多變量製程監控之研究

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摘　　要

本論文內容分成三個主題。

在第一個主題，我們提出一多變量管制圖來偵測多變量製程變異降低的方法，此為根據多變量單邊檢定 $H_0:\Sigma=\Sigma_0\ vs.\ H_1:\Sigma\leq\Sigma_0\ and\ \Sigma\neq\Sigma_0$ 所建立之管制圖，其中 Σ 和 Σ_0 分別為所監控品質特性目前的和在控制狀態下的共變異數矩陣,考慮 Σ_0 已知或未知兩種情形，我們分別導出概似比檢定統計量，並以此建立管制圖。透過統計模擬對幾種 Σ 的變化來比較平均連串長度，證實所提出的管制圖對多變量製程變異降低的問題比現有基於雙邊概似比檢定所建立的管制圖，在偵測能力上有相當不錯的效率。並以一個實例和模擬例子，證實所提出的管制圖具有應用性及有效性。

在第二個主題，我們結合第一個主題所提出的多變量管制圖及先前 Yen and

Shiau (2008) 所提的一個偵測多變量製程變異增加的管制圖建立一結合性多變量管制圖來偵測多變量製程變異增加或降低的方法。並且考慮 Σ_0 為已知或未知兩種情形。透過統計模擬對幾種 Σ 的變化來比較平均連串長度，說明所提出的基於不均等尾端機率管制界限所建立之結合管制圖，對多變量製程變異增加或降低的問題，比現有基於雙邊檢定所建立的管制圖在偵測能力上也有相當不錯的效率。並以兩個實例和模擬例子，證實所提出的管制圖具有應用性及有效性。

此外，對監控多變量常態製程平均值向量，T^2 管制圖是一被廣泛使用的統計製程管制工具，有一個主要缺點：當 T^2 管制圖偵測到製程為失控狀態時並無法直接提供那一個品質特性或那幾個品質特性是造成製程失控原因的資訊。第三個主題的目的則為提出一個根據概似比原理的方法，當 T^2 管制圖發出失控訊號時，來找出那一個個別的品質特性平均值最有可能發生改變而不是試著決定那一個個別的品質特性是否失控。此方法對現行所使用的 T^2 管制圖方法為一個診斷輔助工具而不是替代工具。

A Study on Multivariate Process Monitoring

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Abstract

The contents of this dissertation are divided into three main subjects.

In the first subject, a multivariate control chart for detecting decreases in process dispersion is proposed. The proposed chart is constructed based on the one-sided likelihood ratio test (LRT) for testing $H_0 : \Sigma = \Sigma_0$ *vs.* $H_1 : \Sigma \le \Sigma_0$ *and* $\Sigma \ne \Sigma_0$, where Σ and Σ_0 are respectively the current and the in-control process covariance matrix of the distribution of the quality characteristic vector of interest. Both cases of known and unknown Σ_0 are considered. For each case, the LRT statistic is derived and then used to construct the control chart. A comparative simulation study is conducted and shows that the proposed control chart outperforms the existing two-sided-test-based control charts in terms of the average run length. The applicability and effectiveness of the proposed control chart are demonstrated through two real examples and two simulated examples.

By combining the above mentioned one-sided LRT-based control chart and the

one-sided LRT-based control chart for detecting dispersion increases proposed by Yen and Shiau (2008), we propose a combined chart scheme for detecting both cases of dispersion increases and decreases. Both cases of known and unknown Σ_0 are considered. It is found that a combined chart using an equal tail probability to construct a control limit is biased. By simulation studies, the proposed combined chart scheme when using a set of unequal tail probabilities for the two charts outperforms the existing two-sided-test-based control charts in terms of the average run length, when the process dispersion increases or decreases. Two real examples and two simulated examples are used to illustrate the applicability and effectiveness of our proposed combined chart.

About the third subject, Hotelling's T^2 chart is a well-known statistical process control tool for simultaneously monitoring elements of the mean vector of a multivariate normal process. But it has a drawback that an out-of-control (or a significant) T^2 value does not gives us *direct* information as to which variables in T^2 are likely to have caused the out-of-control condition. We propose a method, based on likelihood principle, for identifying a variable or a group of variables in a multivariate normal process with an unknown covariance matrix that is likely to be responsible for the out-of-control condition signaled by a significant T^2 value. Unlike certain existing methods, our method is not a control/monitoring but a diagnostic tool. Two examples from earlier literatures and one based on simulation are used to illustrate the proposed method. Finally, we compare our results with that of other existing methods for these three examples.

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Chapter 1

Introduction

Control charts is one of the basic quality improvement tools in statistical process control (SPC) for a process to identify special causes of variation and signal the need to take necessary corrective actions. When special causes are present, the process is said to be out of control. If the variation is due to common causes alone, the process is said to be in statistical control.

In many situations, a process needs to monitor two or more quality characteristics, in which some are correlated. Multivariate SPC methods are designed to account for the correlations among the variables and to simultaneously monitor the variables through time. Hence, multivariate SPC is a broad field of research as well as applications devoted to the improvement of products and processes.

There are two phases of SPC or multivariate SPC, namely, Phase I and Phase II. In Phase I analysis, historical observations are analyzed for determining whether the process is in control, to understand the variation in the process, and to estimate the in-control parameters of the process. In contrast, Phase II aims at on-line monitoring of future observations by using the control limits calculated from the in-control data obtained from Phase I to determine if the process continues to be in-control. The objective of Phase II analysis is to quickly detect a process change. It is obvious that a successful Phase II analysis depends on a successful Phase I analysis. Although both of the two phases are devoted to identifying out-of-control situations, each phase has its own purpose.

The purpose of the control charts is to test whether the process is in statistical control. In the past, substantial work on multivariate SPC has focused on monitoring/controlling the process mean vector and many of them are based on Hotelling's T^2 statistic (Hotelling, 1947), one of the most popular statistics for monitoring multivariate normal processes. However, process dispersion is such an important issue in process improvement that in recent years there have been more and more studies on monitoring multivariate process dispersion in the literature. Many of these techniques are centered on two-sided tests of $H_0: \mathbf{\Sigma} = \mathbf{\Sigma}_0$ *vs.* $H_1: \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$ so that both increases and decreases in multivariate process dispersion can be monitored simultaneously.

The study on this thesis addresses three main subjects. In monitoring multivariate process dispersion, a one-sided control chart designed for detecting decreases in multivariate process dispersion is proposed in Chapter 2 and a scheme combining two one-sided control charts, one for monitoring increases and one for decreases in multivariate process dispersion, is developed in Chapter 3. As a diagnostic and complementary tool for monitoring multivariate process mean, we propose in Chapter 4 a method for identifying influential univariate variables in multivariate process control when an out-of-control is signaled by a T^2 chart. We introduce the three subjects in the following.

In most processes, more attention has been paid to the case when the process dispersion increases. However, it is also important to detect decreases in dispersion, that is, detecting process improvement. Consider the hypotheses H_0: $\mathbf{\Sigma} = \mathbf{\Sigma}_0$ *vs.* H_1: $\mathbf{\Sigma} \geq \mathbf{\Sigma}_0$ *and* $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$,, where $\mathbf{\Sigma} \geq \mathbf{\Sigma}_0$ means that $\mathbf{\Sigma} - \mathbf{\Sigma}_0$ are positive semidefinite. Yen and Shiau (2008) considered both cases of known and unknown $\mathbf{\Sigma}_0$ and proposed for each case a simple and yet effective *one-sided* likelihood-ratio-test-based (LRT-based) control chart for the detection of increases in dispersion. In this dissertation, Chapter 2 presents and evaluates two similar *one-sided* LRT-based control charts (derived based on the hypotheses H_0: $\mathbf{\Sigma} = \mathbf{\Sigma}_0$ *vs.* H_1: $\mathbf{\Sigma}_0 \geq \mathbf{\Sigma}$ *and* $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$,) for detecting decreases in process dispersion in the cases of known and unknown $\mathbf{\Sigma}_0$ respectively. The techniques of Chapter 2 is similar to that in Yen and Shiau (2008). Presumably, the detecting power of a *one-sided* test should be larger than that of the corresponding *two-sided* test if the process dispersion indeed increases (or decreases).

Yen and Shiau (2008) reported that their one-sided chart outperforms the existing control charting techniques based on some two-sided tests in terms of the average run length. In Chapter 2, we show by simulation that the proposed control chart for monitoring dispersion decreases has better performance than the existing two-sided control charts. Thus, for simultaneously monitoring both increases and decreases in process dispersion, it is natural to consider combining the two one-sided LRT-based control charts mentioned above to construct a combined-chart scheme. Pachares (1961) showed the two-sided confidence interval based on a equal-tailed probability (i.e., $\alpha/2$) leads to a biased test for a two-sided hypothesis testing on the variance of a normal population. Lowry, Champ, and Woodall (1995) reported that when using R or S chart, it is much more difficult to detect decreases than increases in variance. Hence, we anticipate that the proposed combined chart using unequal tail probabilities instead of an equal tail probability to construct the control limits will get a satisfactory performance.

Hotelling's T^2 chart is a well-known SPC tool for simultaneously monitoring elements of the mean vector of a multivariate normal process. It is a powerful tool because the T^2 statistic is the likelihood-ratio test statistic for testing $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ and, if $\boldsymbol{\mu}_0 = \mathbf{0}$, it is the uniformly most powerful and invariant (under scalar transformations) test among all tests of population (process) mean vector that are based on the sample mean vector and sample covariance matrix (Anderson, 2003, p. 192).

However, Hotelling's T^2 control chart has several drawbacks in practice. The major drawback of a T^2 chart is that it does not provide *direct* information as to which individual variable(s) may be responsible for the out-of-control condition (i.e., rejection of H_0) when the chart signals out of control (by a significant T^2 value). Since T^2 is an aggregated statistic, an out-of-control T^2 value does not necessarily imply that at least one individual variable is out-of-control. Thus, further diagnostic work is needed.

Another practical drawback of the T^2 chart is the potential confounding. Because T^2 is invariant under scalar transformations (e.g., the T^2 statistic will have the same value when all observations from the process become twice larger) it can not detect the change in scale (and hence variation). In other words, when all observations are shifted by a constant, the value of T^2 remains the same. For this reason, T^2 is not recommended for simultaneously detecting changes in the mean vector and the covariance matrix. Hence, before using a T^2 chart to monitor and control the mean of the process, we must assure that the covariance matrix of the process has not changed over time, as we normally do in a univariate $\bar{X}$-chart.

Hence, in Chapter 4, we propose a method, which is based on the likelihood principle, for identifying a variable or a group of variables in a multivariate normal process with an unknown covariance matrix $\mathbf{\Sigma}$ that is likely to be responsible for the out-of-control condition signaled by a significant T^2 value. Unlike certain existing methods, our method is not a control/monitoring but a diagnostic and complementary tool.

Finally, in Chapter 5, we give some concluding remarks for this dissertation. And some discussion for possible future research issues and generalizations are given for the three subjects addressed in this dissertation.

Chapter 2

A Multivariate Control Chart for Detecting Decreases In Multivariate Process Dispersion

2.1 Background

Control charting is one of fundamental quality improvement tools in SPC for detecting process changes and signalling possible needs for corrective actions. This chapter proposes and studies a multivariate control chart particularly designed for detecting decreases in dispersion for processes in which two or more possibly correlated quality characteristics need to be monitored simultaneously.

Although substantial work on SPC methods has been devoted to monitoring the process mean, Montgomery (2008) pointed out that monitoring the process dispersion is just as, or even more, important. When monitoring a process, it is important to quickly detect increases and decreases in dispersion. Increases in dispersion will possibly cause some level of deterioration in the quality of the process output and lead to an increasing number of defective units. On the contrary, decreases in dispersion eventually result in improved product performance, fewer defects, and lower manufacturing cost. While detecting increases in process dispersion is necessary for preventing more defective product items to be produced, detecting decreases has its own merits. First, a search for special causes may lead to a substantial process improvement. Second, if the process indeed has a lower dispersion, the control limits of the monitoring statistic need to be adjusted accordingly.

Several control charts have been suggested for the detection of increases in process dispersion, see, for example, Page (1963), Tuprah and Ncube (1987), Crowder and Hamilton (1992), and Lowry, Champ, and Woodall (1995) for the univariate case, and Sakata (1987), Calvin (1994), and Yen and Shiau (2008) for the multivatiate case. However, monitoring of decreases in dispersion has received a lot less attention and only a few control schemes for the univariate case have been proposed. Nelson (1990) suggested a runs rule for R chart to detect decreases in process standard deviation σ but did not evaluate its performance. Acosta-Mejia (1998) proposed a CUSUM chart based on the subgroup range when small subgroups are considered to detect decreases in σ. Its performance is evaluated and compared to that of the R chart with probability limits, the R chart with Nelson's runs rule, and the CUSUM chart based on the natural log of the subgroup variance suggested by Chang and Gan (1995). Nevertheless, multivariate control charts specifically designed for detecting decreases in process dispersion is basically non-existing in the literature. Therefore, our motivation for this chapter is to develop a control scheme particularly designed for monitoring decreases in process dispersion for multivariate processes.

Consider a multivariate process with p possibly correlated quality characteristics of interest. Suppose that the $p \times 1$ quality characteristic vector $\boldsymbol{X}$ is distributed as a multivariate normal distribution (denoted by $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$) with unknown mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In the literature, a substantial amount of research work has focused on monitoring/controlling the process mean vector $\boldsymbol{\mu}$ and many of them are based on the Hotelling's T^2 statistic (Hotelling, 1947), one of the most popular statistics for monitoring multivariate normal processes. For SPC applications, Mason and Young (2002) studied the T^2 statistic and its variation in great detail. However, the T^2 chart has a notorious drawback that it is sensitive not only to shifts in the mean $\boldsymbol{\mu}$ but also to changes in the covariance matrix $\boldsymbol{\Sigma}$.

For example, see Hawkins (1991, 1993), Mason, Tracy, and Young (1995, 1996, 1997), and Montgomery (2008). This confounding of "location" and "scale" shifts is not desirable for the diagnostic purpose. It would be more effective for quality improvement to have separate diagnostics for changes in process mean and process dispersion.

Although developing methods for monitoring multivariate process dispersion is more difficult than that for process mean due to the more complicated distribution theory involved, there have been more researches devoted to multivariate process dispersion monitoring in the literature in recent years. See, for example, Wierda (1994), Mason, Champ, Tracy, Wierda, and Young (1997), and Woodall and Montgomery (1999).

Alt (1985) provided a summary of techniques for monitoring multivariate process dispersion. Alt and Bedewi (1986) proposed two control charting techniques for the covariance matrix: one based on the likelihood ratio principle and the other making use of the so-called sample generalized variance, $|S|$, which is the determinant of the sample covariance matrix S. Alt and Smith (1998) proposed a $\sqrt{|S|}$ chart. Djauhari (2005) proved that the standard way of using $|S|$ and $\sqrt{|S|}$ for multivariate process dispersion control charting leads to a biased estimate of the control limits and presented an improved control chart with unbiased control limits and a smaller average run length (ARL) than that of the standard chart. Reynolds and Cho (2006) proposed two new multivariate exponentially weighted moving average (MEWMA) control charts for monitoring $\mathbf{\Sigma}$. The proposed control charts were intended to be used in conjunction with the standard MEWMA control chart designed for monitoring $\boldsymbol{\mu}$. They showed that the proposed combined control chart, using a combination of two MEWMA charts with one based on the sample means and the other based on the sum of squared regression adjusted deviations from the target, had the best overall performance when compared with other existing control charts for simultaneous monitoring of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$. Tang and Barnett (1996a, 1996b), utilizing some independent statistics resulting from decomposing the covariance matrix $\mathbf{\Sigma}$, proposed some Shewhart procedures for monitoring and detecting changes in $\mathbf{\Sigma}$, which they claimed are more effective in quality improvement. Levinson, Holmes, and Mergen (2002) proposed a control chart, called G chart, that can detect changes in $\mathbf{\Sigma}$. Yeh, Lin, Zhou, and Venkataramani (2003) developed a multivariate EWMA control chart for detecting changes from $|\mathbf{\Sigma}_0|$, the determinant of the in-control variance-covariance matrix. Yeh, Huwang, and Wu (2004), using the unbiased likelihood ratio test for testing $H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0$ $vs.$ $H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$, developed a multivariate control chart for detecting changes from $\mathbf{\Sigma}_0$ and reported that it can effectively monitor small changes from $\mathbf{\Sigma}_0$. Yeh, Lin, and McGrath (2006) reviewed multivariate control charts designed for monitoring changes in $\mathbf{\Sigma}$ that have been developed in the last 15 years and proposed a new multivariate EWMA control chart. They also discussed the performance comparisons in the literature, which were scattered and limited in the scopes. One important concern worthy of future investigation is to compare all the existing charts in a systematic, organized, and thorough manner.

The afore-mentioned techniques are all centered on two-sided tests of $H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0$ $vs.$ $H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$. These techniques can detect either increases or decreases in multivariate process dispersion. In most processes, more attention has been paid to the case when the process dispersion increases. However, it is also important to detect decreases in dispersion, that is, signaling process improvement. Thus, the following two one-sided tests for monitoring respectively increases and decreases in dispersion are both considered:

$$H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0 \ \ vs. \ \ H_1\colon \mathbf{\Sigma} \geq \mathbf{\Sigma}_0 \ and \ \mathbf{\Sigma} \neq \mathbf{\Sigma}_0, \tag{2.1.1}$$

and

$$H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0 \ \ vs. \ \ H_1\colon \mathbf{\Sigma} \leq \mathbf{\Sigma}_0 \ and \ \mathbf{\Sigma} \neq \mathbf{\Sigma}_0, \tag{2.1.2}$$

where $\mathbf{\Sigma} \geq \mathbf{\Sigma}_0$ and $\mathbf{\Sigma} \leq \mathbf{\Sigma}_0$ mean that $\mathbf{\Sigma} - \mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ are positive semidefinite respectively. We say that the dispersion increases (decreases) if $\mathbf{\Sigma} - \mathbf{\Sigma}_0$ ($\mathbf{\Sigma}_0 - \mathbf{\Sigma}$) is positive semidefinite. This means that, when the process is out-of-control for every $p \times 1$ vector $\boldsymbol{a} \neq \mathbf{0}$, the variance of every possible linear combination is greater than or equal to (less than or equal to) that when the process is in control, i.e., $\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a} \geq \boldsymbol{a}'\boldsymbol{\Sigma}_0\boldsymbol{a}$ ($\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a} \leq \boldsymbol{a}'\boldsymbol{\Sigma}_0\boldsymbol{a}$) for all $\boldsymbol{a} \neq \mathbf{0}$. Presumably, the detecting power of a *one-sided* test should be larger than that of the corresponding *two-sided* test if the process dispersion indeed increases or decreases. That is, a one-sided test would be more sensitive to the detection of an out-of-control condition. However, the testing of the equality of covariance matrices against various types of one-sided alternatives has not yet been fully developed. The difficulty is that the one-sided nature of the hypotheses leads to a restricted parameter space and techniques from order restricted inference need to be used.

The following is a brief literature review of one-sided tests for the hypotheses (2.1.1). Assuming that $\boldsymbol{\Sigma}_0$ is known, Calvin (1994) divided (2.1.1) into two sequential testing hypotheses and constructed a two-stage control charting scheme. The process dispersion is claimed to have increased only when both control charts of the two stages are out of control. In practice, Calvin's method is more complicated than the usual one-chart scheme. Assuming that $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_0$ are unknown, Sakata (1987) derived the likelihood ratio test (LRT) of (2.1.1) using a "modified" likelihood function instead of the exact likelihood function. Based on the *exact* likelihood function, Yen and Shiau (2008) proposed a simple and yet effective *one-sided* LRT-based control chart based on the hypotheses (2.1.1) for detecting increases in dispersion for each of the cases when both $\boldsymbol{\Sigma}_0$ is known and unknown. They also evaluated the performance of the proposed charts based on simulations of the average run length and showed that it outperforms the existing two-sided control charts for various settings under their study.

The purpose of this chapter is to present and evaluate a simple and yet effective *one-sided* LRT-based control chart derived based on the hypotheses (2.1.2) for detecting decreases in process dispersion. The technique of this chapter is similar to that in Yen and Shiau (2008).

The rest of the chapter is organized as follows. Section 2.2 describes the proposed one-sided LRT-based control chart in detail. Both cases of known and unknown $\mathbf{\Sigma}_0$ are considered. Section 2.3 provides a method for computing the control limit. Section 2.4 describes the existing competing techniques based on the two-sided tests of $H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0 \; vs. \; H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$ for both cases of known and unknown $\mathbf{\Sigma}_0$. From the perspective of the ARL, the performance of the proposed control chart is compared with that of the existing two-sided control charts based on simulation studies. Section 2.5 discusses the application of the proposed chart to some real-life and simulated examples to demonstrate the applicability and effectiveness of the proposed chart. Section 2.6 concludes the chapter with a discussion on some related issues. Appendices A.1 and A.2 provide respectively our derivations of the LRT statistics of the cases of known and unknown $\mathbf{\Sigma}_0$.

2.2 One-Sided LRT-based Control Chart

Let $\boldsymbol{X}=(X_1, \cdots, X_p)'$ be a p-dimensional random vector representing the p possibly correlated quality characteristics of interest. Assume that $\boldsymbol{X}$ has a p-dimensional multivariate normal distribution $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$. Similar to Yen and Shiau (2008), in order to derive the LRT statistic for testing (2.1.2), we borrow some techniques from Anderson, Anderson, and Olkin (1986), Anderson (1989), and Kuriki (1993), in which the authors focused on multivariate components of variance models under the condition that the effect covariance matrix is positive semidefinite with a maximum rank. We remark that the model considered in these papers is different from the one considered in this chapter.

Suppose that the in-control covariance matrix $\mathbf{\Sigma}_0$ is positive definite and $\mathbf{\Sigma} \leq \mathbf{\Sigma}_0$. Let $\mathbf{\Theta} = \mathbf{\Sigma}_0 - \mathbf{\Sigma}$ with $rank(\mathbf{\Theta}) = k$, $0 \leq k \leq p$. Consider a more general setting of hypotheses than (2.1.2):

$$H_0^*\colon \mathbf{\Theta} \geq 0 \; with \; k \leq k_0 \;\; vs. \;\; H_1^*\colon \mathbf{\Theta} \geq 0 \; with \; k \leq k_1, \tag{2.2.1}$$

where k_0 and k_1 are two given integers and $k_0 < k_1$. When $k_0 = 0$ and $k_1 = p$, H_0^* and H_1^* in

(2.2.1) are equivalent to H_0 and H_1 in (2.1.2), respectively. Hence, (2.1.2) is a special case of (2.2.1).

2.2.1 One-Sided LRT Statistic When $\mathbf{\Sigma}_0$ is Known

Assume that $\mathbf{\Sigma}_0$ is known. At time t, a subgroup of n random $p \times 1$ vectors, $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$, is sampled from the process. Each $\boldsymbol{X}_{tj}$ is distributed as a p-dimensional normal distribution $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ with both $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ unknown. Let the sample mean and sample covariance matrix of the n observations obtained at time t be, respectively,

$$\begin{aligned} \bar{\boldsymbol{X}}_t &\equiv \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{X}_{tj} \ , \\ \boldsymbol{S}_t &\equiv \frac{1}{n}\sum_{j=1}^{n} (\boldsymbol{X}_{tj} - \bar{\boldsymbol{X}}_t)(\boldsymbol{X}_{tj} - \bar{\boldsymbol{X}}_t)'. \end{aligned} \tag{2.2.2}$$

Then $\boldsymbol{B}_t \equiv n\boldsymbol{S}_t$ has a Wishart distribution (denoted by $W_p(n-1, \mathbf{\Sigma})$) with $n-1$ degrees of freedom and covariance matrix $\mathbf{\Sigma}$. We remark that the reason for using n instead of the usual $n-1$ in the sample covariance matrix $\boldsymbol{S}_t$ is to simplify the derivation of the LRT of (2.2.1). It is interesting to know whether $\boldsymbol{S}_t$ is positive definite, like $\mathbf{\Sigma}$. Dykstra (1970) proved that $\boldsymbol{S}_t$ is positive definite with probability 1 if and only if $n > p$.

Theorem 2.1. *The LRT statistic for testing (2.2.1) is*

$$\tilde{\lambda} \equiv \begin{cases} \prod_{i=k_0^*+1}^{k_1^*} \{d_i \exp[-(d_i - 1)]\}^{\frac{n}{2}} & , for\ k_0^* < k_1^* \\ 1 & , for\ k_0^* = k_1^* \end{cases} , \tag{2.2.3}$$

where $d_1 \geq \cdots \geq d_p > 0$ *are the roots of* $|\boldsymbol{S}_t - d\mathbf{\Sigma}_0| = 0$, $k_0^* = \min(k_0, p_D^*)$, $k_1^* = \min(k_1, p_D^*)$, *and* p_D^* *is the number of* $0 < d_i < 1$.

The proof is given in Appendix A.1. Based on Theorem 2.1, we have the following corollary.

Corollary 2.1. *When* $k_0 = 0$ *and* $k_1 = p$, *the LRT statistic for testing (2.1.2) is*

$$\lambda = \begin{cases} \prod_{i=1}^{p_D^*} \{d_i \exp[-(d_i - 1)]\}^{\frac{n}{2}} & , for\ p_D^* > 0 \\ 1 & , for\ p_D^* = 0 \end{cases} . \tag{2.2.4}$$

The testing procedure is typically performed by the statistic

$$T_D \equiv -2\log\lambda$$
$$= \begin{cases} n\sum_{i=1}^{p_D^*}[(d_i-1)-\log d_i] & ,for\ p_D^*>0 \\ 0 & ,for\ p_D^*=0 \end{cases}. \tag{2.2.5}$$

The rejection region of the test is $\{T_D > T_D(\alpha)\}$, where the critical value $T_D(\alpha)$ is chosen such that the significance level equals α. In other words, $T_D(\alpha)$ is the $(1-\alpha)th$ quantile of the distribution of T_D. Since the distribution of T_D is not easy to derive, we obtain $T_D(\alpha)$, the critical value of the test, based on Monte Carlo simulation.

Since $\boldsymbol{\Sigma}_0$ is assumed symmetric positive definite, there exists a unique symmetric positive definite matrix $\boldsymbol{\Sigma}_0^{1/2}$ such that $\boldsymbol{\Sigma}_0$=$(\boldsymbol{\Sigma}_0^{1/2})(\boldsymbol{\Sigma}_0^{1/2})$ (Golub and Van Loan (1989), p. 395). To simplify the notation, $(\boldsymbol{\Sigma}_0^{1/2})^{-1}$ is denoted by $\boldsymbol{\Sigma}_0^{-1/2}$. Let $\boldsymbol{Z}_{tj} \equiv \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{X}_{tj}$. Then $\{\boldsymbol{Z}_{tj}, j = 1, \cdots, n\}$ is a random sample of size n from $N_p(\boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu}, \boldsymbol{I}_p)$, if $\boldsymbol{X}_{tj}$ follows $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_0)$. Thus, $\bar{\boldsymbol{Z}}_t \equiv \boldsymbol{\Sigma}_0^{-1/2}\bar{\boldsymbol{X}}_t$ and $\boldsymbol{S}_t^{(z)} \equiv \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{S}_t\boldsymbol{\Sigma}_0^{-1/2}$ are the sample mean and sample covariance matrix of the transformed sample, respectively. First, note that $n\boldsymbol{S}_t^{(z)}$ is distributed as $W_p(n-1, \boldsymbol{I}_p)$, which does not depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_0$. Second, since $|\boldsymbol{S}_t - d\boldsymbol{\Sigma}_0| = |\boldsymbol{\Sigma}_0|\left|\boldsymbol{S}_t^{(z)} - d\boldsymbol{I}_p\right|$ and $\boldsymbol{\Sigma}_0$ is assumed positive definite, $|\boldsymbol{S}_t - d\boldsymbol{\Sigma}_0| = 0$ and $\left|\boldsymbol{S}_t^{(z)} - d\boldsymbol{I}_p\right| = 0$ have the same roots, which implies that, when the process is in control, the distribution of T_D based on the eigenvalues of $\boldsymbol{S}_t\boldsymbol{\Sigma}_0^{-1}$ is the same as that based on the eigenvalues of $\boldsymbol{S}_t^{(z)}$. Thus, without loss of generality, when the process is in control, we can assume that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{I}_p$ when studying the distribution of T_D. This invariance property greatly simplifies the control limit computation.

2.2.2 One-Sided LRT Statistic When $\boldsymbol{\Sigma}_0$ is Unknown

Assume that both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_0$ are unknown. Suppose that we have obtained m training samples, $\{\boldsymbol{X}_{1j}\}_{j=1}^n, \cdots, \{\boldsymbol{X}_{mj}\}_{j=1}^n$, each of size n, from the in-control process to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_0$ in Phase I of the control charting process. In Phase II, when the on-line process monitoring begins, a random sample of size n, $\{\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}\}$, is taken from the process at time t. To avoid confusion, let $t > m$. Assume that $\boldsymbol{X}_{ij}$, $j = 1, \cdots, n$, $i = 1, \cdots, m$, are independent and identically distributed ($i.i.d.$) as $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma_0})$ and $\boldsymbol{X}_{tj}$, $j = 1, \cdots, n$, are $i.i.d.$ as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}_0$, and $\boldsymbol{\Sigma}_0$ are all unknown. To test if the process dispersion increases at time t, we derive the LRT statistic in the following. Let the sample mean and the sample covariance of the mn observations in the m training samples be

$$\begin{aligned} \bar{\bar{\boldsymbol{X}}} &\equiv \tfrac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \boldsymbol{X}_{ij} \ , \\ \boldsymbol{S}_0 &\equiv \tfrac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}})(\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}})', \end{aligned} \tag{2.2.6}$$

respectively. Thus, $\boldsymbol{A} \equiv mn\boldsymbol{S}_0$ is distributed as $W_p(mn-1, \boldsymbol{\Sigma}_0)$. In our derivation of the LRT, we need the condition that $\boldsymbol{S}_0$ is positive definite. Note that this condition holds with probability 1 if and only if $mn > p$.

Theorem 2.2. *Assume $\boldsymbol{S}_0$ is positive definite, the LRT statistic for the hypothesis (2.2.1) is*

$$\tilde{\lambda}' \equiv \begin{cases} \prod_{i=k_0^\star+1}^{k_1^\star} \left[\frac{\beta_i^w}{(w\beta_i+1-w)} \right]^{\frac{mn+n}{2}} & , for\ k_0^\star < k_1^\star \\ 1 & , for\ k_0^\star = k_1^\star \end{cases} , \tag{2.2.7}$$

where $w = \frac{1}{m+1}$, $\beta_1 \geq \cdots \geq \beta_p > 0$ are the roots of $|\boldsymbol{S}_t - \beta\boldsymbol{S}_0| = 0$, $k_0^\star = \min(k_0, p_D^\star)$, $k_1^\star = \min(k_1, p_D^\star)$, and $p_D^\star$ is the number of $0 < \beta_i < 1$.

The proof is given in Appendix A.2. The following corollary immediately follows:

Corollary 2.2. *When $k_0 = 0$ and $k_1 = p$, the LRT statistic for hypotheses (2.1.2) is*

$$\lambda' = \begin{cases} \prod_{i=1}^{p_D^\star} \left[\frac{\beta_i^w}{(w\beta_i+1-w)} \right]^{\frac{mn+n}{2}} & , for\ p_D^\star > 0 \\ 1 & , for\ p_D^\star = 0 \end{cases} . \tag{2.2.8}$$

As before, the testing procedure is usually performed by the statistic

$$\begin{aligned} T_D' &\equiv -2\log\lambda' \\ &= \begin{cases} (mn+n)\sum_{i=1}^{p_D^\star} [\log(w\beta_i + 1 - w) - w\log\beta_i] & , for\ p_D^\star > 0 \\ 0 & , for\ p_D^\star = 0 \end{cases} . \end{aligned} \tag{2.2.9}$$

The rejection region of the test is $\{T'_D > T'_D(\alpha)\}$, where $T'_D(\alpha)$ is the $(1-\alpha)$th quantile of the distribution of T'_D.

Let $\boldsymbol{Z}_{ij} \equiv \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{X}_{ij}$, $j = 1, \cdots, n$, $i = 1, \cdots, m$. Then $\{\boldsymbol{Z}_{1j}\}_{j=1}^n, \cdots, \{\boldsymbol{Z}_{mj}\}_{j=1}^n$ are m random samples, each of size n, from $N_p(\boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{\mu}_0, \boldsymbol{I}_p)$. Further, $\bar{\bar{\boldsymbol{Z}}} \equiv \boldsymbol{\Sigma}_0^{-1/2}\bar{\bar{\boldsymbol{X}}}$ and $\boldsymbol{S}_0^{(z)} \equiv \boldsymbol{\Sigma}_0^{-1/2}\boldsymbol{S}_0\boldsymbol{\Sigma}_0^{-1/2}$ are the corresponding sample mean and sample covariance matrix, respectively. When the process is in control, by the same argument in the previous subsection, the distribution of T'_D based on the eigenvalues of $\boldsymbol{S}_t\boldsymbol{S}_0^{-1}$ is the same as that based on the eigenvalues of $\boldsymbol{S}_t^{(z)}(\boldsymbol{S}_0^{(z)})^{-1}$, and hence independent of $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$.

2.2.3 The Proposed Control Charts

It is easy to construct the control charts based on the LRT statistics derived in the previous two subsections by taking the critical values $T_D(\alpha)$ and $T'_D(\alpha)$ as the control limits of the control charts for cases of known and unknown $\boldsymbol{\Sigma}_0$, respectively. Thus, if the monitoring statistic T_D (T'_D) is greater than the control limit $T_D(\alpha)$ ($T'_D(\alpha)$), then the process is considered out of control. The control limits can be obtained by an empirical approach to be described in the next section.

2.3 The Control Limits

As discussed earlier, the control limits are independent of the in-control process mean $\boldsymbol{\mu}_0$ and covariance matrix $\boldsymbol{\Sigma}_0$. Thus, without loss of generality, we can assume that $\boldsymbol{\mu}_0 = \boldsymbol{0}$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{I}_p$ when obtaining the control limits by Monte Carlo simulation.

2.3.1 The Control Limit When $\boldsymbol{\Sigma}_0$ is Known

When $\boldsymbol{\Sigma}_0$ is known, the control limit $T_D(\alpha)$ can be estimated by the sample $(1-\alpha)th$ quantile of the empirical distribution of T_D computed based on the eigenvalues of $\boldsymbol{S}_t^{(z)}$. The following procedure is used for computing the control limit. Let N be the number of simulated values of T_D in one simulation and b be the number of replications.

Procedure 3.1. (Computing the control limit when Σ_0 is known)

Step 1. Input p, n, α, N, and b.

Step 2. For t=1 to N, do

i. Generate n i.i.d. random vectors $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$ from $N_p(\mathbf{0}, \boldsymbol{I}_p)$;

ii. Compute $\boldsymbol{S}_t$ by (2.2.2) ;

iii. Compute the eigenvalues of $\boldsymbol{S}_t$, $d_1 \geq \cdots \geq d_p$;

iv. Compute $T_{D,t}$ by (2.2.5).

Step 3. Compute the $(1-\alpha)th$ sample quantile of $\{T_{D,1}, \cdots, T_{D,N}\}$.

Step 4. Repeat Steps 2-3 b times. Take the average of the b quantiles as the control limit $CL_{p,n,\alpha}$.

The purpose of *Step* 4 of Procedure 3.1 is to give a more precise quantile estimate and to provide information on the precision of the computed control limit $CL_{p,n,\alpha}$.

For $p = 2, 3, 4$, $n = 5, 10, 15, 20, 25, 30, 35, 40$, $\alpha = 0.05, 0.01, 0.0027$, $N = 1,000,000$, and $b = 100$, Table 2.1 gives $CL_{p,n,\alpha}$ and its standard error (in parentheses). We observe the followings from this table:

- For the same p and α, the larger the n is, the smaller the $CL_{p,n,\alpha}$ is.
- For the same n and α, the larger the p is, the larger the $CL_{p,n,\alpha}$ is.
- The smaller the α is, the larger the standard error is. This is typical for quantile estimators, especially when the tail is long.

2.3.2 The Control Limit When Σ_0 is Unknown

Similarly, when $\boldsymbol{\Sigma}_0$ is unknown, the control limit can be constructed as the sample $(1-\alpha)th$ quantile of the empirical distribution of T'_D based on the eigenvalues of $\boldsymbol{S}_t^{(z)}(\boldsymbol{S}_0^{(z)})^{-1}$. Procedure 3.2 computes the control limit.

Procedure 3.2. (Computing the control limit when Σ_0 is unknown)

Step 1. *Input* p, m, n, α, N, *and* b.

Step 2. *For t=m+1 to* $m+N$, *do*

i. *Generate mn i.i.d random vectors* $\boldsymbol{X}_{11}, \cdots, \boldsymbol{X}_{mn}$ *from* $N_p(\mathbf{0}, \boldsymbol{I}_p)$;

ii. *Generate n i.i.d random vectors* $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$ *from* $N_p(\mathbf{0}, \boldsymbol{I}_p)$;

iii. *Compute* $\boldsymbol{S}_0$ *and* $\boldsymbol{S}_t$ *by (2.2.6) and (2.2.2), respectively;*

iv. *Compute the eigenvalues of* $\boldsymbol{S}_t\boldsymbol{S}_0^{-1}$, $\beta_1 \geq \cdots \geq \beta_p$;

v. *Compute* $T'_{D,t-m}$ *by (2.2.9).*

Step 3. *Compute the* $(1-\alpha)$*th sample quantile of* $\{T'_{D,1}, \cdots, T'_{D,N}\}$.

Step 4. *Repeat Steps 2-3 b times. Take the average of the b quantiles as the control limit* $CL_{p,m,n,\alpha}$.

For $p = 2, 3, 4$, $m = 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 80, 90, 100$, $n = 5$, $\alpha = 0.05, 0.01$, 0.0027, $N = 1,000,000$, and $b = 100$, Table 2.2 gives $CL_{p,m,n,\alpha}$ and its standard error (in parentheses). The followings are observed from the table:

- For the same p, α, n, the larger the m is, the larger the $CL_{p,m,n,\alpha}$ is.
- For the same m, n, α, the larger the p is, the larger the $CL_{p,m,n,\alpha}$ is.
- The smaller α is, the larger the standard error is.

The MATLAB programs used for computing the control limits, which can be useful for practitioners, are available at *http://www.stat.nctu.edu.tw/subhtml/source/teachers/jyhjen.htm*. Users can compute the control limit by inputting the appropriate parameters, p, n, m, α, N, and b according to their applications.

2.4 A Comparative Study

In this section, we describe three existing techniques based on the two-sided tests of $H_0\colon \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ *vs.* $H_1\colon \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$ for both cases of known and unknown $\boldsymbol{\Sigma}_0$. Furthermore, we compare our proposed one-sided LRT-based control chart with these competing control charts in terms of out-of-control ARL.

2.4.1 Two-sided LRT and Two-sided *Modified* LRT

We first describe the two-sided LRT for testing H_0: $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ *v.s.* H_1: $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$. When $\boldsymbol{\Sigma}_0$ is known, the two-sided LRT statistic given in Anderson (2003, p. 439) is

$$\begin{aligned}\lambda^* &= n^{-\frac{pn}{2}}\left|\boldsymbol{B}_t\boldsymbol{\Sigma}_0^{-1}\right|^{n/2}\exp\{-\frac{1}{2}tr\left(\boldsymbol{B}_t\boldsymbol{\Sigma}_0^{-1}\right)+\frac{pn}{2}\}\\ &= \left|\boldsymbol{S}_t\boldsymbol{\Sigma}_0^{-1}\right|^{n/2}\exp\{-\frac{n}{2}tr\left(\boldsymbol{S}_t\boldsymbol{\Sigma}_0^{-1}\right)+\frac{pn}{2}\},\end{aligned} \tag{2.4.1}$$

where $\boldsymbol{B}_t = n\boldsymbol{S}_t$ is defined earlier. Since $\boldsymbol{S}_t$ and $\boldsymbol{\Sigma}_0$ are symmetric and positive definite, from Theorem 4.14 of Schott (2005) and a simple transformation, there exists a nonsingular matrix $\boldsymbol{Z}$ such that $\boldsymbol{S}_t = \boldsymbol{Z}\boldsymbol{D}_d\boldsymbol{Z}'$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{Z}\boldsymbol{Z}'$, where $\boldsymbol{D}_d = diag(d_1, \cdots, d_p)$ with $d_1 \geq \cdots \geq d_p$ being the roots of $|\boldsymbol{S}_t - d\boldsymbol{\Sigma}_0| = 0$. Then

$$\begin{aligned}\lambda^* &= \left|\,\boldsymbol{Z}\boldsymbol{D}_d\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{Z}')^{-1}\right|^{n/2}\exp\{-\frac{n}{2}tr\left(\boldsymbol{Z}\boldsymbol{D}_d\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{Z}')^{-1}\right)+\frac{pn}{2}\}\\ &= |\,\boldsymbol{D}_d|^{n/2}\exp\{-\frac{n}{2}tr\boldsymbol{D}_d+\frac{pn}{2}\} = \{\,|\,\boldsymbol{D}_d|\exp\left[-tr(\boldsymbol{D}_d-\boldsymbol{I}_p)\right]\}^{n/2}\\ &= \prod_{i=1}^{p}\{\,d_i\exp\left[-(d_i-1)\right]\}^{\,n/2}.\end{aligned} \tag{2.4.2}$$

Note that (2.4.2) (for the two-sided LRT) and (2.2.4) (for the one-sided LRT) are of the same form. The difference is that the one-sided LRT only includes those $0 < d_i < 1$ in the product while the two-sided LRT uses all d_i's.

Moreover, Sugiura and Nagao (1968) showed that the two-sided LRT based on λ^* is biased, but the two-sided modified LRT based on

$$\lambda^{*(Mod)} = (\frac{e}{n-1})^{\frac{p(n-1)}{2}}\left|\boldsymbol{B}_t\boldsymbol{\Sigma}_0^{-1}\right|^{(n-1)/2}\exp\{-\frac{1}{2}tr\left(\boldsymbol{B}_t\boldsymbol{\Sigma}_0^{-1}\right)\} \tag{2.4.3}$$

is unbiased. Note that when replacing n by $n-1$ (2.4.1) becomes (2.4.3).

Anderson (2003, p.413) gave the LRT test of the equality of q covariance matrices of q sets of random samples, each from a multivariate normal distribution, without specifying a common covariance matrix. For our problem with $\boldsymbol{\Sigma}_0$ unknown, by treating the training samples $\{\boldsymbol{X}_{11}, \cdots, \boldsymbol{X}_{mn}\}$ as the first set and the current sample $\{\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}\}$ as the second set of random samples (i.e., $q = 2$), the two-sided LRT statistic is

$$\lambda^{\star} = \frac{\left|\frac{\boldsymbol{A}}{mn}\right|^{mn/2}\left|\frac{\boldsymbol{B}_t}{n}\right|^{n/2}}{\left|\frac{\boldsymbol{A}+\boldsymbol{B}_t}{mn+n}\right|^{(mn+n)/2}}, \tag{2.4.4}$$

where $A = mn\boldsymbol{S}_0$ and $\boldsymbol{B}_t = n\boldsymbol{S}_t$ are defined in Section 2.2. Assuming $\boldsymbol{S}_0$ is positive definite, by Theorem 4.14 of Schott (2005), there exists a nonsingular matrix $\boldsymbol{Y}$ such that $\boldsymbol{S}_t = \boldsymbol{Y}\boldsymbol{D}_{\beta}\boldsymbol{Y}'$ and $\boldsymbol{S}_0 = \boldsymbol{Y}\boldsymbol{Y}'$, where $\boldsymbol{D}_{\beta} = diag(\beta_1, \cdots, \beta_p)$ with $\beta_1 \geq \cdots \geq \beta_p$ being the roots of $|\boldsymbol{S}_t - \beta\boldsymbol{S}_0| = 0$. Then (4.4) becomes

$$\lambda^{\star} = \frac{|\boldsymbol{Y}\boldsymbol{Y}'|^{mn/2}|\boldsymbol{Y}\boldsymbol{D}_{\beta}\boldsymbol{Y}'|^{n/2}}{\left|\frac{m\boldsymbol{Y}\boldsymbol{Y}'+\boldsymbol{Y}\boldsymbol{D}_{\beta}\boldsymbol{Y}'}{m+1}\right|^{(mn+n)/2}} = \frac{|\boldsymbol{Y}\boldsymbol{Y}'|^{(mn+n)/2}|\boldsymbol{D}_{\beta}|^{n/2}}{|\boldsymbol{Y}\boldsymbol{Y}'|^{(mn+n)/2}\left|\frac{(m\boldsymbol{I}_p+\boldsymbol{D}_{\beta})}{m+1}\right|^{(mn+n)/2}}$$

$$= \prod_{i=1}^{p} \frac{\beta_i^{\frac{n}{2}}}{\left(\frac{m+\beta_i}{m+1}\right)^{\frac{mn+n}{2}}} = \prod_{i=1}^{p}\left[\frac{\beta_i^{w}}{(w\beta_i + 1 - w)}\right]^{\frac{mn+n}{2}}, \tag{2.4.5}$$

where $w = \frac{1}{m+1}$. Again, the difference between the one-sided LRT statistic (2.2.8) and the two-sided LRT statistic (2.4.5) lies on whether or not $0 < \beta_i < 1$.

