	1.841351
	50	0.349838	0.349896	0.349890	1.846241
4.251523	25	0.449991	0.450116	0.450116	1.916361
	50	0.449970	0.449987	0.449987	1.921880
4.513044	25	0.549681	0.549611	0.549647	1.871391
	50	0.549849	0.549818	0.549824	1.876739
4.789864	25	0.649437	0.649152	0.649204	1.712146
	50	0.649806	0.649722	0.649735	1.716465
5.106096	25	0.749394	0.748904	0.748978	1.435130
	50	0.749945	0.749813	0.749832	1.437538
5.511143	25	0.849149	0.848519	0.848605	1.027061
	50	0.849807	0.849644	0.849666	1.026800
6.217922	25	0.949411	0.948872	0.948938	0.438777
	50	0.949942	0.949808	0.949825	0.435943

mation results match the exact cdf and pdf well. An interesting numerical result is that $RE(\tilde{f}(t))$ is much smaller than n^{-1}, the convergence rate of $\tilde{f}(t)$, as shown in theorem 5.

Example 2: (Independent and identically distributed r.v.'s) The second includes five i.i.d. Weibull r.v.'s with shape parameter 2 and scale parameter 1. The results are listed in table 5. Although relative error is not available in table 5. We see that $n = 25$ is good enough when compared to double the section size from $n = 25$ to $n = 50$. The t's chosen in table 5 are estimated quantiles. It's also shown that the three different hypercube approximations display a good performance in precision.

Table 6: Sum of dependent random variables with joint survival function $\exp(-\sum_{i=1}^{5} t_i - 0.5\Pi_{i=1}^{5} t_i)$

t	n	$\hat{F}(t)$	$\hat{F}_{(1)}(t)$	$\hat{F}_{(2)}(t)$	$\tilde{f}(t)$
1	25	0.004369	0.004367	0.004367	0.017522
	50	0.004371	0.004370	0.004371	0.017534
3	25	0.183282	0.183133	0.183153	0.161367
	50	0.183434	0.183396	0.183401	0.161537
5	25	0.555022	0.554577	0.554635	0.183577
	50	0.555484	0.555369	0.555384	0.183728
7	25	0.833752	0.833208	0.833275	0.085304
	50	0.834315	0.834177	0.834194	0.085122
9	25	0.941615	0.941326	0.941356	0.034339
	50	0.941860	0.941794	0.941802	0.034266
11	25	0.984243	0.984160	0.984174	0.011102
	50	0.984368	0.984334	0.984338	0.011050
13	25	0.996348	0.996298	0.996305	0.002749
	50	0.996409	0.996403	0.996405	0.002726

In Santus Filho and Yacoub (2006), some moments of sum of Weibull r.v.'s are needed to solve some parameters, and moreover a simple and closed approximation form can be obtained. In stead of Santus Filho and Yacoub (2006), our method is based on decomposing $B^1_{t,\,\mathbf{0}}$ and making some estimates on type-j set, for $j = 0, 1, ..., k-1$. We also offer a simple and closed approximated distribution function and density function of sum of not only Weibull but also arbitrary positive random variables while their joint distribution function or joint survival function are given.

Example 3: (Dependent r.v.'s) In the third example, we present four kinds of dependence r.v.'s. These examples can be found in Nelson (1999) p20, p29, p46 and p51, and involve five dependent r.v.'s with joint survival functions, $S(t_1, t_2, t_3, t_4, t_5) = P(X_1 > t_1, X_2 > t_2, X_3 > t_3, X_4 > t_4, X_5 > t_5)$, each of which is a simple imitation

Table 7: Sum of dependent random variables with joint survival function $\exp(-\sum_{i=1}^{5} t_i - 0.1 \max_{i=1\sim 5} t_i)$.

t	n	$\hat{F}(t)$	$\tilde{f}(t)$
1	25	0.026341	0.041180
	50	0.026410	0.041218
5	25	0.629634	0.163108
	50	0.630152	0.163180
9	25	0.960979	0.025592
	50	0.961216	0.025492
13	25	0.997747	0.001714
	50	0.997780	0.001692
17	25	0.999905	0.000078
	50	0.999908	0.000076
21	25	0.999997	0.000003
	50	0.999997	0.000003
25	25	1.000000	0.000000
	50	1.000000	0.000000

of two dimensional copula.

(1) $S(t_1, t_2, t_3, t_4, t_5) = \exp(-\sum_{i=1}^{5} t_i - 0.5\Pi_{i=1}^{5} t_i)$,

(2) $S(t_1, t_2, t_3, t_4, t_5) = \exp(-\sum_{i=1}^{5} t_i - 0.1 \max_{i=1\sim 5} t_i)$,

(3) $S(t_1, t_2, t_3, t_4, t_5) = \left(1 + \sum_{i=1}^{5} t_i 1_{\{t_i \geq 0\}}\right)^{-1.5}$,

(4) $S(t_1, t_2, t_3, t_4, t_5) = \exp(-\max_{i=1\sim 5} t_i) + \frac{1}{3} \exp(-2 \sum_{i=1}^{5} t_i) \left[1 - \exp(3 \min_{i=1\sim 5} t_i)\right]$.

The results are listed in table 6 to table 9, respectively. Note that we only present the optimal approximations of $F(t)$ and $f(t)$ in table 7 to table 9. Some dependent examples about the Weibull model are provided in Lai and Xie (2006) p164 and p165, and thus our hypercube method can be also used to obtain the approximation of distribution function and density function of sum of dependent Weibull r.v.'s.

Table 8: Sum of dependent random variables with joint survival function $\left(1+\sum_{i=1}^{5} t_i 1_{\{t_i \geq 0\}}\right)^{-1.5}$

t	n	$\hat{F}(t)$	$\tilde{f}(t)$
1	25	0.064330	0.018105
	50	0.064482	0.018121
5	25	0.591137	0.080680
	50	0.591353	0.080733
9	25	0.775959	0.085038
	50	0.776119	0.085083
13	25	0.854939	0.081448
	50	0.855056	0.081485
17	25	0.896567	0.076873
	50	0.896656	0.076905
21	25	0.921547	0.072559
	50	0.921617	0.072588
25	25	0.937890	0.068724
	50	0.937947	0.068750

6. DISCUSSION

Some numerical results in section 5 show that the more overlapping with these hypercubes , our estimators are, the more accurate our approximation reach. In fact, $\bar{B}^1_{t,\,\mathbf{0}}$ in the proof of theorem 1 can be used to be our estimator of $B^1_{t,\,\mathbf{0}}$, yet this kind of estimator provides a less precision unless the section size n is huge. Each type-j set, for $j = 1, 2, ..., k-1$, hides behind k type-$(j-1)$ sets, hence the roofs belong to the out most hypercubes of $\bar{B}^1_{t,\,\mathbf{0}}$ leaving others unattended. We alternatively estimate all type-j sets individually and thus there are some "holes" in our estimate $\hat{B}^1_{t,\,\mathbf{0}}$, for $j = 1, 2, ..., k-1$.

The volume-invariant set A_j in (8) can be extended to

$$A^*_j = \Bigg\{ \left(B_{l,\,\mathbf{0}} \setminus B_{d,\,(l-d)\mathbf{1}}\right) \bigcup \left(\bigcup_{m=1}^{k} B_{k^{-\frac{1}{k}}d,\; l\mathbf{l}_m} \right) :\; 0 < d < l \leq 1$$

$$\text{and } m_k\left(\left(B_{l,\,\mathbf{0}} \setminus B_{d,\,(l-d)\mathbf{1}}\right) \bigcup \left(\bigcup_{m=1}^{k} B_{k^{-\frac{1}{k}}d,\; l\mathbf{l}_m} \right) \right) = m_k\left(B^j_{1,\,\mathbf{0}}\right) \Bigg\},$$

Table 9: Sum of dependent random variables with joint survival function $\exp(-\max_{i=1\sim 5} t_i) + \frac{1}{3}\exp(-2\sum_{i=1}^{5} t_i)\,[1-\exp(3\min_{i=1\sim 5} t_i)]$.

t	n	$\hat{F}(t)$	$\tilde{f}(t)$
1	25	0.176449	0.181858
	50	0.175551	0.182028
5	25	0.631992	0.074024
	50	0.631940	0.074011
9	25	0.834713	0.033051
	50	0.834712	0.033050
13	25	0.925727	0.014855
	50	0.925727	0.014855
17	25	0.966627	0.006675
	50	0.966627	0.006675
21	25	0.985004	0.002999
	50	0.985004	0.002999
25	25	0.993262	0.001348
	50	0.993262	0.001348

where $\mathbf{l}_m = (0, ..., 0, 1, 0, ..., 0)$ the mth component is 1 and the others are zero. For instance, we take $l_{3j} = \left(m_k\left(B^j_{1,\,\mathbf{0}}\right)\right)^{\frac{1}{k}}$ and $d_{3j} = l_{3j} - \frac{j}{k}$, and