Furthermore, it was shown in Das Gupta (1969) that the two-sided LRT based on (2.4.4) is biased. An unbiased two-sided LRT was derived in Sugiura and Nagao (1968) based on

$$\lambda^{\star(Mod)} = \frac{|\boldsymbol{A}|^{(mn-1)/2}|\boldsymbol{B}_t|^{(n-1)/2}}{|\boldsymbol{A}+\boldsymbol{B}_t|^{(mn+n-2)/2}}, \tag{2.4.6}$$

which can be obtained from (2.4.4) with mn replaced by $mn-1$ and n by $n-1$.

In this chapter, the control charts based on (2.4.3) and (2.4.6) are referred to as the two-sided "Modified-LRT" control charts.

2.4.2 A Control Chart Based on Decomposition Method

Assuming that $\mathbf{\Sigma}_0$ is known, Tang and Barnett (1996a, 1996b) proposed a multivariate Shewhart chart for monitoring $H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0$ $vs.$ $H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0$ that is based on decomposing $\boldsymbol{B}_t/(n-1)$ into a sum of a series of independent χ^2 statistics. Decompose $\mathbf{\Sigma}$ and $\boldsymbol{B}_t/(n-1)$ the same way and define $\sigma^2_{i\cdot 1,\cdots,i-1}$ and $s^2_{i\cdot 1,\cdots,i-1}$ respectively as the conditional population and sample variance of the ith variable given the first $i-1$ variables. Also, define $\mathbf{\Sigma}_{i,i+1,\cdots,p\cdot 1,\cdots,i-1}$ as the conditional population covariance matrix of the last $p-i+1$ variables given the first $i-1$ variables. σ_1^2 and s_1^2 are respectively defined as the population and sample variance of the first variable. In addition, let $\boldsymbol{\vartheta}_i$ and $\boldsymbol{R}_i$ $(i = 2, \cdots, p)$ denote respectively the $(p-i+1)\times 1$ vectors of population and sample regression coefficients when each of the last $p-i+1$ variables is regressed on the $(i-1)$-th variable while the first $i-2$ variables are held fixed. Note that $\boldsymbol{\vartheta}_2$ and $\boldsymbol{R}_2$ should be interpreted as the $(p-1)\times 1$ vectors of unconditional population and sample regression coefficients when each of the last $p-1$ variables is regressed on the first variable. When the current sample of n observations is drawn, an appropriate statistic based on a decomposition is given by

$$T^{(decom)} = \sum_{j=1}^{2p-1} Z_j^2, \tag{2.4.7}$$

where

$$Z_1^2 = \Phi^{-1}\left\{\chi^2_{n-1}\left[\frac{(n-1)s_1^2}{\sigma_1^2}\right]\right\},$$

$$Z_j^2 = \Phi^{-1}\left\{\chi^2_{n-j}\left[\frac{(n-1)s^2_{j\cdot 1,2,\cdots,j-1}}{\sigma^2_{j\cdot 1,2,\cdots,j-1}}\right]\right\},\ for\, j = 2, 3, \cdots, p,$$

$$Z^2_{p+1} = \Phi^{-1}\left\{\chi^2_{p-1}\left[(n-1)s_1^2(\boldsymbol{R}_2-\boldsymbol{\vartheta}_2)'\mathbf{\Sigma}^{-1}_{2,\cdots,p\cdot 1}(\boldsymbol{R}_2-\boldsymbol{\vartheta}_2)\right]\right\},$$

and, for $j = 3, \cdots, p$,

$$Z^2_{p+j-1} = \Phi^{-1}\left\{\chi^2_{p-j+1}\left[(n-1)s^2_{j-1\cdot 1,2,\cdots,j-2}(\boldsymbol{R}_j-\boldsymbol{\vartheta}_j)'\mathbf{\Sigma}^{-1}_{2,\cdots,p\cdot 1,\cdots,j-1}(\boldsymbol{R}_j-\boldsymbol{\vartheta}_j)\right]\right\}.$$

Note that $\Phi^{-1}\{\cdot\}$ is the inverse of the distribution function of $N(0,1)$ and $\chi^2_v[x] = P(\chi^2_v \leq x)$ is the distribution function of a χ^2 distribution with v degrees of freedom. When the process is in control, Z_j's are $i.i.d.$ as $N(0,1)$ and hence $T^{(decom)}$ is distributed as χ^2_{2p-1}. Thus the control chart can be established by plotting $T^{(decom)}$'s against the sampling sequence and an out-of-control alarm is signaled soon as $T^{(decom)}$ exceeds the upper control limit (UCL), $\chi^2_{2p-1}[\alpha]$. Note that the decomposition is not unique since it depends on how the p variables are arranged. Tang and Barnett (1996a) suggested the variables should be arranged in decreasing order of importance from 1 to p to reflect the relative importance of variables involved in particular.

In this chapter, the control charts based on (2.4.7) is referred to as the "TB-dcecomposed" control chart.

2.4.3 The G Chart for Covariance Matrices

Assume that $\boldsymbol{\Sigma}_0$ is unknown. Denote $\bar{\boldsymbol{S}} \equiv \frac{1}{m(n-1)} \sum_{i=1}^{m} \sum_{j=1}^{n} (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i)(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i)'$, where $\bar{\boldsymbol{X}}_i \equiv \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{X}_{ij}$. For monitoring changes in $\boldsymbol{\Sigma}$, Levinson, Holmes, and Mergen (2002) proposed a multivariate Shewhart chart, called the G-chart, based on the following statistic:

$$G = C_g \ln \left[\frac{|\boldsymbol{S}_p|^{(m+1)(n-1)}}{\left|\bar{\boldsymbol{S}}\right|^{m(n-1)} \left|\frac{\boldsymbol{B}_t}{n-1}\right|^{(n-1)}} \right], \tag{2.4.8}$$

where $C_g = 1 - \left[\frac{1}{m(n-1)} + \frac{1}{n-1} - \frac{1}{(m+1)(n-1)}\right] \times \left[\frac{2p^2+3p-1}{6(p+1))}\right]$ and $\boldsymbol{S}_p = \frac{m(n-1)\bar{\boldsymbol{S}}+\boldsymbol{B}_t}{(m+1)(n-1)}$. When the process is in control, the approximate distribution of G follows $\chi^2_{p(p+1)/2}$ and Levinson, Holmes, and Mergen (2002) suggested that $\chi^2_{p(p+1)/2,\alpha/2}$ and $\chi^2_{p(p+1)/2,1-\alpha/2}$ are used as the UCL and lower control limit (LCL).

However, when n is small, our empirical studies show that the control limits suggest by Levinson, Holmes, and Mergen (2002) are not accurate (for example, if $\alpha = 0.0027$, the in-control ARL is a lot less than 370). Therefore, we then simulate the distribution of G and fund that the value of G, is larger than that for the in-control process dispersion when the process dispersion increases or decreases. Thus, the suggested control limits by Levinson, Holmes, and Mergen (2002) are not appropriate for our purpose. Based on these reasons, we obtain the control limits for the statistic G by simulation and only consider the UCL since the G statistic is analogous to LRT. We refer to the control charts based on (2.4.8) as the "LHM-G" control chart.

2.4.4 Comparisons

In this subsection, we compare the ARL performance of the proposed procedure with that of the existing techniques described in the previous three subsections using Monte Carlo simulation.

Denote the average run length of the in-control process and out-of-control process by ARL_0 and ARL_1, respectively. Let T_D (T'_D) denote the test statistic for the case of known (unknown) $\mathbf{\Sigma}_0$. To estimate ARL, we first generate N statistics $T_{D,1}, \cdots, T_{D,N}$ ($T'_{D,1}, \cdots, T'_{D,N}$) for a very large number N and compute the proportion of $T_{D,i}$'s ($T'_{D,i}$'s) that exceed the control limit at a significance level α as described in Section 2.3. After repeating the above steps b times, we obtain b proportions. To estimate ARL, two procedures can be considered: (i) take the reciprocal of each proportion as an estimate of ARL and then take the average of these b ARL estimates to be the final ARL estimate; (ii) average the b proportions and take the reciprocal of the average as the ARL estimate. For the first ARL estimator, the standard error can be obtained easily by taking the sample standard deviation of the b ARL estimates and then dividing it by $\sqrt{b}$. The standard error of the second estimator can be obtained by the following argument. Note that, multiplying each proportion by N, we have b statistics that are *i.i.d.* as $binomial(N, \theta)$, where θ is the detecting power, the probability that T_D (T'_D) statistic of a randomly selected sample exceeds the control limit. When the process is in control, the detecting power is equal to α. Denote the second ARL estimator by $\widehat{ARL}$. Since $\widehat{ARL}$ is the reciprocal of the maximum likelihood estimator (MLE) of θ, then, by the asymptotic efficiency property of MLE, it can easily be shown that $\widehat{ARL}$ has an asymptotic normal distribution with mean $1/\theta$ and standard deviation

$$\left(\frac{1-\theta}{Nb\theta^3}\right)^{\frac{1}{2}}.$$

Then the standard error of this ARL estimator can be calculated by

$$\left[\widehat{ARL}^2(\widehat{ARL}-1)/(Nb)\right]^{\frac{1}{2}}. \tag{2.4.9}$$

It is found from our simulation study that the difference between the results of the two estimating procedures is negligible. We thus only report the results of the second approach in this chapter.

Assume that the covariance matrix has been "decreased" from $\mathbf{\Sigma_0}$ to $\mathbf{\Sigma}$, i.e., $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ is positive semidefinite and $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$. As shown earlier, the distribution of T_D (T'_D) is invariant in $\mathbf{\Sigma}_0$. Thus, without loss of generality, we can assume that $\mathbf{\Sigma}_0 = I_p$ when simulating the distribution of T_D (T'_D). For simplicity of discussion, we consider $p = 2$. To create out-of-control scenarios, express $\mathbf{\Sigma}$ as $\begin{bmatrix} \Delta_1 & \rho\sqrt{\Delta_1\Delta_2} \\ \rho\sqrt{\Delta_1\Delta_2} & \Delta_2 \end{bmatrix}$, where $\Delta_i \leq 1$, $i = 1, 2$. This means the variance of the ith quality characteristic has been decreased by a factor of Δ_i, $i = 1, 2$ and ρ is the correlation coefficient. We consider studying ARL_1 for various combinations of Δ_1, Δ_2, and ρ. It can be easily shown that the eigenvalues of $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ are

$$\frac{1}{2}\left[(2-\Delta_1-\Delta_2) \pm \sqrt{(\Delta_1-\Delta_2)^2 + 4\rho^2\Delta_1\Delta_2}\right] \tag{2.4.10}$$

and, under the condition that $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ is positive semidefinite, the restricted range of ρ is

$$|\rho| \leq \left[\frac{(1-\Delta_1)(1-\Delta_2)}{\Delta_1\Delta_2}\right]^{\frac{1}{2}}. \tag{2.4.11}$$

Note that (2.4.11) implies that we cannot consider the case when $\Delta_1 = \Delta_2 = 1$ and $\rho \neq 0$, i.e., the case when only the correlation changes.

In our comparative study, setting $\alpha = 0.0027$ which results in $ARL_0 = 370.4$, we consider the following cases: $p = 2$; $n = 5$, 10 for known $\mathbf{\Sigma}_0$; $m = 25$, 50 and $n = 5$ for unknown $\mathbf{\Sigma}_0$. The in-control covariance matrix is $\mathbf{\Sigma}_0 = I_p$ and the out-of-control covariance matrix is $\mathbf{\Sigma}$. The following three scenarios of $\mathbf{\Sigma}$ are considered:

(1) $\Delta_1 = \Delta_2 = c$ and $\rho = 0$, (that is, $\mathbf{\Sigma} = c\mathbf{\Sigma}_0$), for $c = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1$.

(2) $\Delta_1 \neq \Delta_2$ and $\rho = 0$. for the following 8 combinations: (Δ_1, Δ_2) = (0.8,1), (0.6,1), (0.4,1), (0.2,1), (0.8,0.6), (0.6,0.4), (0.4,0.2), (0.2,0.8).

(3) For $\rho \neq 0$, under the condition (2.4.11), we choose $|\rho|$ =0.2 and 0.4 for the following 4 combinations: (Δ_1, Δ_2) =(0.6,0.6), (0.6,0.4), (0.4,0.4), (0.4,0.2). Note that these 4 combinations are selected from scenarios (1) and (2) so that we can study the effect of ρ on ARL performance.

In the simulation study, we take $N = 1,000,000$ and $b = 100$ to obtain the ARL estimate along with its standard error for each scenario. When $\mathbf{\Sigma}_0$ is known, for $\alpha = 0.0027$, the one-sided, two-sided LRT, and two-sided Modified-LRT control limits obtained are respectively 22.23621, 22.68151, and 17.67692 for $n = 5$; 16.84193, 17.53596, and 15.45388 for $n = 10$. It is found that, with the afore-mentioned control limits obtained from empirical study, $ARL_0 \approx 370$ with standard error $\sqrt{\frac{1-0.0027}{Nb \cdot 0.0027^3}}$ around 0.7118152. Moreover, the control limit for both $n = 5, 10$ based on TB-decomposed control chart, is $\chi^2_3(0.9973)$=14.15625. Table 2.3 gives the estimates of ARL_1 and their standard errors (in parentheses) of the one-sided, two-sided LRT, two-sided Modified-LRT, and TB-decomposed control charts for the scenarios (1)-(3) described above. The following results are observed:

(i) The ARL_1 value of the one-sided control chart is *much* smaller than that of the other three control charts for all cases tested. This shows that the one-sided control chart greatly outperforms the three compering control charts when dispersion decreases, especially for the smaller subgroup size $n = 5$. Furthermore, it is surprising that the two-sided LRT control chart is unbiased in all cases tested and has a better ARL_1 performance than the Modified-LRT and TB-decomposed. In addition, the TB-decomposed control chart is biased in the sense that some of its ARL_1 values are greater than 370.

Hence, the TB-decomposed control chart has the worst performance.

(ii) Given c, the ARL_1 for $n = 10$ is smaller than that for $n = 5$. This result also confirms the general observation that the detecting power gets larger when the subgroup size gets larger. For fixed n, ARL_1 is smaller when c is smaller. This again is not surprising since it is easier to detect larger shifts.

(iii) For all the combinations of Δ_1 and Δ_2 in scenarios (1)-(3), the ARL_1 for $n = 10$ is smaller than that for $n = 5$. For fixed n, ρ, and one of Δ_1 and Δ_2, say Δ_2, the ARL_1 decreases when Δ_1 decreases. These results are expected. Also, a smaller ARL_1 results in a smaller standard error due to (2.4.9).

(iv) For the effect of ρ, we first observe that, by (2.4.10), the eigenvalues of $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ depend on ρ through ρ^2. Hence the sign of ρ does not play any role in ARL_1 performance as found in our simulation study. The ARL_1 decreases when $|\rho|$ increases from 0 to 0.4. This means that the ability of the proposed chart detecting a decrease in dispersion gets better when the correlation (positive or negative) between the two quality characteristics becomes stronger.

For the case of unknown $\mathbf{\Sigma}_0$, the one-sided, two-sided LRT, two-sided Modified-LRT, and LHM-G control limits obtained respectively are 22.07988, 22.58894, 53.27833, and 14.48746 for $m = 25$; 22.16664, 22.66328, 58.79951, and 14.49071 for $m = 50$. The ARL_1 values along with their standard errors are given in Table 2.4. Similar observations as those discussed earlier for the case of known $\mathbf{\Sigma}_0$ can also be made here. It is interesting to observe that the ARL performance of the two-sided Modified-LRT control chart and the LHM-G control chart are very close, with the former being slightly better.

2.4.5 Comparing the Detecting Power for Increases Versus Decreases in Dispersion

Lowry, Champ, and Woodall (1995) reported that detecting decreases in variance is much harder than detecting increases in the univariate case. In this subsection, we study this same issue for the multivariate case with the control chart presented in this chapter and the control chart designed for detecting dispersion increases proposed in Yen and Shiau (2008). Yen and Shiau (2008) proposed the following statistics for testing (2.1.1) for detecting increases in dispersion are, respectively:

$$T_I = \begin{cases} n\sum_{i=1}^{p_I^*}[(d_i-1)-\log d_i] & ,for\ p_I^*>0 \\ 0 & ,for\ p_I^*=0 \end{cases} \tag{2.4.12}$$

for the case of known $\mathbf{\Sigma}_0$ and

$$T_I' = \begin{cases} (mn+n)\sum_{i=1}^{p_I^\star}[\log(w\beta_i+1-w)-w\log\beta_i] & ,for\ p_I^\star>0 \\ 0 & ,for\ p_I^\star=0 \end{cases} \tag{2.4.13}$$

for the case of $\mathbf{\Sigma}_0$ unknown, where p_I^* is the number of $d_i>1$ and $p_I^\star$ is the number of $\beta_i>1$. For the case of $\mathbf{\Sigma}=c\mathbf{\Sigma}_0$ with $c=1.25,1.5,1.75,2,2.25,\ 2.5,2.75,3$, Tables 2.5 presents the ARL_1 values of the one-sided chart proposed in Yen and Shiau (2008) and the charts described earlier in Subsections 2.4.1-2.4.3. Note that the ranges of the changes in covariance matrix for increases and decreases are different, that is, (1,∞) for increases and (0,1) for decreases. Thus, for a fair comparison, it is more reasonable to take a log transformation on the sign of changes, i.e., $log(c)$, so that the two ranges become (0,∞) and ($-\infty$,0). Each of the Figures 2.1-2.4 depicts the ARL_1 curves of all the control charts under study for both cases of monitoring increases and decreases in dispersion and in both of the original and logarithm scales for c. Figures 2.1 (for $n=5$) and 2.2 (for $n=10$) are for the case of known $\mathbf{\Sigma}_0$ and Figures 2.3 (for $m=25$ and $n=5$) and 2.4 (for $m=50$ and $n=5$) are for the case of unknown $\mathbf{\Sigma}_0$. By comparing two control charts of Yen and Shiau (2008) and that of this chapter, the following results are observed:

(i) By the statistics (2.2.5) and (2.4.12) ((2.2.9) and (2.4.13)), it can be seen that 1 separates the eigenvalues into two sets, one for (2.2.5) ((2.2.9)) and the other for (2.4.12) ((2.4.13)), for all d_i's (β's). Hence, the rejection regions of testing (2.1.1) and (2.1.2) are disjoint.

(ii) Observed from the ARL_1's, we find that the power of the one-sided chart in Yen and Shiau (2008) for monitoring increases in dispersion is better than that of the proposed chart in this Chapter for monitoring decreases in dispersion. The comparison is based on the same absolute logarithm scale of values c (for example, $log(1.25) = |log(0.8)|$, $log(2) = |log(0.5)|$, $log(2.5) = |log(0.4)|$). For example, from the cases of $n = 5$ and $n = 10$ for known $\mathbf{\Sigma}_0$ shown in Tables 3 and 5, it can be observed that the ARL_1's for detection increases with $c = 1.25, 2, 2.5$ are, respectively, 69.2106, 6.91187, and 4.73748 for $n = 5$ and 43.9506, 3.21706, and 1.77936 for $n = 10$. They are all a lot smaller than those for detecting decreases with $c = 0.8, 0.5, 0.4$, which are, respectively, 233.475, 82.6634, and 49.4864 for $n = 5$ and 135.346, 16.9510, and 7.11778 for $n = 10$. Similarly, for the cases of $m = 25$ and $m = 50$ when $\mathbf{\Sigma}_0$ is unknown, we can observe from Tables 4 and 5, the outcomes are in a similar manner.

For the control charts described in Subsections 2.4.1-2.4.3, except the case of the two-sided LRT control chart, the results all lead to the same conclusion presented above. The power of the two-sided LRT control chart for monitoring increases in dispersion is worse than that for monitoring increases in dispersion. This is becomes the two-sided LRT control chart is biased for detecting dispersion increases. We also observe that the two-sided Modified-LRT and LHM-G control charts are both unbiased for detecting either dispersion increases or decreases and the TB-decomposed control chart is biased for detecting dispersion decreases.

2.5 Examples

In this section, the application of the proposed control chart is illustrated by a real-life example. In addition, two simulated examples are presented to demonstrate the better detecting power of the one-sided control chart over the existing two-sided control charts when the dispersion decreases.

2.5.1 A real example

A set of real data for assuming the semiconductor failure rate was taken in a semiconductor company. The failure rates are strongly correlated to the process stability, unsuitable personal operations, and so on. In order to get the failure rate of the process, the wafer acceptance tests are taken. Two quality characteristics monitored are the functions of "write" and "erase", which are two test items related to a wafer acceptance test and failure rate calculation for each wafer. These two values are strongly related to the stability of the process measurement technique. The two failure rates are measured on each wafer after the write-action and erase-action, respectively. Thus the two variables are correlated. Let X_1 and X_2 be the averages of the five successive measurements for the write-action and erase-action, respectively. Denote $\boldsymbol{X} = (X_1, X_2)$. 50 subgroups of random samples, each of size 5, were taken from the original process. The sample mean is $\bar{\bar{\boldsymbol{X}}} = \begin{pmatrix} 1.98920 \\ 6.14052 \end{pmatrix}$ and sample covariance matrix is $\boldsymbol{S} = \frac{1}{(50\times5-1)}\sum_{i=1}^{50}\sum_{j=1}^{5}(\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}})(\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}})' = \begin{pmatrix} 0.84598 & 0.54288 \\ 0.54288 & 5.46428 \end{pmatrix}$. Data are given in Table 2.6. First, we use the usual $|\boldsymbol{S}|$ and T^2 control charts to inspect whether these data were taken from an in-control process. Figure 2.5 depicts the $|\boldsymbol{S}|$ and T^2 control charts, under an in-control false-alarm rate of $\alpha = 0.0027$ so that indicating the process were stable that these samples can be used as the training samples. By (2.2.6), we have $\boldsymbol{S}_0 = \begin{pmatrix} 0.84260 & 0.54071 \\ 0.54071 & 5.44242 \end{pmatrix}$ and the sample correlation coefficient between the 250 X_1's and X_2's is $\hat{\rho} = 0.2525$.

For $p=2$, $m=50$, $n=5$, and $\alpha=0.0027$, by the procedure described before, the one-sided, two-sided, two-sided Modified-LRT, and LHM-G control limits are 22.16664, 22.66328, 58.79951, and 14.49071, respectively. We use these control limits to monitor another 21 subgroups of samples, each of size 5, taken on-line from the process. Table 2.7 gives these data. The control charts are displayed in Figure 2.6. It is found that the 9th, 11th, 14th, and 15th subgroups exceed the control limit of the one-sided control chart, while the 9th, 11th, and 15th subgroups exceed the control limit of the two-sided control chart, and the 9th and 15th subgroups exceed the control limit of the two-sided modified-LRT and LHM-G control chart. This demonstrates that the one-sided control chart is more sensitive than the other three control charts.

2.5.2 Simulated examples

Consider the real-life example described in the previous subsection. Suppose the random vector $\boldsymbol{X}$ from the in-control process is distributed as $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ with $\boldsymbol{\mu}_0 = \bar{\bar{\boldsymbol{X}}}$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{S}_0$. We simulate some in-control data and out-of-control data to investigate the effectiveness of the proposed control chart.

100 subgroups of samples, each of size 5, are generated. The first ten subgroups and the 51st to 60th subgroups are from the in-control process. 11th to 50th subgroups are from $N_p(\bar{\bar{\boldsymbol{X}}}, \boldsymbol{\Sigma}_1)$ and the 61st to 100th subgroups are from $N_p(\bar{\bar{\boldsymbol{X}}}, \boldsymbol{\Sigma}_2)$ with $\boldsymbol{\Sigma}_1 \leq \boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma}_2 \leq \boldsymbol{\Sigma}_0$. Let $\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sqrt{\Delta_1} & 0 \\ 0 & \sqrt{\Delta_2} \end{bmatrix} \boldsymbol{S_0} \begin{bmatrix} \sqrt{\Delta_1} & 0 \\ 0 & \sqrt{\Delta_2} \end{bmatrix}$ and $\boldsymbol{\Sigma}_2 = \begin{bmatrix} \sqrt{\Delta'_1} & 0 \\ 0 & \sqrt{\Delta'_2} \end{bmatrix} \boldsymbol{S_0} \begin{bmatrix} \sqrt{\Delta'_1} & 0 \\ 0 & \sqrt{\Delta'_2} \end{bmatrix}$, where Δ_1, Δ_2, Δ'_1, and Δ'_2 are all smaller than 1. Two scenarios are considered for $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ corresponding to: (i)$(\Delta_1, \Delta_2) = (0.6, 0.4)$ and $(\Delta'_1, \Delta'_2) = (0.2, 0.4)$ and (ii) $(\Delta_1, \Delta_2) = (0.4, 0.4)$ and $(\Delta'_1, \Delta'_2) = (0.2, 0.2)$.

Figures 2.7-2.8 depict all of the one-sided, two-sided, two-sided Modified-LRT, and LHM-G control charts for scenario (i) and (ii), respectively. It is quite striking to observe that, for scenario (i)((ii)), the one-sided control chart effectively picks 2 (3) and 4 (8) out-of-control points from the first and the second out-of-control regions, respectively, while the two-sided, two-sided Modified-LRT, and LHM-G chart pick 1, 0, 0 (2, 1, 1) and 3, 2, 2 (6, 4, 4) out-of-control points from the first and the second out-of-control regions for scenario (i)((ii)), respectively. The figures also demonstrate that the second out-of-control region is easier to detect than the first out-of-control region and scenario (ii) is easier to detect than scenario (i), as one would expect.

2.6 Discussions

In this chapter, we have proposed and studied a control chart based on the one-sided likelihood ratio test that is specifically designed for detecting decreases in dispersion for multivariate processes. Both cases when the in-control covariance matrix $\boldsymbol{\Sigma}_0$ is known or unknown are considered. It was shown that the control limit does not depend on $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$. The performance study showed that the proposed control chart indeed outperforms the existing control charts based on the two-sided tests of $H_0: \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ $vs.$ $H_1: \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$ in terms of the ARL, when the process dispersion decreases.

The proposed control chart is a Shewhart-type chart. It is well known that EWMA and CUSUM charts are more sensitive to small changes. An EWMA extension of the proposed chart will be reported in a follow-up study. A combination of a Shewhart and an EWMA (or CUSUM) chart can provide a more effective control since a wider range of dispersion decreases will be covered. This is definitely worthy of future investigations.

The proposed control chart is analogous to those of Yen and Shiau (2008), which is a one-sided LRT-based control chart for detecting dispersion increases for multivariate processes. For a more effective monitoring, one can consider using the control chart of Yen and Shiau (2008) or the proposed chart in the current chapter for monitoring only increases or only decreases in dispersion. Furthermore, when detecting both increases and decreases in dispersion, a combined chart based on these two one-sided LRT-based control charts could potentially outperform a two-sided LRT-based control chart. This is the subject of the next chapter.

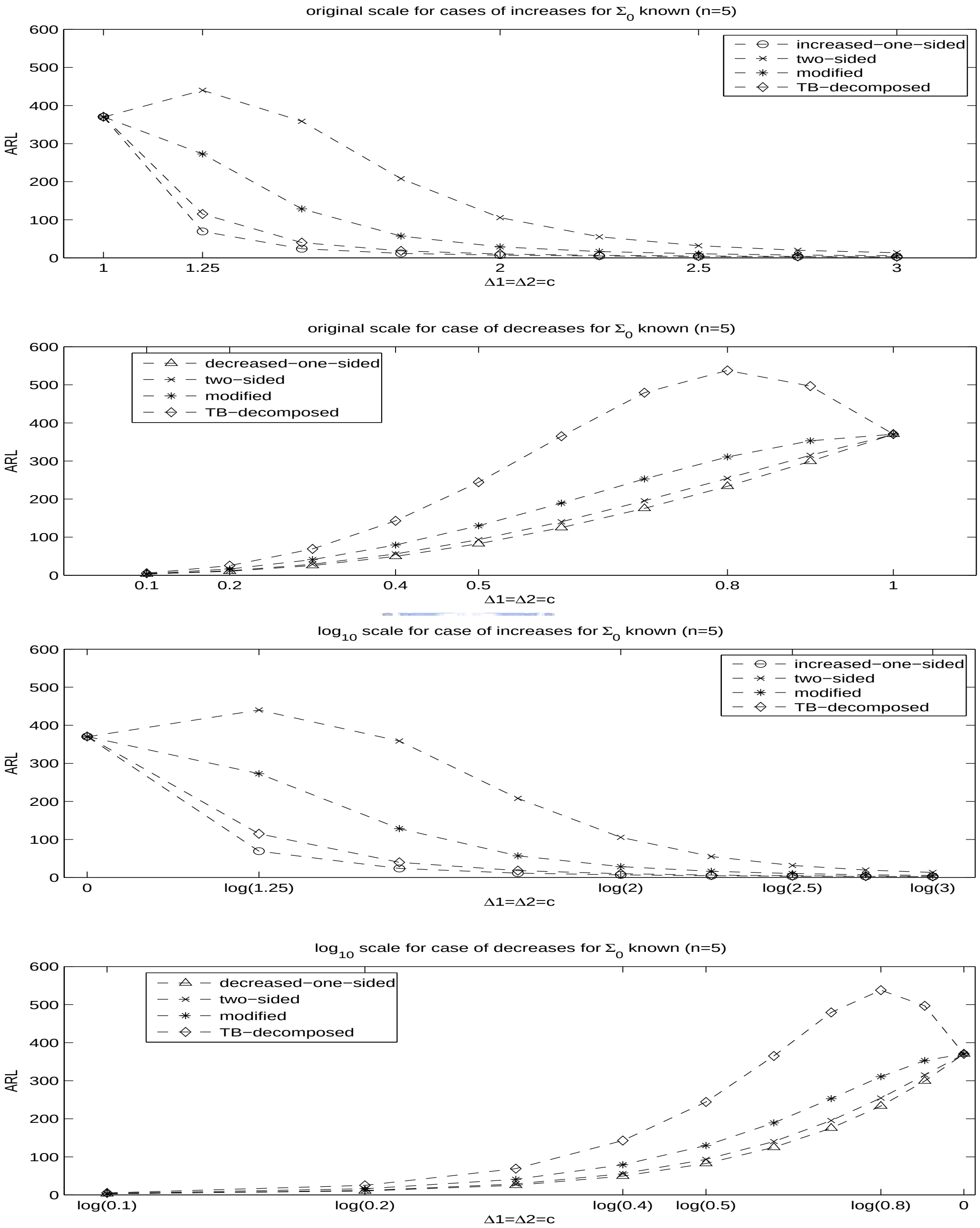

Figure 2.1: The ARL curves of four control charts for both cases of increases and decreases, including the original and logarithm scales for c, when $\mathbf{\Sigma}_0$ is known ($n = 5$).

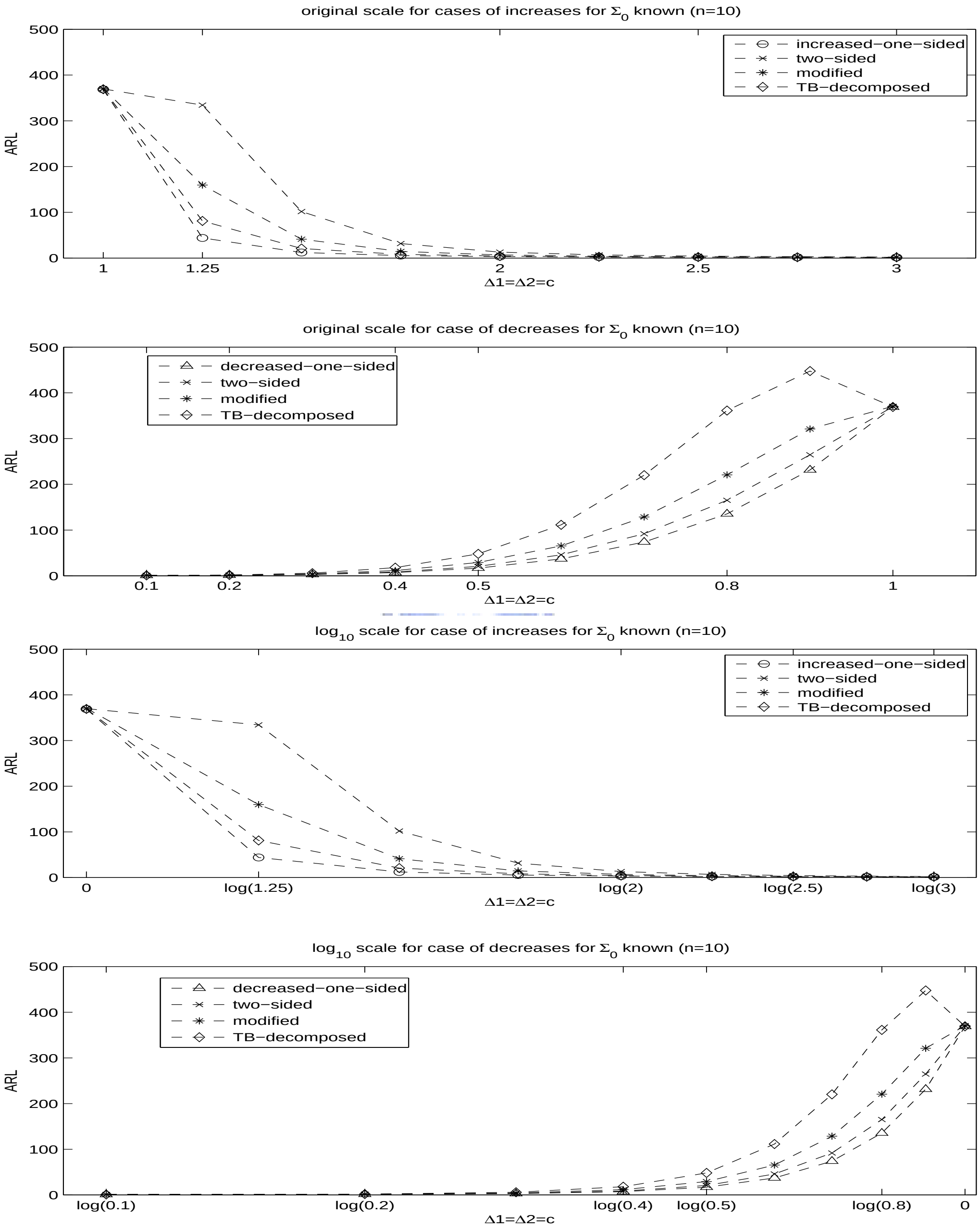

Figure 2.2: The ARL curves of four control charts for both cases of increases and decreases, including the original and logarithm scales for c, when $\mathbf{\Sigma}_0$ is known ($n = 10$).

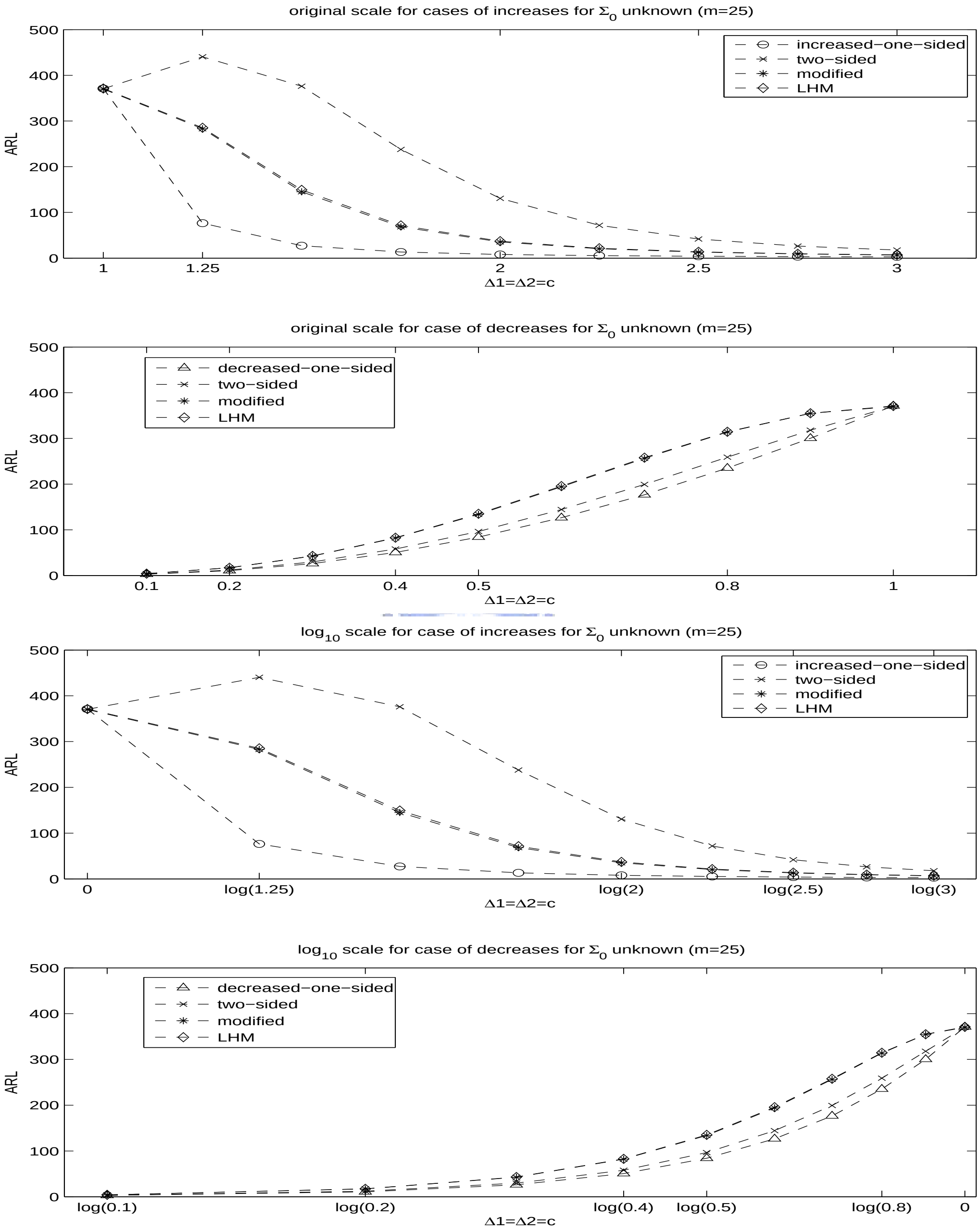

Figure 2.3: The *ARL* curves of four control charts for both cases of increases and decreases, including the original and logarithm scales for c, when $\mathbf{\Sigma}_0$ is unknown ($m = 25$,$n = 5$).

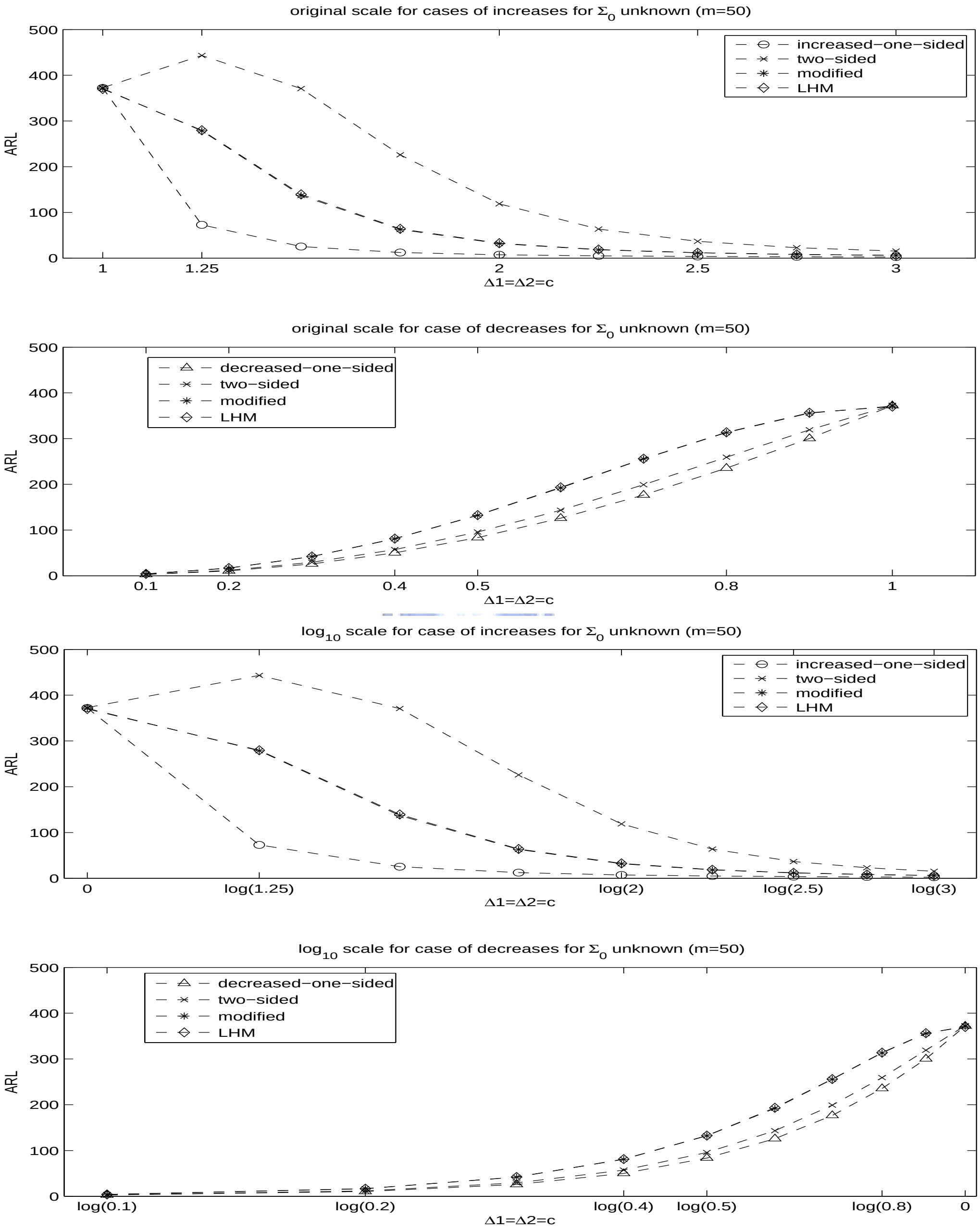

Figure 2.4: The *ARL* curves of four control charts for both cases of increases and decreases, including the original and logarithm scales for c, when $\mathbf{\Sigma}_0$ is unknown ($m = 50$,$n = 5$).

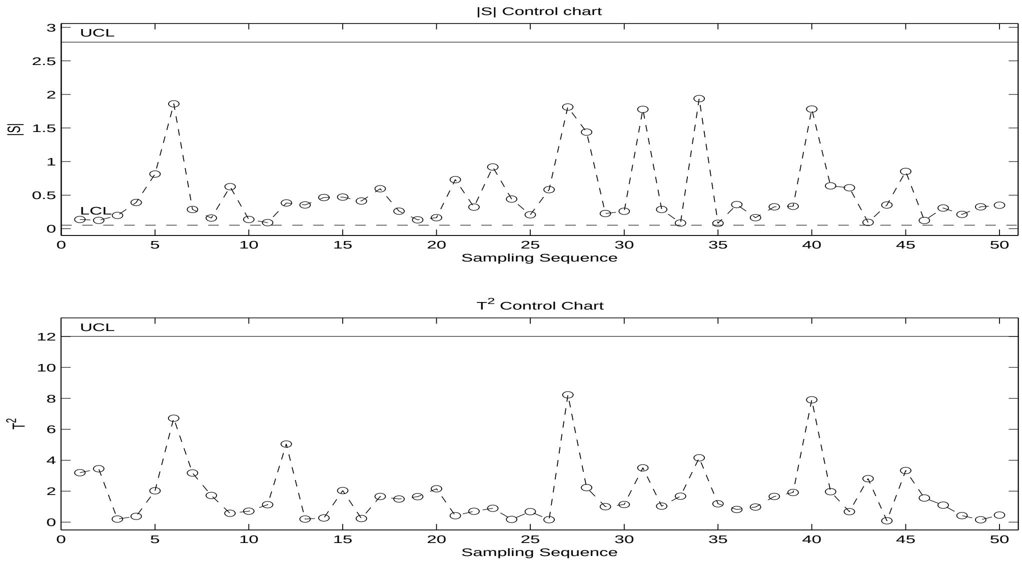

Figure 2.5: $|\boldsymbol{S}|$ and T^2 control chart on the 50 training samples of the real-life example, indicating the process is stable.

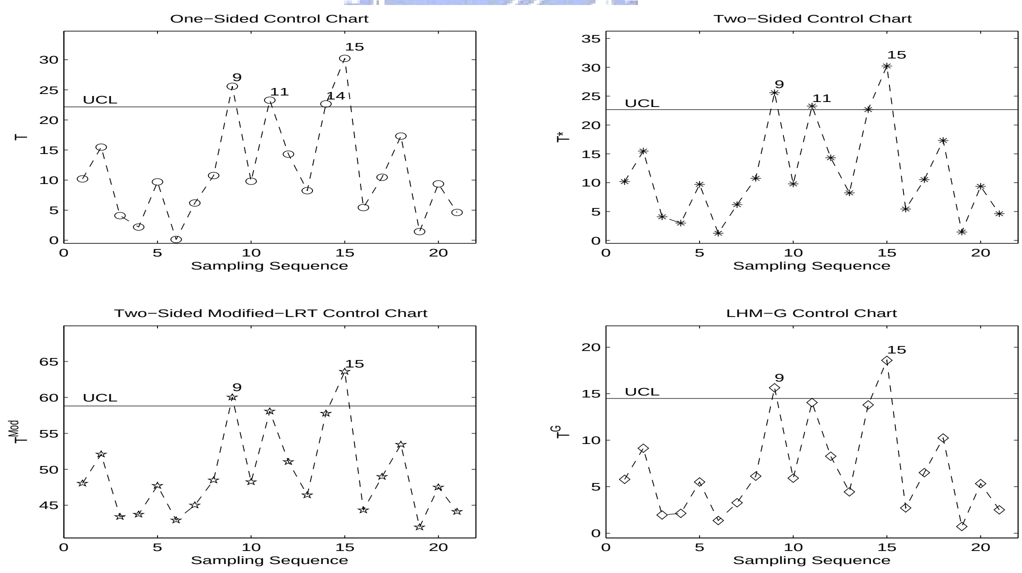

Figure 2.6: One-sided, two-sided, two-sided Modified-LRT, and LHM-G control charts on 21 new samples of the real-life example. One-sided control chart outperforms the other three control charts.

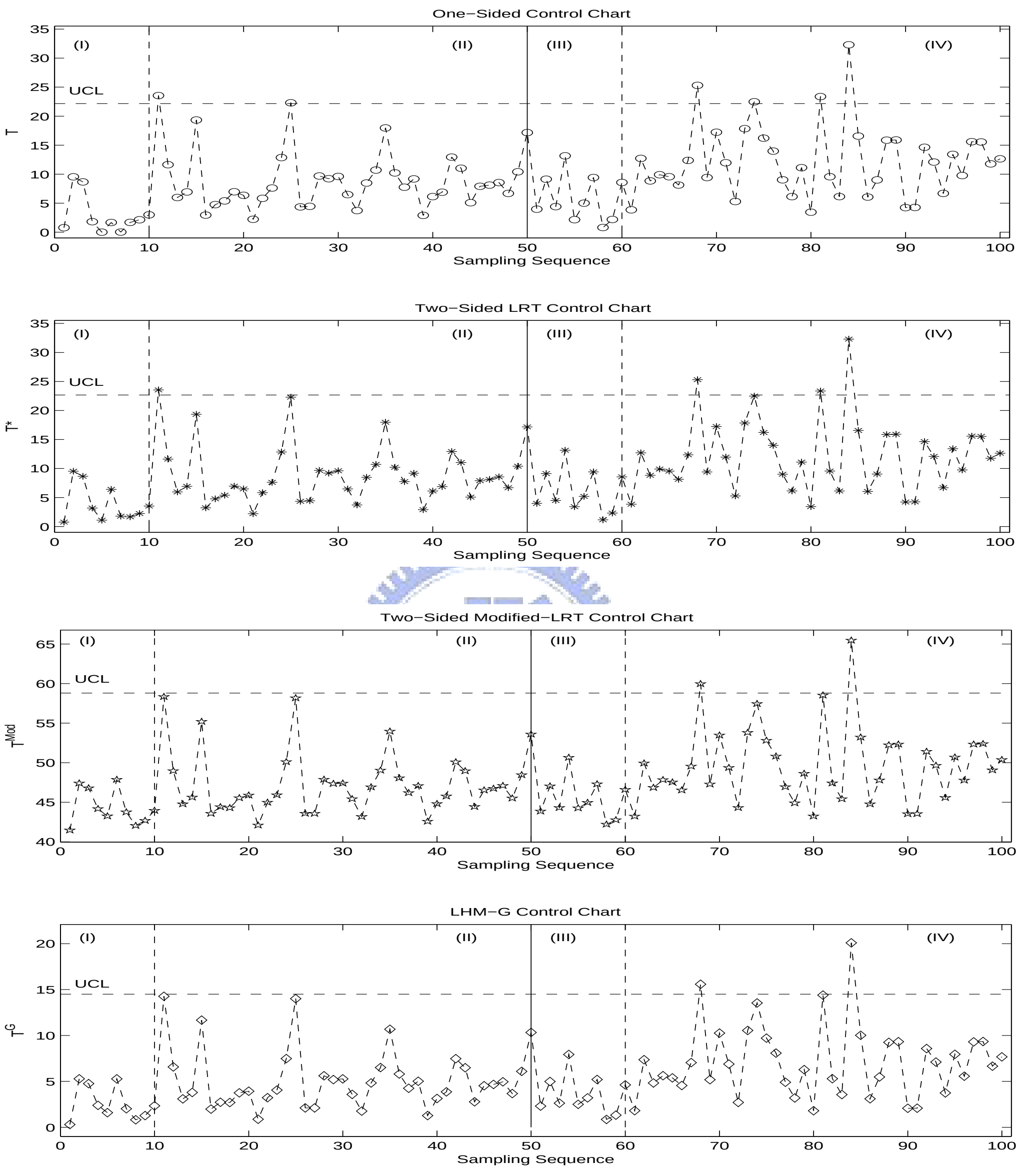

Figure 2.7: One-sided, two-sided, two-sided Modified-LRT, and LHM-G control charts for scenario (i) of the hypothetical example.

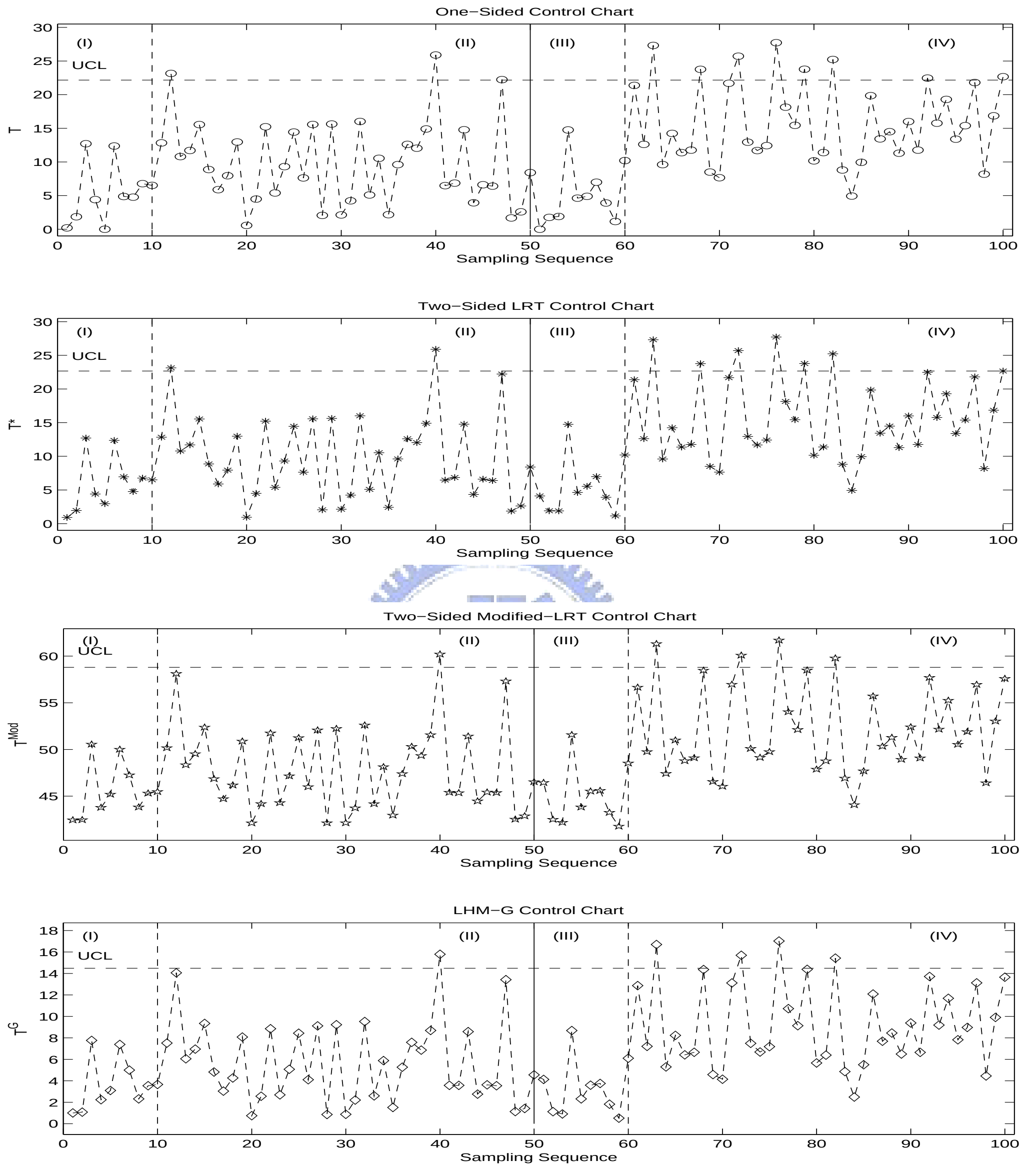

Figure 2.8: One-sided, two-sided, two-sided Modified-LRT, and LHM-G control charts for scenario (ii) of the hypothetical example.

TABLES

Table 2.1 The control limits and their standard errors (in parentheses) for various p, n, and α when $\mathbf{\Sigma}_0$ is known.

		$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.0027$
	$n = 5$	12.07387 (.00157)	17.73936 (.00372)	22.23621 (.00650)
	$n = 10$	8.897327 (.00125)	13.32618 (.00297)	16.84193 (.00504)
$p = 2$	$n = 15$	8.005460 (.00115)	12.12254 (.00248)	15.39511 (.00496)
	$n = 20$	7.550712 (.00099)	11.52044 (.00228)	14.67947 (.00447)
	$n = 25$	7.272272 (.00120)	11.15209 (.00253)	14.23805 (.00427)
	$n = 30$	7.078561 (.00106)	10.89676 (.00228)	13.94104 (.00444)
	$n = 35$	6.934560 (.00102)	10.70614 (.00226)	13.71810 (.00405)
	$n = 40$	6.819769 (.00106)	10.55508 (.00248)	13.54144 (.00442)
	$n = 5$	22.90575 (.00229)	31.43728 (.00525)	38.17811 (.00972)
	$n = 10$	14.71335 (.00158)	20.27104 (.00314)	24.54849 (.00568)
$p = 3$	$n = 15$	12.91980 (.00143)	17.94842 (.00297)	21.82590 (.00521)
	$n = 20$	12.06823 (.00150)	16.87110 (.00302)	20.57219 (.00490)
	$n = 25$	11.55274 (.00127)	16.21673 (.00261)	19.81813 (.00590)
	$n = 30$	11.20298 (.00114)	15.77728 (.00290)	19.32009 (.00515)
	$n = 35$	10.94578 (.00112)	15.44731 (.00248)	18.93584 (.00504)
	$n = 40$	10.74509 (.00134)	15.19751 (.00251)	18.64674 (.00543)
	$n = 5$	46.32313 (.00392)	62.63170 (.00997)	75.76703 (.01772)
	$n = 10$	22.33401 (.00201)	29.18483 (.00416)	34.37385 (.00783)
$p = 4$	$n = 15$	19.04439 (.00178)	25.05275 (.00380)	29.59728 (.00602)
	$n = 20$	17.57767 (.00163)	23.24285 (.00323)	27.51586 (.00578)
	$n = 25$	16.72193 (.00165)	22.18989 (.00330)	26.32282 (.00612)
	$n = 30$	16.14666 (.00138)	21.48316 (.00293)	25.53406 (.00590)
	$n = 35$	15.72819 (.00145)	20.98150 (.00283)	24.95429 (.00600)
	$n = 40$	15.40851 (.00128)	20.59187 (.00272)	24.52203 (.00621)

Table 2.2 The control limits and their standard errors (in parentheses) for various n=5 and p, m, α when $\boldsymbol{\Sigma}_0$ is unknown.

		$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.0027$
	$m = 25$	11.92729 (.00166)	17.58328 (.00369)	22.07988 (.00679)
	$m = 30$	11.94909 (.00148)	17.60504 (.00333)	22.10407 (.00628)
	$m = 35$	11.97148 (.00165)	17.63610 (.00374)	22.13565 (.00747)
	$m = 40$	11.98110 (.00165)	17.64386 (.00307)	22.14124 (.00647)
	$m = 45$	11.99234 (.00147)	17.65663 (.00373)	22.15535 (.00680)
	$m = 50$	12.00225 (.00151)	17.66655 (.00366)	22.16664 (.00623)
$p = 2$	$m = 55$	12.00584 (.00158)	17.67245 (.00384)	22.16867 (.00632)
$(n = 5)$	$m = 60$	12.01185 (.00136)	17.67432 (.00314)	22.17503 (.00594)
	$m = 65$	12.01496 (.00147)	17.68237 (.00301)	22.17948 (.00611)
	$m = 70$	12.02254 (.00151)	17.68813 (.00297)	22.18749 (.00607)
	$m = 80$	12.02643 (.00162)	17.69250 (.00340)	22.19493 (.00646)
	$m = 90$	12.03081 (.00138)	17.69783 (.00348)	22.19774 (.00691)
	$m = 100$	12.03676 (.00150)	17.70677 (.00329)	22.21453 (.00584)
	$m = 25$	22.66663 (.00247)	31.18786 (.00587)	37.90671 (.01013)
	$m = 30$	22.70397 (.00240)	31.23109 (.00535)	37.96759 (.01007)
	$m = 35$	22.72928 (.00239)	31.25447 (.00556)	37.96784 (.00985)
	$m = 40$	22.75709 (.00240)	31.28772 (.00566)	37.99968 (.01074)
	$m = 45$	22.77365 (.00255)	31.29524 (.00471)	38.00987 (.01079)
	$m = 50$	22.78922 (.00267)	31.31723 (.00435)	38.05282 (.00813)
$p = 3$	$m = 55$	22.79606 (.00230)	31.33093 (.00471)	38.04586 (.01016)
$(n = 5)$	$m = 60$	22.80513 (.00283)	31.32847 (.00535)	38.04199 (.01049)
	$m = 65$	22.81211 (.00228)	31.33656 (.00459)	38.06254 (.00975)
	$m = 70$	22.82176 (.00218)	31.35669 (.00561)	38.07508 (.00948)
	$m = 80$	22.82944 (.00250)	31.35271 (.00537)	38.05986 (.00904)
	$m = 90$	22.83594 (.00256)	31.37675 (.00593)	38.09044 (.01058)
	$m = 100$	22.84349 (.00225)	31.37371 (.00480)	38.10482 (.00782)
	$m = 25$	45.97473 (.00432)	62.29930 (.01021)	75.41831 (.02022)
	$m = 30$	46.02699 (.00448)	62.34284 (.01002)	75.50858 (.02075)
	$m = 35$	46.06585 (.00458)	62.36199 (.00963)	75.51993 (.01836)
	$m = 40$	46.10179 (.00447)	62.39709 (.01035)	75.53980 (.02111)
	$m = 45$	46.12730 (.00486)	62.43510 (.01036)	75.59381 (.01772)
	$m = 50$	46.14573 (.00472)	62.46074 (.00916)	75.57842 (.01859)
$p = 4$	$m = 55$	46.15985 (.00451)	62.46803 (.01013)	75.60569 (.01877)
$(n = 5)$	$m = 60$	46.17025 (.00493)	62.47965 (.00967)	75.60087 (.01921)
	$m = 65$	46.17819 (.00544)	62.48028 (.00928)	75.61300 (.02026)
	$m = 70$	46.19099 (.00438)	62.51232 (.01000)	75.64397 (.01940)
	$m = 80$	46.21218 (.00465)	62.52291 (.01128)	75.65437 (.01847)
	$m = 90$	46.21615 (.00463)	62.53290 (.01026)	75.67351 (.01673)
	$m = 100$	46.23938 (.00391)	62.55555 (.01099)	75.67159 (.01945)

Table 2.3 (a) ARL_1 and their standard errors (in parentheses) of the one-sided and the three two-sided control charts when $\mathbf{\Sigma}_0$ is known and $n = 5$.

$p = 2$		n=5			
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	TB-decomposed
[$\rho = 0$]					
0.9	0.9	298.983 (.51611)	314.880 (.55786)	353.000 (.66229)	496.845 (1.1064)
0.8	0.8	233.475 (.35598)	254.553 (.40533)	310.482 (.54620)	537.819 (1.2461)
0.7	0.7	175.510 (.23185)	195.367 (.27237)	252.949 (.40150)	479.370 (1.0485)
0.6	**0.6**	124.864 (.13897)	140.358 (.16569)	189.433 (.26004)	364.898 (.69608)
0.5	0.5	82.6634 (.07470)	93.2962 (.08963)	129.793 (.14730)	244.197 (.38082)
0.4	**0.4**	49.4864 (.03446)	55.8158 (.04132)	79.3682 (.07026)	142.781 (.17001)
0.3	0.3	25.5262 (.01264)	28.6359 (.01505)	41.0587 (.02599)	69.1756 (.05712)
0.2	0.2	10.3866 (.00318)	11.5322 (.00374)	16.3339 (.00640)	25.0818 (.01231)
0.1	0.1	2.81299 (.00038)	3.03468 (.00043)	3.99012 (.00069)	5.27597 (.00109)
0.8	1	292.969 (.50060)	305.901 (.53415)	338.103 (.62077)	399.009 (.79603)
0.6	1	211.343 (.30651)	225.849 (.33866)	263.982 (.42809)	313.678 (.55467)
0.4	1	128.709 (.14545)	138.392 (.16221)	166.308 (.21383)	184.734 (.25040)
0.2	1	52.6129 (.03780)	56.6234 (.04223)	68.9434 (.05683)	67.5800 (.05514)
0.8	0.6	170.073 (.22114)	188.601 (.25832)	242.327 (.37645)	472.288 (1.0253)
0.6	**0.4**	77.8419 (.06824)	87.8123 (.08182)	121.782 (.13384)	236.375 (.36265)
0.4	**0.2**	21.8535 (.00998)	24.4871 (.01187)	34.8693 (.02029)	60.2424 (.04637)
0.2	0.8	43.6235 (.02848)	48.4802 (.03341)	64.2977 (.05116)	81.0882 (.07257)
[$\rho = 0.2$]					
0.6	**0.6**	117.658 (.12708)	131.958 (.15101)	176.398 (.23362)	353.679 (.66420)
0.6	**0.4**	73.3948 (.06245)	82.6247 (.07465)	114.051 (.12127)	229.145 (.34611)
0.4	**0.4**	46.7310 (.03160)	52.6636 (.03785)	74.6288 (.06404)	137.985 (.16150)
0.4	**0.2**	20.7560 (.00923)	23.2491 (.01097)	33.0219 (.01869)	58.3553 (.04419)
[$\rho = 0.4$]					
0.6	**0.6**	96.2540 (.09394)	106.761 (.10979)	139.138 (.16353)	315.542 (.55962)
0.6	**0.4**	60.5702 (.04675)	67.8713 (.05550)	92.2621 (.08814)	206.056 (.29507)
0.4	**0.4**	38.8405 (.02389)	43.6855 (.02854)	61.3143 (.04762)	123.844 (.13726)
0.4	**0.2**	17.4736 (.00709)	19.5435 (.00842)	27.5368 (.01419)	52.5819 (.03776)

Table 2.3 (continued) (b) ARL_1 and their standard errors (in parentheses) of the one-sided and the three two-sided control charts when $\mathbf{\Sigma}_0$ is known and $n = 10$.

$p=2$		n=10			
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	TB-decomposed
[$\rho = 0$]					
0.9	0.9	231.379 (.35119)	264.566 (.42952)	320.872 (.57388)	447.483 (.94554)
0.8	0.8	135.346 (.15688)	165.032 (.21136)	220.625 (.32696)	361.246 (.68565)
0.7	0.7	73.8704 (.06306)	91.9113 (.08764)	128.586 (.14524)	220.068 (.32572)
0.6	**0.6**	37.0430 (.02224)	45.9560 (.03081)	65.3752 (.05245)	111.343 (.11696)
0.5	0.5	16.9510 (.00677)	20.6787 (.00917)	29.1816 (.01549)	48.0918 (.03300)
0.4	**0.4**	7.11778 (.00176)	8.43177 (.00230)	11.5158 (.00373)	17.8846 (.00735)
0.3	0.3	2.86157 (.00039)	3.24077 (.00049)	4.13835 (.00073)	5.83895 (.00128)
0.2	0.2	1.30750 (.00007)	1.38223 (.00009)	1.56255 (.00012)	1.88248 (.00018)
0.1	0.1	1.00114 ($<10^{-5}$)	1.00195 ($<10^{-5}$)	1.00496 ($<10^{-5}$)	1.01318 ($<10^{-5}$)
0.8	1	218.002 (.32114)	241.937 (.37554)	282.737 (.47457)	330.960 (.60118)
0.6	1	102.165 (.10276)	116.061 (.12449)	142.131 (.16885)	168.282 (.21765)
0.4	1	33.7151 (.01928)	38.1904 (.02329)	46.9748 (.03185)	53.6292 (.03891)
0.2	1	5.92378 (.00131)	6.54248 (.00154)	7.74691 (.00201)	8.33526 (.00226)
0.8	0.6	67.9924 (.05565)	83.8212 (.07628)	115.561 (.12369)	204.400 (.29151)
0.6	**0.4**	14.9257 (.00557)	18.1365 (.00751)	25.3001 (.01247)	42.7605 (.02763)
0.4	**0.2**	2.34435 (.00027)	2.62424 (.00033)	3.26253 (.00049)	4.60519 (.00087)
0.2	0.8	4.90073 (.00097)	5.66636 (.00122)	7.09728 (.00175)	8.54222 (.00235)
[$\rho = 0.2$]					
0.6	**0.6**	31.9588 (.01778)	39.3761 (.02439)	55.0788 (.04050)	102.047 (.10258)
0.6	**0.4**	13.1879 (.00460)	15.9629 (.00617)	21.9999 (.01008)	39.8826 (.02487)
0.4	**0.4**	6.42251 (.00150)	7.58589 (.00195)	10.2652 (.00312)	16.8478 (.00671)
0.4	**0.2**	2.19874 (.00024)	2.45094 (.00030)	3.01835 (.00043)	4.44193 (.00082)
[$\rho = 0.4$]					
0.6	**0.6**	20.1843 (.00884)	24.1900 (.01165)	32.0725 (.01788)	72.8507 (.06175)
0.6	**0.4**	8.95023 (.00252)	10.6660 (.00332)	14.1725 (.00514)	30.7313 (.01676)
0.4	**0.4**	4.67435 (.00090)	5.45385 (.00115)	7.14850 (.00177)	13.5552 (.00480)
0.4	**0.2**	1.81738 (.00016)	1.99743 (.00020)	2.38678 (.00028)	3.87256 (.00066)

Table 2.4 (a) ARL_1 and their standard errors (in parentheses) of the one-sided and the three two-sided control charts when $\mathbf{\Sigma}_0$ is unknown, $m = 25$, and $n = 5$.

p=2, n=5		$m = 25$			
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	LHM-G
[$\rho = 0$]					
0.9	0.9	300.251 (.51940)	317.853 (.56579)	354.692 (.66706)	354.932 (.66774)
0.8	0.8	235.100 (.35971)	258.969 (.41594)	313.832 (.55508)	314.597 (.55711)
0.7	0.7	176.663 (.23414)	199.286 (.28062)	256.435 (.40984)	257.768 (.41305)
0.6	**0.6**	126.486 (.14169)	144.363 (.17285)	194.099 (.26972)	195.610 (.27288)
0.5	0.5	83.9913 (.07652)	96.4599 (.09424)	133.833 (.15425)	135.165 (.15656)
0.4	**0.4**	50.5025 (.03553)	57.9341 (.04371)	82.0479 (.07387)	82.8456 (.07495)
0.3	0.3	26.1424 (.01311)	29.8303 (.01602)	42.6076 (.02748)	43.0115 (.02788)
0.2	0.2	10.6687 (.00332)	12.0253 (.00399)	16.9773 (.00679)	17.1750 (.00691)
0.1	0.1	2.89294 (.00040)	3.15760 (.00046)	4.14626 (.00074)	4.18972 (.00075)
0.8	1	293.752 (.50261)	308.764 (.54167)	339.894 (.62571)	340.328 (.62691)
0.6	1	212.269 (.30854)	228.781 (.34528)	266.099 (.43326)	267.060 (.43561)
0.4	1	129.483 (.14677)	140.726 (.16635)	168.520 (.21812)	169.368 (.21977)
0.2	1	53.1694 (.03840)	57.7512 (.04351)	69.9848 (.05813)	70.5043 (.05878)
0.8	0.6	171.540 (.22402)	192.841 (.26710)	246.668 (.38662)	247.790 (.38927)
0.6	**0.4**	79.1554 (.06998)	90.7739 (.08601)	125.299 (.13969)	126.240 (.14128)
0.4	**0.2**	22.3772 (.01035)	25.4904 (.01261)	36.1465 (.02143)	36.5399 (.02178)
0.2	0.8	44.1723 (.02902)	49.8000 (.03479)	65.7451 (.05290)	66.3392 (.05362)
[$\rho = 0.2$]					
0.6	**0.6**	118.830 (.12899)	135.289 (.15678)	180.368 (.24156)	181.286 (.24341)
0.6	**0.4**	74.3733 (.06371)	85.1623 (.07813)	117.014 (.12604)	118.198 (.12796)
0.4	**0.4**	47.6579 (.03255)	54.6337 (.04001)	77.1867 (.06737)	77.9034 (.06832)
0.4	**0.2**	21.2264 (.00955)	24.1688 (.01163)	34.1903 (.01970)	34.5395 (.02000)
[$\rho = 0.4$]					
0.6	**0.6**	97.2977 (.09548)	109.372 (.11386)	141.832 (.16832)	142.733 (.16993)
0.6	**0.4**	61.4337 (.04776)	69.9163 (.05804)	94.5625 (.09147)	95.3010 (.09255)
0.4	**0.4**	39.5626 (.02457)	45.2396 (.03009)	63.2515 (.04991)	63.8848 (.05066)
0.4	**0.2**	17.8613 (.00733)	20.3000 (.00892)	28.4961 (.01494)	28.7979 (.01518)

Table 2.4 (continued) (b) ARL_1 and their standard errors (in parentheses) of the one-sided and the three two-sided control charts when $\mathbf{\Sigma}_0$ is unknown, $m = 50$, and $n = 5$.

p=2, n=5		$m = 50$			
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	LHM-G
[$\rho = 0$]					
0.9	0.9	300.256 (.51941)	319.078 (.56907)	355.630 (.66971)	356.410 (.67191)
0.8	0.8	235.460 (.36054)	259.269 (.41666)	313.185 (.55336)	313.845 (.55511)
0.7	0.7	176.523 (.23387)	199.084 (.28020)	255.558 (.40774)	256.112 (.40907)
0.6	**0.6**	125.881 (.14067)	143.369 (.17107)	192.197 (.26576)	193.403 (.26827)
0.5	0.5	83.4934 (.07583)	95.6114 (.09300)	132.205 (.15143)	132.645 (.15219)
0.4	**0.4**	50.1027 (.03511)	57.2803 (.04297)	80.8473 (.07224)	81.2036 (.07272)
0.3	0.3	25.8802 (.01291)	29.4328 (.01569)	41.9253 (.02682)	42.0998 (.02699)
0.2	0.2	10.5447 (.00326)	11.8503 (.00390)	16.6923 (.00661)	16.7629 (.00666)
0.1	0.1	2.85666 (.00039)	3.11066 (.00045)	4.07562 (.00071)	4.09409 (.00072)
0.8	1	293.583 (.50218)	309.474 (.54354)	340.219 (.62661)	340.244 (.62668)
0.6	1	212.496 (.30903)	228.879 (.34551)	264.982 (.43053)	265.262 (.43121)
0.4	1	129.315 (.14648)	140.637 (.16619)	167.787 (.21669)	168.097 (.21729)
0.2	1	53.0649 (.03829)	57.6590 (.04340)	69.7328 (.05781)	69.8278 (.05793)
0.8	0.6	171.314 (.22357)	192.250 (.26587)	244.673 (.38193)	245.632 (.38419)
0.6	**0.4**	78.7650 (.06946)	90.0171 (.08493)	124.023 (.13756)	124.163 (.13779)
0.4	**0.2**	22.1775 (.01021)	25.1889 (.01239)	35.6433 (.02098)	35.7590 (.02108)
0.2	0.8	44.0036 (.02886)	49.5413 (.03452)	65.2279 (.05228)	65.4355 (.05253)
[$\rho = 0.2$]					
0.6	**0.6**	118.565 (.12856)	134.681 (.15572)	179.101 (.23902)	179.712 (.24024)
0.6	**0.4**	74.1842 (.06346)	84.6650 (.07744)	115.953 (.12432)	116.257 (.12481)
0.4	**0.4**	47.3171 (.03220)	54.0651 (.03938)	76.0597 (.06590)	76.3590 (.06629)
0.4	**0.2**	21.0257 (.00941)	23.8594 (.01141)	33.6936 (.01927)	33.8194 (.01937)
[$\rho = 0.4$]					
0.6	**0.6**	96.9475 (.09496)	108.827 (.11301)	140.659 (.16623)	141.060 (.16694)
0.6	**0.4**	61.0850 (.04735)	69.3900 (.05738)	93.6283 (.09011)	93.7191 (.09024)
0.4	**0.4**	39.2932 (.02432)	44.7793 (.02963)	62.4353 (.04894)	62.5788 (.04911)
0.4	**0.2**	17.7091 (.00724)	20.0673 (.00876)	28.1036 (.01463)	28.1960 (.01470)

Table 2.5 ARL_1 and their standard errors (in parentheses) of the one-sided and the three two-sided control charts when $\mathbf{\Sigma}_0$ is known or unknown.

Δ_1	Δ_2	One-sided	Two-sided		
			LRT	Modified-LRT	TB-decomposed
$[\rho = 0]$			$[n = 5]$		
1.25	1.25	69.2106 (.05716)	440.129 (.92231)	272.795 (.44973)	115.050 (.12287)
1.5	1.5	23.8224 (.01138)	358.564 (.67802)	128.429 (.14498)	39.9944 (.02497)
1.75	1.75	11.5632 (.00376)	208.000 (.29926)	57.1190 (.04279)	18.1724 (.00753)
2	2	6.91187 (.00168)	105.702 (.10816)	28.6055 (.01503)	10.1564 (.00307)
2.25	2.25	4.73748 (.00092)	55.3383 (.04079)	16.4050 (.00644)	6.57161 (.00155)
2.5	2.5	3.56269 (.00057)	31.7439 (.01760)	10.5358 (.00325)	4.71030 (.00091)
2.75	2.75	2.85965 (.00039)	19.8941 (.00865)	7.37932 (.00186)	3.63512 (.00059)
3	3	2.40662 (.00029)	13.4635 (.00475)	5.52952 (.00118)	2.96044 (.00041)
$[\rho = 0]$			$[n = 10]$		
1.25	1.25	43.9506 (.02880)	334.228 (.61012)	159.591 (.20098)	81.0847 (.07256)
1.5	1.5	12.2232 (.00409)	101.751 (.10213)	41.2284 (.02615)	20.8143 (.00927)
1.75	1.75	5.46448 (.00115)	31.6456 (.01752)	14.4928 (.00532)	8.26446 (.00223)
2	2	3.21706 (.00048)	13.0682 (.00454)	6.90922 (.00168)	4.40205 (.00081)
2.25	2.25	2.26244 (.00025)	6.84932 (.00166)	4.09836 (.00072)	2.86533 (.00039)
2.5	2.5	1.77936 (.00016)	4.27350 (.00077)	2.82486 (.00038)	2.13220 (.00023)
2.75	2.75	1.50672 (.00011)	3.02003 (.00043)	2.16195 (.00023)	1.73069 (.00015)
3	3	1.34228 (.00008)	2.33100 (.00027)	1.77936 (.00016)	1.49254 (.00010)

Δ_1	Δ_2	One-sided	Two-sided		
			LRT	Modified-LRT	LHM-G
$[\rho = 0]$			$[m = 25\ n = 5]$		
1.25	1.25	76.3834 (.06632)	440.498 (.92347)	282.889 (.47496)	285.185 (.48076)
1.5	1.5	27.1195 (.01386)	375.955 (.72799)	145.116 (.17421)	149.254 (.18173)
1.75	1.75	13.2349 (.00463)	237.788 (.36591)	68.4090 (.05617)	71.4286 (.05994)
2	2	7.88379 (.00207)	130.624 (.14872)	35.2062 (.02059)	37.0370 (.02223)
2.25	2.25	5.35542 (.00112)	71.7085 (.06030)	20.2915 (.00891)	21.3624 (.00964)
2.5	2.5	3.98820 (.00069)	41.7738 (.02667)	12.9662 (.00449)	13.6377 (.00485)
2.75	2.75	3.16790 (.00047)	26.2012 (.01315)	8.99172 (.00254)	9.43396 (.00274)
3	3	2.64070 (.00034)	17.6311 (.00719)	6.66174 (.00159)	6.97367 (.00170)
$[\rho = 0]$			$[m = 50\ n = 5]$		
1.25	1.25	72.9696 (.06190)	443.137 (.93179)	278.493 (.46392)	279.565 (.46660)
1.5	1.5	25.4780 (.01261)	370.844 (.71318)	137.053 (.15986)	139.435 (.16406)
1.75	1.75	12.3797 (.00418)	225.850 (.33866)	62.8724 (.04945)	64.1026 (.05092)
2	2	7.38949 (.00187)	118.806 (.12895)	31.8424 (.01768)	32.6797 (.01839)
2.25	2.25	5.03749 (.00101)	63.7458 (.05049)	18.2982 (.00761)	18.7605 (.00791)
2.5	2.5	3.76963 (.00063)	36.7302 (.02196)	11.7049 (.00383)	11.9904 (.00398)
2.75	2.75	3.01011 (.00043)	22.9691 (.01077)	8.15272 (.00218)	8.35205 (.00226)
3	3	2.51985 (.00031)	15.4818 (.00589)	6.06996 (.00137)	6.21263 (.00142)

TABLE 2.6 50 Training Samples of Wafer Data

	WRITE (X_1)					ERASE (X_2)				
K	n=1	n=2	n=3	n=4	n=5	n=1	n=2	n=3	n=4	n=5
1	2.630	4.000	2.920	0.948	2.220	5.190	5.220	4.890	5.950	5.260
2	0.747	1.750	1.240	0.863	1.580	6.720	4.380	5.510	4.920	4.790
3	1.450	1.380	3.470	1.580	1.740	6.400	6.460	6.740	5.810	7.180
4	1.390	2.900	2.370	0.958	1.560	4.860	5.110	3.840	7.370	6.530
5	1.840	1.320	1.950	2.530	2.250	3.950	8.300	13.50	7.170	5.000
6	3.900	1.180	0.973	1.320	2.220	24.20	3.650	5.380	3.870	6.550
7	2.460	2.450	1.660	2.000	2.530	9.590	5.510	7.560	9.090	8.350
8	1.350	1.340	1.150	1.830	1.580	4.340	7.280	5.630	6.460	4.400
9	2.910	1.570	0.857	1.850	1.390	5.500	6.100	3.070	5.260	8.000
10	1.460	2.030	2.250	1.460	2.000	6.270	4.220	5.950	5.130	4.770
11	1.780	2.020	1.330	1.340	1.720	6.400	8.540	4.660	5.460	7.730
12	1.850	1.810	1.830	1.980	2.470	5.390	7.100	6.060	3.700	19.90
13	1.170	2.270	1.920	2.810	0.958	6.370	5.710	6.030	8.600	4.480
14	1.410	1.740	1.940	3.330	1.490	4.260	5.660	6.450	9.170	7.780
15	1.510	3.720	1.330	3.230	1.640	3.860	5.110	5.840	6.200	4.380
16	2.310	3.790	1.840	0.898	2.060	7.180	5.500	6.830	5.520	5.520
17	2.350	3.550	2.090	2.560	1.550	8.120	11.80	6.360	4.700	5.010
18	3.270	2.660	2.160	1.700	1.670	7.160	4.850	4.550	5.600	4.560
19	1.430	1.700	1.290	1.100	1.960	5.470	4.720	4.400	6.740	5.320
20	1.420	1.070	1.810	1.780	2.030	4.130	7.140	4.300	3.550	4.410
21	4.150	1.390	1.520	0.878	3.030	6.280	7.540	4.860	4.660	5.930
22	0.967	2.030	2.280	1.580	1.780	4.620	7.790	5.090	3.740	5.800
23	0.928	1.920	1.120	6.250	1.160	4.280	6.350	5.220	6.210	6.320
24	1.620	1.710	1.220	3.350	2.220	5.510	7.510	5.070	4.260	6.380
25	2.820	1.670	2.270	1.080	2.420	5.060	5.280	5.500	4.410	6.540
26	2.880	1.960	1.560	1.640	1.580	5.970	9.790	3.670	6.030	6.860
27	1.300	1.610	5.010	3.190	4.770	6.340	11.70	4.110	6.970	5.800
28	1.620	2.500	3.350	2.890	1.670	8.400	5.220	5.030	14.10	4.950
29	1.150	1.990	1.380	1.960	1.400	6.810	4.850	4.300	7.180	6.480
30	1.950	1.020	1.330	1.880	1.680	5.300	3.410	5.780	5.220	7.850
31	5.700	3.060	1.020	2.350	1.670	5.920	7.770	5.270	4.170	9.140
32	1.810	2.300	3.490	2.100	2.190	7.650	5.530	6.030	5.390	5.400
33	0.986	1.280	1.870	1.510	1.760	5.870	6.080	6.910	5.380	6.880
34	1.560	1.970	1.600	1.290	7.050	5.960	4.080	7.590	4.620	16.50
35	1.300	1.510	1.670	1.640	1.680	5.550	7.290	5.130	4.140	5.470
36	1.040	1.250	1.030	3.370	1.400	5.890	4.590	3.910	9.310	5.260
37	2.140	0.960	1.830	1.980	0.999	6.270	6.150	6.570	6.180	4.560
38	1.650	1.920	0.968	1.660	1.130	4.450	6.410	4.880	5.580	8.880
39	2.310	1.550	2.710	1.290	2.440	6.500	5.490	3.600	4.570	3.780
40	3.620	1.180	1.760	2.100	1.290	7.930	8.180	13.10	3.530	12.30

TABLE 2.6 (Continuted) 50 Training Samples of Wafer Data

	WRITE (X_1)					ERASE (X_2)				
K	n=1	n=2	n=3	n=4	n=5	n=1	n=2	n=3	n=4	n=5
41	3.890	1.240	2.510	1.230	3.240	4.590	6.060	7.130	4.400	5.190
42	1.860	3.080	1.710	1.260	1.050	9.560	6.360	6.340	5.340	5.860
43	2.280	2.420	2.660	2.610	2.830	4.700	4.850	7.430	5.220	5.540
44	1.600	2.590	0.813	3.000	1.620	5.390	5.560	7.440	6.960	6.400
45	1.840	3.130	2.150	4.070	1.320	5.110	8.000	4.630	4.250	3.570
46	1.550	1.800	1.930	2.010	1.290	3.500	4.870	4.850	5.260	5.910
47	1.380	2.580	1.200	4.780	1.840	5.600	5.200	5.270	7.500	5.500
48	2.380	1.130	1.540	1.060	2.500	7.970	5.180	4.820	4.860	6.810
49	2.520	1.790	1.210	2.560	1.170	4.660	5.550	7.000	7.250	6.750
50	1.610	2.060	2.230	1.640	1.010	7.770	3.770	6.960	5.070	5.990

TABLE 2.7 On-line Samples of Wafer Data

	WRITE (X_1)					ERASE (X_2)				
t	n=1	n=2	n=3	n=4	n=5	n=1	n=2	n=3	n=4	n=5
1	2.260	2.310	2.740	2.120	1.990	6.220	4.840	6.520	5.120	8.970
2	1.850	1.580	1.520	1.820	1.980	7.290	6.570	6.180	3.950	4.840
3	1.230	2.650	2.630	1.330	1.670	5.500	8.220	5.010	4.510	4.080
4	2.090	2.800	1.700	3.530	1.630	5.520	5.870	4.490	12.80	8.250
5	2.280	1.760	2.570	1.120	1.610	5.930	7.170	4.240	6.060	6.600
6	2.150	1.720	1.610	4.040	1.910	12.70	8.650	4.600	6.170	6.370
7	2.990	1.820	2.130	2.650	2.310	4.720	6.280	8.050	7.490	4.230
8	2.730	1.730	2.120	1.500	3.100	5.580	5.920	5.620	4.380	4.920
9	1.470	2.060	2.480	1.560	1.860	4.340	5.620	6.000	4.870	4.930
10	2.040	1.960	1.640	1.520	2.050	4.690	6.100	7.510	5.470	11.20
11	1.660	1.720	1.410	1.770	1.480	4.480	6.150	5.450	5.650	5.700
12	1.720	2.210	1.480	1.830	1.630	4.610	6.880	4.990	5.380	6.870
13	2.010	1.680	2.740	1.210	1.520	8.650	7.220	7.850	8.480	6.470
14	2.330	2.760	2.290	2.480	2.020	5.350	7.910	5.030	5.800	4.570
15	1.950	1.830	1.680	1.950	2.160	5.730	5.020	5.280	5.270	6.100
16	1.340	2.290	2.100	2.640	2.870	4.730	5.890	5.710	6.760	3.440
17	1.840	2.170	1.470	1.850	1.920	8.520	7.340	4.270	4.260	10.70
18	1.850	1.840	1.240	2.330	1.490	4.200	4.200	5.190	5.010	4.940
19	1.440	1.600	3.160	2.450	1.570	8.460	5.820	10.90	4.520	5.220
20	2.940	2.500	1.320	2.620	1.540	7.250	4.750	3.820	6.440	4.810
21	2.010	2.250	1.190	1.190	1.190	7.240	5.520	10.20	8.010	4.890

Chapter 3

Combining Two One-sided Control Charts for Monitoring Multivariate Process Dispersion

3.1 Background

Over the last two decades, the problem of multivariate quality control has received considerable attention and, as a result, there have been increasing research works in multivariate control charts published in statistical and quality journals. When monitoring a multivariate process, while it is important to monitor the process mean, it is just as important to monitor both increases and decreases in process dispersion. In this chapter, we propose a combined chart that incorporates two one-sided likelihood ratio test based control charts for detecting both increases and decreases in process dispersion.

Consider a multivariate process with p possibly correlated quality characteristics. Suppose that the $p \times 1$ quality characteristic vector $\boldsymbol{X}$ is distributed as a multivariate normal distribution (denoted by $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$) with unknown mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In the literature, a substantial amount of research work has focused on developing multivariate control charts for monitoring the process mean vector $\boldsymbol{\mu}$. Excellent reviews of these developments can be found in, for example, Wierda (1994), Lowry and Montgomery (1995), Mason, Champ, Tracy, Wierda, and Young (1997), and Montgomery (2008).

Although developing methods for monitoring process dispersion is more difficult than that for process mean due to the more complicated distribution theory involved, there has been some research in recent years devoted to multivariate process dispersion monitoring in the literature. Particularly, papers by Wierda (1994), Mason, Tracy, and Young (1997), and Woodall and Montgomery (1999) mentioned the importance of controlling the process dispersion. Also, the development and discussions of multivariate control charts for monitoring changes in the covariance matrix can be found in, for example, Alt (1984), Alt and Bedewi (1986), Sakata (1987), Alt and Smith (1998), Calvin (1994), Tang and Barnett (1996a, 1996b), Levinson, Holmes, and Mergen (2002), Yeh, Lin, Zhou, and Venkataramani (2003), Yeh, Huwang, and Wu (2004), Djauhari (2005), Reynolds and Cho (2006), and Huwang, Yeh, and Wu (2007). Recently, Yeh, Lin, and McGrath (2006) reviewed multivariate control charts designed for monitoring changes in the covariance matrix that have been developed in the last 15 years and proposed a new multivariate EWMA control chart. They also discussed comparisons of chart performance between their chart and charts in the literature.

In most processes, more attention would be paid to investigate if the process dispersion increases. However, it is also important to detect decreases in dispersion, which in essence leads to process improvement. When a chart is used to simultaneously monitor for both increases and decreases in process dispersion a desirable feature is to have all of its out-of-control average run length (ARL) values smaller than the in-control ARL. Such a chart was defined as being ARL-unbiased by Pignatiello, Acosta-Mejia, and Rao (1995). Pachares (1961) showed the two-sided confidence interval based on an equal tail probability, say, $\alpha/2$, leads to a biased test for a two-sided hypothesis test for the variance of a normal population. Consequently, a chart uses an equal tail probability to construct a limit for detecting changes in variance, it will be ARL-biased. For example, Pignatiello, Acosta-Mejia, and Rao (1995) reported that an R chart with equal-tail-probability limits is ARL-biased. Lowry, Champ, and Woodall (1995) reported that when using R or S chart, it is much more difficult to detect decreases than increases in variance. As variance decreases, the out-of-control ARL value increases, thus becoming larger than the in-control ARL. Acosta-Mejia (1998) proposed combining two one-sided CUSUM chart based on the subgroup range for monitoring both increases and decreases in dispersion. Acosta-Mejia, Pignatiello, and Rao (1999) discussed and compared several control charts for monitoring increases and decreases in the variance of a normal process and demonstrated their proposed chart is superior to other procedures.