$$\hat{B}^1_{t,\,\mathbf{0}(3)} = \left(\bigcup_{j=0}^{n-k}\bigcup_{\boldsymbol{y}\in\omega^{(j)}} B_{\frac{t}{n},\frac{t}{n}\boldsymbol{y}}\right)$$
$$\bigcup\left(\bigcup_{j=0}^{k-1}\bigcup_{\boldsymbol{y}\in\omega^{(n-j)}}\left(\left(B_{\frac{t}{n}l_{3j},\frac{t}{n}\boldsymbol{y}} \setminus B_{\frac{t}{n}d_{3j},\frac{t}{n}[\boldsymbol{y}+(l_{3j}-d_{3j})\mathbf{1}]}\right)\bigcup\left(\bigcup_{m=1}^{k} B_{\frac{t}{n}k^{-\frac{1}{k}}d_{3j},\ \frac{t}{n}(\boldsymbol{y}+l_{3j}\mathbf{1}_m)}\right)\right)\right)$$

be our estimate of $B^j_{1,\,\mathbf{0}}$. Figure 6 shows the alternative volume-invariant set in two-dimension. The dark part represents the estimate of $B^j_{1,\,\mathbf{0}}$.

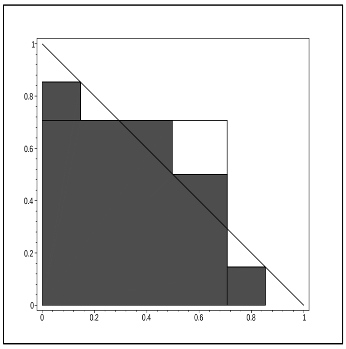

Figure 6: The alternative volume-invariant set

Table 10: Sum of independent Gamma random variables, each with scale parameter 1 and shape parameter 1, 1, 2, 3 and 4, respectively. (alternative volume-invariant set)

t	n	$F(t)$	$\hat{F}_{(3)}(t)$	$RE(\hat{F}_{(3)}(t))$
3	25	0.000292	0.000294	0.005141
	50		0.000293	0.001331
6	25	0.042621	0.042618	0.000082
	50		0.042620	0.000010
9	25	0.294012	0.293313	0.003366
	50		0.292836	0.000848
12	25	0.652771	0.651124	0.007265
	50		0.652356	0.001829
15	25	0.881536	0.880092	0.013825
	50		0.881174	0.003463
18	25	0.969634	0.968913	0.024473
	50		0.969455	0.006087
21	25	0.993749	0.993501	0.039914
	50		0.993688	0.009841

Let $\hat{F}_{(3)}(t)$, based on this kind of estimate, be the approximation of $F(t)$, and

$$
\begin{aligned}
\hat{F}_{(3)}(t) &= \int_{\hat{B}^1_{t,\ \mathbf{0}(3)}} g(\boldsymbol{x}) m_k(d\boldsymbol{x}) \\
&= \sum_{j=0}^{n-k} \sum_{\boldsymbol{y}\in\omega^{(j)}} P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+1),\ i=1,2,...,k\right) \\
&+ \sum_{j=0}^{k-1} \sum_{\boldsymbol{y}\in\omega^{(n-j)}} \left[P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}(y_i+1_{3j}),\ i=1,2,...,k\right)\right. \\
&- P\left(\frac{t}{n}[y_i+(l_{3j}-d_{2j})] \le X_i \le \frac{t}{n}(y_i+l_{3j}),\ i=1,2,...,k\right) \\
&+ \sum_{m=1}^{k} P\left(\frac{t}{n}y_i \le X_i \le \frac{t}{n}\left(y_i+k^{-\frac{1}{k}}d_{3j}\right),\ i\ne m,\right. \\
&\qquad\qquad \left.\left.\frac{t}{n}\left(y_m+l_{3j}\right) \le X_m \le \frac{t}{n}\left(y_m+l_{3j}+k^{-\frac{1}{k}}d_{3j}\right)\right)\right].
\end{aligned}
$$

Some numerical results are presented in table 10. The results, which compare with $\hat{F}_{(1)}(t)$ or $\hat{F}_{(2)}(t)$, show a good performance in precision. Therefore, many kinds of the volume-invariant sets can be considered, and in each case the common goal is to increase the overlapping part of the estimate and $B^1_{t,\ \mathbf{0}}$.

Note that our method is suitable for small sample size k. For large k, there are many well-known methods that can be used to obtain the approximated distribution function of sum of r.v.'s, such as the central limit theorem and Edgeworth expansion. This paper present an innovative idea which combines probability and geometry to approximate the distribution function and density function of sum of positive r.v.'s while the their joint distribution function G is given. Further analysis will be concentrated on applying our method to some relative fields such as biology, reliability and so on.

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