Extending to the multivariate case, most of techniques for monitoring the covariance matrix are all centered on two-sided tests of

$$H_0\colon \mathbf{\Sigma} = \mathbf{\Sigma}_0 \ \ vs. \ \ H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0. \tag{3.1.1}$$

These techniques can detect either increases or decreases in multivariate process dispersion. In current chapter, we consider the two one-sided tests, (2.1.1) and (2.1.2), for monitoring increases and decreases in dispersion. Yen and Shiau (2008) proposed a simple and yet effective *one-sided* LRT-based control chart based on the hypotheses (2.1.1) for detecting increases in dispersion for each of the cases of known and unknown $\mathbf{\Sigma}_0$. They reported that their one-sided chart outperforms the existing control charting techniques based on the two-sided tests in terms of the average run length. An analogous one-sided LRT- based control chart based on the hypotheses (2.1.2) for detecting decreases in dispersion has been proposed and studied in Chapter 2.

The purpose of this chapter is to combine the two one-sided LRT-based control charts mentioned above to construct a scheme for monitoring both increases and decreases in dispersion simultaneously. In addition, we also show the control limits of the proposed chart are better suited when using a set of unequal tail probabilities instead of equal tail probabilities.

The rest of the chapter is organized as follows. Section 3.2 describes in detail the proposed LRT-based combined control chart. Both cases of known and unknown $\mathbf{\Sigma}_0$ are considered. Section 3.3 describes the existing techniques based on the two-sided tests of (3.1.1) for both cases of known and unknown $\mathbf{\Sigma}_0$. From the perspective of the ARL, the performance of the proposed combined control chart is compared with that of the existing two-sided control charts based on simulation studies. Section 3.4 discusses the application of the proposed chart to some real-life data to demonstrate the applicability and the effectiveness of the proposed chart. Section 3.5 concludes the chapter with a brief summary and further discussion.

3.2 A Combined Chart Based On Two One-Sided LRT-based Control Charts

3.2.1 A Combined LRT-based Control Chart When $\mathbf{\Sigma}_0$ is Known

As described in Subsection 2.2.1, when monitoring increases in dispersion, the one-sided control chart based on the LRT statistic for testing hypothesis (2.1.1), proposed by Yen and Shiau (2008), is constructed based on calculating the statistic T_I in (2.4.12). When monitoring decreases in dispersion, the one-sided control chart based on the LRT statistic for testing hypothesis (2.1.2) is constructed based on calculating the statistic T_D in (2.2.5). The rejection region for the test (2.1.1) ((2.1.2)) is $\{T_I > T_I(\alpha_I)\}$ ($\{T_D > T_D(\alpha_D)\}$), where the critical value $T_I(\alpha_I)$ ($T_D(\alpha_D)$) is chosen such that the significance level equals α_I (α_D). In other words, $T_I(\alpha_I)$ ($T_D(\alpha_D)$) is the $(1-\alpha_I)$th ($(1-\alpha_D)$th) quantile of the distribution of T_I (T_D). As before, without loss of generality, we can assume that $\boldsymbol{\mu}_0 = 0$ and $\mathbf{\Sigma}_0 = \boldsymbol{I}_p$ when studying the distribution of T_I (T_D).

When $\mathbf{\Sigma}_0$ is known, a combined LRT-based control chart can be constructed by combining the two one-sided control charts described above. Hence, the combined LRT-based control chart for monitoring both increases and decreases in dispersion simultaneously signals an out-of-control alarm if

$$T_I > T_I(\alpha_I) \quad or \quad T_D > T_D(\alpha_D), \tag{3.2.1}$$

where the critical values $T_I(\alpha_I)$ and $T_D(\alpha_D)$ are taken as the control limits and obtained by controlling the type I error probability. Similar to Yen and Shiau (2008) and Chapter 2, we use Monte Carlo simulation to obtain $T_I(\alpha_I)$ and $T_D(\alpha_D)$, the critical values of the two tests. By the nature that each test takes care of one side of the alternative and the two test test statistics, T_I and T_D, are calculated with two disjoint sets of eigenvalues, $\{d_i|d_i > 1\}$ and $\{d_i|0 < d_i < 1\}$, we would expect these two regions, $\{T_I > T_I(\alpha_I)\}$ and $\{T_D > T_D(\alpha_D)\}$, are disjoint. Unfortunately, this is not true theoretically. However, our simulation study indicates that the two rejection regions are disjoint in practice. Hence, the type I error probability for the combined LRT-based control chart is practically $\alpha_I + \alpha_D$. The control limits, $T_I(\alpha_I)$ and $T_D(\alpha_D)$, can be obtained by an empirical approach to be described in Subsection 2.3.1.

3.2.2 A Combined LRT-based Control Chart When $\mathbf{\Sigma}_0$ is Unknown

As described in Subsection 2.2.2, the LRT statistic to test if the process dispersion increases at time t is T'_I in (2.4.13). See Yen and Shiau (2008). And the LRT statistic for testing decreases in dispersion is T'_D in (2.2.9). See Chapter 2. The rejection region for the test (2.1.1) ((2.1.2)) is $\{T'_I > T'_I(\alpha'_I)\}$ ($\{T'_D > T'_D(\alpha'_D)\}$), where the critical value $T'_I(\alpha'_I)$ ($T'_D(\alpha'_D)$) is chosen such that the significance level equals α'_I (α'_D).

Similar to the case of known $\mathbf{\Sigma}_0$ given in the previous subsection, we can construct a combined chart for detecting changes in process dispersion by combining the two one-sided charts constructed with the LRT statistics (2.4.13) and (2.2.9) respectively.

3.2.3 The Control Limits

As shown earlier, the control limits does not depend on the in-control process mean $\boldsymbol{\mu}_0$ and covariance matrix $\mathbf{\Sigma}_0$. Thus, without loss of generality, we can assume that $\boldsymbol{\mu}_0 = 0$ and $\mathbf{\Sigma}_0 = \boldsymbol{I}_p$ when obtaining the control limits by Monte Carlo simulation.

For the case of known $\mathbf{\Sigma}_0$, to construct the control limits, we generate $N(= 1,000,000)$ samples of size n from $N_p(0, \boldsymbol{I}_p)$. For each sample, compute the eigenvalues of the sample covariance matrix $\boldsymbol{S}_t$ and then the LRT statistics T_I and T_D. Then, for given α_I (α_D), the control limit is the 100(1-α_I) (100(1-α_D)) percentile of the N simulated values of T_I (T_D). For the case of unknown $\boldsymbol{\Sigma}_0$, the control limits can be obtained in the same way. The only difference is that we need to generate a historical data set to compute the sample covariance matrix $\boldsymbol{S}_0$ in addition to $\boldsymbol{S}_t$ and find the eigenvalues of $\boldsymbol{S}_t\boldsymbol{S}_0^{-1}$. Furthermore, α_I and α_D (α'_I and α'_D), satisfying $\alpha = \alpha_I + \alpha_D$ ($=\alpha'_I + \alpha'_D$), are chosen by a search algorithm.

3.2.4 Unequal-Tail-Probability Control Limits

As mentioned before, in the univariate case, a chart with equal-tail-probability limits for detecting changes in dispersion is ARL-biased. Extending to the multivariate case, consider $p = 2$ and using α_I=α_D=$\alpha/2$, the equal-tail probability, for the proposed combined chart for detecting increases and decreases in dispersion. The out-of-control ARL (denoted by ARL_1) values, for variance changes in covariance matrix, are presented in Table 3.1 and Figure 3.1. Since some of these values are greater than 370, this demonstrates that using equal-tail probabilities for the proposed chart also leads to an ARL-biased procedure in the multivariate case. Hence, we suggest using unequal tail probabilities to construct the control limits of the proposed combined control chart, i.e., $\alpha_I \neq \alpha_D$ ($\alpha'_I \neq \alpha'_D$). But how to split α between α_I and α_D (α'_I and α'_D) is a question. We conduct a simulation study and find that the power of the one-sided chart based on (2.2.5) ((2.2.9)) for monitoring decreases in dispersion is worse than that of the one-sided chart based on (2.4.12) ((2.4.13)) for monitoring increases in dispersion. Therefore, to achieve the ARL-unbiasedness, it is necessary to set

$$\alpha_I < \alpha/2 \text{ and } \alpha_D = \alpha - \alpha_I \;\; (\alpha'_I < \alpha/2 \text{ and } \alpha'_D = \alpha - \alpha'_I). \tag{3.2.2}$$

Although there might be plenty of such the combinations of (α_I,α_D) ((α'_I,α'_D)), in this study, we just consider these five following combinations of (α_I,α_D) and (α'_I,α'_D) for $p = 2$ and $\alpha = 0.0027$. For $n = 5$, consider (α_I,α_D)=(0.000515,0.002185), (0.000415,0.002285), (0.000395,0.002305), (0.000375,0.002325), and (0.000275,0.002425). These five combinations of (α_I, α_D) are also considered for the combinations of (α'_I, α'_D) for $m = 25, 50$. Also, for $n = 10$, consider (α_I, α_D)= (0.000715,0.001985), (0.000635,0.002065), (0.000615, 0.002085), (0.000595,0.002105), and (0.000515,0.002185). These corresponding combinations chosen for their ARL_1 values are smaller than the two-sided Modified-LRT control chart described in Subsection 2.4.1 for all cases tested. We obtain the control limits for each of (α_I, α_D) $((\alpha'_I, \alpha'_D))$ as described in the previous subsection. For each control chart, we consider N=200,000, the number of simulated values of T_I (T_D, T'_I, T'_D) in each simulation, to get one control limit and repeat the procedure b(=100) times to get b control limits. Then the average of these b control limits is taken as the final control limit and the sample standard deviation divided by $\sqrt{b}$ is the standard error of the final control limit. Table 3.2 gives the control limits and the corresponding standard error (in parentheses) for considered (α_I, α_D) $((\alpha'_I, \alpha'_D))$.

3.3 A Comparative Study

In this section, we compare our proposed chart, which combines two one-sided LRT-based control charts, with the existing control charts based on the two-sided test (3.1.1) for both cases of known and unknown $\mathbf{\Sigma}_0$ in terms of the out-of-control ARL. The existing two-sided-tested-based control charts considered in this chapter are the two-sided, the two-sided Modified-LRT, and the TB-decomposed control charts described in the previous subsections 2.4.1-2.4.2.

The LHM-G control chart described in Subsection 2.4.3 is not considered in this comparative study. Yen and Shiau (2008) and Chapter 2 together showed by simulation studies that the ARL performance of the two-sided Modified-LRT control chart and the LHM-G control chart, either for detecting increases or for detecting decreases in dispersion, are very close with the former slightly better. Hence, we do not include the LHM-G-chart in comparison.

3.3.1 Comparisons

In this subsection, we compare the performances of the proposed procedure with the existing techniques described in Chapter 2 and the performance is evaluated in terms of ARL with an empirical study.

Yen and Shiau (2008) (Chapter 2) defined that the covariance matrix has been "increased" ("decreased") from $\mathbf{\Sigma_0}$ to $\mathbf{\Sigma}$, i.e., $\mathbf{\Sigma} - \mathbf{\Sigma}_0$ ($\mathbf{\Sigma}_0 - \mathbf{\Sigma}$) is positive semidefinite and $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$. As shown in Yen and Shiau (2008) (Chapter 2), the distributions of T_I and T_D (T_I' and T_D') are independent of $\mathbf{\Sigma}_0$. Thus, as shown previously, we can assume that $\mathbf{\Sigma}_0 = I_p$ without loss of generality, when simulating the distributions of T_I and T_D (T_I' and T_D'). For easy to discuss, we consider $p = 2$. As in Chapter 2, to create out-of-control scenarios, express $\mathbf{\Sigma}$ as $\begin{bmatrix} \Delta_1 & \rho\sqrt{\Delta_1\Delta_2} \\ \rho\sqrt{\Delta_1\Delta_2} & \Delta_2 \end{bmatrix}$, $i = 1, 2$.

In our comparative study, setting $\alpha = 0.0027$ which results in $ARL_0 = 370$, we consider the following cases: $p = 2$; $n = 5$, 10 for known $\mathbf{\Sigma}_0$; $m = 25$, 50 and $n = 5$ for unknown $\mathbf{\Sigma}_0$. The in-control covariance matrix is $\mathbf{\Sigma}_0 = I_p$ and the out-of-control covariance matrix is $\mathbf{\Sigma}$. The following three scenarios for $\mathbf{\Sigma}$ are considered:

(1) $\Delta_1 = \Delta_2 = c$ and $\rho = 0$ (that is, $\mathbf{\Sigma} = c\mathbf{\Sigma}_0$), for c=1.25,1.35,1.5,1.75,2,2.25,2.5,2.75,3 for increases and 0.9,0.8,0.7,0.6,0.5,0.4,0.3,0.2,0.1 for decreases.

(2) $\Delta_1 \neq \Delta_2$ and $\rho = 0$. for the following 8 combinations: (Δ_1, Δ_2) = (1.25,1), (1.75,1), (2.25,1), (2.75,1), (1.25,1.75), (1.75,2.25), (2.75,1.25), (2.25,2.75) for increases and (0.8,1), (0.6,1), (0.4,1), (0.2,1), (0.8,0.6), (0.6,0.4), (0.4,0.2), (0.2,0.8) for decreases.

(3) For $\rho \neq 0$, under condition (2.4.11), we choose $|\rho|$ =0.2 and 0.4 for the following 8 combinations: (Δ_1, Δ_2) =(1.75,1.75), (1.75,2.25), (2.25,2.25), (2.25,2.75) for increases and (0.6,0.6), (0.6,0.4), (0.4,0.4), (0.4,0.2) for decreases. Note that these 8 combinations are selected from scenarios (1) and (2) so that we can study the effect of ρ on ARL performance.

In the simulation study, we take $N = 200,000$ and $b = 100$ to obtain the ARL estimate along with its standard error for each scenario. When $\mathbf{\Sigma}_0$ is known, for $\alpha = 0.0027$, two-sided LRT, and two-sided Modified-LRT control limits obtained are respectively 22.68151 and 17.67692 for $n = 5$; 17.53596 and 15.45388 for $n = 10$. In particular, the control limits of the proposed combined chart for five combinations of (α_I,α_D) considered are given in Table 3.2 for $n = 5, 10$. It is found that, with the afore-mentioned control limits obtained from empirical study, $ARL_0 \approx 370$ with standard error $\left(\sqrt{\frac{1-0.0027}{Nb \cdot 0.0027^3}}\right)$ around 1.591667. Moreover, for both $n = 5, 10$, the control limit based on TB-decomposed control chart is $\chi^2_3(0.9973)$=14.15625. Tables 3.3 and 3.4 give the estimates of ARL_1 and their standard errors (in parentheses) of the two one-sided combined-LRT, two-sided LRT, two-sided Modified-LRT, and TB-decomposed control charts for the scenarios (1)-(3) described above. The following results are observed:

(i) For the two one-sided combined control chart, considering the five combinations (α_I,α_D), the larger the α_I is (especially for α_I=0.000515 ($n = 5$) and 0.000715 ($n = 10$)), the smaller the ARL_1 value for detecting dispersion increases is and also the larger the ARL_1 value for detecting dispersion decreases is. Similarly, the larger the α_D is (especially for α_D=0.002425 ($n = 5$) and 0.002185 ($n = 10$)), the smaller the ARL_1 value for detecting dispersion decreases is and also the larger the ARL_1 value for detecting dispersion increases is. Regardless of whether the dispersion increases or decreases, the proposed combined control chart is ARL-unbiased for the five combinations (α_I,α_D) and for all cases tested. It outperforms, for all cases tested, the two-sided Modified-LRT control chart, which is also ARL-unbiased. It is interesting to note that the TB-decomposed control chart performs the worst when detecting dispersion decreases but performs the best for most of the cases tested when detecting dispersion increases. On the other hand, the two-sided LRT control chart performs the worst when detecting dispersion increases but performs the best for most of the cases tested when detecting dispersion decreases. This may link to the ARL-biasedness or unbiasedness of the control chart. An ARL-biased chart usually performs poorly compared to an ARL-unbiased chart. Furthermore, consider the ARL_1 performance for all cases tested, we suggest setting α_I=0.000415, 0.000395, 0.000375 for $n = 5$, and 0.000635, 0.000615, 0.000695 for $n = 10$ or other α_I around these values for the proposed combined chart to have a more satisfactory performance than the existing two-sided control charts. Figures 3.2-3.5, in the original and logarithm scale for $\Delta_i = c$, show the ARL curves of the combined chart for α_I=0.000515, 0.000395, 0.000275 for $n = 5$, and 0.000715, 0.000615, 0.000515 for $n = 10$ and the existing control charts for the cases tested in scenario (1). Through these Figures, it is easier to comprehend what we have discussed here.

(ii) Given c, the ARL_1 for $n = 10$ is smaller than that for $n = 5$. On the other hand, for fixed n, ARL_1 is smaller when c (> 1) is larger or c (< 1) is smaller. For all the combinations of Δ_1 and Δ_2 in scenarios (1)-(3), the ARL_1 for $n = 10$ is smaller than that for $n = 5$. For fixed n, ρ, and one of Δ_1 and Δ_2, say Δ_2, the ARL_1 decreases when Δ_1 (> 1) increases or Δ_1 (< 1) decreases. These results are all expected. For the effect of ρ, we first observe that, by (2.4.11), the eigenvalues of $\mathbf{\Sigma} - \mathbf{\Sigma_0}$ or $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ depend on ρ through ρ^2, hence the sign of ρ does not play any role in ARL_1 performance as seen in our simulation study. The ARL_1 decreases when $|\rho|$ increases from 0 to 0.4. This means that the ability of the proposed chart detecting a decrease in dispersion gets better when the correlation (positive or negative) between the two quality characteristics becomes stronger.

For the case of unknown $\mathbf{\Sigma}_0$, two-sided LRT and two-sided Modified-LRT control limits obtained respectively are 22.58894 and 53.27833 for $m = 25$; 22.66328 and 58.79951 for $m = 50$. Similarly, the control limits for the combined chart for the five combinations of (α'_I,α'_D) considered for $m = 25, 50$ are given in Table 3.2. The ARL_1 values along with their standard errors are given in Tables 3.4-3.5. And Figures 3.6-3.9 show the ARL curves of the two one-sided combined chart for α'_I=0.000515, 0.000395, 0.000275 ($m = 25, 50$) and the existing control charts for the cases tested in scenario (1). Similar observations as those discussed earlier for the case of known $\mathbf{\Sigma}_0$ can also be made here.

3.3.2 Discussion

The two one-sided combined control charts are based on the two one-sided hypotheses (2.1.1) and (2.1.2) respectively, while the existing two-sided control charts in this chapter are all based on the two-sided hypothesis (3.1.1). In this subsection, we briefly discuss the alternative hypotheses of (3.1.1), (2.1.1), and (2.1.2). It is easy to show that the union of the sets of the alternative hypotheses of (2.1.1) and (2.1.2) is not equal to the set of the alternative hypothesis of (3.1.1), that is,

$$\{H_1\colon \mathbf{\Sigma} \geq \mathbf{\Sigma}_0 \text{ } and \text{ } \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\} \cup \{H_1\colon \mathbf{\Sigma} \leq \mathbf{\Sigma}_0 \text{ } and \text{ } \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\} \neq \{H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\}. \qquad (3.3.1)$$

The former set is smaller than the latter one because there are many $\mathbf{\Sigma} - \mathbf{\Sigma}_0$ that are neither positive semidefinite nor negative semidefinite. The combined control chart is not designed for all possible dispersion changes if the out-of-control scenario considered is outside of the alternative hypotheses of (2.1.1) and (2.1.2), the proposed combined chart might not give a better performance. Nevertheless, the combined control chart with unequal-tail-probability control limits has a better performance than the existing control charts based on the two-sided hypotheses (3.1.1) for all cases tested for dispersion shifts in Subsection 3.3.1 for alternative hypothesis of (2.1.1) and (2.1.2),

However, if the two-sided LRT and TB-decomposed control charts are used to monitor all possible dispersion changes, that is, $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$, the former is ARL-biased for detecting dispersion increases and the latter is ARL-biased for detecting dispersion decreases. The two-sided modified control chart based on the alternative hypothesis $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$ is ARL-unbiased for detecting dispersion increases or decreases. In general, when detecting dispersion increases or decreases and the change range of ρ satisfying (2.4.11), our proposed combined chart can give a satisfactory performance than the existing two-sided charts considered in this study.

3.4 Example

In this section, the two real-life examples given respectively in Yen and Shiau (2008) and Chapter 2 are used to illustrate the application of the proposed combined control chart. Moreover, two simulated examples are presented to demonstrate the detecting power of the combined control chart and the existing two-sided control charts when the dispersion increases or decreases.

3.4.1 Two real examples

The first example is related to a metal layer process for the semiconductor elements of a wafer and the data were given in Yen and Shiau (2008). The two quality characteristics monitored are "after-develop-inspection-critical-dimension (ADICD)" and "after-etch-inspection-critical-dimension (AEICD)". The two critical-dimension are measured at five points on each wafer size after the develop-action and etch-action, respectively. Let X_1 and X_2 be the averages of the five ADICD and AEICD measurements on a wafer, respectively. Let $\boldsymbol{X}= (X_1, X_2)$. Five such $\boldsymbol{X}$ samples from five wafers are considered as a subgroup of samples with size 5. The first example describes an application in which process dispersion is increased. For this example, 50 sets of random samples, each of size 5, were taken from the original process. The sample mean $\bar{\bar{\boldsymbol{X}}}$ and sample covariance matrix $\boldsymbol{S}_0$, calculated by (2.2.6), for the first example are respectively $\begin{pmatrix} 0.79966 \\ 0.85744 \end{pmatrix}$ and $\begin{pmatrix} 3.70395 \times 10^{-4} & 1.38183 \times 10^{-4} \\ 1.38183 \times 10^{-4} & 4.95859 \times 10^{-4} \end{pmatrix}$. For $p = 2$, $m = 50$, $n = 5$, and $\alpha = 0.0027$, the two-sided LRT and two-sided Modified-LRT control limits are 22.66328 and 58.79951, respectively. The control limits of the proposed combined chart for (α'_I,α'_D)=(0.000395,0.002305) are 11.7444 and 22.7055. Moreover, 25 sets of on-line samples, each of size 5, are monitored by using these control limits. Figure 3.10 shows these control charts for the first example.

The second example is the example described in Chapter 2, which illustrates a case process dispersion is decreased. The second example is describes a decreased case. Similarly, we also took 50 sets of random samples, each of size 5, from the original process. The sample mean $\bar{\bar{\boldsymbol{X}}}$ and sample covariance matrix $\boldsymbol{S}_0$ for the second example are respectively $\begin{pmatrix} 1.98920 \\ 6.14052 \end{pmatrix}$ and $\begin{pmatrix} 0.84260 & 0.54071 \\ 0.54071 & 5.44242 \end{pmatrix}$. The control limits of the control charts under study are the same as described in the first example. Moreover, 21 additional sets for the second example are also monitored by using the above control limits. Figure 3.11 shows these control charts for the second example.

From Figures 3.10 and 3.11, regardless of whether the dispersion increases or decreases, we observe that the detecting power of the proposed combined chart is indeed a bit weaker than that of the one-sided chart but still better than that of the two-sided modified-LRT chart. The detecting power of the two-sided control chart is the worst for detecting dispersion increases, but is as well as the proposed combined chart for detecting dispersion decreases. Moreover, we also observe one thing interesting for the proposed combined chart that, for the same sample, the larger the value of T_I' is, the smaller the corresponding value of T_D' is. And hence the proposed combined chart, in detecting dispersion decreases in the first example and in detecting dispersion increases in the second example, does not detect any. For the second example, it is also found that the T_I' values by (2.4.13) for the proposed combined chart are mostly close to 0 since most of the eigenvalues, β_i's, from the data are less than 1. These two examples demonstrate that the out-of-control conditions are successfully picked up by the right individual control chart in the combined scheme. This demonstrates that the proposed combined control chart is more powerful than the two-sided control charts.

3.4.2 Simulated examples for second example

Consider the second example in the previous section. Take the in-control distribution as $N_p(\bar{\bar{\boldsymbol{X}}}, \boldsymbol{S}_0)$ where $\bar{\bar{\boldsymbol{X}}}$ and $\boldsymbol{S}_0$ are from the second example. In order to study the effectiveness of the proposed combined chart, we generate 100 subgroups of samples, each of size 5, in which the first ten subgroups and the 51st to 60th subgroups are from the in-control process $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, the 11th to 50th subgroups are from $N_p(\bar{\bar{\boldsymbol{X}}}, \boldsymbol{\Sigma}_1)$, and the 61st to 100th subgroups are from $N_p(\bar{\bar{\boldsymbol{X}}}, \boldsymbol{\Sigma}_2)$ with $\boldsymbol{\Sigma}_1 \geq \boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma}_2 \leq \boldsymbol{\Sigma}_0$. Let $\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sqrt{\Delta_1} & 0 \\ 0 & \sqrt{\Delta_2} \end{bmatrix} \boldsymbol{S_0} \begin{bmatrix} \sqrt{\Delta_1} & 0 \\ 0 & \sqrt{\Delta_2} \end{bmatrix}$ and $\boldsymbol{\Sigma}_2 = \begin{bmatrix} \sqrt{\Delta_1'} & 0 \\ 0 & \sqrt{\Delta_2'} \end{bmatrix} \boldsymbol{S_0} \begin{bmatrix} \sqrt{\Delta_1'} & 0 \\ 0 & \sqrt{\Delta_2'} \end{bmatrix}$, where Δ_1 and Δ_2 are greater than 1 and Δ_1', and Δ_2' are less than 1. Two scenarios are considered for $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ with (i)$(\Delta_1, \Delta_2) = (1.75, 2.25)$ and $(\Delta_1', \Delta_2') = (0.2, 0.4)$ and (ii) $(\Delta_1, \Delta_2) = (1.75, 1.75)$ and $(\Delta_1', \Delta_2') = (0.4, 0.4)$.

The results of the proposed combined, two-sided, and two-sided Modified-LRT control charts of scenario (i) and (ii) are, respectively, presented in Figures 3.12-3.13. From Figure 3.12, we first observe that all three charts pick none in the two in-control zones. The "increase-chart" of the combined chart picks five out-of-control samples in the increase-zone and none in the decrease-zone. The "decrease-chart" picks three in the decrease-zone and none in the increase-zone. Since the two-sided chart is strong on detecting decreases and weak on increases, it picks three from the increase-zone (same as the combined chart) and none from the decrease-zone. Since two-sided modified control chart is unbiased, it picks three in the increase-zone and two from the decrease-zone, not as many as that for the combined chart. This demonstrates that the combined chart has a stronger detecting power than the two-sided modified chart. Moreover, similar results are observed for the scenario (ii) presented in Figure 3.13. The figures also demonstrate that the first out-of-control region is easier to detect than the second out-of-control region and scenario (i) is easier to detect than scenario (ii) because of the larger shift size, which is expected.

3.5 Discussion

In this chapter, we propose a combined control chart constructed by combining the two one-sided likelihood-ratio-test-based control charts that are specifically designed for detecting dispersion increases and decreases respectively for multivariate processes. The proposed combined chart is in essence a multivariate extension of the two-sided unequal tail test for univariate variance. Both cases when the in-control covariance matrix $\mathbf{\Sigma}_0$ is known or unknown are considered. It is shown that the control limits do not depend on $\boldsymbol{\mu}_0$ and $\mathbf{\Sigma}_0$ and can be constructed with unequal tail probabilities to achieve ARL-unbiased. The simulation study demonstrates that the proposed combined control chart outperforms the existing control charts based on the two-sided tests of (3.1.1) in terms of the ARL, when the process dispersion is indeed increases or decreases.

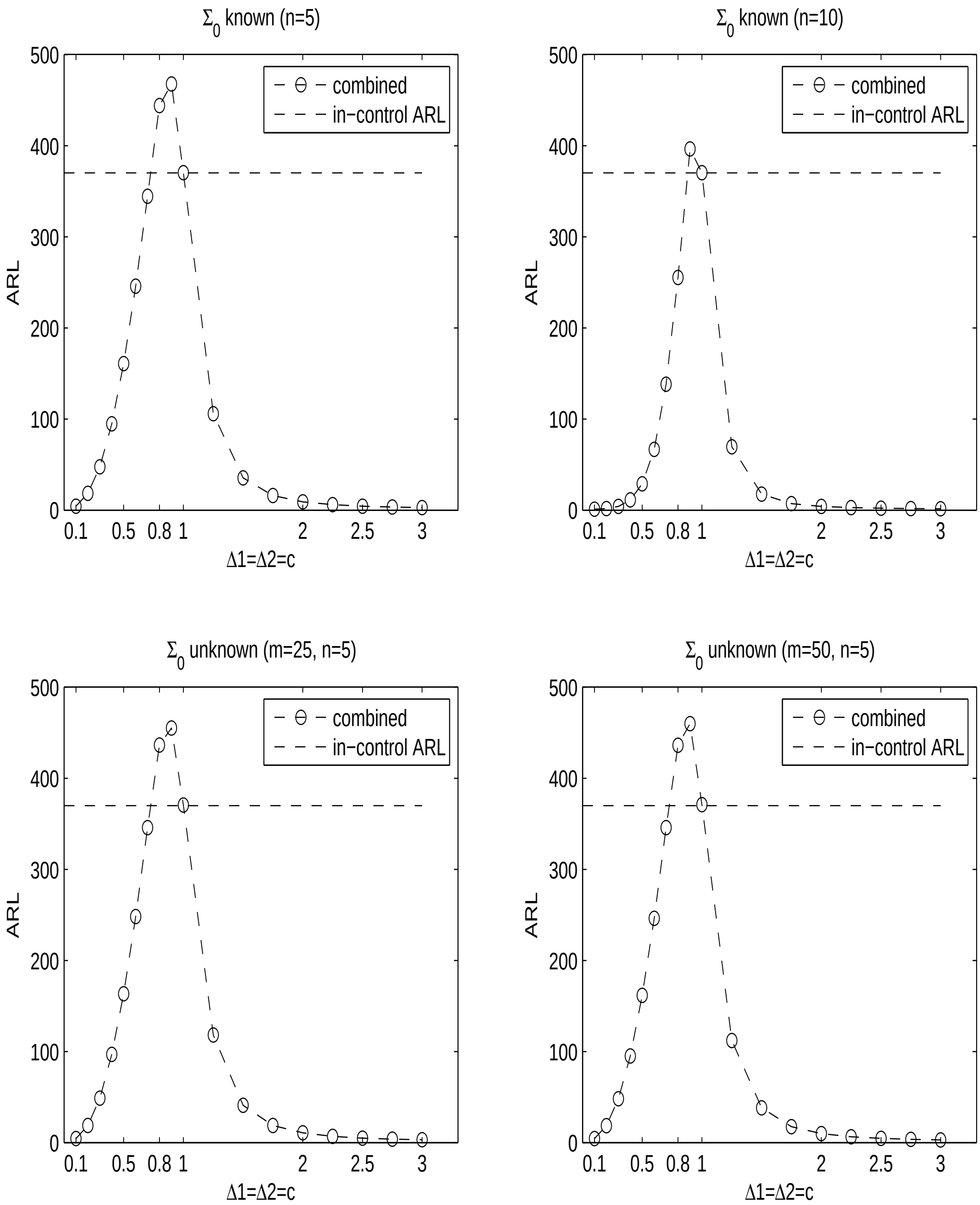

Figure 3.1: The ARL curves of the proposed combined chart for equal tail probability (α_I=α_D=0.00135) when $p = 2$ and $\mathbf{\Sigma}_0$ is known or unknown.

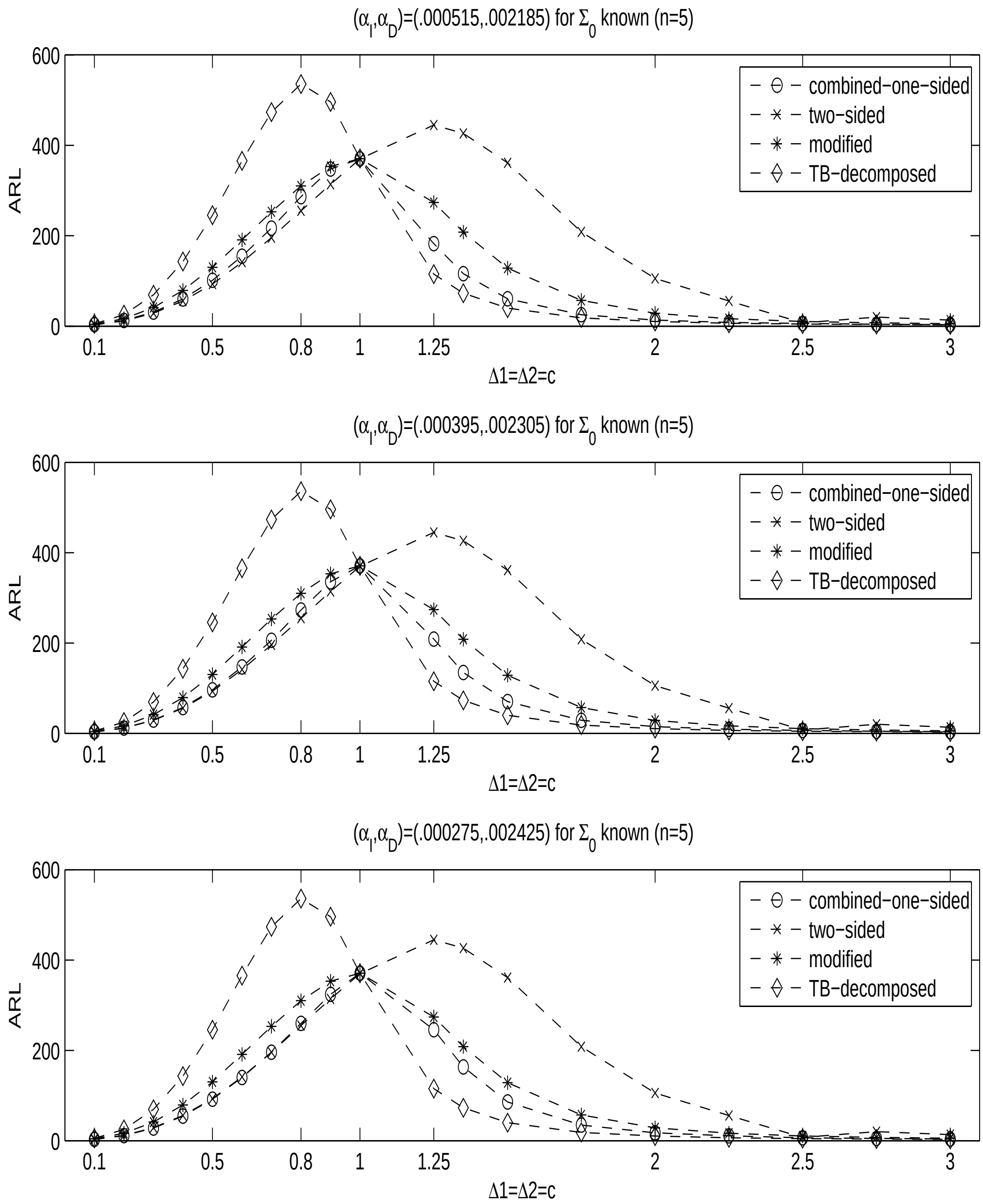

Figure 3.2: The ARL curves of the control charts under study for (α_I,α_D) in the original scale when $p = 2$ and $\mathbf{\Sigma}_0$ is known $(n = 5)$.

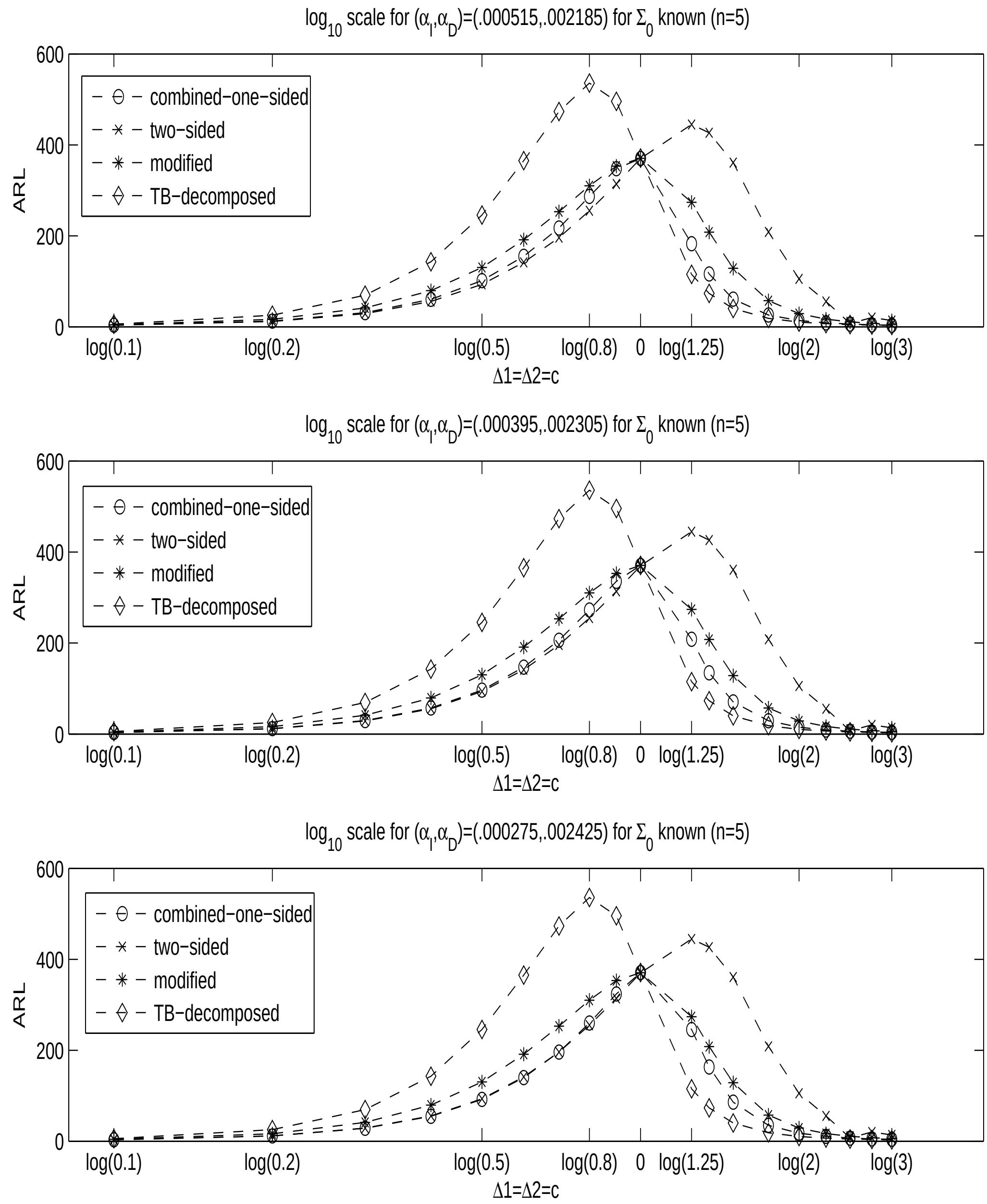

Figure 3.3: The ARL curves of the control charts under study for (α_I,α_D) in the logarithm scale when $p = 2$ and $\mathbf{\Sigma}_0$ is known ($n = 5$).

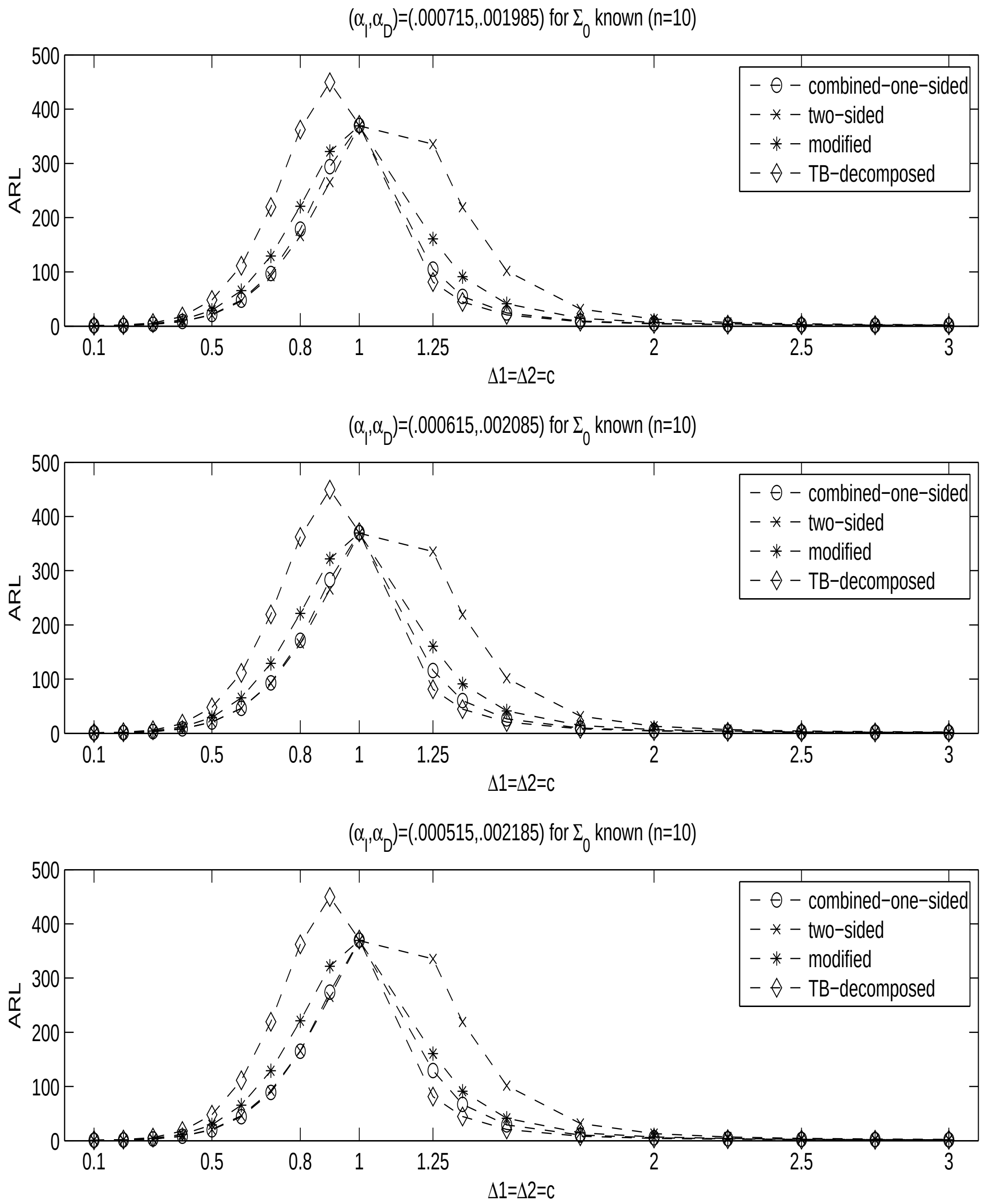

Figure 3.4: The ARL curves of the control charts under study for (α_I,α_D) in the original scale when $p = 2$ and $\boldsymbol{\Sigma}_0$ is known ($n = 10$).

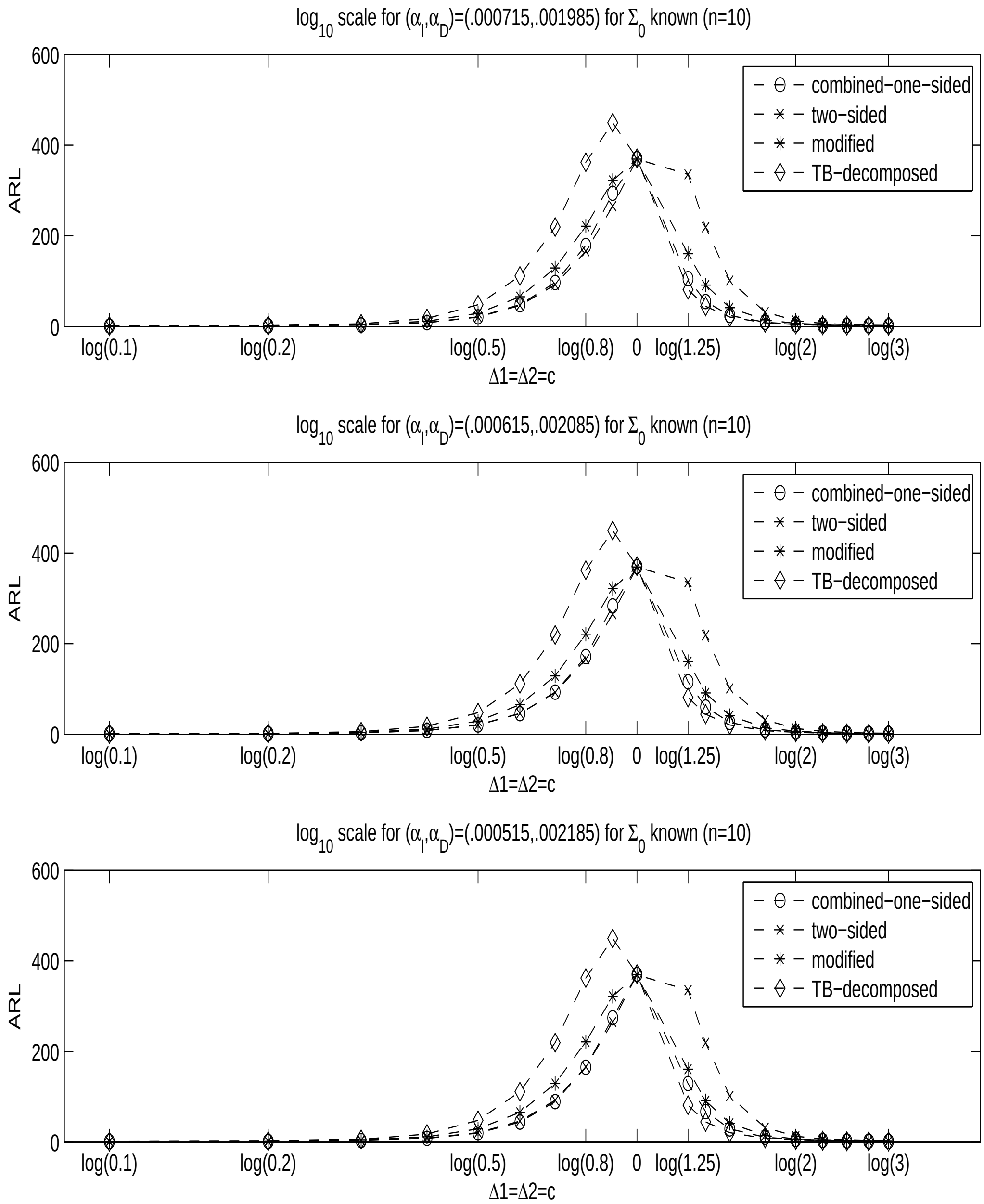

Figure 3.5: The ARL curves of the control charts under study for (α_I,α_D) in the logarithm scale for c when $p = 2$ and $\mathbf{\Sigma}_0$ is known ($n = 10$).

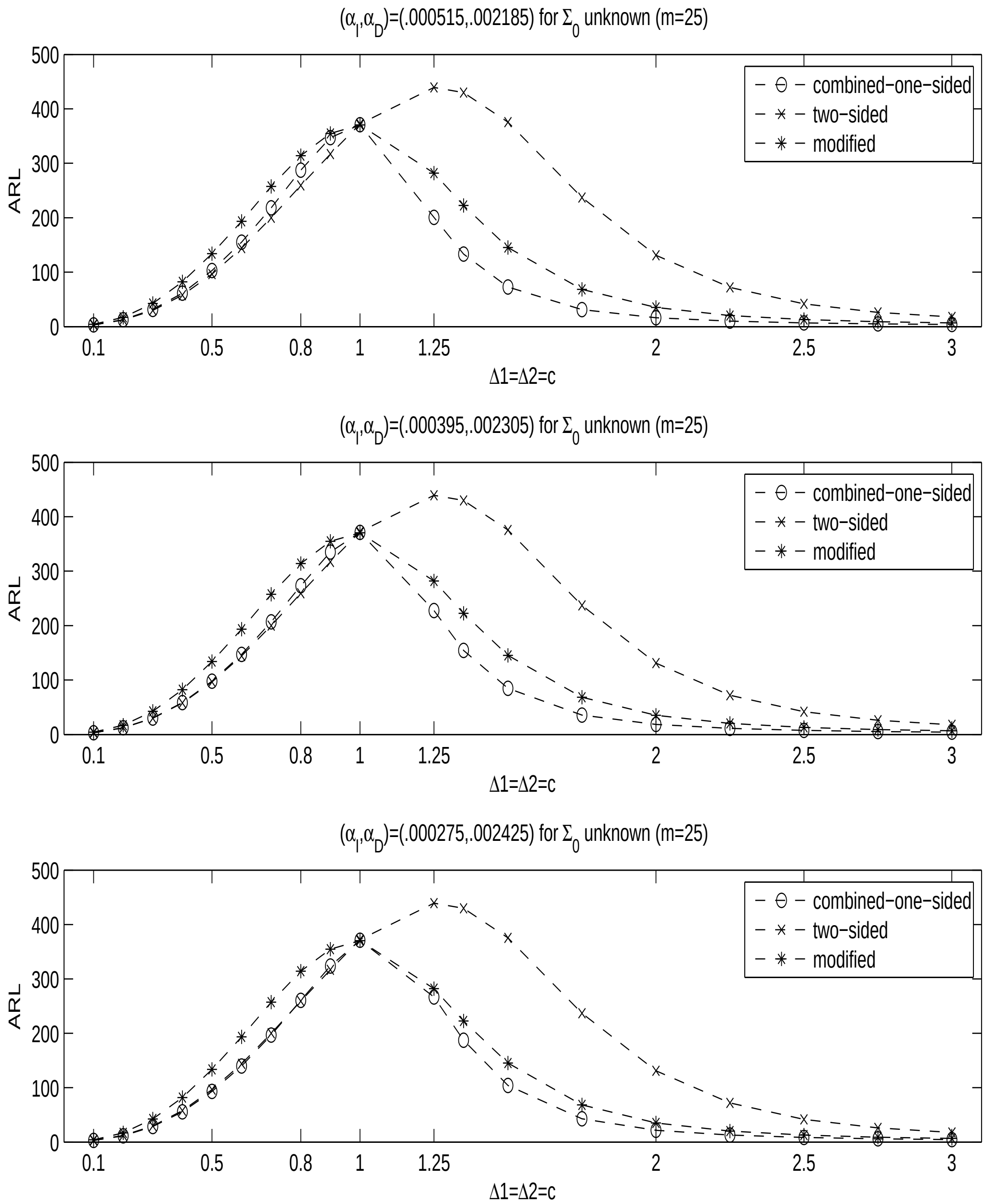

Figure 3.6: The ARL curves of the control charts under study for (α_I,α_D) in the original scale for c when $p = 2$ and $\mathbf{\Sigma}_0$ is unknown ($m = 25$,$n = 5$).

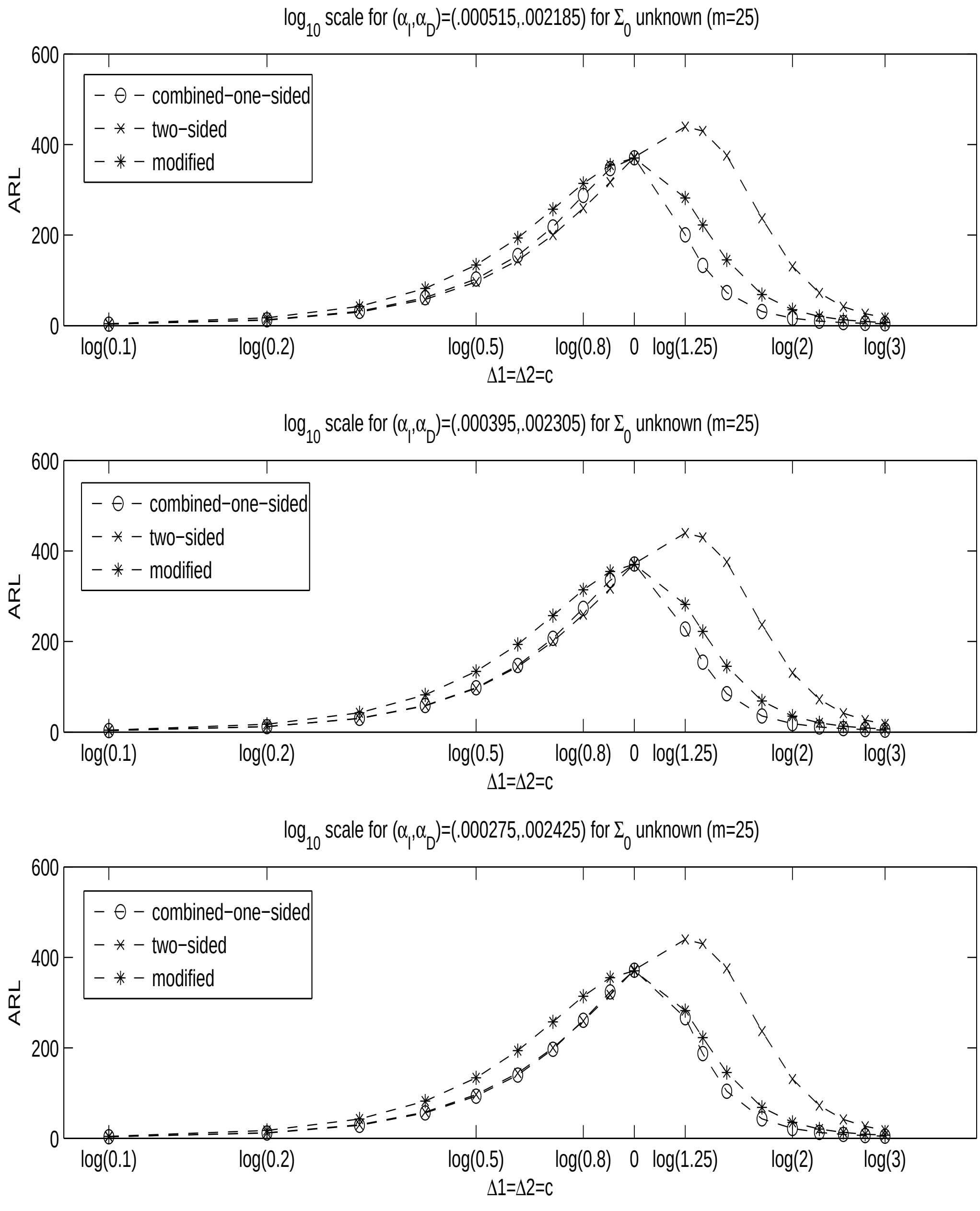

Figure 3.7: The ARL curves of the control charts under study for (α_I,α_D) in the logarithm scale for c when $p = 2$ and $\mathbf{\Sigma}_0$ is unknown ($m = 25$,$n = 5$).

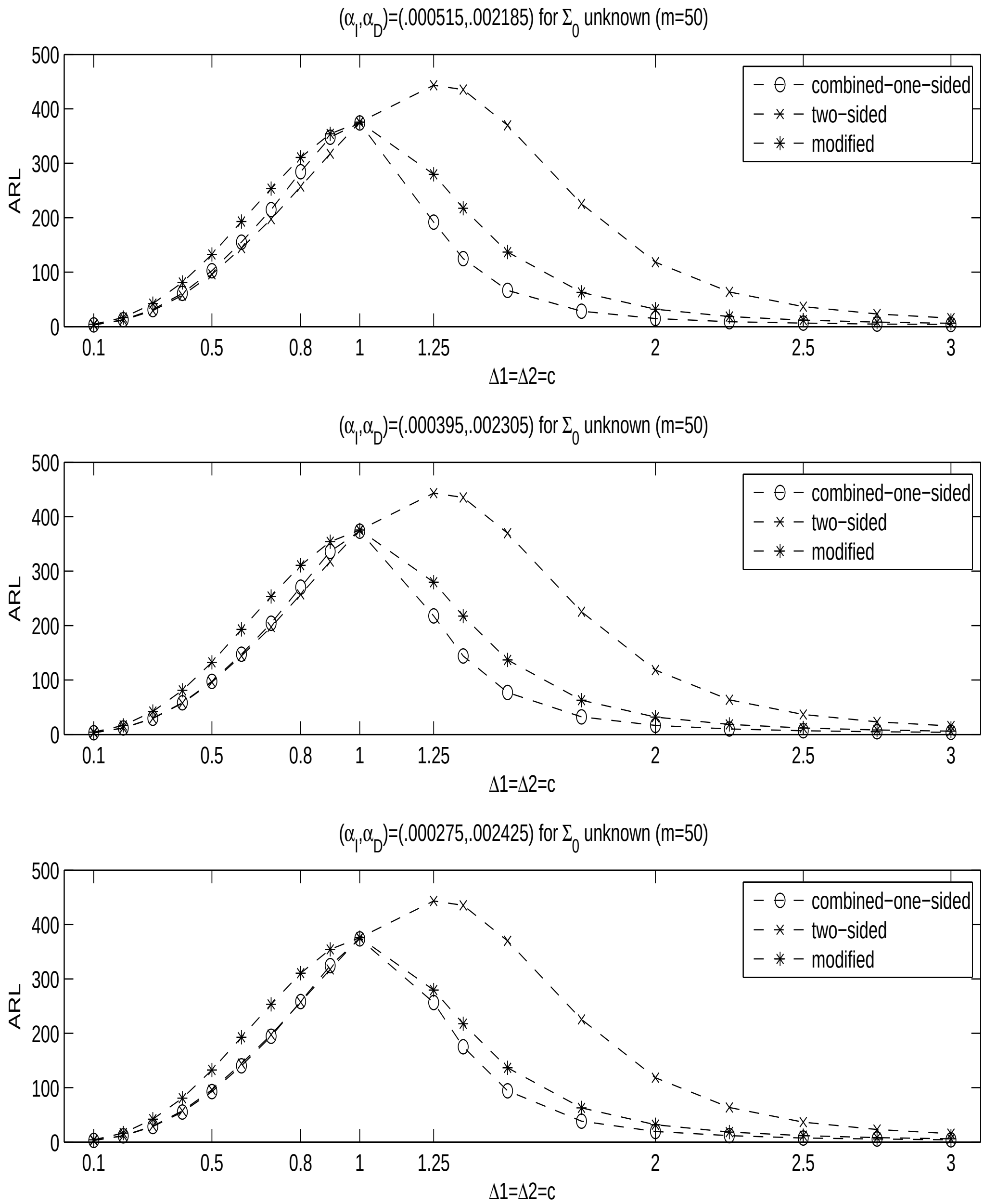

Figure 3.8: The ARL curves of the control charts under study for (α_I,α_D) in the original scale for c, when $p = 2$ and $\mathbf{\Sigma}_0$ is unknown ($m = 50$,$n = 5$).

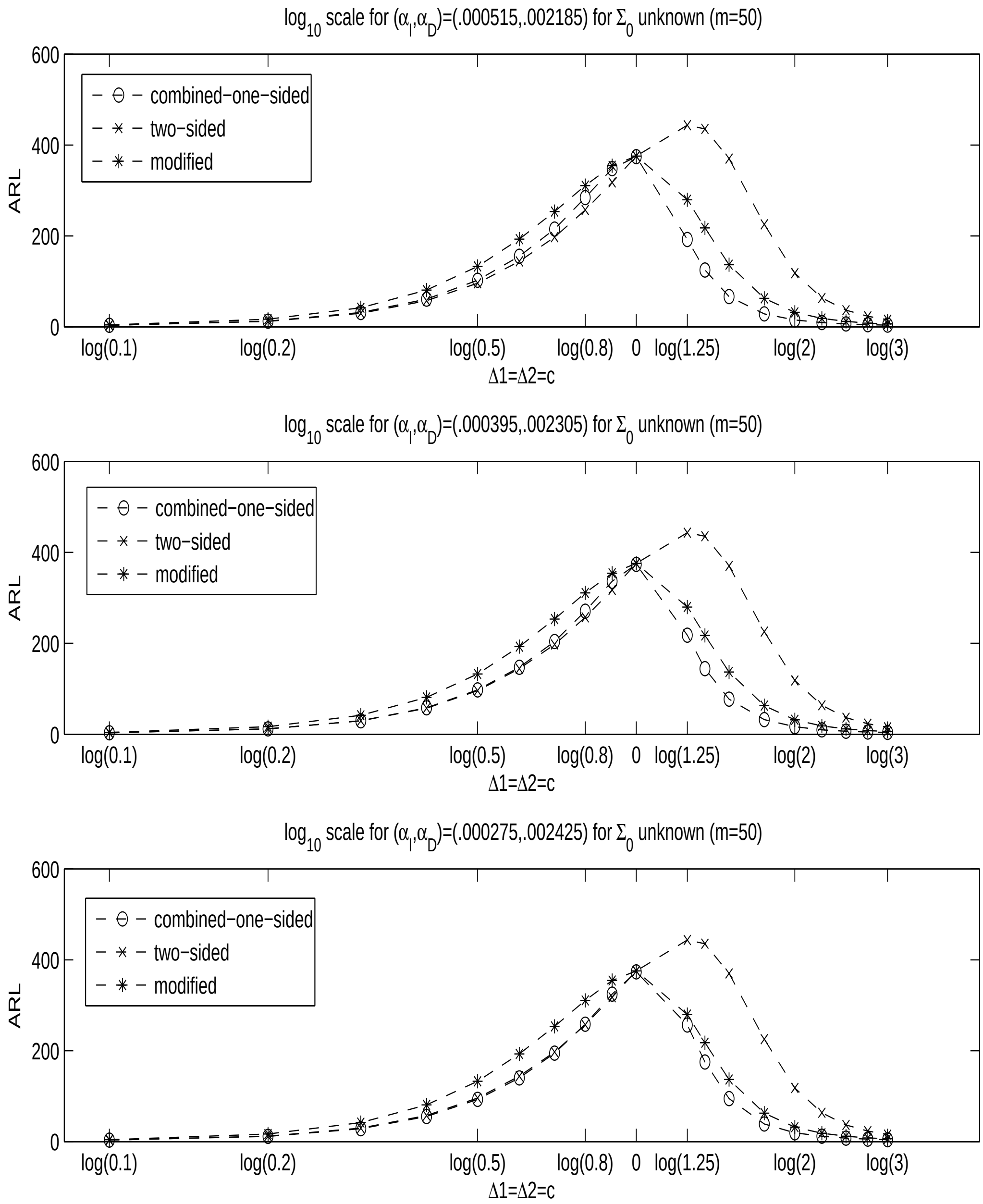

Figure 3.9: TThe *ARL* curves of the control charts under study for (α_I,α_D) in the logarithm scale for c when $p = 2$ and $\mathbf{\Sigma}_0$ is unknown ($m = 50$,$n = 5$).

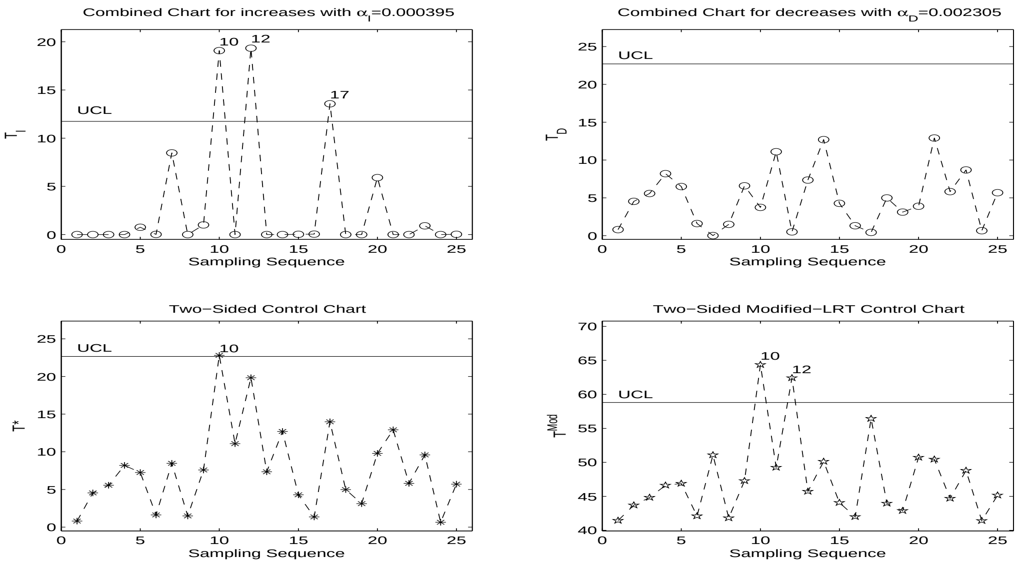

Figure 3.10: The combined, two-sided, and two-sided Modified-LRT control charts on 25 new samples of the first example.

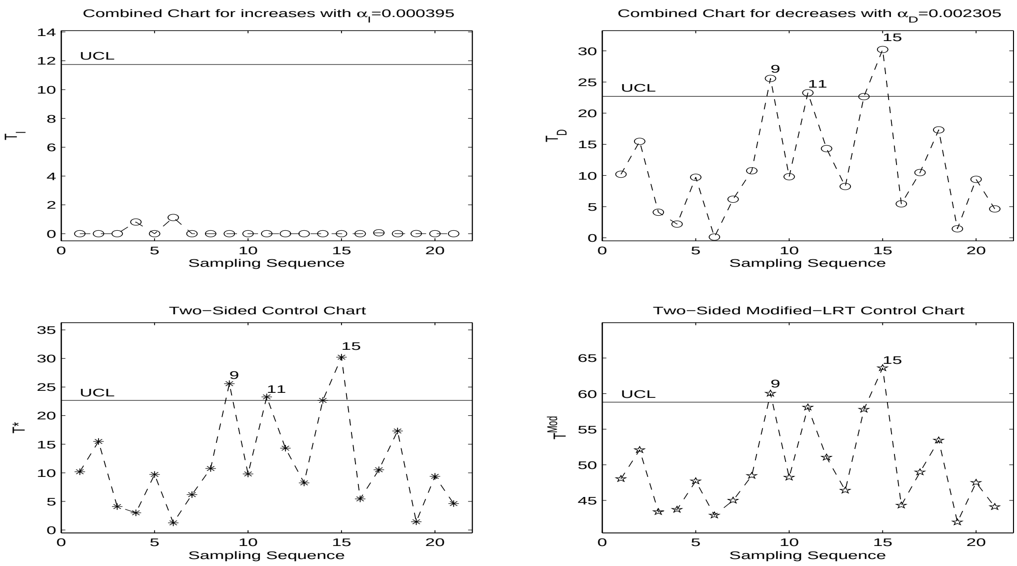

Figure 3.11: The combined, two-sided, and two-sided Modified-LRT control charts on 21 new samples of the second example.

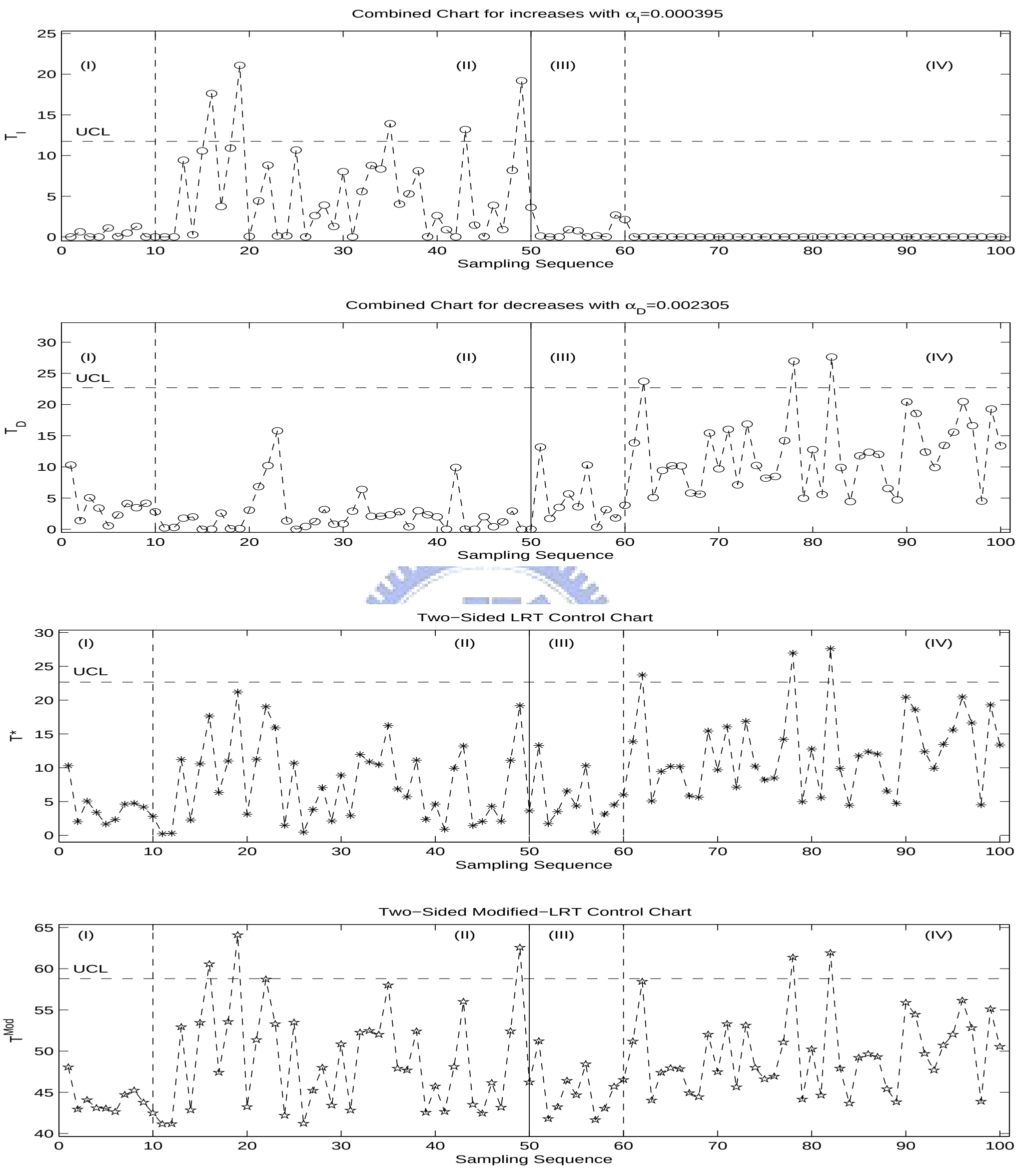

Figure 3.12: The combined, two-sided, and two-sided Modified-LRT control charts for scenario (i) of the hypothetical example.

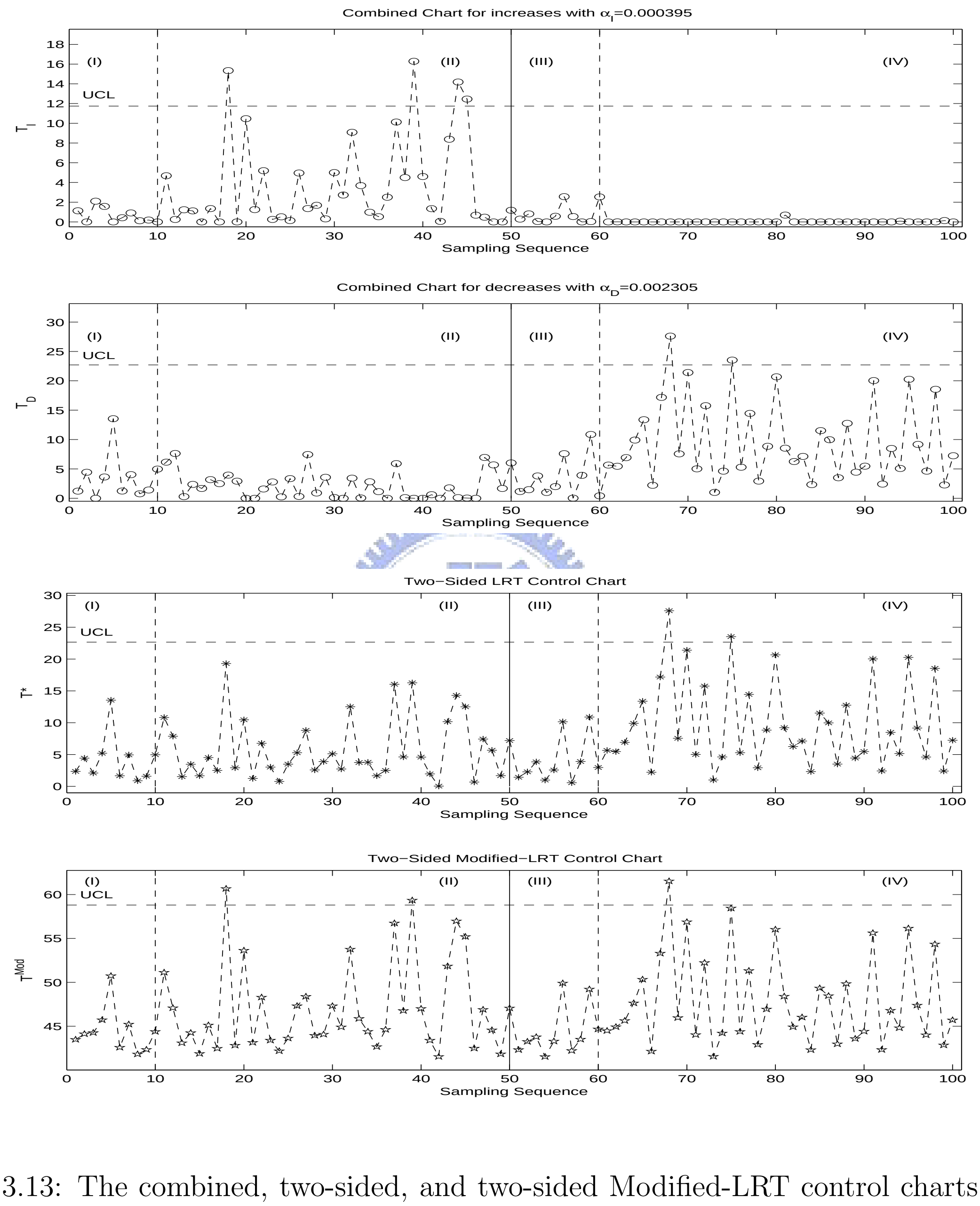

Figure 3.13: The combined, two-sided, and two-sided Modified-LRT control charts for scenario (ii) of the hypothetical example.

TABLES

Table 3.1 ARL_1 and their standard errors (in parentheses) of the two one-sided combined control charts with the equal tail probability (α_I=α_D=0.00135).

Δ_1	Δ_2	$\boldsymbol{\Sigma}_0$ known ($p=2$)		$\boldsymbol{\Sigma}_0$ unknown ($p=2,\ n=5$)	
		$n=5$	$n=10$	$m=25$	$m=50$
1.25	1.25	105.839 (.24232)	69.4879 (.12859)	118.236 (.28626)	112.116 (.26427)
1.35	1.35	65.6866 (.11813)	37.3194 (.05029)	74.8315 (.14378)	70.0771 (.13024)
1.5	1.5	35.3531 (.04633)	17.5094 (.01591)	41.0454 (.05808)	38.0502 (.05179)
1.75	1.75	16.0562 (.01393)	7.14902 (.00396)	18.8476 (.01780)	17.4070 (.01577)
2	2	9.09603 (.00579)	3.94574 (.00151)	10.6591 (.00741)	9.82699 (.00653)
2.25	2.25	5.96982 (.00298)	2.63968 (.00076)	6.92311 (.00377)	6.41319 (.00334)
2.5	2.5	4.33605 (.00177)	2.00210 (.00045)	4.96734 (.00221)	4.63533 (.00198)
2.75	2.75	3.38186 (.00117)	1.64956 (.00030)	3.83278 (.00144)	3.59575 (.00130)
3	3	2.78195 (.00083)	1.43883 (.00021)	3.11680 (.00101)	2.94284 (.00092)
0.9	0.9	**467.716** (2.2594)	**396.432** (1.7628)	**455.135** (2.1688)	**459.950** (2.2033)
0.8	0.8	**444.000** (2.0896)	255.252 (.91009)	**436.348** (2.0358)	**436.243** (2.0351)
0.7	0.7	344.394 (1.4270)	138.152 (.36178)	345.847 (1.4361)	345.698 (1.4352)
0.6	0.6	245.809 (.86000)	66.5321 (.12043)	248.124 (.87219)	246.096 (.86151)
0.5	0.5	160.730 (.45423)	28.8944 (.03412)	163.281 (.46511)	161.632 (.45807)
0.4	0.4	94.6172 (.20471)	11.2477 (.00805)	96.7717 (.21176)	95.0810 (.20622)
0.3	0.3	47.4876 (.07240)	4.02466 (.00157)	48.8658 (.07560)	48.1095 (.07384)
0.2	0.2	18.3683 (.01712)	1.53430 (.00025)	18.8980 (.01788)	18.6323 (.01749)
0.1	0.1	4.31953 (.00176)	1.00434 (.00001)	4.45286 (.00185)	4.38605 (.00180)

Table 3.2 The control limits and their standard errors (in parentheses) for five combinations of (α_I,α_D) when $p=2$ and $\boldsymbol{\Sigma}_0$ is known or unknown.

α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425
CL for $\boldsymbol{\Sigma}_0$ known (n=5)	11.0242 22.9694 (.00825,.00760)	11.4214 22.8159 (.00888,.00736)	11.5120 22.7870 (.00895,.00724)	11.6090 22.7574 (.00909,.00717)	12.1762 22.6138 (.01146,.00707)
CL for $\boldsymbol{\Sigma}_0$ unknown (m=25)	11.4782 22.8063 (.00911,.00763)	11.8849 22.6530 (.01047,.00731)	11.9749 22.6227 (.01070,.00723)	12.0712 22.5936 (.01109,.00712)	12.6561 22.4485 (.012592,.00715)
CL for $\boldsymbol{\Sigma}_0$ unknown (m=50)	11.2603 22.8869 (.00877,.00810)	11.6547 22.7351 (.00967,.00806)	11.7444 22.7055 (.00971,.00805)	11.8378 22.6771 (.00974,.00806)	12.4128 22.5327 (.011138,.00793)
α_I	.000715 .001985	.000635 .002065	.000615 .002085	.000595 .002105	.000515 .002185
CL for $\boldsymbol{\Sigma}_0$ known (n=10)	11.3660 17.6482 (.00679,.00546)	11.5879 17.5438 (.00697,.00537)	11.6478 17.5187 (.00714,.00535)	11.7108 17.493 (.00724,.00542)	11.9770 17.3956 (.00797,.00501)

Table 3.3 ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 5$.

$n = 5$			combined chart					two-sided chart		
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
$[\rho = 0]$										
1.25	1.25		182.427	203.422	208.390	213.792	245.670	444.612	273.774	115.093
			(.54945)	(.64716)	(.67105)	(.69735)	(.85927)	(2.0940)	(1.0111)	(.27489)
1.35	1.35		116.011	131.017	134.632	138.466	163.251	426.376	207.989	73.0108
			(.27820)	(.33405)	(.34801)	(.36302)	(.46498)	(1.9664)	(.66911)	(.13854)
1.5	1.5		60.6654	68.4594	70.3712	72.4871	85.7688	361.089	128.347	39.8668
			(.10478)	(.12573)	(.13106)	(.13704)	(.17658)	(1.5322)	(.32387)	(.05558)
1.75	**1.75**		25.4883	28.3380	29.0164	29.7647	34.5769	208.336	57.1628	18.1600
			(.02820)	(.03313)	(.03434)	(.03570)	(.04480)	(.67079)	(.09579)	(.01682)
2	2		13.3973	14.6453	14.9445	15.2717	17.3485	105.350	28.5795	10.1459
			(.01055)	(.01210)	(.01248)	(.01290)	(.01569)	(.24064)	(.03356)	(.00686)
2.25	**2.25**		8.30019	8.53714	9.02943	9.27666	10.6090	55.5172	16.4173	6.57604
			(.00501)	(.00524)	(.00572)	(.00597)	(.00735)	(.09166)	(.01441)	(.00347)
2.5	2.5		5.74057	5.87977	6.16803	6.31514	7.08263	31.7536	10.5236	4.70923
			(.00279)	(.00290)	(.00314)	(.00326)	(.00391)	(.03938)	(.00726)	(.00203)
2.75	2.75		4.31196	4.55855	4.61730	4.68115	5.07374	19.9077	7.38124	3.63466
			(.00175)	(.00192)	(.00196)	(.00201)	(.00229)	(.01936)	(.00417)	(.00132)
3	3		3.43157	3.60228	3.64242	3.68653	3.95505	13.4558	5.52709	2.95963
			(.00120)	(.00130)	(.00132)	(.00135)	(.00152)	(.01062)	(.00263)	(.00093)
0.9	0.9		347.554	337.405	335.616	333.511	323.881	313.745	353.070	495.872
			(1.4468)	(1.3838)	(1.3728)	(1.3599)	(1.3014)	(1.2407)	(1.4814)	(2.4666)
0.8	0.8		286.393	274.959	272.769	270.519	259.642	255.060	310.039	535.705
			(1.0819)	(1.0177)	(1.0055)	(.99306)	(.93371)	(.90906)	(1.2187)	(2.7699)
0.7	0.7		217.002	207.723	206.130	204.459	195.982	195.731	252.963	473.350
			(.71315)	(.66783)	(.66015)	(.65212)	(.61193)	(.61075)	(.89786)	(2.3004)
0.6	**0.6**		154.820	147.997	146.793	145.605	139.847	141.401	191.037	365.430
			(.42936)	(.40123)	(.39633)	(.39152)	(.36848)	(.37465)	(.58887)	(1.5599)
0.5	0.5		101.368	97.1628	96.3721	95.5804	91.8113	93.2975	130.250	245.405
			(.22708)	(.21305)	(.21045)	(.20785)	(.19564)	(.20042)	(.33111)	(.85787)
0.4	**0.4**		60.1249	57.6616	57.2145	56.7611	54.5995	55.5833	79.0986	142.800
			(.10338)	(.09705)	(.09592)	(.09478)	(.08938)	(.09182)	(.15631)	(.38023)
0.3	0.3		30.9166	29.7078	29.4825	29.2566	28.1810	28.6862	41.1490	69.3633
			(.03781)	(.03559)	(.03518)	(.03478)	(.03285)	(.03375)	(.05830)	(.12824)
0.2	0.2		12.3451	11.9033	11.8211	11.7371	11.3442	11.5274	16.3257	25.1002
			(.00930)	(.00879)	(.00870)	(.00860)	(.00816)	(.00836)	(.01429)	(.02755)
0.1	0.1		3.19401	3.10859	3.09301	3.07707	3.00166	3.03688	3.99371	5.28138
			(.00106)	(.00101)	(.00100)	(.00099)	(.00095)	(.00097)	(.00155)	(.00244)
1	1		370.117	370.233	370.727	370.700	370.837	369.898	370.501	370.693
			(1.5901)	(1.5908)	(1.5939)	(1.5938)	(1.5947)	(1.5886)	(1.5925)	(1.5938)

Table 3.3 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 5$.

$n = 5$			combined chart					two-sided chart		
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
$[\rho = 0]$										
1.25	1		269.847	288.679	292.864	297.486	323.478	407.598	317.541	206.849
			(.98936)	(1.0949)	(1.1188)	(1.1454)	(1.2989)	(1.8378)	(1.2633)	(.66361)
1.75	1		66.3940	74.0113	75.8098	77.7820	90.5563	270.128	112.686	48.8971
			(.12006)	(.14141)	(.14662)	(.15240)	(.19162)	(.99091)	(.26629)	(.07567)
2.25	1		22.7755	24.9236	25.4409	25.9992	29.5393	109.515	39.5475	18.1628
			(.02376)	(.02726)	(.02812)	(.02907)	(.03529)	(.25509)	(.05490)	(.01683)
2.75	1		11.2772	12.1208	12.3222	12.5417	13.8990	46.4183	18.3754	9.49832
			(.00808)	(.00904)	(.00927)	(.00953)	(.01116)	(.06995)	(.01713)	(.00619)
1.25	1.75		52.2713	58.5448	60.0815	61.7412	72.4315	293.798	104.098	32.2161
			(.08369)	(.09931)	(.10326)	(.10760)	(.13689)	(1.1241)	(.23635)	(.04025)
1.75	**2.25**		12.9404	14.1077	14.3900	14.6988	16.6359	93.7726	26.8216	9.58347
			(.01000)	(.01142)	(.01177)	(.01216)	(.01471)	(.20196)	(.03048)	(.00628)
2.25	**2.75**		5.69492	6.07014	6.15910	6.25599	6.85672	30.4234	10.3371	4.62303
			(.00276)	(.00306)	(.00313)	(.00321)	(.00371)	(.03690)	(.00706)	(.00197)
2.75	1.25		10.7817	11.0454	11.1395	12.2696	12.5781	48.2621	17.8314	8.61868
			(.00754)	(.00783)	(.00793)	(.00921)	(.00957)	(.07419)	(.01636)	(.00532)
$[\rho = 0.2]$										
1.75	**1.75**		22.1800	24.4308	24.9756	25.5726	29.3464	151.037	45.5483	17.0671
			(.02282)	(.02644)	(.02735)	(.02835)	(.03494)	(.41368)	(.06798)	(.01530)
1.75	**2.25**		11.9014	12.9049	13.1473	13.4113	15.0608	72.9049	23.1441	9.26786
			(.00879)	(.00996)	(.01025)	(.01056)	(.01263)	(.13824)	(.02435)	(.00596)
2.25	**2.25**		7.83989	8.41990	8.55829	8.70950	9.64438	44.4463	14.7104	6.43987
			(.00458)	(.00513)	(.00526)	(.00541)	(.00634)	(.06551)	(.01218)	(.00336)
2.25	**2.75**		5.49399	5.83908	5.92113	6.00971	6.55506	25.8864	9.58968	4.57673
			(.00260)	(.00287)	(.00294)	(.00301)	(.00345)	(.02888)	(.00628)	(.00194)
$[\rho = 0.4]$										
1.75	**1.75**		16.2703	17.6750	18.0106	18.3719	20.6587	77.2699	28.1743	14.3516
			(.01422)	(.01614)	(.01661)	(.01712)	(.02048)	(.15089)	(.03284)	(.01173)
1.75	**2.25**		9.69349	10.3984	10.5669	10.7508	11.8847	42.4239	16.4006	8.38560
			(.00639)	(.00713)	(.00731)	(.00751)	(.00877)	(.06106)	(.01439)	(.00510)
2.25	**2.25**		6.75567	7.18696	7.28901	7.40011	8.08238	27.6544	11.2275	6.03661
			(.00362)	(.00400)	(.00409)	(.00419)	(.00481)	(.03193)	(.00803)	(.00303)
2.25	**2.75**		5.18476	5.27819	5.31128	5.70572	5.81142	17.9707	7.90921	4.41903
			(.00237)	(.00244)	(.00247)	(.00277)	(.00285)	(.01655)	(.00465)	(.00183)

Table 3.3 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 5$.

$n = 5$			combined chart					two-sided chart		
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
$[\rho = 0]$										
0.8	1		339.173	330.087	328.526	326.947	318.497	308.718	341.507	400.986
			(1.3949)	(1.3390)	(1.3295)	(1.3199)	(1.2690)	(1.2109)	(1.4091)	(1.7932)
0.6	1		252.691	243.739	242.286	240.662	232.504	226.116	263.286	313.716
			(.89641)	(.84914)	(.84155)	(.83309)	(.79103)	(.75861)	(.95346)	(1.2405)
0.4	1		155.491	149.553	148.423	147.243	141.941	138.060	165.710	185.099
			(.43216)	(.40759)	(.40297)	(.39816)	(.37680)	(.36142)	(.47555)	(.56159)
0.2	1		64.1978	61.6052	61.1174	60.6404	58.3473	56.6815	68.9822	67.5459
			(.11412)	(.10724)	(.10596)	(.10472)	(.09880)	(.09458)	(.12718)	(.12321)
0.8	0.6		211.506	202.298	200.706	199.017	190.934	190.309	244.445	478.538
			(.68618)	(.64180)	(.63422)	(.62622)	(.58840)	(.58551)	(.85284)	(2.3383)
0.6	**0.4**		95.8447	91.7705	91.0365	90.2845	86.6889	88.0406	122.245	236.041
			(.20872)	(.19551)	(.19316)	(.19076)	(.17944)	(.18367)	(.30099)	(.80918)
0.4	**0.2**		26.4296	25.3954	25.2060	25.0166	24.1059	24.5286	34.9046	60.3489
			(.02980)	(.02805)	(.02773)	(.02741)	(.02591)	(.02660)	(.04545)	(.10396)
0.2	0.8		51.5425	50.9160	50.7174	48.7413	48.3682	48.5092	64.3588	81.0461
			(.08194)	(.08044)	(.07996)	(.07531)	(.07444)	(.07476)	(.11455)	(.16214)
$[\rho = 0.2]$										
0.6	**0.6**		145.378	139.029	137.895	136.797	131.286	132.330	177.552	353.989
			(.39060)	(.36524)	(.36077)	(.35646)	(.33508)	(.33910)	(.52753)	(1.4872)
0.6	**0.4**		89.9337	86.1954	85.5180	84.8162	81.4684	82.6078	114.030	228.686
			(.18964)	(.17790)	(.17580)	(.17363)	(.16341)	(.16687)	(.27108)	(.77160)
0.4	**0.4**		56.7609	54.4676	54.0523	53.6075	51.5534	52.4787	74.3677	138.276
			(.09478)	(.08906)	(.08803)	(.08694)	(.08196)	(.08419)	(.14244)	(.36227)
0.4	**0.2**		25.0526	24.0846	23.9109	23.7274	22.8649	23.2634	33.0781	58.4811
			(.02747)	(.02588)	(.02559)	(.02529)	(.02391)	(.02454)	(.04189)	(.09914)
$[\rho = 0.4]$										
0.6	**0.6**		118.665	113.662	112.769	111.829	107.365	107.054	139.789	316.416
			(.28783)	(.26977)	(.26659)	(.26325)	(.24760)	(.24652)	(.36824)	(1.2566)
0.6	**0.4**		74.1372	71.1023	70.5440	69.9567	67.2527	67.8900	92.4313	206.264
			(.14177)	(.13312)	(.13155)	(.12990)	(.12240)	(.12416)	(.19763)	(.66079)
0.4	**0.4**		47.3060	45.3854	45.0403	44.6784	43.0012	43.7086	61.3450	123.548
			(.07198)	(.06761)	(.06684)	(.06603)	(.06232)	(.06387)	(.10656)	(.30582)
0.4	**0.2**		21.0468	20.2362	20.0880	19.9395	19.2246	19.5465	27.5379	52.7095
			(.02107)	(.01985)	(.01962)	(.01940)	(.01835)	(.01882)	(.03172)	(.08475)

Table 3.4 ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 10$.

$n = 10$			combined chart					two-sided chart		
Δ_1	Δ_2	α_I α_D	.000715 .001985	.000635 .002065	.000615 .002085	.000595 .002105	.000515 .002185	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
$[\rho = 0]$										
1.25	1.25		105.209	113.404	115.722	118.205	129.266	335.402	160.786	81.2546
			(.24015)	(.26885)	(.27716)	(.28615)	(.32736)	(1.37146)	(.45447)	(.16277)
1.35	1.35		55.0231	59.1732	60.3196	61.5572	67.1666	219.135	91.1760	44.4824
			(.09043)	(.10092)	(.10388)	(.10711)	(.12217)	(.72370)	(.19360)	(.06559)
1.5	1.5		24.5234	26.1133	26.5648	27.0406	29.1209	101.643	41.2840	20.8205
			(.02660)	(.02926)	(.03003)	(.03086)	(.03453)	(.22801)	(.05859)	(.02073)
1.75	**1.75**		9.20924	9.66587	9.80172	9.93430	10.5325	31.6083	14.4905	8.23782
			(.00590)	(.00636)	(.00650)	(.00664)	(.00727)	(.03910)	(.01190)	(.00496)
2	2		4.79554	4.97644	5.02645	5.08029	5.30837	13.0719	6.90898	4.40110
			(.00209)	(.00222)	(.00226)	(.00229)	(.00246)	(.01016)	(.00376)	(.00181)
2.25	**2.25**		3.06618	3.15531	3.18006	3.20617	3.31821	6.86255	4.09890	2.86838
			(.00099)	(.00104)	(.00105)	(.00106)	(.00113)	(.00372)	(.00161)	(.00088)
2.5	2.5		2.24655	2.29704	2.31093	2.32574	2.38851	4.27830	2.82483	2.13099
			(.00056)	(.00058)	(.00059)	(.00060)	(.00063)	(.00173)	(.00085)	(.00051)
2.75	2.75		1.80297	1.83423	1.84288	1.85200	1.89135	3.01917	2.16227	1.73099
			(.00036)	(.00037)	(.00038)	(.00038)	(.00040)	(.00096)	(.00052)	(.00033)
3	3		1.54194	1.56286	1.56861	1.57464	1.60058	2.33018	1.77936	1.49271
			(.00025)	(.00026)	(.00026)	(.00027)	(.00028)	(.00060)	(.00035)	(.00023)
0.9	0.9		294.014	285.343	283.274	281.314	274.190	265.305	321.807	449.549
			(1.1254)	(1.0759)	(1.0642)	(1.0532)	(1.0134)	(.96446)	(1.2889)	(2.1290)
0.8	0.8		178.674	172.282	171.724	170.227	165.160	165.899	221.080	362.207
			(.53255)	(.50417)	(.50172)	(.49516)	(.47318)	(.47636)	(.73337)	(1.5393)
0.7	0.7		97.1147	93.7383	92.9303	92.1540	89.0777	92.1222	129.160	219.474
			(.21289)	(.20185)	(.19924)	(.19674)	(.18693)	(.19664)	(.32696)	(.72538)
0.6	**0.6**		47.8917	46.3034	45.9410	45.5804	44.1263	46.0766	65.4834	111.350
			(.07333)	(.06969)	(.06887)	(.06805)	(.06480)	(.06917)	(.11758)	(.26155)
0.5	0.5		21.4046	20.7646	20.6149	20.4664	19.8653	20.6785	29.1854	48.0834
			(.02162)	(.02064)	(.02042)	(.02019)	(.01929)	(.02051)	(.03465)	(.07378)
0.4	**0.4**		8.66646	8.44613	8.39414	8.34225	8.14508	8.43772	11.5247	17.8969
			(.00537)	(.00515)	(.00510)	(.00505)	(.00487)	(.00515)	(.00836)	(.01645)
0.3	0.3		3.30745	3.24439	3.22941	3.21468	3.15809	3.24126	4.13852	5.83729
			(.00112)	(.00109)	(.00108)	(.00107)	(.00104)	(.00109)	(.00164)	(.00287)
0.2	0.2		1.39559	1.38317	1.38023	1.37732	1.36595	1.38217	1.56256	1.88200
			(.00020)	(.00019)	(.00019)	(.00019)	(.00018)	(.00019)	(.00026)	(.00040)
0.1	0.1		1.00212	1.00197	1.00193	1.00189	1.00174	1.00197	1.00497	1.01321
			(.00001)	(.00001)	(.00001)	(.00001)	(.00001)	(.00001)	(.00002)	(.00003)
1	1		370.076	370.055	370.343	369.365	370.028	369.174	369.365	370.892
			(1.5898)	(1.5896)	(1.5915)	(1.5852)	(1.5895)	(1.5840)	(1.5852)	(1.5950)

Table 3.4 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 10$.

$n = 10$			combined chart					two-sided chart		
Δ_1	Δ_2	α_I α_D	.000715 .001985	.000635 .002065	.000615 .002085	.000595 .002105	.000515 .002185	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
$[\rho = 0]$										
1.25	1		198.961	211.195	214.484	218.012	233.825	351.130	230.862	160.799
			(.62596)	(.68467)	(.70075)	(.71814)	(.79779)	(1.4692)	(.78265)	(.45452)
1.75	1		27.0185	28.6382	29.0923	29.5761	31.7195	71.0162	36.3381	23.4215
			(.03082)	(.03367)	(.03448)	(.03535)	(.03931)	(.13287)	(.04830)	(.02480)
2.25	1		8.23820	8.58353	8.67989	8.78198	9.22167	17.9978	10.6023	7.60635
			(.00496)	(.00529)	(.00538)	(.00548)	(.00591)	(.01659)	(.00735)	(.00437)
2.75	1		4.14600	4.27165	4.30609	4.34255	4.50061	7.57549	5.06239	3.94313
			(.00164)	(.00173)	(.00175)	(.00178)	(.00188)	(.00434)	(.00228)	(.00151)
1.25	1.75		20.4036	21.6194	21.9563	22.3140	23.9191	69.5681	31.7445	16.4998
			(.02010)	(.02195)	(.02248)	(.02304)	(.02561)	(.12881)	(.03936)	(.01453)
1.75	**2.25**		4.64911	4.81889	4.86590	4.91602	5.13046	12.0845	6.59164	4.21806
			(.00199)	(.00211)	(.00214)	(.00218)	(.00233)	(.00900)	(.00349)	(.00169)
2.25	**2.75**		2.23557	2.28515	2.29877	2.31323	2.37566	4.21104	2.80315	2.11340
			(.00056)	(.00058)	(.00059)	(.00059)	(.00062)	(.00169)	(.00084)	(.00050)
2.75	1.25		3.77392	3.88611	3.91677	3.94973	4.09223	7.54359	4.85401	3.64604
			(.00141)	(.00158)	(.00150)	(.00152)	(.00161)	(.00431)	(.00213)	(.00133)
$[\rho = 0.2]$										
1.75	**1.75**		8.02191	8.38021	8.47991	8.58753	9.05147	23.0817	11.8257	7.62386
			(.00475)	(.00509)	(.00519)	(.00529)	(.00574)	(.02425)	(.00870)	(.00439)
1.75	**2.25**		4.32733	4.47424	4.51484	4.55788	4.74279	10.1944	5.93403	4.07143
			(.00177)	(.00186)	(.00189)	(.00192)	(.00205)	(.00691)	(.00295)	(.00160)
2.25	**2.25**		2.94260	3.02277	3.04487	3.06829	3.17050	6.13899	3.84906	2.82340
			(.00092)	(.00096)	(.00097)	(.00099)	(.00104)	(.00311)	(.00145)	(.00085)
2.25	**2.75**		2.19997	2.23491	2.24767	2.26117	2.31848	3.94749	2.70873	2.10082
			(.00056)	(.00054)	(.00056)	(.00057)	(.00060)	(.00152)	(.00079)	(.00049)
$[\rho = 0.4]$										
1.75	**1.75**		5.87187	6.08734	6.14681	6.21045	6.48241	12.2542	7.53406	6.03458
			(.00307)	(.00290)	(.00312)	(.00317)	(.00339)	(.00919)	(.00431)	(.00303)
1.75	**2.25**		3.60654	3.70965	3.73789	3.76781	3.89789	6.84123	4.53419	3.61577
			(.00137)	(.00130)	(.00138)	(.00140)	(.00148)	(.00370)	(.00191)	(.00131)
2.25	**2.25**		2.62804	2.68862	2.70521	2.72282	2.79960	4.65980	3.24360	2.65407
			(.00078)	(.00075)	(.00079)	(.00080)	(.00084)	(.00199)	(.00109)	(.00076)
2.25	**2.75**		2.05332	2.09106	2.10139	2.11234	2.15975	3.31618	2.44877	2.04582
			(.00048)	(.00047)	(.00049)	(.00050)	(.00052)	(.00113)	(.00066)	(.00047)

Table 3.4 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is known and $p = 2$, $n = 10$.

$n = 10$		combined chart					two-sided chart		
Δ_1	Δ_2	α_I .000715 α_D .001985	.000635 .002065	.000615 .002085	.000595 .002105	.000515 .002185	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
[$\rho = 0$]									
0.8	1	273.538	265.827	264.016	262.412	254.130	242.757	284.139	333.934
		(1.0098)	(.96731)	(.95743)	(.94871)	(.90409)	(.84401)	(1.0691)	(1.3625)
0.6	1	132.423	128.083	127.139	126.135	122.724	116.120	142.207	168.913
		(.33946)	(.32286)	(.31929)	(.31551)	(.30276)	(.27859)	(.37786)	(.48943)
0.4	1	43.5446	42.1234	41.7933	41.4533	40.2210	38.2751	47.0963	53.7468
		(.06351)	(.06040)	(.05969)	(.05895)	(.05632)	(.05225)	(.07150)	(.08728)
0.2	1	7.27234	7.07771	7.03223	6.98670	6.81005	6.54454	7.74979	8.34102
		(.00407)	(.00390)	(.00386)	(.00382)	(.00367)	(.00345)	(.00450)	(.00505)
0.8	0.6	89.2351	86.1638	85.4672	84.7537	81.6257	83.8086	115.485	203.845
		(.18743)	(.17780)	(.17564)	(.17344)	(.16389)	(.17053)	(.27630)	(.64918)
0.6	**0.4**	18.7702	18.2198	18.0888	17.9601	17.4446	18.1194	25.2685	42.6998
		(.01769)	(.01691)	(.01672)	(.01654)	(.01582)	(.01676)	(.02783)	(.06166)
0.4	**0.2**	2.67239	2.62582	2.61479	2.60387	2.56413	2.62463	3.26293	4.60380
		(.00077)	(.00075)	(.00074)	(.00074)	(.00072)	(.00075)	(.00110)	(.00195)
0.2	0.8	5.93490	5.78753	5.75277	5.71763	5.58357	5.66841	7.10218	8.54148
		(.00295)	(.00283)	(.00280)	(.00278)	(.00267)	(.00274)	(.00392)	(.00525)
[$\rho = 0.2$]									
0.6	**0.6**	41.1430	39.8038	39.4928	39.1850	38.0150	39.4933	55.1601	102.277
		(.05829)	(.05544)	(.05479)	(.05414)	(.05172)	(.05479)	(.09077)	(.23015)
0.6	**0.4**	16.5463	16.0612	15.9483	15.8380	15.4047	15.9709	22.0159	39.9115
		(.01459)	(.01394)	(.01379)	(.01364)	(.01307)	(.01382)	(.02257)	(.05567)
0.4	**0.4**	7.79599	7.59940	7.55307	7.50742	7.33827	7.59217	10.2691	16.8506
		(.00454)	(.00437)	(.00432)	(.00428)	(.00413)	(.00436)	(.00699)	(.01500)
0.4	**0.2**	2.49762	2.45554	2.44546	2.43557	2.39656	2.45186	3.01971	4.44298
		(.00068)	(.00066)	(.00066)	(.00065)	(.00063)	(.00066)	(.00096)	(.00184)
[$\rho = 0.4$]									
0.6	**0.6**	25.8062	25.0026	24.8133	24.6240	23.8554	24.1704	32.0291	72.9145
		(.02874)	(.02739)	(.02708)	(.02676)	(.02550)	(.02602)	(.03989)	(.13826)
0.6	**0.4**	11.0973	10.7890	10.7167	10.6444	10.3733	10.6836	14.1897	30.7822
		(.00789)	(.00755)	(.00747)	(.00739)	(.00710)	(.00743)	(.01152)	(.03756)
0.4	**0.4**	5.59687	5.46460	5.43325	5.40232	5.28963	5.45465	7.15180	13.5648
		(.00268)	(.00258)	(.00256)	(.00253)	(.00245)	(.00257)	(.00397)	(.01075)
0.4	**0.2**	2.02996	2.00005	1.99295	1.98583	1.95885	1.99766	2.38692	3.87459
		(.00046)	(.00045)	(.00044)	(.00044)	(.00043)	(.00045)	(.00063)	(.00147)

Table 3.5 ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 25$, $n = 5$.

$m = 25$			combined chart					two-sided chart	
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
[$\rho = 0$]									
1.25	1.25		200.499	222.591	227.622	233.106	266.628	439.396	281.980
			(.63324)	(.74091)	(.76621)	(.79411)	(.97169)	(2.0572)	(1.0569)
1.35	1.35		133.020	150.210	154.264	158.725	187.163	429.941	222.474
			(.34176)	(.41028)	(.42704)	(.44574)	(.57102)	(1.9911)	(.74033)
1.5	1.5		72.7614	82.3635	84.6887	87.2201	104.005	375.523	145.211
			(.13783)	(.16612)	(.17324)	(.18109)	(.23603)	(1.6250)	(.38993)
1.75	**1.75**		31.0075	34.7489	35.6274	36.6065	43.0258	236.908	68.4088
			(.03798)	(.04514)	(.04688)	(.04884)	(.06237)	(.81365)	(.12559)
2	2		16.3170	17.9813	18.3734	18.8022	21.6389	130.931	35.1455
			(.01428)	(.01657)	(.01712)	(.01774)	(.02198)	(.33372)	(.04592)
2.25	**2.25**		9.99249	10.8705	11.0738	11.2974	12.7661	71.9735	20.3119
			(.00670)	(.00764)	(.00786)	(.00811)	(.00979)	(.13558)	(.01996)
2.5	2.5		6.82658	7.34211	7.46218	7.59337	8.44271	41.7899	12.9767
			(.00368)	(.00413)	(.00424)	(.00436)	(.00515)	(.05968)	(.01004)
2.75	2.75		5.05063	5.38297	5.45996	5.54412	6.08316	26.2249	8.99630
			(.00227)	(.00252)	(.00258)	(.00264)	(.00307)	(.02945)	(.00569)
3	3		3.96910	4.19808	4.25079	4.30795	4.67427	17.6264	6.66151
			(.00153)	(.00168)	(.00171)	(.00175)	(.00200)	(.01607)	(.00354)
0.9	0.9		346.963	336.984	334.986	332.696	323.515	316.822	354.736
			(1.4431)	(1.3812)	(1.3689)	(1.3549)	(1.2991)	(1.2590)	(1.4919)
0.8	0.8		287.501	275.505	273.153	271.065	260.739	259.245	314.041
			(1.0882)	(1.0207)	(1.0076)	(.99608)	(.93964)	(.93156)	(1.2424)
0.7	0.7		218.105	208.712	206.962	205.153	196.870	199.994	257.314
			(.71860)	(.67261)	(.66416)	(.65545)	(.61610)	(.63084)	(.92116)
0.6	**0.6**		154.902	148.312	147.037	145.752	139.821	143.974	193.470
			(.42970)	(.40251)	(.39732)	(.39212)	(.36837)	(.38494)	(.60018)
0.5	0.5		102.866	98.5222	97.6968	96.8800	93.0527	96.4325	133.752
			(.23215)	(.21756)	(.21482)	(.21212)	(.19963)	(.21065)	(.34459)
0.4	**0.4**		61.5152	58.9564	58.4563	58.0065	55.7891	57.8865	82.0116
			(.10700)	(.10036)	(.09908)	(.09793)	(.09234)	(.09763)	(.16506)
0.3	0.3		31.5413	30.3098	30.0758	29.8473	28.7406	29.8083	42.6108
			(.03898)	(.03669)	(.03626)	(.03585)	(.03385)	(.03578)	(.06146)
0.2	0.2		12.6829	12.2296	12.1438	12.0597	11.6513	12.0464	17.0133
			(.00969)	(.00916)	(.00906)	(.00897)	(.00850)	(.00895)	(.01522)
0.1	0.1		3.27715	3.18968	3.17270	3.15667	3.07807	3.15405	4.14050
			(.00111)	(.00106)	(.00105)	(.00104)	(.00099)	(.00104)	(.00164)
1	1		370.727	371.175	371.264	371.195	371.243	372.252	370.268
			(1.5940)	(1.5969)	(1.5974)	(1.5970)	(1.5973)	(1.6038)	(1.5910)

Table 3.5 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 25$, $n = 5$.

$m = 25$			combined chart					two-sided chart	
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
[$\rho = 0$]									
1.25	1		284.079	302.133	306.157	310.429	333.868	403.405	320.431
			(1.0688)	(1.1724)	(1.1959)	(1.2210)	(1.3621)	(1.8095)	(1.2806)
1.75	1		80.0894	89.7215	91.9841	94.4952	110.547	286.402	128.348
			(.15926)	(.18897)	(.19619)	(.20431)	(.25872)	(1.08190)	(.32387)
2.25	1		27.9277	30.8116	31.4929	32.2341	37.1230	129.328	47.6161
			(.03241)	(.03762)	(.03889)	(.04028)	(.04989)	(.32760)	(.07270)
2.75	1		13.6857	14.8421	15.1082	15.4029	17.3047	58.2318	22.4305
			(.01090)	(.01235)	(.01269)	(.01307)	(.01562)	(.09851)	(.02322)
1.25	1.75		62.8243	70.8752	72.7866	74.8503	88.5763	313.770	119.810
			(.11046)	(.13248)	(.13790)	(.14383)	(.18535)	(1.24082)	(.29201)
1.75	**2.25**		15.7444	17.3026	17.6732	18.0764	20.7448	117.160	33.0484
			(.01352)	(.01562)	(.01614)	(.01670)	(.02061)	(.28235)	(.04183)
2.25	**2.75**		6.75602	7.26380	7.38047	7.50821	8.33781	39.9575	12.7054
			(.00362)	(.00407)	(.00417)	(.00428)	(.00505)	(.05577)	(.00972)
2.75	1.25		12.3292	13.3748	13.6197	13.8861	15.6135	61.4383	21.9094
			(.00928)	(.01052)	(.01082)	(.01115)	(.01335)	(.10680)	(.02240)
[$\rho = 0.2$]									
1.75	**1.75**		27.1138	30.1417	30.8498	31.6375	36.7901	177.888	55.2789
			(.03098)	(.03638)	(.03769)	(.03916)	(.04921)	(.52903)	(.09107)
1.75	**2.25**		14.4603	15.8180	16.1345	16.4835	18.7616	92.1056	28.5232
			(.01186)	(.01362)	(.01404)	(.01450)	(.01768)	(.19658)	(.03346)
2.25	**2.25**		9.41033	10.1926	10.3735	10.5710	11.8650	57.8854	18.1660
			(.00610)	(.00691)	(.00710)	(.00731)	(.00875)	(.09762)	(.01683)
2.25	**2.75**		6.50558	6.97235	7.07953	7.19664	7.95403	34.0574	11.7639
			(.00341)	(.00381)	(.00390)	(.00401)	(.00469)	(.04379)	(.00863)
[$\rho = 0.4$]									
1.75	**1.75**		19.8762	21.7790	22.2223	22.7040	25.8709	94.8222	34.2368
			(.01931)	(.02220)	(.02289)	(.02365)	(.02885)	(.20538)	(.04414)
1.75	**2.25**		12.4651	12.7890	12.9015	14.3655	14.7313	54.2985	20.1510
			(.00944)	(.00982)	(.00995)	(.01174)	(.01221)	(.08864)	(.01972)
2.25	**2.25**		8.50375	8.69962	8.76775	9.64686	9.86456	35.7937	13.7297
			(.00521)	(.00540)	(.00546)	(.00634)	(.00657)	(.04721)	(.01095)
2.25	**2.75**		5.84587	6.21665	6.30251	6.39501	6.99025	23.4306	9.60358
			(.00288)	(.00317)	(.00325)	(.00332)	(.00383)	(.02481)	(.00630)

Table 3.5 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 25$, $n = 5$.

$m = 25$		combined chart					two-sided chart	
Δ_1	Δ_2	α_I .000515 α_D .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
$[\rho = 0]$								
0.8	1	336.757	327.402	325.600	323.892	315.244	309.210	339.386
		(1.3798)	(1.3226)	(1.3117)	(1.3014)	(1.2496)	(1.2138)	(1.3960)
0.6	1	252.873	243.950	242.140	240.359	232.461	228.828	266.383
		(.89739)	(.85025)	(.84079)	(.83151)	(.79081)	(.77232)	(.97035)
0.4	1	156.583	150.491	149.324	148.178	142.680	140.735	168.175
		(.43673)	(.41144)	(.40665)	(.40197)	(.37975)	(.37200)	(.48622)
0.2	1	64.5503	61.9510	61.4347	60.9583	58.6429	57.7277	69.9864
		(.11506)	(.10815)	(.10679)	(.10555)	(.09956)	(.09722)	(.12998)
0.8	0.6	210.844	202.143	200.451	198.835	190.951	193.147	246.024
		(.68296)	(.64105)	(.63301)	(.62536)	(.58847)	(.59867)	(.86112)
0.6	**0.4**	96.8092	92.7584	91.9819	91.2396	87.6524	90.6738	125.196
		(.21189)	(.19868)	(.19618)	(.19381)	(.18245)	(.19200)	(.31198)
0.4	**0.2**	26.9682	25.9164	25.7182	25.5251	24.5888	25.4884	36.1412
		(.03073)	(.02893)	(.02859)	(.02827)	(.02670)	(.02820)	(.04791)
0.2	0.8	53.9347	51.7114	51.2838	50.8881	48.9078	49.8988	65.8365
		(.08775)	(.08234)	(.08132)	(.08037)	(.07569)	(.07802)	(.11854)
$[\rho = 0.2]$								
0.6	**0.6**	145.558	139.371	138.249	137.117	131.617	134.935	179.567
		(.39133)	(.36659)	(.36216)	(.35771)	(.33635)	(.34919)	(.53655)
0.6	**0.4**	91.5101	87.7093	86.9890	86.2683	82.8370	85.5857	117.224
		(.19467)	(.18263)	(.18037)	(.17813)	(.16757)	(.17601)	(.28258)
0.4	**0.4**	58.1791	55.7922	55.3400	54.9019	52.7828	54.7642	77.3416
		(.09837)	(.09235)	(.09122)	(.09013)	(.08493)	(.08979)	(.15111)
0.4	**0.2**	25.5653	24.5813	24.3894	24.2103	23.3178	24.1731	34.1936
		(.02833)	(.02669)	(.02638)	(.02608)	(.02463)	(.02602)	(.04405)
$[\rho = 0.4]$								
0.6	**0.6**	118.665	113.662	112.743	111.861	107.399	108.957	141.525
		(.28783)	(.26977)	(.26649)	(.26336)	(.24772)	(.25314)	(.37514)
0.6	**0.4**	75.1611	72.0163	71.3924	70.8200	67.9766	69.9328	94.7024
		(.14473)	(.13570)	(.13394)	(.13232)	(.12440)	(.12983)	(.20498)
0.4	**0.4**	48.1467	46.1807	45.8005	45.4351	43.6725	45.2577	63.3346
		(.07392)	(.06941)	(.06855)	(.06772)	(.06379)	(.06732)	(.11181)
0.4	**0.2**	20.7759	20.5384	20.4606	19.7248	19.5790	20.2724	28.4625
		(.02066)	(.02030)	(.02018)	(.01909)	(.01887)	(.01990)	(.03335)

Table 3.6 ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 50$, $n = 5$.

$m = 50$			combined chart					two-sided chart	
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
$[\rho = 0]$									
1.25	1.25		191.885 (.59281)	213.172 (.69432)	217.855 (.71736)	223.072 (.74332)	256.417 (.91634)	443.321 (2.0848)	279.572 (1.0434)
1.35	1.35		124.714 (.31018)	140.451 (.37087)	144.250 (.38605)	148.355 (.40269)	175.283 (.51743)	435.303 (2.0285)	217.540 (.71580)
1.5	1.5		66.5779 (.12056)	75.1241 (.14463)	77.1900 (.15066)	79.3878 (.15717)	94.4118 (.20404)	369.891 (1.5886)	136.712 (.35612)
1.75	**1.75**		28.1902 (.03287)	31.3760 (.03867)	32.1508 (.04012)	32.9747 (.04169)	38.5342 (.05279)	225.385 (.75493)	62.7544 (.11027)
2	2		14.8217 (.01232)	16.2408 (.01418)	16.5785 (.01463)	16.9406 (.01512)	19.3620 (.01855)	118.166 (.28601)	31.7730 (.03941)
2.25	**2.25**		9.10594 (.00580)	9.37756 (.00607)	9.93657 (.00664)	10.2127 (.00693)	11.7978 (.00867)	63.6343 (.11261)	18.2903 (.01701)
2.5	2.5		6.26270 (.00321)	6.70090 (.00358)	6.80442 (.00367)	6.91518 (.00376)	7.63516 (.00440)	36.6982 (.04903)	11.7054 (.00856)
2.75	2.75		4.67360 (.00200)	4.95547 (.00220)	5.02233 (.00225)	5.09270 (.00230)	5.55313 (.00265)	23.00390 (.02413)	8.16346 (.00489)
3	3		3.69135 (.00135)	3.88586 (.00148)	3.93135 (.00151)	3.97969 (.00154)	4.29254 (.00174)	15.5088 (.01321)	6.07394 (.00306)
0.9	0.9		347.826 (1.4485)	338.306 (1.3893)	336.225 (1.3765)	334.236 (1.3643)	324.259 (1.3036)	317.450 (1.2627)	354.202 (1.4885)
0.8	0.8		284.321 (1.0701)	272.740 (1.0053)	270.482 (.99286)	268.428 (.98156)	258.371 (.92685)	257.122 (.92013)	310.704 (1.2227)
0.7	0.7		214.767 (.70214)	205.749 (.65831)	204.102 (.65042)	202.476 (.64265)	194.416 (.60459)	197.338 (.61830)	253.415 (.90027)
0.6	**0.6**		155.110 (.43057)	148.569 (.40356)	147.314 (.39845)	146.095 (.39350)	140.183 (.36980)	144.116 (.38552)	192.924 (.59763)
0.5	0.5		102.476 (.23083)	98.2391 (.21662)	97.4189 (.21390)	96.6599 (.21140)	92.8311 (.19892)	95.9702 (.20913)	132.515 (.33981)
0.4	**0.4**		60.9355 (.10549)	58.4934 (.09917)	58.0205 (.09797)	57.5654 (.09681)	55.3099 (.09114)	57.3087 (.09616)	80.9415 (.16182)
0.3	0.3		31.2162 (.03837)	30.0031 (.03613)	29.7743 (.03571)	29.5561 (.03532)	28.4740 (.03337)	29.4511 (.03513)	41.9432 (.06001)
0.2	0.2		12.5058 (.00949)	12.0617 (.00897)	11.9776 (.00887)	11.8959 (.00878)	11.4981 (.00833)	11.8566 (.00874)	16.6807 (.01477)
0.1	0.1		3.23682 (.00108)	3.15152 (.00103)	3.13511 (.00102)	3.11970 (.00102)	3.04252 (.00097)	3.11219 (.00101)	4.07814 (.00160)
1	1		371.140 (1.5966)	371.175 (1.5969)	371.230 (1.5972)	371.264 (1.5974)	371.195 (1.5969)	371.278 (1.5975)	371.243 (1.5973)

Table 3.6 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 50$, $n = 5$.

$m = 25$			combined chart					two-sided chart	
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
$[\rho = 0]$									
1.25	1		277.158 (1.0299)	295.849 (1.1359)	299.437 (1.1567)	303.767 (1.1819)	328.240 (1.3277)	405.293 (1.8222)	318.258 (1.2676)
1.75	1		73.3280 (.13945)	81.8046 (.16443)	83.7910 (.17048)	86.0026 (.17730)	100.294 (.22347)	281.369 (1.05348)	120.923 (.29610)
2.25	1		25.2745 (.02784)	27.7197 (.03204)	28.3086 (.03308)	28.9397 (.03421)	33.0475 (.04183)	119.985 (.29266)	43.4631 (.06333)
2.75	1		12.4522 (.00942)	13.4257 (.01058)	13.6568 (.01086)	13.9011 (.01116)	15.5133 (.01322)	52.6348 (.08457)	20.3831 (.02007)
1.25	1.75		57.2764 (.09608)	64.1894 (.11410)	65.8762 (.11865)	67.7025 (.12364)	79.7324 (.15820)	305.087 (1.1896)	112.012 (.26390)
1.75	**2.25**		14.2761 (.01163)	15.6065 (.01334)	15.9270 (.01376)	16.2633 (.01421)	18.5268 (.01734)	106.390 (.24422)	29.8108 (.03578)
2.25	**2.75**		6.20664 (.00317)	6.63506 (.00352)	6.73701 (.00361)	6.84511 (.00370)	7.55161 (.00432)	35.1280 (.04589)	11.4778 (.00831)
2.75	1.25		11.2331 (.00804)	12.1185 (.00904)	12.3292 (.00928)	12.5519 (.00954)	14.0170 (.01131)	55.0549 (.09051)	19.8609 (.01929)
$[\rho = 0.2]$									
1.75	**1.75**		24.6422 (.02679)	27.2082 (.03115)	27.8341 (.03224)	28.5040 (.03343)	32.9069 (.04156)	165.061 (.47275)	50.4163 (.07925)
1.75	**2.25**		13.1347 (.01023)	14.2879 (.01165)	14.5658 (.01200)	14.8592 (.01237)	16.7860 (.01491)	83.1107 (.16840)	25.7950 (.02872)
2.25	**2.25**		8.59901 (.00530)	9.25769 (.00595)	9.41647 (.00611)	9.58298 (.00628)	10.6802 (.00743)	51.0732 (.08081)	16.3612 (.01434)
2.25	**2.75**		5.97932 (.00298)	6.37322 (.00330)	6.46699 (.00338)	6.56532 (.00346)	7.20754 (.00402)	29.9202 (.03598)	10.6341 (.00738)
$[\rho = 0.4]$									
1.75	**1.75**		18.0225 (.01663)	19.6303 (.01895)	20.0160 (.01952)	20.4234 (.02013)	23.1171 (.02431)	86.6559 (.17933)	31.1987 (.03834)
1.75	**2.25**		11.2814 (.00809)	11.5589 (.00840)	11.6541 (.00851)	12.8700 (.00991)	13.1932 (.01030)	48.3187 (.07432)	18.1926 (.01687)
2.25	**2.25**		7.36170 (.00415)	7.85470 (.00460)	7.97130 (.00471)	8.09509 (.00482)	8.89838 (.00559)	31.7279 (.03933)	12.4124 (.00938)
2.25	**2.75**		5.39056 (.00253)	5.70325 (.00277)	5.77640 (.00282)	5.85366 (.00288)	6.35850 (.00329)	20.6197 (.02042)	8.72164 (.00542)

Table 3.6 (continuted) ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts when $\mathbf{\Sigma}_0$ is unknown and $p = 2$, $m = 50$, $n = 5$.

$m = 25$			combined chart					two-sided chart	
Δ_1	Δ_2	α_I α_D	.000515 .002185	.000415 .002285	.000395 .002305	.000375 .002325	.000275 .002425	Two-sided LRT chart	Modified-LRT chart
$[\rho = 0]$									
0.8	1		335.818	327.070	325.272	323.803	315.070	309.645	339.368
			(1.3740)	(1.3206)	(1.3097)	(1.3009)	(1.2485)	(1.2164)	(1.3959)
0.6	1		252.153	243.007	241.394	239.802	231.946	228.652	265.143
			(.89355)	(.84532)	(.83690)	(.82862)	(.78818)	(.77143)	(.96357)
0.4	1		156.785	150.848	149.651	148.575	142.990	141.156	168.370
			(.43758)	(.41290)	(.40799)	(.40359)	(.38099)	(.37367)	(.48707)
0.2	1		64.3318	61.7692	61.2882	60.8223	58.5175	57.6449	69.6282
			(.11448)	(.10767)	(.10641)	(.10519)	(.09924)	(.09701)	(.12898)
0.8	0.6		208.705	200.200	198.622	197.007	189.138	191.518	243.873
			(.67258)	(.63182)	(.62435)	(.61674)	(.58010)	(.59110)	(.84984)
0.6	**0.4**		96.6660	92.6866	91.9088	91.1569	87.5507	90.4049	124.563
			(.21142)	(.19845)	(.19595)	(.19354)	(.18213)	(.19114)	(.30961)
0.4	**0.2**		26.7261	25.7049	25.5070	25.3212	24.3906	25.2238	35.6460
			(.03031)	(.02857)	(.02824)	(.02792)	(.02638)	(.02776)	(.04692)
0.2	0.8		53.5835	51.3757	50.9585	50.5575	48.5995	49.5200	65.2633
			(.08688)	(.08154)	(.08054)	(.07958)	(.07498)	(.07713)	(.11699)
$[\rho = 0.2]$									
0.6	**0.6**		144.985	138.871	137.707	136.630	131.160	134.355	178.597
			(.38902)	(.36461)	(.36003)	(.35580)	(.33460)	(.34693)	(.53220)
0.6	**0.4**		90.6438	86.8236	86.1319	85.4387	82.0466	84.6095	115.692
			(.19190)	(.17986)	(.17770)	(.17555)	(.16516)	(.17299)	(.27705)
0.4	**0.4**		57.4389	55.1326	54.6920	54.2881	52.2017	54.0254	75.9250
			(.09649)	(.09070)	(.08961)	(.08861)	(.08352)	(.08797)	(.14695)
0.4	**0.2**		25.2759	24.3102	24.1246	23.9475	23.0814	23.8541	33.6628
			(.02785)	(.02625)	(.02594)	(.02565)	(.02425)	(.02550)	(.04302)
$[\rho = 0.4]$									
0.6	**0.6**		118.970	114.091	113.103	112.243	107.843	109.311	141.239
			(.28894)	(.27130)	(.26777)	(.26472)	(.24926)	(.25438)	(.37400)
0.6	**0.4**		74.5590	71.4429	70.8564	70.2943	67.5185	69.3373	93.7106
			(.14299)	(.13408)	(.13242)	(.13084)	(.12313)	(.12817)	(.20176)
0.4	**0.4**		48.1467	46.1807	45.8005	45.4351	43.6725	45.2577	63.3346
			(.07392)	(.06941)	(.06855)	(.06772)	(.06379)	(.06732)	(.11181)
0.4	**0.2**		20.6114	20.3837	20.3060	19.5704	19.4368	20.0780	28.0969
			(.02041)	(.02007)	(.01995)	(.01886)	(.01866)	(.01961)	(.03270)

Chapter 4

Methods for Identifying Influential Univariate Variables in Multivariate Process Control

4.1 Background

To test the hypothesis about the mean vector of a multivariate normal process (population), Hotelling (1947) proposed a T^2statistic, which has received increasing attention in the area of SPC. Consider a process with $p$quality characteristics (variables) in $X = (X_1, \ldots, X_p)'$, which are assumed to jointly follow a multivariate normal distribution with unknown mean vector μ and covariance matrix Σ (denoted by $X \sim N_p(\mu, \Sigma)$). The process is said to be in-control in mean (or simply in-control) at a given time if the hypothesis $H_0 : \mu = \mu_0$ can not be rejected based on a random sample taken from the process at that time, where $\mu_0 \equiv (\mu_{01}, \ldots, \mu_{0p})'$ is a constant vector representing the in-control mean. On the other hand, μ can shift (drift) to a value other than μ_0 at an unknown time, and the main purpose of SPC is to detect this shift as soon as possible.

Since in SPC practice, at least initially (in Phase I), we are concerned with the stability of the mean vector (not the capability of the process), μ_0 is typically not specified so we need some independent reference (training) samples, say $X_i^{(R)} = (X_{i1}^{(R)}, \ldots, X_{ip}^{(R)})'$, $i = 1, \ldots, N$, from an in-control process to estimate the unspecified in-control mean and the unknown covariance matrix by $\bar{X} = \sum_{i=1}^{N} X_i^{(R)}/N$ and $S = \sum_{i=1}^{N}(X_i^{(R)} - \bar{X}^{(R)})(X_i^{(R)} - \bar{X}^{(R)})'/(N-1)$, respectively. Now, to monitor the process mean at time t, we take an observation $X_t = (X_{t1}, \ldots, X_{tp})'$ from the process. Then the following statistic is commonly used (cf. Anderson (2003) and Hotelling (1947)):

$$T^2 = \frac{N}{N+1}(X_t - \bar{X})'S^{-1}(X_t - \bar{X}). \tag{4.1.1}$$

Note that the T^2 statistic in (4.1.1) is slightly different from that of Hotelling (1947). The latter assumes μ_0 is known and all (training and current) data are assumed to have the same mean vector μ. The Hotelling's T^2 is the likelihood-ratio test statistic for testing H_0 and, if $\mu_0 = 0$, it is the uniformly most powerful, invariant test (under scalar transformation) among all tests of population (process) mean vector that are based on sample mean vector and sample covariance matrix (Anderson, 2003, p. 192). Hence we expect the test statistic in (4.1.1) will perform well under the current SPC setting.

One drawback of a T^2statistic, however, is it does not provide *direct* information as to which individual variable(s) may be responsible for the out of control condition (rejection of H_0) when an out-of-control condition is signaled (by a significant T^2 value). An approach to overcoming this difficulty is to use a system of individual, univariate Shewart $\bar{X}$charts (by testing the individual univariate means separately) based on the Bonforrni inequality. This approach, however, ignores the covariance structure and may give an actual overall type-I risk much smaller than the "desirable" risk level α. In SPC terminology, this means that the actual in-control average run length (ARL) can be much larger than the intended value, making the out-of-control ARL smaller than necessary. There are many other techniques and they will be discussed in detail in Section 4.2 when we conduct a literature review.

In this chapter we will propose a method for identifying a variable or a group of variables that is mostly likely to be responsible for the out-of-control condition when $H_0 : \mu = \mu_0$ is rejected because of a significant T^2 value. Our method will compute the conditional likelihood that an individual mean (or a group of means) is out-of-control, given $H_0 : \mu = \mu_0$ is rejected. By ranking these likelihoods, we will identify the mean that is mostly likely to be out-of-control. We use the words "most likely" because T^2 is an aggregate statistic and hence a significant T^2 value does not necessarily imply any out-of-control individual means.

Our method, which is a diagnostic tool, is different from most of the existing methods such as Mason, Tracy, and Young (1995), because their methods did not assume the rejection of $H_0 : \mu = \mu_0$ as given and this is why their critical values for their statistics are all based on certain central (instead of noncentral) distributions. That is, these methods basically assume that all the other individual means are in-control when they are testing a particular univariate mean. Nevertheless, our method should be considered as a complementary tool, not a substitution, to the existing methods.

This chapter is organized as follows. Section 4.2 gives a brief review of the current literature. We describe our methods in detail in Section 4.3, while Section 4.4 provides an alternative derivation of the distributions of the components in the decomposition of the overall T^2 statistic that we gave in the previous section. And, in Section 4, we also give a note for the T^2 decomposition method of Mason, Tracy, and Young (1995). We illustrate our methods in Section 4.5 using examples taken from earlier literatures and show that our method may produce results different from others. Section 4.6 gives a more extensive numerical study and provides some comparisons with the existing methods. Concluding remarks are given in Section 4.7.

4.2 Literature Review

Univariate control charts are often used for monitoring individual process parameters. For example, one may monitor mean shifts with an$\bar{X}$chart and variation change with an R chart, as both procedures are capable of detecting deviations from the historical baseline. When a univariate control chart gives an out-of-control signal, one can easily determine the problem and give a solution for that variable. In a multivariate control chart the problem is more complicated because the aggregate statistic is related to the measurements of p correlated variables. In this section, we review several methods for detecting which of the p variables is out of control when the aggregate statistic gives an overall out-of-control signal.

4.2.1 Univariate Control Charts based on Bonferroni Inequality

Checking p univariate control charts separately gives us the first evidence as to which of the p variables is responsible for an out-of-control signal. Alt (1985) proposed the following control limits based on the Bonferroni inequality: $UCL = \mu_i + z_{1-\alpha/(2p)}\sigma_i/\sqrt{n}$ and $LCL = \mu_i - z_{1-\alpha/(2p)}\sigma_i/\sqrt{n}$, for the ith variable. Experience shows that the actual overall Type-I risk is much smaller than the intended value of α.

Hayter and Tsui (1994) improved the Bonferroni-type control limits by giving a procedure for constructing exact simultaneous control limits for each of the variable means so that the procedure will maintain a desired value of α. The control procedure operates as follows. For a *known* variance-covariance matrix Σ and a pre-determined Type I risk, α, one will first evaluate the critical point $C_{R,\alpha}$, where R is the correlation matrix generated from Σ. Then, given any observation $x = (x_1, \ldots, x_p)'$, one constructs confidence intervals $(x_i - \sigma_i C_{R,\alpha},\ \ x_i + \sigma_i C_{R,\alpha})$ for each of the p variables. The process is considered to be in-control as long as each of these confidence intervals contains the respective standard value μ_{0i}; otherwise, the process is considered to be out-of-control. The variable or variables whose confidence intervals do not contain μ_{0i}, are identified as responsible for the out-of-control signal. That is, this procedure signals when $M = \max_{1 \le i \le p} |x_i - \mu_{0i}|/\sigma_i > C_{R,\alpha}$.

4.2.2 Methods Based on T^2 Decomposition

To improve on the interpretability of T^2, some methodologies are based on decompositions of T^2, as follows.

4.2.2.1 Roy's Step-Down Procedure

The step-down reasoning for solving multivariate hypothesis-testing problems was formally initiated by Roy and Bargmann (1958) in the context of testing multiple independence. Roy (1958) extended it to the problem of testing a multivariate general linear hypothesis, which includes Hotelling's problem as a special case. The step-down solutions assume an a priori ordering of the variables and involve a sequence of tests of significance. The step-down procedure proposed by Roy (1958) for the MANOVA problem, when applied to test $H_0 : \mu = 0$, consists of a sequence of tests based on statistics

$$F_i = \frac{T_i^2 - T_{i-1}^2}{1 + T_{i-1}^2/(n-1)}, \quad i = 1,\ 2,\ \ldots,\ p,$$

where $T_0^2 = 0$, and T_i^2 denotes Hotelling's T^2 statistic for testing $(\mu_1, \ldots, \mu_i)' = 0$ for the first i variables, and is also called the unconditional T^2. The null hypothesis H_0 is rejected when a component test in the sequence shows significance. It is well-known (Roy, Gnanadesikan, and Srivastava (1971), p. 473) that under H_0, the F_i's are independently distributed according to F distribution with degrees of freedom 1 and $n-i$. Consequently the size α of the overall procedure is related to the levels α_i of the component tests by $1 - \alpha = \prod_{i=1}^{p} (1 - \alpha_i)$. Subbaiah and Mudholkar (1978) studied the power functions of the two methods, namely, Hotelling's T^2 and Roy's step-down procedure.

In the setting of a multivariate control chart, F_i would be the charting statistic and this procedure can be considered as an alternative to the regular T^2 chart, not only as supplement.

4.2.2.2 Murphy's Out-of-Control Variable Selection Method

Murphy (1987) proposed a procedure for selecting out-of-control variables and provided an interpretation of T^2 values, assuming that the covariance matrix is known. Murphy's method is based on a discriminant analysis. Given that a particular $\bar{x}^*$ signals an out-of-control condition on a T^2chart, i.e., $T^2(\bar{x}^*) > K$, one partitions $\bar{x}^*$ as $\bar{x}^* = (\bar{x}^{(1)*},\ \bar{x}^{(2)*})$ where $\bar{x}^{(1)*}$ contains the p_1 subset of the $p$variables, which are suspected to have caused the signal, and $\bar{x}^{(2)*}$ contains the remaining p_2 variables $(p_1 + p_2 = p)$.

Define the full squared Mahalanobis distance of $\bar{x}^*$ from the in-control mean as $T_p^2 = T^2(\bar{x}^*) = n(\bar{x}^* - \mu_0)'\Sigma^{-1}(\bar{x}^* - \mu_0)$ and the corresponding distance for the p_1 subset: $T_{p1}^2 = T^2(\bar{x}^{*(1)}) = n(\bar{x}^{*(1)} - \mu_0^{(1)})'\Sigma_{11}^{-1}(\bar{x}^{*(1)} - \mu_0^{(1)})$. Also define the expected full squared distance as $\Delta_p^2 = n(\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0)$ and the expected reduced squared distance as $\Delta_{p1}^2 = n(\mu^{(1)} - \mu_0^{(1)})'\Sigma^{-1}(\mu^{(1)} - \mu_0^{(1)})$. Then, to test $H_0 : \Delta_p^2 - \Delta_{p1}^2 = 0$, which is equivalent to testing that the p_1 subset discriminates just as well as the full set of $p$variables, the difference $D = T_p^2 - T_{p1}^2$ can be used as a test statistic. If D is large and then we reject the hypothesis that the p_1 subset had caused the signal. To determine the critical value, it is proved that, under the null hypothesis, D follows a Chi-square distribution with p_2 degrees of freedom.

4.2.2.3 Doganaksoy, Faltin and Tucker's Out-of-Control Variable Selection Method

Doganaksoy, Faltin, and Tucker (1991) proposed ranking the $p$univariate t statistics and computing the probability P of accepting the hypothesis. If the P value for any variable is less than the Bonferroni's risk, α'/p, we may conclude that the mean of that variable has changed. This method is based on use of the p unconditional T^2 terms. The diagnostic approach is triggered by an out-of-control signal from a T^2-Chart.

4.2.2.4 Interpretation Algorithm of Timm

Timm (1996) proposed a step-down procedure of using Finite Intersections Tests (*FIT*) for the case where the variance-covariance matrix Σ is assumed known. It also assumes that there is an a priori ordering among the means of the p variables and it sequentially tests subsets using this ordering to determine the sequence. Timm (1996) stated that when shifts occur for some subset of variables, the optimal procedure is to utilize a finite intersection test.

A process is in-control if hypotheses $H_i : \mu_i = \mu_{0i},\ i = 1, 2, \ldots, p$, or equivalently the intersection of the H_i's, $H_0 : \bigcap_{i=1}^{p} H_i$, is true, where μ_i is the mean of the ith variable. To test H_0, one can use the FIT procedure. To construct a FIT of H_0, one defines a likelihood ratio test statistic for each elementary hypothesis H_i and determine the joint distribution of the test statistics. Let the test statistics be defined as

$$z_i^2 = \frac{(x_i - \mu_{0i})^2}{\sigma_{ii}}, \quad i = 1,\ 2,\ \ldots,\ p.$$

Krishnaiah and Rao (1961) show that the joint distribution of $z_1^2, z_2^2, \ldots, z_p^2$ follows a noncentral p-dimensional χ^2 distribution with 1 degree of freedom. Consider that the known variance-covariance matrix $\Sigma = \sigma^2\Omega$, where $\Omega = (\rho_{ij})$ is the correlation matrix of the accompanying multivariate normal. Note that $\chi_p^2(1, \Omega; 1-\alpha)$ is the $1-\alpha$ percentage value of the p-dimensional χ^2 distribution with 1 degree of freedom and the accompanying correlation matrix, Ω. A multivariate quality control process is in-control if $z_i^2 < \chi_p^2(1, \Omega; 1-\alpha)$, where $P(z_i^2 < \chi_p^2(1, \Omega; 1-\alpha),\ i = 1, 2, \ldots, p|H_0) = 1-\alpha$. The process is out-of-control if $\max\limits_{1\leq i\leq p} z_i^2 > \chi_p^2(1, \Omega; 1-\alpha)$, where $P(\max\limits_{1\leq i\leq p} z_i^2 < \chi_p^2(1, \Omega; 1-\alpha)|H_0) = \alpha$. Since a non-central p-dimensional χ^2 distribution with 1 degree of freedom is a special case of the multivariate t distribution with infinite degrees of freedom, the process may alternatively be judged as out-of-control if $P(\max\limits_{1\leq i\leq p}|T_i| = |x_i - \mu_{0i}| \big/ \sqrt{\sigma_{ii}} > T_{p,1-\alpha}^2) = \alpha$. Hence, $P(|T_i| \leq T_{p,1-\alpha}^2,\ i = 1, 2, \ldots, p|H_0) = 1-\alpha$.

The $1-\alpha$ level simultaneous confidence sets are easily established for each variable

$$x_i - T_{p,1-\alpha}^2\sqrt{\sigma_{ii}} \leq \mu_i \leq x_i + T_{p,1-\alpha}^2\sqrt{\sigma_{ii}}.$$

The process is said to be out-of-control if at least one confidence interval does not contain its in-control mean. Timm (1996) also gave a step down *FIT* procedure for the case that the variance-covariance matrix Σ is unknown.

4.2.2.5 Mason, Tracy and Young's (MTY) Decomposition Method

Mason, Tracy, and Young (1995, 1996, 1997) presented an approach in which the Hotelling's T^2 statistic is decomposed into p orthogonal components. One form of the MTY decomposition can be expressed as (Rencher (1993))

$$T^2 = T_1^2 + T_{2.1}^2 + T_{3.1,2}^2 + \cdots + T_{p.1,2,\ldots,p-1}^2 = T_1^2 + \sum_{j=1}^{p-1} T_{j\cdot 1,2,\ldots,j-1}^2.$$

The first term T_1^2 is an unconditional Hotelling's T^2 for the first variable of the observation vector$x$. The rest are referred as conditional terms. The ordering of the p components is not unique and there are $p!$ possibilities of decompositions of T^2. Each ordering generates the same overall T^2 value, but provides a distinct partitioning of T^2 into p orthogonal, conditional terms. If we exclude duplications, there are $p \times 2^{p-1}$ distinct components among the $p \times p!$ possible terms that should be evaluated for the potential contribution of the p variables to signal.

An unconditional term has a same statistic of a univariate Shewhart control chat. It calculates the squared standardized deviation of the observed value from the in-control mean value for the jth variable. A signal will occur if this value is too far away from the in-control mean. The p unconditional, univariate T^2 terms based on squaring a univariate t statistic can be computed and then compared to the appropriate F distribution. Specifically, under the hypothesis $H_0 : \mu_j = \mu_{0j}$, T_j^2 will follow an F distribution which can be used to determine the upper control limit; i.e.,

$$T_j^2 = \frac{n}{n+1} \frac{(x_j - \bar{x}_j)^2}{s_j^2} \sim F_{1,\, n-1}, \quad j = 1, \ldots, p. \tag{4.2.1}$$

where $\bar{x}_j$ and s_j are respectively the mean and standard deviation of variableX_j, obtained from the training data.

The jth conditional term $T^2_{j\cdot 1,\,\ldots,\,j-1}$ is a standardized observation for the$j$thvariable adjusted by estimates of the mean and variance from the conditional distribution associated with $x_{j\cdot 1,\,\ldots,\,j-1}$. The most important function of the conditional term is that it determines whether the jth variable is consistent with the relationship pattern with other variables established from historical in-control data.

The general form of the conditional terms is given as

$$T^2_{j\cdot 1,\,\ldots,\,j-1} = \frac{n}{n+1}\frac{(x_j - \bar{x}_{j\cdot 1,\,\ldots,\,j-1})^2}{s^2_{j\cdot 1,\,\ldots,\,j-1}}, \quad j = 2, \ldots, p \tag{4.2.2}$$

where

$$\bar{x}_{j\cdot 1,\,\ldots,\,j-1} = \bar{x}_j + b'_j(X^{(j-1)} - \bar{X}^{(j-1)})$$

and $X^{(j-1)}$ is the vector containing the first $j \; - 1$ variables, $\bar{x}_j$ is the sample mean of the$j$th variable, b_j is a $(j-1)$-dimensional vector estimating the regression coefficients of the jth variable regressed on the first $(j-1)$ variables. In addition, it can be shown that the numerator of F_i, the statistic based on Roy's step down procedure, is a conditional T^2 value:

$$T^2_i - T^2_{i-1} = T^2_{i\cdot 1,\,2,\,\ldots,\,i-1}.$$

Consequently, the $T^2_{j\cdot 1,\ldots,j-1}$ value is the squared deviation of the value on$j$thvariable from the conditional in-control mean, which is the mean of the conditional distribution of X_j given the values of X_1,$X_2, \ldots, X_{j-1}$. The exact distribution is

$$T^2_{j\cdot 1,\ldots,j-1} \sim \frac{(n-1)}{(n-k-1)} F_{1,\,n-k-1}, \; where \; k = j-1. \tag{4.2.3}$$

Thus, this statistic can also be used to determine whether the jth variable is consistent with the relationship pattern with other variables established from the historical in-control data in the training data set. Moreover, the distances $D_i = T^2 - T^2_i$ can be computed and compared to the F distribution.

$T^2_{j\cdot 1,2,\ldots,j-1}$ can be re-expressed as (Mason, Tracy, and Young (1997)),

$$T^2_{j\cdot 1,2,\ldots,j-1} = \frac{(x_j - \bar{x}_{j\cdot 1,2,\ldots,j-1})^2}{s_j^2(1 - R^2_{j\cdot 1,2,\ldots,j-1})}.$$

The numerator is the squared residual between the observed and the predicted value from regressing x_j on the variables $x_1, \ldots, x_{j-1}$. $R^2_{j\cdot 1,\ldots,j-1}$ is the squared multiple correlation coefficient between x_j and $x_1, \ldots, x_{j-1}$. From the above equation, it is noted that the conditional T^2 will become large if x_j is significantly far from what is predicted value, unless the $R^2_{j.1,2,\ldots,j-1}$ is close to 1.

The following is a sequential computational scheme that has the potential of further reducing the computations to a reasonable number when the overall T^2 signals, as proposed by Mason, Tracy, and Young (1997).

Step 0: Conduct a T^2 test with a specified nominal confidence level α. If an out-of-control condition is signaled then continue with the step 1; otherwise move to the next period.

Step 1: Compute the individual T^2 statistic for every component of the X vector. Remove variables whose observations produce a significant T_i^2. The observations on these variables are out of individual control and it is not necessary to check how they relate to the other observed variables. With significant variables removed we have a reduced set of variables. Check the subvector of the remaining k variables for a signal. If you do not receive one, we have located the source of the problem.

Step 2: (Optional) Examine the correlation structure of the reduced set of variables. Remove any variable having a very weak correlation (0.3 or less) with all the other variables. The contribution of a variable that falls in this category is measured by the T_i^2 component.

Step 3: If a signal remains in the subvector of k variables not deleted, compute all $T^2_{i\cdot j}$ terms. Remove from the study all pairs of variables, (X_i, X_j), that have a significant $T^2_{i\cdot j}$

term. This indicates that something is wrong with the bivariate relationship. When this occurs it will further reduce the set of variables under consideration. Examine all removed variables for the cause of the signal. Compute the T^2 terms for the remaining subvector. If no signal is present, the source of the problem is with the bivariate relationships and those variables were out of individual control.

Step 4: If the subvector of the remaining variables still contains a signal, compute all $T^2_{i \cdot j,k}$ terms. Remove any triple, (X_i, X_j, X_k), of variables that show significant results and check the remaining subvector for a signal.

Step 5: Continue computing the higher order terms in this fashion until there are no variables left in the reduced set. The worst case situation is that all unique terms will have to be computed.

Generally, the T^2 statistic associated with an observation from a multivariate problem is a function of the residuals taken from a set of linear regressions among the various process variables. These residuals are contained in the conditional T^2 terms of the orthogonal decomposition of the statistic. Mason and Young (1999) showed that a large residual in one of these fitted models can be due to incorrect model specification. By improving the model specification at the time when the historical data set is constructed, it may be possible to increase the sensitivity of the T^2 statistic to signal detection. Also, they showed that the resulting regression residuals can be used to improve the sensitivity of the T^2 statistic to small but consistent process shifts, using plots that are similar to cause-selecting charts.

The productivity of an industrial processing unit often depends on equipment that changes over time. These changes may not be stable, and, in many cases, may appear to occur in stages. Although changes in the process levels within each stage may appear insignificant, they can be substantial when monitored across the various stages. Standard process control procedures do not perform well in the presence of these step-like changes, especially when the observations from stage to stage are correlated. Mason, Tracy, and Young (1996) present an alternative control procedure for monitoring a process under these conditions, which is based on a double decomposition of Hotelling's T^2 statistic.

4.2.3 Methods Based on Principal Components

Another diagnostic approach is based on principal components analysis (*PCA*) method. The most noticeable advantages of *PCA* is it can provide the most important principal directions of variability in data and its ability to transform the original set of correlated variables into a new set of uncorrelated variables. These principal components are linear combinations of the original variables and thus the dimensionality can often be reduced. Once a *PCA* model is constructed based on an in-control historical data set, the original variables are considered simultaneously and the relation between variables are also captured. A new observation can be detected as out-of-control if it significantly deviates from the (in-control) *PCA* model. The principal components do not provide an overall measure of departure of the multivariate data from the norms. It can be difficult to assign a meaningful interpretation to these principal components.

Jackson (1991) recommended using the *PCA* for improving on the interpretation of T^2. When given an out-of-control signal in a T^2 control chart, the most common practice is to use the first k most significant principal components for further investigation. The basic assumption is that the first k principal components can be physically interpreted, and named. Consequently, if the T^2 chart gives an out-of-control signal and, for example, the second principal component chart also gives an out-of-control signal, then from the interpretation of this component, a direction to the variables which are suspect to be out-of-control can be figured. The problem is that the principal components do not always have a physical interpretation.

Kourti and MacGregor (1996) provided a different approach based on *PCA* and partial least squares. The T^2 statistic is expressed in terms of normalized principal components scores of the multinormal variables. When an out-of-control signal occurs, contribution plots (Wasterhuis, Gurden, and Smilde (2000)) can be constructed for the normalized scores with high values to find the variables responsible for the signal. A contribution plot indicates how each variable involved in the calculation of that score contributes to it. The computation of variable contributions showed that principal components can actually have physical interpretation. This approach is particularly applicable to large ill-conditioned data sets.

Maravelakis, Bersimis, Panaretos, and Psarakis (2002) proposed a new method based on PCA to identify the variable or variables that are responsible when an out-of-control signal in the χ^2 control chart. Theoretical control limits were derived and a detailed investigation of the properties and the limitations of the new method were given. They considered two different cases– one where the covariance matrix has only positive correlations and the second with both positive and negative correlations–and also provided their algorithms. Furthermore, a graphical technique which can be applied in some of these limiting situations was provided.

4.2.4 Hawkins's Regression-Adjustment Method

Hawkins (1991, 1993) discussed the multivariate quality control based on regression-adjusted variables for individual variables to help identify the single variable that is responsible for a shift in mean. It is a method that uses the vector of scaled residuals from the regression of each variable on all others. The purpose of this technique is to make the controls more effective than those based on individual variables. He defined a set of regression-adjusted variables, Z, and its ith element Z_i, which is the residual when X_i is regressed on all other components of X . Apart from their good performance characteristics, each Z_i has the desirable property of being on the same scale asX_i. He proposed to chart a CUSUM statistic using data onZ_i. Hence, $p$separate cumulative Z-charts can be used to monitor the individual means. The ith chart tests whether the variable i conforms to its in-control conditional distribution given the other variables. His test statistic involved the p-adjusted values, which can be shown to be related to the statistics presented in Mason, Tracy, and Young (1995) decomposition. This method is capable of separating shifts in the mean from shifts in the covariance structure and makes signal identification a less involved task. Hawkins pointed out that his method is suited mainly for cases in which only one or two of the quality variables may shift out of control at any time.

4.2.5 The Cause-Selection Chart and The Minimax Control Chart

The cause-selecting chart (CSC), proposed by Zhang (1980, 1984, 1992), is a different approach to solving the problem of interpreting an out-of-control signal in the T^2chart. The idea of the CSC is similar to the regression control chart of Mandel (1969) in terms of charting a variable after the observations have been adjusted for the effect of an outside covariate. This idea has also been used by Hawkins (1991, 1993). Wade and Woodall (1993) proposed a modification of the CSC chart to investigate the relationship between cause-selection control and the multivariate T^2 chart for the diagnostic purpose. A CSC charting the outputs adjusted by the effect of input can effectively distinguish between incoming quality problems and operation problems in the current stage. The construction of a CSC involves a two-step procedure based on a simple linear regression model. Once the CSC is set up, it can be used to indicate whether the current process stage is out of control. The cause-selecting value measures the specific quality at the current process step. As a result, an out-of-control signal on the CSC indicates that the current process step is out of control.

Sepulveda (1996) developed the Minimax control chart that can give evidence about which variable cause the out-of-control signal. The Minimax control chart was again discussed in Sepulveda and Nachlas (1997) and is similar to the charts proposed by Hater and Tsui (1994) and Timm (1996). The basic idea of the Minimax control chart was based on monitoring the maximum standardized sample mean ($Z_{[p]}$) and the minimum standardized sample mean ($Z_{[1]}$) of samples, taken from a multivariate process, to determine if the process should be considered in control or out of control. It is assumed that the data are normally distributed and that the variance-covariance matrix is known and constant over time. Therefore, by monitoring the maximum and the minimum standardized sample mean, an out-of-control signal is directly connected with the corresponding out-of-control variable. Sepuldveda and Nachlas (1997) also discussed the statistical properties and the ARL performance of the Minimax control chart.

4.2.6 Dummy Variable Multiple Regression and Neural Network

Kalagonda and Kulkarni (2003) proposed a diagnostic procedure called 'D-technique,' using the dummy variable multiple regression equation. Use of this technique leads to the identification of the causes such as the mean shift and/or relationship shift. Also, their proposed method can indicate the direction of the shift, that is, whether the mean is increased or decreased.

Aparisi, Avendano, and Sanz(2006) presented a method, using a neural network approach, for indicating which variable or group of variables had caused the problem when an out-of-control signal occurs. They evaluated the effectiveness of the proposed approach in terms of the correct classification percentage of the variables (shift, no shift) and compared their results with those obtained using the MTY decomposition approach. They showed that the results obtained using the neural network method are in general better than those obtained using the decomposition approach.

4.2.7 Graphical Techniques

There are various statistical approaches for identifying the out-of-control variables. An important adjunct to the statistical procedures is a suitable graphical scheme that can display the basic features of the data.

Iglewicz and Hoaglin (1987) used different techniques of displaying boxplots to signal out-of-control situation due to shift in mean/variance and presence of an outlier.

The multivariate profile (MP) chart proposed by Fuchs and Benjamini (1994) is probably one of the best ways for simultaneously displaying univariate and multivariate SPC information. It can simultaneously control a process and interpret an out-of-control signal. An MP chart is a plot of the standardized values of the p process variables using a bar graph with the reference level represented by a T^2statistic. Specifically, every bar graph represents summaries of data for an individual variable, and the location of the bar graph on the plot is used for providing global information about the group of observations. Each group is displayed by one symbol and this symbol of the profile plot enables the user to get a clear view of the size and sign of each variable from its reference value. Therefore, the MP chart can be used to identify easily the cause of a shift because all the individual variables are displayed in a common scale within a group. However, one major disadvantage in using the MP chart is that when p increases, say more than seven, it becomes cumbersome to plot. For the purpose for reducing dimensionality, Fuchs and Benjamini suggested the MP chart for the principal components in higher dimensional SPC's.

Sparks, Adolphson, and Phatak (1997) proposed a graphical method for monitoring multivariate process data based on the Gabriel biplot. In contrast to existing methods that are based on some form of dimension reduction, they use reduction to two dimensions for displaying the state of statistical control. This approach can detect changes in location, variation, and correlation structure accurately and yet display a large amount of information concisely. They illustrated the use of the biplot on an example of industrial data.

Atienza, Tang, and Ang (1998) proposed a graphical procedure for simultaneously monitoring univariate and multivariate process information. The proposed chart, called the multivariate boxplot-T^2 control chart (MBTCC), displays both types of process information using a modified boxplot. It is particularly useful when the monitoring process involves a large number of variables. The MBTCC is not meant to replace the statistical procedures of isolating variables affected by an out-of-control signal in multivariate control charting. It acts as a diagnostic tool for assessing the nature of an out-of-control situation.

However, these graphical methods have several drawbacks including that their operation is tedious and cumbersome due to their graphical nature. The main problem is that this type of approach requires the user to interpret the results. This means that it is a complex task to measure their effectiveness in an objective manner.

4.3 The Proposed Method Based On Likelihood Principle

In this section we will propose a new method for identifying a variable or a group of variables that are most likely to be responsible for an out-of-control signal indicated by a significant T^2 value when testing $H_0 : \mu = \mu_0$.

4.3.1 Variables Likely Responsible for an Overall Out-of-control Condition

First note that sample mean $\bar{X}$ from the training samples follows$N_p(\mu_0, (1/N)\Sigma)$ and sample covariance S independently follows a p-dimensional Wishart distribution with degrees of freedom $N-1$ and covariance matrix $\frac{1}{N-1}\Sigma$ (write S $\sim W_p(N-1, \frac{1}{N-1}\Sigma)$). Then, from Anderson (2003), $\frac{N-p}{p}\frac{T^2}{N-1}$ follows a noncentral $F_{p,N-p,\lambda}$ distribution with noncentrality $\lambda = \frac{N}{N+1}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)$, which becomes central ($\lambda = 0$) under$H_0 : \mu = \mu_0$. A process is out of control (i.e., if H_0 is false) if λ is greater than zero. The hypothesis $H_0 : \mu = \mu_0$ is rejected if the observed T^2 is greater than $\frac{(N-1)p}{N-p}F_{p,N-p,0}(\alpha)$, where $F_{p,N-p,0}(\alpha)$ is the $(1-\alpha)$ percentile of $F_{p,N-p,0}$.

We now partition the mean vector into k subvectors as $\mu = (\mu^{(1)'}, \mu^{(2)'}, ..., \mu^{(k)'})'$, where $\mu^{(j)}$ is a p_j-dimensional subvector of p_j individual means and $\sum_{j=1}^{k} p_j = p$. Let the given vector μ_0 be partitioned similarly, then the null hypothesis can be written as $H_0 : \cap_{j=1}^{k}(H_{j0} : \mu^{(j)} = \mu_0^{(j)})$. In some cases, the mean vector can shift to a certain known value (e.g., by one or two standard deviations). So, we will consider the following two cases - one with (Case (A)) and the other without (Case (B)) a specified alternative hypothesis:

(A). $H_a : \cap_{j=1}^{k}(H_{ja} : \mu_a^{(j)} - \mu_0^{(j)} = \mu_a^{*(j)} - \mu_0^{(j)})$, where $\mu_a = (\mu_a^{(1)'}, \ldots, \mu_a^{(k)'})'$ is the true shifted mean vector and $\mu_a^* = (\mu_a^{*(1)'}, \ldots, \mu_a^{*(k)'})'$ is a specified vector;

(B). H_a : not H_0, or equivalently, $H_a : \cup_{j=1}^{k}(H_{ja} : \mu^{(j)} \neq \mu_0^{(j)})$.

For case (A), the direction and magnitude of the shift in the jth subvector of the mean is denoted by $\Delta_j = \mu_a^{(j)} - \mu_0^{(j)}$ and that we would like to detect is denoted by $\Delta_j^* = \mu_a^{*(j)} - \mu_0^{(j)}$.

Assume at time t , the observed T^2 in (4.1.1), from the current sample $X_t = (X_{t1}, \ldots, X_{tp})'$, had signaled an overall out-of-control condition, i.e., H_0 is rejected because the observed $T^2 = t^2 > \frac{(N-1)p}{N-p} F_{p,N-p,0}(\alpha)$. For simplicity, let's write $t_0^2(\alpha) \equiv \frac{(N-1)p}{N-p} F_{p,N-p,0}(\alpha)$. For case (A), the question is: Which alternative hypothesis,$H_{ja} : \mu^{(j)} = \mu_a^{(j)}$ is mostly likely to be true, given H_0 is rejected. Our method is based on the likelihood principle and will consider the dependence structure among the variables through T^2. We will calculate the conditional (maximum) likelihood, $\ell_j(H_{ja}|H_a)$, of H_{ja} for each j in Case (A) and $\ell'_j(H_{j0}|H_a)$ in Case (B). By ranking these likelihood values, one can identify which variables are most likely to be responsible for the out-of-control condition.

The conditional maximum likelihood of H_{ja} is calculated as follows:

$$\ell_j(H_{ja}|H_a) = \underset{\mu^{(j)} \in H_{ja}}{Max} f_{\mu^{(j)}}(X_t^{(j)} = x_t^{(j)} | T^2 = t^2 > t_0^2(\alpha)), \tag{4.3.1}$$

where $f_{\mu^{(j)}}(X_t^{(j)} = x_t^{(j)} | T^2 = t^2 > t_0^2(\alpha))$ is the conditional pdf of $X_t^{(j)}$ at $x_t^{(j)}$. For Case (A) where H_{ja} is a simple alternative, (4.3.1) reduces to

$$\ell_j(H_{ja}|H_a) = f_{\mu_a^{(j)}}(X_t^{(j)} = x_t^{(j)} | T^2 = t^2 > t_0^2(\alpha)), \quad j = 1, \cdots, k. \tag{4.3.2}$$

Then, we will identify the subvector with the largest $\ell-$value as the most likely mean to be responsible for the overall out-of-control condition.

For Case (B), our question becomes: Which H_{j0} is the least likely to be true, given H_0 is rejected. The conditional likelihood of H_{j0} is similar to that for Case (A) and can be calculated as (since H_{j0} is simple)

$$\ell'_j(H_{j0}|H_a) = f_{\mu_0^{(j)}}(X_t^{(j)} = x_t^{(j)} | T^2 = t^2 > t_0^2(\alpha)) \tag{4.3.3}$$

In this case, we will identify the subvector with the smallest ℓ'-value will as a potentially problematic subvector.

4.3.2 Calculations of the Conditional Likelihood Functions

To calculate the condition pdf's in (4.3.2) and (4.3.3), we will consider only $\ell_1(H_{10}|H_a)$ and $\ell_1'(H_{1a}|H_a)$ for the first subvector$X_t^{(1)}$ (i.e., j =1) because conditional likelihoods for the other subvectors can be computed similarly by rearranging and renaming the variables inX_t. Now partitionX_t, $\bar{X}$, and S as

$$X_t = (X_t^{(1)'}, X_t^{(2)'})', \quad \bar{X} = (\bar{X}^{(1)'}, \bar{X}^{(2)'})' \text{ , and } S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{4.3.4}$$

where $X_t^{(1)}$ and $\bar{X}^{(1)}$ are $p_1 \times 1$ vectors and $X_t^{(2)}$ and $\bar{X}^{(2)}$ are $q_1 \times 1$ vectors containing the remaining elements in X_t and $\bar{X}$, respectively, so $q_1 = p - p_1$. Similarly, μ, μ_0, and Σ are partitioned as follows:

$$\mu = (\mu^{(1)'}, \mu^{(2)'})', \quad \mu_0 = (\mu_0^{(1)'}, \mu_0^{(2)'})', \text{ and } \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}. \tag{4.3.5}$$

Define the partial (i.e., conditional) covariance matrices:

$$\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \text{ , and } S_{22.1} = S_{22} - S_{21}S_{11}^{-1}S_{12}. \tag{4.3.6}$$

To compute $f_{\mu^{(1)}}(X_t^{(1)} = x_t^{(1)}|T^2 = t^2 > t_0^2(\alpha))$ in (4.3.2), first we see that T^2 can be decomposed as

$$\begin{aligned} T^2 &= \frac{N}{N+1}(X_t - \bar{X})'S^{-1}(X_t - \bar{X}) \\ &= \frac{N}{N+1} \cdot (X_t^{(1)} - \bar{X}^{(1)})'S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)}) \\ &+\frac{N}{N+1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})]'S_{22.1}^{-1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})] \\ &= T_1^2 + T_2^2 \text{ , say} \end{aligned} \tag{4.3.7}$$

Theorem 3.3. *(Proof in Appendix B.1)*

(i). T_1^2 follows a noncentral $\frac{(N-1)p_1}{N-p_1}F_{p_1,N-p_1,\lambda_1}$ distribution with noncentrality

$$\lambda_1 = \frac{N}{N+1}(\mu^{(1)} - \mu_0^{(1)})'\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)}). \tag{4.3.8}$$

(ii). The conditional distribution of $T_2^{2(*)} \equiv \frac{T_2^2}{(1+T_1^2/(N-1))}$*, given* $T_1^2 = t_1^2 = \frac{N}{N+1}(x_t^{(1)} - \bar{x}^{(1)})' s_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)})$ *is a noncentral* $\frac{(N-1)q_1}{N-p} F_{q_1,N-p,\lambda_2}$ *distribution with noncentrality*

$$\lambda_2 = \frac{N}{N+1} \frac{[(\mu^{(2)} - \mu_0^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)})]'\Sigma_{22.1}^{-1}[(\mu^{(2)} - \mu_0^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)})]}{(1 + t_1^2/(N-1))}$$

$$= \frac{\lambda - \lambda_1}{(1 + t_1^2/(N-1))}. \tag{4.3.9}$$

Note that the unconditional distribution of T_2^2 can be obtained from (i) and (ii) as $\int\limits_{t_1^2>0} f_{T_2^2|T_1^2}(t|t_1^2) f_{T_1^2}(t_1^2) dt_1^2$.

Now from (i) and (ii) above, for any$\mu = (\mu'^{(1)}, \mu'^{(2)})'$, the conditional distribution of T^2 given$(X_t^{(1)}, \bar{X}^{(1)}, S_{11}) = (x_t^{(1)}, \bar{x}^{(1)}, s_{11})$, is $(1 + \frac{t_1^2}{N-1})\frac{(N-1)q_1}{N-p} F_{q_1,N-p,\lambda_2} + t_1^2$, which depends on $(x_t^{(1)}, \bar{x}^{(1)}, s_{11})$but through $t_1^2 = \frac{N}{N+1}(x_t^{(1)} - \bar{x}^{(1)})' s_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)})$. From the fact that the distribution of $X_t^{(1)}$, $\bar{X}^{(1)}$, and S_{11} are independently $N_{p_1}(\mu^{(1)}, \Sigma_{11})$, $N_{p_1}(\mu_0^{(1)}, \frac{\Sigma_{11}}{N})$, and $W_{p_1}(N-1, \frac{\Sigma_{11}}{N-1})$, respectively, the conditional pdf of $X_t^{(1)}$, given $T^2 = t^2 > t_0^2(\alpha)$ (i.e.,H_0 is rejected), is

$$f_{\mu^{(1)}}(X_t^{(1)} = x_t^{(1)}|t^2) = \frac{f_{(X_t^{(1)},T^2)}(x_t^{(1)}, t^2)}{f_{T^2}(t^2)}$$

$$= \frac{1}{f_{T^2}(t^2)} \int\limits_{s_{11}>0} \int\limits_{\bar{x}^{(1)}} f_{(X_t^{(1)},T^2)|(\bar{X}^{(1)},S_{11})}(x_t^{(1)}, t^2|\bar{x}^{(1)}, s_{11}) f_{\bar{X}^{(1)}}(\bar{x}^{(1)}) f_{S_{11}}(s_{11}) d\bar{x}^{(1)} ds_{11}$$

$$= \frac{f_{X_t^{(1)}}(x_t^{(1)})}{f_{T^2}(t^2)} \cdot \int\limits_{s_{11}>0} \int\limits_{\bar{x}^{(1)}} f_{T^2|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(t^2|x_t^{(1)}, \bar{x}^{(1)}, s_{11}) \cdot f_{\bar{X}^{(1)}}(\bar{x}^{(1)}) \cdot f_{S_{11}}(s_{11}) d\bar{x}^{(1)} ds_{11}.$$

Note that $f_{T^2|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(t^2|x_t^{(1)}, \bar{x}^{(1)}, s_{11})= 0$ if $t_1^2 \equiv \frac{N}{N-1}(x_t^{(1)} - \bar{x}^{(1)})' s_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)}) > t^2$. Furthermore, this conditional distribution is the conditional distribution of $T^2 \equiv T_1^2 + T_2^2$ or $t_1^2 + T_2^2$, and, from the proof of Theorem 1, it depends on $x_t^{(1)}, \bar{x}^{(1)}, s_{11}$ but through t_1^2. Simple transformation give

$$f_{T^2|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(t^2|x_t^{(1)}, \bar{x}^{(1)}, s_{11}) = \quad f_{T_2^2|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(t^2 - t_1^2|x_t^{(1)}, \bar{x}^{(1)}, s_{11})$$

$$= f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2 - t_1^2}{(1 + t_1^2/(N-1))}|x_t^{(1)}, \bar{x}^{(1)}, s_{11}) \frac{1}{(1 + t_1^2/(N-1))},$$

where the conditional distribution of $T_2^{2(*)}$, given $T_1^2 = t_1^2$, can be found in Theorem 1 (*ii*). So we have

$$f_{\mu^{(1)}}(X_t^{(1)} = x_t^{(1)}|t^2) = \frac{f_{X_t^{(1)}}(x_t^{(1)})}{f_{T^2}(t^2)} \int\limits_{s_{11}>0} \int\limits_{\bar{x}^{(1)}} f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|x_t^{(1)},\bar{x}^{(1)},s_{11})$$
$$\times \frac{1}{(1+t_1^2/(N-1))} f_{\bar{X}^{(1)}}(\bar{x}^{(1)}) f_{S_{11}}(s_{11}) d\bar{x}^{(1)} ds_{11}$$
$$= \frac{f_{X_t^{(1)}}(x_t^{(1)})}{f_{T^2}(t^2)} E_{(\bar{X}^{(1)},S_{11})} \left(f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|x_t^{(1)},\bar{x}^{(1)},s_{11}) \frac{1}{(1+t_1^2/(N-1))} \right). \tag{4.3.10}$$

Note that inside the expectation in (4.3.10) $x_t^{(1)}$ is fixed while $\bar{X}^{(1)}$ and S_{11} are random such that $t_1^2 = \frac{N}{N+1}(x_t^{(1)} - \bar{X}^{(1)})' S_{11}^{-1}(x_t^{(1)} - \bar{X}^{(1)})$ and $t_1^2 \leq t^2$. To simplify the computation, the expectation in (4.3.10) can be approximated as follows (proof in Appendix B.2).

$$E_{(\bar{X}^{(1)},S_{11})} \left(f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|x_t^{(1)},\bar{x}^{(1)},s_{11}) \cdot \frac{1}{(1+t_1^2/(N-1))} \right)$$
$$= \iint\limits_{s_{11}>0,\bar{x}^{(1)},t_1^2\leq t^2} f_{asymp}(t^2,t_1^2|x_t^{(1)},\bar{x}^{(1)},s_{11}) \cdot f_{\bar{X}^{(1)}}(\bar{x}^{(1)}) \cdot f_{S_{11}}(s_{11}) d\bar{x}^{(1)} ds_{11}$$
$$= \iint\limits_{s_{11}>0,\bar{x}^{(1)},t_1^2\leq t^2} \left((\frac{1}{N})^0 f_{asymp}^{(0)} + (\frac{1}{N}) f_{asymp}^{(1)} + (\frac{1}{N})^2 f_{asymp}^{(2)} + O_3(\frac{1}{N}) \right) \cdot f_{\bar{X}^{(1)}}(\bar{x}^{(1)}) \cdot f_{S_{11}}(s_{11}) d\bar{x}^{(1)} ds_{11}, \tag{4.3.11}$$

where

$$f_{asymp}(t^2,t_1^2|x_t^{(1)},\bar{x}^{(1)},s_{11}) = (\frac{1}{N})^0 f_{asymp}^{(0)} + (\frac{1}{N}) f_{asymp}^{(1)} + (\frac{1}{N})^2 f_{asymp}^{(2)} + O_3(\frac{1}{N}),$$

$$f_{asymp}^{(0)} = e^{-\frac{(\lambda-\lambda_1)}{2} + \frac{t_1^2-t^2}{2}} \left(\sum_{\beta=0}^{\infty} \frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!} \cdot \left(\frac{\lambda-\lambda_1}{2} \right)^{\beta} 2^{-(\frac{q_1}{2}+\beta)} \right),$$

$$f_{asymp}^{(1)} = e^{-\frac{(\lambda-\lambda_1)}{2} + \frac{t_1^2-t^2}{2}} \sum_{\beta=0}^{\infty} \frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!} \cdot \left(\frac{\lambda-\lambda_1}{2} \right)^{\beta} 2^{-(\frac{q_1}{2}+\beta)} \times (\frac{(\lambda-\lambda_1)t_1^2}{2}$$
$$-\frac{(1-t_1^2)(1-t_1^2-2p)}{4} + \frac{(1-t^2)(1-t^2-2p_1)}{4} + \frac{q_1}{2}(\frac{q_1}{2}-p-1) - (t^2+t_1^2+p_1-\beta)\beta),$$

and $f_{asymp}^{(2)}$ is defined in Appendix B.2.

We provide the following interpretation for our results. In addition to the usual interpretation of a likelihood function, it can be considered as a measure of the consistency of the observed data to the value(s) – assumed or estimated – of the population parameter(s). Note that ℓ_1' in (4.3.3) for $H_{10} : \mu^{(1)} = \mu_0^{(1)}$ can be obtained by letting $\mu^{(1)} = \mu_0^{(1)}$ in (4.3.10). The first likelihood (pdf) on the right side of (4.3.10) shows the conditional likelihood of observing $x_t^{(1)}$ for $X_t^{(1)}$ if $H_{10} : \mu^{(1)} = \mu_0^{(1)}$ is true; i.e., how consistent this observation is with the hypothesized mean $\mu^{(1)} = \mu_0^{(1)}$ of $X_t^{(1)}$. If this likelihood is small, then the hypothesis $H_{10} : \mu^{(1)} = \mu_0^{(1)}$ is likely to be false. Secondly, note from (4.3.10) that the conditional likelihood inside the expectation on the right side of (4.3.10) depends on λ_2. So from Theorem 1, the conditional likelihood function in (4.3.10) shows, after removing the direct effect t_1^2 of the first subvector from t^2, how well $t^2 - t_1^2$, as an observed value for T_2^2, explains (or, is consistent with) the estimated noncentrality $\lambda_2 = \lambda$ if H_0 is true. Again if this is small, we tend to believe that$H_{10} : \mu^{(1)} = \mu_0^{(1)}$ is false. Therefore, if ℓ_1' is small, then $H_{10} : \mu^{(1)} = \mu_0^{(1)}$ is likely to be false. The interpretation for ℓ_1 is similar.

4.3.3 The Computation Procedure for Case (A) and Case (B)

When an observed overall T^2 is significant, the following is a sequential computational procedure for both Cases (A) and (B). Note that when computing $f_{\mu^{(1)}}(X_t^{(1)} = x_t^{(1)}|t^2)$ on the left side of (4.3.9), we need to compute $f_{T^2}(t^2)$, $f_{X_t^{(1)}}(x_t^{(1)})$, and

$$E_{(\bar{X}^{(1)},S_{11})}\left(f_{T_2^{2(*)}|T_1^2}\left(\frac{t^2 - t_1^2}{(1 + t_1^2/(N-1))}|t_1^2\right) \cdot \frac{1}{(1 + t_1^2/(N-1))}\right)$$

, for all combinations of the p_1 component of X_t, where $p_1 < p$. λ and λ_1 are the parameters in (4.3.9), which need to be estimated. Let$\hat{\lambda}$ and $\hat{\lambda}_1$ be the estimates.

Procedure 1: (For Case (A))

Step 1: For a given observation X_t, which is from the current process at time t, and the usual sample mean vector $\bar{X}$ and the sample covariance matrix$S$, which are obtained from the reference sample, $X_i = (X_{i1}, \ldots, X_{ip})'$, $i = 1, \ldots, N$, from the in-control process (or Historical Data Set (HDS)), compute the T^2 value. If an out-of-control condition is *signaled at a significant level α, then go to Step 2; otherwise move to the next sampling period $t + 1$.*

Step 2: *Consider $p_1 = 1$ and computing three functions in (4.3.10) for every component of X_t. The parameters in $f_{T^2}(t^2)$ and $f_{T_2^{2(*)}|T_1^2}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|t_1^2)$ of (4.3.10) are λ and $\frac{\lambda-\lambda_1}{(1+t_1^2/(N-1))}$, respectively. Computing both of densities with the estimated noncentrality parameters $\hat{\lambda} = \frac{N}{N+1}(\mu_a - \bar{X})'S^{-1}(\mu_a - \bar{X})$ and $\hat{\lambda}_1 = \frac{N}{N+1}(\mu_a^{(1)} - \bar{X}^{(1)})'S_{11}^{-1}(\mu_a^{(1)} - \bar{X}^{(1)})$. Note that $X_t^{(1)} \sim N_{p_1}(\mu_a^{(1)}, s_{11})$ when computing $f_{X_t^{(1)}}(x_t^{(1)})$. Then using numerical method to compute $E_{(\bar{X}^{(1)},S_{11})}\left(f_{T_2^{2(*)}|T_1^2}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|t_1^2) \cdot \frac{1}{(1+t_1^2/(N-1))}\right)$ from the fact that $\bar{X}^{(1)} \sim N_{p_1}(\bar{x}^{(1)}, s_{11})$ and $S_{11} \sim W_{p_1}(N-1, \frac{s_{11}}{N-1})$.*

Step 3: *Based on results of Step 2, the variable corresponding to the largest value is most likely source of the out-f-control condition.*

Step 4: *When considering the correlation between variables, we can repeat Step 2 and Step3 for $p_1 \geq 2$. Note that in Step 3, the set of variables with the largest ℓ value is most likely the problematic variables.*

Procedure 2: (For Case (B))

Step 1: *Same as Step 1 for Case (A).*

Step 2: *It is similar to Step 2 for Case (A). Consider $p_1 = 1$ and computing three functions of (4.3.9) for every component of X_t. The parameters in $f_{T^2}(t^2)$ and $f_{T_2^{2(*)}|T_1^2}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|t_1^2)$ of (4.3.9) are λ and $\frac{\lambda-\lambda_1}{(1+t_1^2/(N-1))}$, respectively. Compute both densities with the noncentrality parameter estimates $\hat{\lambda} = \frac{N}{N+1}(X_t - \bar{X})'S^{-1}(X_t - \bar{X})$ and $\hat{\lambda}_1 = \frac{N}{N+1}(X_t^{(1)} - \bar{X}^{(1)})'S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})$ for case (B). Use the fact that $X_t^{(1)} \sim N_{p_1}(\bar{x}^{(1)}, s_{11})$ when computing $f_{X_t^{(1)}}(x_t^{(1)})$. Then obtain $E_{(\bar{X}^{(1)},S_{11})}\left(f_{T_2^{2(*)}|T_1^2}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|t_1^2) \cdot \frac{1}{(1+t_1^2/(N-1))}\right)$ knowing $\bar{X}^{(1)} \sim N_{p_1}(\bar{x}^{(1)}, s_{11})$ and $S_{11} \sim W_{p_1}(N-1, \frac{s_{11}}{N-1})$.*

Step 3: *Rank the ℓ'_j values obtained from Step 2. The variable corresponding to the smallest value is most likely the problematic variable.*

Step 4: *When considering the correlation between variables, we can repeat Step 2 and Step 3 but with $p_1 \geq 2$ variables in the first vector.*

4.4 The Alternative of The Distribution of T^2 Based On The Multivariate t Distribution

In this section we introduce multivariate t distribution to obtain the distribution of T^2 in (4.3.7).

4.4.1 Definition of Multivariate t Distribution

A p-dimensional random vector $T = (T_1, \ldots, T_p)'$ is said to have the p-variate t distribution with degrees of freedom ν, mean vector η, and correlation matrix R (denoted by $t_p(\nu, \eta, R)$) if its joint probability density function (pdf) is given by (Kotz (2004))

$$f(T) = \frac{\Gamma(\frac{\nu+p}{2})}{(\pi\nu)^{\frac{p}{2}}\Gamma(\frac{\nu}{2})}|R|^{-\frac{1}{2}}(1 + \frac{1}{\nu}(T-\eta)'R^{-1}(T-\eta))^{-\frac{\nu+p}{2}}. \tag{4.4.1}$$

The degrees of freedom parameter ν is also referred to as the shape parameter because the peakedness of (4.4.1) may be diminished, preserved, or increased by varying ν. The distribution is said to be central if $\eta = 0$; otherwise, it is said to be noncentral. It is also known that $T \sim t_p(\nu, \eta, R)$ has the representation $T = (\Phi^{-\frac{1}{2}})'Y + \eta$, where $\Phi = \Phi^{\frac{1}{2}}(\Phi^{\frac{1}{2}})' \sim W_p(\nu + p - 1, R^{-1})$ and $Y \sim N_p(0, \nu I_p)$ are independent.

4.4.2 Quadratic Forms

If T has the p-variate t distribution with degrees of freedom ν, mean vector η, and correlation matrix R, then $\frac{T'R^{-1}T}{p}$ has the noncentral F distribution with degrees of freedom p and ν and noncentrality parameter $\frac{\eta' R^{-1}\eta}{p}$. See Hsu (1990) for a special case of this result. When $\eta = 0$, the distribution is a central F.

4.4.3 Representation and Partition of T

Again, recall that X_t, $\bar{X}$, and S independently follow $N_p(\mu, \Sigma)$, $N_p(\mu_0, \frac{\Sigma}{N})$, and $W_p(N - 1, \frac{\Sigma}{N-1})$, respectively. With the partitions in (4.3.4) and (4.3.5), let $S^{\frac{1}{2}} = \begin{bmatrix} S_{11}^{\frac{1}{2}} & 0 \\ S_{21}S_{11}^{-\frac{1}{2}} & S_{22.1}^{\frac{1}{2}} \end{bmatrix}$, where $S^{\frac{1}{2}}S^{\frac{1}{2}'} = S$. Then the inverse of $S^{\frac{1}{2}}$ is $(S^{-\frac{1}{2}})' = \begin{bmatrix} (S_{11}^{-\frac{1}{2}})' & 0 \\ -(S_{22.1}^{-\frac{1}{2}})'S_{21}S_{11}^{-1} & (S_{22.1}^{-\frac{1}{2}})' \end{bmatrix}$, where $(S^{-\frac{1}{2}})(S^{-\frac{1}{2}})' = S^{-1}$. Let

$$T = (S^{-\frac{1}{2}})'\left[\sqrt{\frac{N}{N+1}}\left(X_t - \bar{X}\right)\right]. \tag{4.4.2}$$

Then

$$T = \sqrt{\frac{N}{N+1}}\begin{bmatrix} (S_{11}^{-\frac{1}{2}})' & 0 \\ -(S_{22.1}^{-\frac{1}{2}})'S_{21}S_{11}^{-1} & (S_{22.1}^{-\frac{1}{2}})' \end{bmatrix}\begin{pmatrix} X_t^{(1)} - \bar{X}^{(1)} \\ X_t^{(2)} - \bar{X}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{N}{N+1}}(S_{11}^{-\frac{1}{2}})'(X_t^{(1)} - \bar{X}^{(1)}) \\ \sqrt{\frac{N}{N+1}}(S_{22.1}^{-\frac{1}{2}})'\left((X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})\right) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix},$$

where

$$T_1 = \sqrt{\frac{N}{N+1}}(S_{11}^{-\frac{1}{2}})'(X_t^{(1)} - \bar{X}^{(1)})$$

$$T_2 = \sqrt{\frac{N}{N+1}}(S_{22.1}^{-\frac{1}{2}})'\left((X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(2)})\right) \tag{4.4.3}$$

is a partition of T with dimensions p_1 and q_1. Hence,

$$T'T = \sqrt{\frac{N}{N+1}}(X_t - \bar{X})'(S^{-\frac{1}{2}})(S^{-\frac{1}{2}})'\sqrt{\frac{N}{N+1}}(X_t - \bar{X})$$

$$= \frac{N}{N+1}(X_t - \bar{X})'S^{-1}(X_t - \bar{X})$$

is the same as T^2 in (4.3.1). Similarly, we can get $T_1'T_1 = \frac{N}{N+1}(X_t^{(1)} - \bar{X}^{(1)})'S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)}) = T_1^2$ and $T_2'T_2 = \frac{N}{N+1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})]'S_{22.1}^{-1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})] = T_2^2$

4.4.4 Distribution of T

The density of the distribution of T in (4.4.2) (proof in Appendix B.3) is given by

$$f_T(T) = \frac{\pi^{-\frac{p}{2}} \cdot \Gamma(\frac{N}{2})}{\Gamma(\frac{N-p}{2})}(N-1)^{-\frac{p}{2}}(1 + (N-1)^{-1}T'T)^{-\frac{N}{2}} \cdot I_T(\mu - \mu_0, T), \tag{4.4.4}$$

where

$$I_T(\mu - \mu_0, T) = etr\{-\frac{1}{2}\frac{N}{N+1}(\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0)\} \cdot \Gamma_p(\frac{N}{2})^{-1}.$$

$$\times \int_{U>0} |U|^{\frac{N-1-p}{2}} etr\{-U + \sqrt{\frac{2N}{N+1}}(\Sigma^{-\frac{1}{2}})'(\mu-\mu_0)T'(TT'+(N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}dU.$$

Hence, if $H_0 : \mu = \mu_0$ is true, then $I_T(0, T) = 1$ and (4.4.4) becomes

$$f_T(T) = \frac{\Gamma(\frac{N}{2})|(N-1)I_p|^{-\frac{1}{2}}}{\pi^{\frac{p}{2}}\Gamma(\frac{N-p}{2})}(1+\frac{T'T)}{N-1})^{-\frac{N}{2}}. \tag{4.4.5}$$

Furthermore, if we make a simple transformation $T^* = \sqrt{N-p}T$, then T^* is a p-variate t distribution with degrees of freedom $N-p$, mean vector 0, and correlation matrix $(N-1)I_p$ (denoted by $t_p(N-p, 0, (N-1)I_p)$) with the density given by

$$f(T^*) = \frac{\Gamma(\frac{N}{2})|(N-1)I_p|^{-\frac{1}{2}}}{(\pi(N-p))^{\frac{p}{2}}\Gamma(\frac{N-p}{2})}(1+\frac{1}{N-p}\frac{T^{*\prime}T^*}{N-1})^{-\frac{N}{2}}. \tag{4.4.6}$$

Hence, using the result in subsection 4.4.2, $\frac{T^{*\prime}(N-1)^{-1}T^*}{p} = \frac{T'T}{N-1}\frac{N-p}{p} = \frac{T^2}{N-1}\frac{N-p}{p}$ has the central F distribution with degrees of freedom p and $N-p$; i.e., from the multivariate t distribution, we can also obtain the distribution of Hotelling' T^2 when $H_0 : \mu = \mu_0$ is true.

Furthermore, when $\mu - \mu_0 = 0$, we can obtain the marginal distributions of T_1 and T_2 in (4.4.3) as (proof in Appendix B.4)

$$f_{T_1}(T_1) = \frac{\pi^{-\frac{p_1}{2}} \cdot \Gamma(\frac{N}{2})}{\Gamma(\frac{N-p_1}{2})}(N-1)^{-\frac{p_1}{2}}(1+(N-1)^{-1}T_1'T_1)^{-\frac{N}{2}},$$

and

$$f_{T_2}(T_2) = \frac{\pi^{-\frac{q_1}{2}}\Gamma(\frac{N-p+q_1}{2})}{\Gamma(\frac{N-p}{2})}(N-1)^{-\frac{q_1}{2}}(1+(N-1)^{-1}T_2'T_2)^{-\frac{N-p+q_1}{2}}, \tag{4.4.7}$$

respectively. Similarly, if we transform from T_1 and T_2 to $T_1^* = \sqrt{N-p_1}T_1$ and $T_2^* = \sqrt{N-p}T_2$, respectively, we can obtain $T_1^* \sim t_{p_1}(N-p_1, 0, (N-1)I_{p_1})$ and $T_2^* \sim t_{q_1}(N-p, 0, (N-1)I_{q_1})$. Again, using the result in subsection 4.4.2, we have $\frac{T_1^{*\prime}(N-1)^{-1}T_1^*}{p_1} = \frac{T_1'T_1}{N-1}\frac{N-p_1}{p_1} = \frac{T_1^2}{N-1}\frac{N-p_1}{p_1}$ and $\frac{T_2^{*\prime}(N-1)^{-1}T_2^*}{q_1} = \frac{T_2'T_2}{N-1}\frac{N-p}{q_1} = \frac{T_2^2}{N-1}\frac{N-p}{q_1}$ have the central F distribution with degrees of freedom p_1 and $N-p_1$ and q_1 and $N-p$, respectively. That is, when $\mu - \mu_0 = 0$,

$$T_1^2 \sim \frac{p_1(N-1)}{N-p_1}F_{p_1,N-p_1} \text{ and } T_2^2 \sim \frac{q_1(N-1)}{N-p}F_{q_1,N-p}. \tag{4.4.8}$$

Moreover, from (4.4.4) and (4.4.7), it is found that $f_T(T) \neq f_{T_1}(T_1) \cdot f_{T_2}(T_2)$ and hence we know that neither T_1 and T_2 nor T_1^2 and T_2^2 are independent since $T_1^2 = T_1'T_1$ and $T_2^2 = T_2'T_2$ are respectively the function of T_1 and T_2.

4.4.5 A Note on MTY Decomposition

In the MTY decomposition, Mason, Tracy, and Young (1995, 1997) (or Manson and Young (1999, 2002)) obtain the unconditional T^2 value, individual T_j^2 for $j = 1, \ldots, p$, (see equation (4.2.1)), and the conditional $T^2_{j\cdot 1,\ldots,j-1}$ values (see (4.2.2) and (4.2.3)). By using the result of (4.4.8), we also can prove the distributions of T_j^2 and $T^2_{j\cdot 1,\ldots,j-1}$. First, under the null hypothesis, from (4.4.8),

$$T_j^2 \sim F_{1,N-1} \text{ when } p_1 = 1 \text{ and } q_1 = p-1 \text{ , and}$$

$$T^2_{j\cdot 1,\ldots,j-1} \sim \frac{(N-1)}{N-(k+1)} F_{1,N-(k+1)} \text{ when } p_1 = k \text{ and } q_1 = 1. \tag{4.4.9}$$

In addition, from previous subsection, we also know that T_j^2 and $T^2_{j\cdot 1,\ldots,j-1}$ are not independent.

When an out-of-control signal occurs in a T^2 chart, it means that the mean is shifted, that is, $H_0 : \mu = \mu_0$ is rejected, and the Hotelling T^2 follows a non-central F distribution. According to MTY decomposition method, a large value for the unconditional univariate T^2 term, T_j^2, in the decomposition indicates that the associated variable is outside its univariate control limits. Significant conditional T^2 terms in the decomposition imply that something is wrong with the relationship between a group of the original variables relative to that displayed in the historical sample.

Notice that, under assuming the Σ is unknown, then under $H_{j0} : \mu_j = \mu_{j0}$, the null distribution of the test statistic $T^2_{j\cdot 1,\ldots,j-1}$ is an independent central or a non-central $\frac{(N-1)}{N-(k+1)} F_{1,N-(k+1)}$ distribution, depending on whether $\lambda = 0$ or not (i.e., depending on whether the means of $X_1, \ldots, X_{j-1}$ have changed or not). If a mean has not changed, let $\mu_j = \mu_{j0}$; but once a mean is declared to have changed, the corresponding sample mean is used to estimate the population mean when calculating the noncentrality λ for checking the remaining variables. It should be pointed out that their distributional result in (4.2.3) is questionable. Firstly, since the family of F-distributions is not infinitely divisible, the F-distributed T^2 (ignoring all adjusting constants) can not be decomposed into a sum of independent, identical, and F-distributed components. Secondly, unless the means of all variables $X_1, ..., X_j$ had not changed, $T^2_{j\cdot 1,...,j-1}$does not have a central distribution (see (4.4.9)).

4.5 Examples

From our earlier discussions in Section 4.3, we consider two cases. In Case (A) the individual alternative hypothesis is assumed to be

$$H_{ja} : \Delta_j = \mu_a^{(j)} - \mu_0^{(j)}.$$

Here $j = 1, \ldots, k$ and $k = C^p_{p_1}$, while in Case (B) no alternative is specified. To illustrate the proposed method, we consider three examples for Case (B) from the literatures and one simulated example for Case (A). In this section, the conditional likelihood is obtained from the procedure given in subsection 4.3.3.

4.5.1 Example 1 for Case (B)

The first example uses the data from Hawkins (1991), which was in turn based on data from Flury and Riedwyl (1988, p151) on five dimensions of switch drums:X_1 is the inside diameter of a drum, and X_2, X_3, X_4, X_5 are the distances from the head to the edges of four sectors cut in the drum. Table 4.1 gives the original data of Flury and Riedwyl. Hawkins (1991) treated the sample mean and covariance matrix as the in-control mean and covariance matrix, respectively, in order to simulate in-control and out-of-control data in his study. We will follow the same assumption and assume the in-control mean and the correlation matrix (from Table 4.1) as

$$\mu_0 = (17.960,10.3, 13.76, 11.08, 8.26)' \tag{4.5.1}$$

$$R_0 = \begin{pmatrix} 1 & 0.1388 & 0.3496 & 0.0829 & 0.2652 \\ 0.1388 & 1 & 0.7324 & 0.9130 & 0.6932 \\ 0.3496 & 0.7324 & 1 & 0.6824 & 0.8214 \\ 0.0829 & 0.9130 & 0.6824 & 1 & 0.7640 \\ 0.2652 & 0.6932 & 0.8214 & 0.7640 & 1 \end{pmatrix}, \tag{4.5.2}$$

with the standard deviations(1.8622,1.7053, 1.7090, 1.8718, 2.2114), respectively. To simulate the training in-control as well as out-of-control data, the first 35 observations in Table 4.2 are sampled from $N_5(\mu_0, \Sigma_0)$ and the remaining 15 observations were taken after adding an upward shift of 0.25 and 0.5 standard deviations to the mean of X_5 and X_1, respectively, with no changes to the other process parameters. The sample mean and covariance matrix based on the first 35 in-control observations in Table 4.2 are respectively (in our notation)

$$\bar{X} = (17.6289, 10.3365, 13.6189, 11.1776, 8.2437)' \tag{4.5.3}$$

and

$$S = \begin{pmatrix} 2.7355 & 0.5193 & 1.3496 & 0.80294 & 1.4865 \\ 0.5193 & 2.4673 & 1.6465 & 2.5275 & 2.0266 \\ 1.3496 & 1.6465 & 2.2259 & 1.9026 & 2.4228 \\ 0.8029 & 2.5275 & 1.9026 & 3.4201 & 2.9601 \\ 1.4865 & 2.0266 & 2.4228 & 2.9601 & 4.5689 \end{pmatrix}. \tag{4.5.4}$$

A T^2chart, using (4.3.1), at a significant level $\alpha = 0.05$ is constructed for each of the observations 36-50. The chart indicates an out-of-control signal for the observation 48, namely, $X_{t(48)} = (13.065, 11.625, 14.923, 12.589, 12.446)'$, with a T^2 value of $22.2447 > (5(35-1)/(35-5))\, F_{5,35-5}(0.95) = 14.3568$. So, according to Hawkins, the mean vector of $X_{t(48)}$ is said to have changed with $\alpha = 0.05$. Note that all correlations are positive and the five variables seem to form two groups, X_1 and (X_2, X_3, X_4, X_5), because X_1 is weakly correlated with the others and the correlations amongX_2, X_3, X_4, X_5 are likely to be significant. It could imply that, if one of four variables,X_2, X_3, X_4, X_5, has a relatively extreme value, the other variables will tend to have a relatively extreme value (in the same direction) as well. And if X_1 has a relatively extreme value, the values of the other variables may not be influenced. The dependence structure of (X_1,X_2, X_3, X_4, X_5) is a key point that can affect us in determining which variable or which set of variables had shifted in mean.

Using our method, we consider $p_1 = 1$ to see which individual variable is most likely to have a shift in mean and $p_1 = 2$ to see which two are most likely to have shifts in mean together. Hence, k values corresponding $p_1 = 1$ and 2 are 5 and 10, respectively.

Now, assuming no alternative is given (i.e., Case (B)), Table 4.3 gives the conditional likelihood,$\ell_j'(H_{j0}|H_a)$that the jth (j = 1, 2, ..., 5) variable is in-control given that the overall process mean vector is out-of-control (with T^2=22.2447), for $X_{t(48)}$ with $p_1 = 1$ and 2. The ranking of the variables, from the smallest to the largest $\ell_j'(H_{j0}|H_a)$ value for $p_1 = 1$, is: X_1,X_5,X_4,X_2, andX_3. It can be observed that the likelihood value of X_1 is about 0.1419 times that of X_5 and also less than 0.0199 times those of other variables. This means H_{10} is the least likely to be true. Furthermore, from the distance between $X_{t(48)}$ and $\bar{X}$ of ($-2.7594\sigma_1$,$0.8203\sigma_2$,$0.8741\sigma_3$,$0.7632\sigma_4$,$1.966\sigma_5$), X_1 has the largest absolute distance and X_5 is the next one. Hence, we can say that X_1 is the most likely variable to have a shift in mean, and X_5 might be the next one to have a shift in mean.

When considering two variables at a time (i.e., $p_1 = 2$), the likelihood values for all 10 combinations are also given in Table 4.3. The two variables corresponding to the smallest conditional likelihood is (X_1,X_5). It is interesting to note that these two variable have largest distances from their respective in-control means ($-2.7594\sigma_1$ and $1.966\sigma_5$), but with different directions. We consider this possible because their correlation of $\rho_{15} = 0.2652$ is rather weak. Furthermore, the likelihood value of $X_1$and X_5 is less than 0.0588 times that of (X_1,X_3), 0.0021 times that of (X_1,X_4), 0.0014 times that of (X_1,X_2), 10^{-4} times that of others. This means $H_{(1,5)0} : (\mu_1, \mu_5)' = (\mu_{10}, \mu_{50})'$ is the least likely to be true and then we can make a reasonable conclusion that (X_1,X_5) are the potentially problematic variables. Both Hawkins (1991) and Mason, Tracy, and Young (1995) gave the same conclusion.

4.5.2 Example 2 for Case (B)

The second example is about testing of ballistic missiles taken from Jackson (1980), where four related thrust measurements (p =4) were obtained from each round tested. There are two different thrust measurements, say A and B, obtained from each strain gauge and there are two gauges attached to each round fired. Then, there are four correlated thrust measurements for each test firing (a measurement using Gauge #1 Method A, Gauge #1 Method B, Gauge #2 Method A, Gauge #2 Method B). The reference mean vector based on 40 training observations is given by $\bar{X}$= (0, 0, 0, 0)$'$, and the variances are σ_1^2=102.74, σ_2^2=142.74, $\sigma_3^2 = 84.57$, and $\sigma_4^2 = 99.06$, respectively, with correlation matrix given by

$$\rho = \begin{pmatrix} 1 & 0.732 & 0.719 & 0.536 \\ 0.732 & 1 & 0.788 & 0.673 \\ 0.719 & 0.788 & 1 & 0.758 \\ 0.536 & 0.673 & 0.758 & 1 \end{pmatrix}.$$

Now, a new individual observation is X_t= (15, 10, 20, -5)$'$. From (4.3.1), the observed $T^2 = 15.921$ is larger than the corresponding critical value $(4(40-1)/(40-4))\,F_{4,40-4}(0.95) = 11.412$, so the mean vector of $X$is said to have changed at $\alpha = 0.05$. Note that all correlations are positive and likely to be significant. As before, with no alternative hypothesis given, Table 4.4 gives the $\ell'_j(H_{j0}|H_a)$ values for each variable, given T^2=15.921, for$p_1 = 1$and 2. Therefore, the ranking of the variables with the smallest to the largest $\ell'_j(H_{j0}|H_a)$ value for $p_1 = 1$ is: X_3,X_1,X_2, and X_4. The conditional likelihood value of X_3 is about 0.2582 times that of X_1 and also less than 0.15 times those of other variables. This means H_{30} is the least likely to be true. The ranking is consistent with the one given in Doganaksoy, Faltin, and Tucker (1991). The variable X_3 is most likely to have a shift in mean and the next variable will be X_1 and (X_3,X_1) are the variables for Method A. When we consider two variables at a time (i.e., $p_1 = 2$), the two variables with the smallest conditional likelihood value is (X_3,X_4), which are the variables for second gauge. The likelihood value for (X_3,X_4) is less than 0.004 times those of other pairs of two variables. Thus, from above results, we can make two reasonable conclusion that Method A is potentially problematic in terms of method, and Gauge #2 is potentially problematic in terms of gauge, which is consistent with the conclusion of Jackson's (1980) based on principal component analysis.

4.5.3 Example 3 for Case (B)

The third example is given in Mason and Young (2002, p. 150). This example is used to illustrate MTY decomposition method for the interpretation of out-of-control signal. The three dimensional in-control history data set (HDS) is represented by 23 observations, and the in-control mean vector is $\bar{X} = (525.435, 513.435, 539.913)'$ and the standard variances are σ_1^2=41.075, σ_2^2=4.984, σ_3^2=12.173, with the correlation matrix given by

$$\begin{pmatrix} 1 & 0.205 & 0.725 \\ 0.205 & 1 & 0.629 \\ 0.725 & 0.629 & 1 \end{pmatrix}.$$

The T^2 value computed from (4.3.1) for a new observation vector $X_{(t)} = (533, 514, 528)'$ is 76.636, which is larger than the T^2 critical value of $3(23-1)/(23-3) \cdot F_{3,23-3}(0.95) = 10.225$, at $\alpha = 0.05$. Hence, the observation produces a signal. As before in Example 1, assume no alternative is given, the $\ell_j'(H_{j0}|H_a)$ values for $X_{(t)}$ given T^2=22.2447 when $p_1 = 1$ are: 3.0948×10^{-2}, 1.7439×10^{-1}, and 2.9209×10^{-4} for variables 1 through 3. The $\ell_j'(H_{j0}|H_a)$ values when $p_1 = 2$ are: 5.6995×10^{-3}, 8.71×10^{-12}, and 4.6472×10^{-7} for the combination of two variables (1,2), (1,3), and (2,3), respectively. Variable 3 has the smallest ℓ' when $p_1 = 1$ and we can see that ℓ' of variable 3 is very close to ℓ' of variable 1 due to a very strong correlation between them. The least ℓ' when $p_1 = 2$ is the combination of variable 1 and variable 3. Thus, we can make a reasonable conclusion that variable 3 is the potentially problematic variable and the next will be variable 1. This conclusion is also consistent with those of Mason and Young (2002).

4.5.4 A Simulated Study for Case (A)

When an out-of-control signal is triggered in a T^2 chart, we want to know if the mean vector has shifted from μ_0 to a vector μ_a in a specific direction, say μ_a^*. Hence, we have Case (A) with the following alternative hypothesis

$$H_a = \cap_{j=1}^{k} (H_{ja} : \Delta_j \equiv \mu_a^{(j)} - \mu_0^{(j)} = \Delta_j^* \equiv \mu_a^{*(j)} - \mu_0^{(j)}),$$

where $\mu_a = (\mu_a^{(1)'}, \ldots, \mu_a^{(k)'})'$, $\mu_a^* = (\mu_a^{*(1)'}, \ldots, \mu_a^{*(k)'})'$, and Δ_j^*'s are to be specified by the user. Since μ_a and hence Δ are unknown, choosing Δ_j^*'s is a difficult task.

To illustrate our method, we again use the data generated following the method of Hawkins (1991) as described in subsection 4.5.1, where the in-control process is assumed to follow$N_5(\mu_0, \Sigma_0)$ with the in-control μ_0 and Σ_0 given in (4.5.1) and (4.5.2), respectively. The reference data still contain the first 35 observations in Table 4.2, with the sample mean and covariance matrix given in (4.5.3) and (4.5.4), respectively. Next, we randomly generated a sample,X_t, from $N_5(\mu_a, \Sigma_0)$ to produce an out-of-control T^2 with a value greater than the corresponding critical value of $5(35-1)/(35-5) \cdot F_{5,35-5}(0.95)$= 14.3568. For example, assume the true mean $\mu_a = \mu_0 + (2.5\sigma_1, 0, 0, 0, 0)'$ after the shift, and let X_t= (23.19104, 10.53652, 13.89620, 11.01731, 9.57183)' be a random sample from$N_5(\mu_a, \Sigma_0)$, with its T^2-value $= 17.99087 > 14.3568$, the critical T^2-value. Hence, an out-of-control signal occurs. Now, assume the direction of mean shift under the alternative hypothesis is consistent with the actual direction of shift (i.e., $\Delta^* = \Delta = (2.5\sigma_1, 0, 0, 0, 0)'$) and is given by $H_a = \cap_{j=1}^{k}(H_{ja} : \Delta_j = \Delta_j^*)$, where $k = C_{p_1}^{p}$. For $p_1 = 1$, the $\ell_j(H_{ja}|H_a)$-values, given T^2=17.99087 ($>$ 14.3568) , are: 0.31519, 0.25097, 0.26626, 0.21544, and 0.15556 for Variables 1 to 5, respectively. Ranking these $\ell_j(H_{ja}|H_a)$- values in a descending order shows $H_{1a} : \Delta_1 = 2.5\sigma_1$ is the most likely to be true. In contrast, if, for the same observationX_t given above, we specify the shifted direction as H_a:$\mu_a^* = \mu_0 + (0, 2.5\sigma_2, 0, 0, 0)'$, the individual $\ell_j(H_{ja}|H_a)$-values for Variables 1 through 5 are 0.00011, 0.03088, 0.31536, 0.25647, and 0.17074 . The ranking of the five variables according to their $\ell_j(H_{ja}|H_a)$- values is X_3,X_4,X_5,X_2, and X_1, and this means $H_{1a} : \Delta_1$= 0 is the least likely to be true. That is, the given observation, generated from $\mu_a = \mu_0 + (2.5\sigma_1, 0, 0, 0, 0)$, is inconsistent with the current hypothesized direction μ_a under H_a for variables X_1 and X_2.

Thus, we conduct a more detailed simulation study under the various types of true shifts, $\mu_a = \mu_0 + \Delta$, and hypothesized shifts, $\mu_a^* = \mu_0 + \Delta^*$, to demonstrate the proposed method and also show that the hypothesized shifts can be effectively detected especially when $\Delta = \Delta^*$. We simulated 500 out-of-control data points, X_t for $t = 1, \ldots, 500$, from $N_5(\mu_0 + \Delta, \Sigma_0)$, all with a significant T^2 value. But our hypothesized shift under $H_a$is Δ^*. First, let $D_\sigma = diag(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$, then, for example, $(2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma = (2.5\sigma_1, 2.5\sigma_2, 2.5\sigma_3, 2.5\sigma_4, 2.5\sigma_5)$. We consider the following four scenarios for Δ and Δ^*:

(1). $\Delta' = (2.5, 0, 0, 0, 0) \cdot D_\sigma$ with $\Delta^{*'} = (2.5, 0, 0, 0, 0) \cdot D_\sigma, (0, 2.5, 0, 0, 0) \cdot D_\sigma, (2.5, 0, 0, 2.5, 0) \cdot D_\sigma$, and $(2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$, respectively.

(2). $\Delta' = (2.5, 0, 0, 0, 2.5) \cdot D_\sigma$ with $\Delta^{*'} = (2.5, 0, 0, 0, 2.5) \cdot D_\sigma, (0, 0, 0, 2.5, 0) \cdot D_\sigma, (2.5, 2.5, 0, 0, 0) \cdot D_\sigma, (2.5, 0, 2.5, 0, 2.5) \cdot D_\sigma$, and $(2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$, respectively.

(3). $\Delta' = (0, 2.5, 0, 2.5, 0) \cdot D_\sigma$ with $\Delta^{*'} = (0, 2.5, 0, 2.5, 0) \cdot D_\sigma, (2.5, 0, 0, 0, 0) \cdot D_\sigma, (2.5, 0, 0, 2.5, 0) \cdot D_\sigma, (0, 2.5, 0, 2.5, 2.5) \cdot D_\sigma$, and $(2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$, respectively.

(4). $\Delta' = (0, 0, 2.5, 0, 2.5) \cdot D_\sigma$ with $\Delta^{*'} = (0, 0, 2.5, 0, 2.5) \cdot D_\sigma, (0, 0, 0, 2.5, 0) \cdot D_\sigma, (2.5, 0, 2.5, 0, 0) \cdot D_\sigma, (2.5, 0, 2.5, 0, 2.5) \cdot D_\sigma$, and $(2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$, respectively.

Scenario (1) above assumes only one variable (Variable 1 in this case) has shifted in mean, while the other three assume shifts in two means. Based on our proposed method, we rank the variables by their $\ell_j(H_{ja}|H_a)$-values. For each scenario Tables 4.5 - 4.8 gives the number of times (out of 500) each ℓ_j-value (for H_{ja}) was ranked first and second. It also gives the number of times (out of 500) that, for each pair of two variables (i.e., $p_1 = 2$), its $\ell_{(j_1,j_2)}$-value was ranked first and second, under each hypothesized alternative with Δ^*. For example, for $p_1 = 1$ in Scenario (1) (simulated under $\Delta = (2.5\sigma_1, 0, 0, 0, 0)$), our method first indicates that $H_{1a} : \Delta_1 = 2.5\sigma$ is likely to be true in 330 of the 500 simulation runs. When the hypothesized $\Delta^* = (2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$ (all five variables shifted simultaneously with the same relative magnitude), among all individual H_{ja}'s, the simulation study first picks $H_{1a} : \Delta_1 = 2.5\sigma_1$ as most likely in 470 of the 500 simulation runs , while the number of times other H_{ja} : $\Delta_j = 2.5\sigma_j$, $j = 2, 3, 4, 5$, were picked are fairly small. This evidence only shows that the mean shifted to $2.5\sigma_1$ in X_1 is most likely to be true. $H_{ja} : \Delta_j = 2.5\sigma_j$, for$j = 2, 3, 4, 5$, were not selected often because their true $\Delta_j = 0$. When the hypothesized $\Delta^* = (0, 2.5\sigma_2, 0, 0, 0)$, the simulation (under $\Delta = (2.5\sigma_1, 0, 0, 0, 0)$) first (second) picks $H_{3a} : \Delta_3 = 0$ in 252 (106) of 500 simulations, which is appropriate because $\Delta_3 = 0$. Same discussion is true for $H_{4a} : \Delta_4 = 0$ and $H_{5a} : \Delta_5 = 0$, respectively, as they were ranked first and second in 321 ($= 142 + 179$) simulation runs for H_{4a} and 224 ($= 45 + 179$) forH_{5a}. The number of times of being ranked first or second for H_{1a} and H_{2a} are relative small, which means $H_{1a} : \Delta_1 = 0$ and H_{2a}: $\Delta_2 = 2.5\sigma_2$ were not likely and this is appropriate because the observations were simulated with $\Delta_1 = 2.5\,\sigma_1$ and $\Delta_2 = 0$.

Let's look at Scenario (2) in more detail. When Δ^* indicates that the mean for all variables had shifted with the same magnitude, for example $\Delta^* = (2.5, 2.5, 2.5, 2.5, 2.5) \cdot D_\sigma$, then $H_{1a} : \Delta_1 = 2.5\sigma_1$ and $H_{5a} : \Delta_5 = 2.5\sigma_5$ receive the high frequencies (292 and 178 of being ranked first and second for H_{1a}, and 196 and 290 for H_{5a}); but the numbers of times the simulation picked other H_{ja}: $\Delta_j = 2.5\sigma_j$, $j = 2, 3, 4$, are all relatively small. This is quite reasonable since $\Delta_j = 0$, for$j = 2, 3, 4$. Furthermore, when $p_1 = 2$, among all $H_{(j_1,j_2)a} : (\mu_{j_1,a}, \mu_{j_2,a})' = (\mu_{j_10}, \mu_{j_20})' + (\Delta^*_{j_1}, \Delta^*_{j_2})'$, the simulation first picks $H_{(1,5)a}$: $(\Delta_1, \Delta_5)' = (2.5\sigma_1, 2.5\sigma_5)'$ to be most likely in 451 of 500 simulation runs. The results show that our method can accurately identify the most likely hypothesis because we assumed $(\Delta_1, \Delta_5)' = (2.5\sigma_1, 2.5\sigma_5)'$ in this scenario. Also in Scenario (2), when the hypothesized $\Delta^* = (0, 0, 0, 2.5\sigma_4, 0)$,$H_{2a} : \Delta_2 = 0$ received 184 and 223 out of 500 for the first and second rankings, and $H_{3a} : \Delta_3 = 0$ received 236 and 185. Again, it is reasonable. If considering two variables at a time ($p_1 = 2$), the simulation first picks $H_{(2,3)a} : (\Delta_2, \Delta_3)' = (0, 0)'$ to be likely in 433 of 500 simulation runs. This is also reasonable because $\Delta_2 = \Delta_3 = 0$. Other alternative hypotheses for pairs of two variables were not selected with high frequencies, but appropriately, since $\Delta_1 = 2.5\sigma_1$, $\Delta_4 = 0$ and $\Delta_5 = 2.5\sigma_5$. When $\Delta^* = (2.5\sigma_1, 2.5\sigma_2, 0, 0, 0)$,

$H_{3a} : \Delta_3 = 0$ and $H_{(3,4)a} : (\Delta_3, \Delta_4)' = (0, 0)'$ are respectively identified with high frequencies (with 198 and 203, respectively) for $p_1 = 1$ and $p_1 = 2$. This is quite reasonable because $\Delta_3 = \Delta_4 = 0$. Furthermore, $H_{1a} : \Delta_1 = 2.5\sigma_1$ and $H_{4a} : \Delta_4 = 0$ for $p_1 = 1$, and $H_{(1,3)a} : (\Delta_1, \Delta_3)' = (2.5\sigma_1, 0)'$ and $H_{(1,4)a} : (\Delta_1, \Delta_4)' = (0, 0)'$ for $p_1 = 2$ are also likely identified. One possible explanation as to why H_{3a} had the highest individual frequency is because its standard deviation is the smallest. For $H_{(3,4)a}$, it may be because X_3 and X_4 have high correlation ($\rho_{34} = 0.6824$) . When the hypothesized $\Delta^* = (2.5\sigma_1, 0, 2.5\sigma_3, 0, 2.5\sigma_5)$, $H_{1a} : \Delta_1 = 2.5\sigma_1$ is the most identified for $p_1 = 1$ because $\Delta_1 = 2.5\sigma_1$ is assumed when generating the data. In addition, $H_{2a} : \Delta_2 = 0$, $H_{4a} : \Delta_4 = 0$, and $H_{5a} : \Delta_5 = 2.5\sigma_5$ are also likely identified, also appropriately since $\Delta_2 = \Delta_4 = 0$ and $\Delta_5 = 2.5\sigma_5$ were assumed. Furthermore, $H_{(2,4)a} : (\Delta_2, \Delta_4)' = (0, 0)'$ is identified with the highest frequency, also appropriately since $\Delta_2 = \Delta_4 = 0$. $H_{(1,2)a} : (\Delta_1, \Delta_2)' = (2.5\sigma_1, 0)'$ and $H_{(4,5)a} : (\Delta_4, \Delta_5)' = (0, 2.5\sigma_5)'$ are also likely identified. Finally, when $\Delta^* = (2.5\sigma_1, 0, 0, 0, 2.5\sigma_5)$ is specified as equal to the true shift Δ, the most likelyH_{ja} and $H_{(j_1,j_2)a}$are all correctly identified: $H_{1a} : \Delta_1 = 2.5\sigma_1$ and $H_{(2,4)a} : (\Delta_2, \Delta_4)' = (0, 0)'$ are respectively identified when$p_1 = 1$ and $p_1 = 2$. This result is accurate because $\Delta_1 = 2.5\sigma_1$ and $\Delta_2 = \Delta_4 = 0$. Moreover, $H_{(2,4)a}$ was identified with high frequencies mainly because their correlation is very high ($\rho_{24} = 0.9130$) so the hypothesis $H_{(2,4),a}$ is basically about consistency (between the true and the hypothesized mean) of one variable instead of 2 variables in other alternative hypotheses. Furthermore, $H_{2a} : \Delta_2 = 0$ and $H_{3a} : \Delta_3 = 0$ for $p_1 = 1$, and $H_{(2,3)a} : (\Delta_2, \Delta_3) = (0, 0)$ and $H_{(3,5)a} : (\Delta_3, \Delta_5) = (0, 2.5\sigma_5)$ for $p_1 = 2$ are also likely identified. In this case, simulated data on five variables should be consistent with the most likely H_{ja}'s and $H_{(j_1,j_2)a}$'s. However, $H_{5a} : \Delta_5 = 2.5\sigma_5$ and $H_{(1,5)a} : (\Delta_1, \Delta_5)' = (2.5\sigma_1, 2.5\sigma_5)'$ received the lowest frequencies mainly because the standard deviation for X_5 is the largest and X_1 and X_5 have low correlation ($\rho_{15} = 0.2652$), resulting in the data as least reliable for X_5 and that we are dealing with two rather uncorrelated variables simultaneously. The results for Scenarios (3) and (4) can be similarly interpreted as in Scenario (2).

To sum up, the above results show that we can give a good detection of out-of-control *individual* variables when the *individual* alternative is consistent with the *individual* true shift. Results of other cases within each scenario indicate that, when considering Case (A), we were actually assessing the consistency between hypothesized shifts and the true shifts through the data we collected. The interpretation of the results in Case (A) is somewhat different from that for the Case (B) where no alternative hypotheses with known directions for the shifts were specified. In this subsection, we only studied the proposed method for positive mean shifts; but we also did a simulation study for negative mean shifts and obtained similar conclusions. Furthermore, we also did the extreme case with independent variables and found the mean shifts can be more precisely identified by the proposed method.

4.6 Comparisons with Results of Other Methods

Mason, Tracy, and Young (1995) present a decomposition of the overall T^2 statistic to help interpret multivariate control charts. Their procedure is illustrated in Section 4.2 (also see Mason and Young (2002)) and their decomposition is called the MTY decomposition. It, along with the regression adjustment method of Hawkins (1991, 1993), are popular techniques for interpretation out-of-control signal in a T^2 chart. In this section, we will compare our proposed method with the MTY and Hawkins' method using the same examples of Flury and Riedwyl (1988) and Jackson (1980), both were discussed in Section 4.5.

4.6.1 Example of Flury and Riedwyl

Flury and Riedwyl (1988, p. 151) gave 50 switch drums data (see Table 4.1). We will treat these data the"reference sample"and their summary statistics are given in (4.5.1). They also gave two observations and one of them is out-of-control in the T^2 chart. This observation is $X=(13, 9, 12, 12, 7)'$ and its observed $T^2= 15.17188$, which is greater than the critical value $5(50-1)/(50-5)F_{5,50-5}(0.95)= 13.18691$, at $\alpha = 0.05$.

For the MTY decomposition method, the unconditional individual T^2 value, T_j^2 for $j = 1, \ldots, 5$, for the five variables of X are 6.95528, 0.56973, 1.03973, 0.23684, and 0.31828, respectively. From their result in (4.2.1) that $T_j^2 \sim F_{1,50-1}$, only the unconditional univariate T^2 value of X_1 is greater that the corresponding critical value of $F_{1,50-1}(0.95)= 4.038$ at $\alpha = 0.05$. Next, since the T^2 value of the subvector $(9, 12, 12, 7)'$ of X is 11.20225, which is greater than the critical value $4(50-1)/(50-4)F_{4,50-4}(0.95)= 10.96763$ at $\alpha = 0.05$, we only need to compute the conditional T^2 terms, $T_{i.j}^2$, for Variables 2 to 5. These values are given in Table 4.9 Since $T_{i.j}^2 \sim (50-1)/(50-1-1)F_{1,50-1-1}$ (see (4.2.3)), then only two conditional terms, namely $T_{2.4}^2$ and $T_{4.2}^2$, are greater than the corresponding critical value $(50-1)/(50-2)F_{1,50-2}= 4.127$. And the decomposition will be stopped according to their proposed procedure (see subsection 4.2.2) because the T^2 value of $(12, 7)'$ is 1.2696, which is less than the critical value of$2(50-1)/(50-2)F_{2,50-2}(0.95)= 6.514$ at $\alpha = 0.05$. Thus, only X_1, X_2, and X_4 are singled out based on the MTY decomposition method.

For the regression adjustment method, Hawkins (1991) proposed a Z-statistic with $Z = [diag(\Sigma^{-1})]^{-1/2}\Sigma^{-1}(X-\mu_0)$. The jth element, Z_j, of Z is the residual when X_j is regressed on all other components of X. However, the assumption that other variables are in-control is a strong one, since we do not know whether that is true or not. We can modify Hawkins' method to allow the situation when this assumption is not satisfied, as follows. In fact, if X follows $N(\mu, \Sigma)$, Z follows

$$N([diag(\Sigma^{-1})]^{-1/2}\Sigma^{-1}(\mu-\mu_0), [diag(\Sigma^{-1})]^{-1/2}\Sigma^{-1}[diag(\Sigma^{-1})]^{-1/2})$$

such that $Var(Z_j) = 1$ for all j. So, if a variable is not in control, its new population mean can be estimated by the sample mean when calculating the mean vector of Z. Let m_j be the jth element of the mean vector of Z assuming $\mu_j = \mu_{j0}$, then the test statistic for testing $H_{j0} : \mu_j = \mu_{j0}$ is $(Z_j - m_j)^2$, which follows $\chi^2(1)$. Of course, m_j depends on whether other variables are in control or not. This approximately true if the reference sample mean and covariance matrix in (4.5.1) obtained from a large sample are used to replaceμ_0 and Σ. The transformed Z for the observation X is (-2.0122, -2.66797, 0.73474, 2.8089, -1.10562)'. Because m_j =0 for j =1, . . .,5 and, under $H_{j0} : \mu_j = \mu_{j0}$, Z_j are observations from $\chi^2(1)$. Also, Z_1^2, Z_2^2, and Z_4^2 are greater than the critical value $\chi^2(1, 0.95)$= 3.841 at $\alpha = 0.05$, we can conclude that X_1, X_2, and X_4 are out-of-control based on the regression adjustment method.

Now, assume no alternative is given in terms of our method, the conditional $\ell'_j(H_{j0}|H_a)$ values of X, given T^2= 15.17188 when $p_1 = 1$, $p_1 = 2$, and $p_1 = 3$, are given in Table 4.10 Therefore, the ranking of five variables from the smallest to the largest according to their $\ell'_j(H_{j0}|H_a)$ values for $p_1 = 1$ is: X_1,X_3,X_5,X_2, and X_4. We observe that the likelihood value of X_1 is less than 0.025 times that of those of other variables and also the differences among the four likelihood values of the remaining variables are not far from 10^{-2}. This means H_{10} is the least likely to be true. It can also be observed that the component-wise distance between X and $\bar{X}$, $-2.6635\sigma_1$, $-0.7623\sigma_2$, $-1.0298\sigma_3$, $0.4915\sigma_4$, $-0.5698\sigma_5$, X_1 has the largest absolute distance and X_3 second. Also, the correlation $\rho_{13} = 0.3496$ is the largest among the correlations with X_1. Hence, it is reasonable to say that X_1 is the most likely variable to have a shift in mean, and X_3 might be the next one to have a shift in mean.

When considering two variables at a time, the likelihood values for all 10 combinations are given in Table 4.10 and the two variables with the least conditional joint likelihood are (X_1,X_4) and (X_2,X_4) , and their values are close. It can be observed that, among all distances of five variables from the estimated in-control means, X_1 has the largest value in the negative direction, $(-2.6635\sigma_1)$, and X_4 has the largest value in the positive direction, $(0.4915\sigma_4)$, but is correlation $\rho_{14} = 0.0829$ is the smallest among the correlations with X_1. Moreover, the distance of X_2 is in the negative direction, $(-0.7623\sigma_2)$, but the distance of X_4 is in the positive direction. It seems a contradiction since $\rho_{24} = 0.913$ is a very strong positive correlation. Although these two pairs include the variableX_4, the deviation in of X_1 seems to be large enough to make the conditional joint likelihood value of (X_1,X_4) the smallest. Furthermore, when considering three variables at a time (i.e.,$p_1 = 3$), the likelihood values for all 10 combinations are also given in Table 4.10 and the three variables corresponding to the smallest likelihood are (X_1,X_2,X_4). From above explanation for the results when$p_1 = 2$, it seems reasonable to conclude that Variables (X_1,X_2,X_4) have shifts in mean when $p_1 = 3$.

4.6.2 Example of Jackson

Jackson's example is illustrated in subsection 4.5.2. Recall that, a new individual observation is X = (15, 10, 20, -5)$'$ with a T^2 value at 15.92142, which is larger than the critical value 11.412 with $\alpha = 0.05$.

For MTY decomposition method, the unconditional individual T^2 value, T_j^2 for $j = 1, \ldots, 4$, for five variables of X are 2.13658, 0.68349, 4.61445, and 0.24622, respectively. Hence, only the unconditional T^2 value of X_3 is greater that the critical value $F_{1,40-1}(0.95)$= 4.091 with $\alpha = 0.05$. Since the T^2? value of (15, 10, -5)$'$ is 4.625, which is less than the critical value $3(40-1)/(40-3)F_{3,40-3}(0.95)$= 9.03998 with $\alpha = 0.05$, then the decomposition will be stopped according to their proposed procedure. Thus, only X_3 is singled out in terms of MTY decomposition method.

For the regression adjustment method of Hawkins (1991), the transformed Z is (0.14569, -0.80709, 3.40281, -3.05744)$'$. Since m_j =0 for j=1,. . .,5 and, under $H_{j0} : \mu_j = \mu_{j0}$, Z_j^2 , $j = 1, \ldots, 5$, are 0.02123, 0.65139, 11.57911, and 9.34796, respectively. Hence Z_3^2 and Z_4^2 are greater than the critical value $\chi^2(1, 0.95)$= 3.84146 with $\alpha = 0.05$. It can be concluded that Variables X_3 and X_4 are out-of-control from the regression adjustment method.

Our result for this example is illustrated in subsection 4.5.2. Recall that X_3 is the most likely to be a problematic variable for $p_1 = 1$ and X_1 might be the next one. When $p_1 = 2$, (X_3, X_4) is a set of two variables to be the most likely to have a shift in mean.

4.7 Discussion

Multivariate statistical process control is an important application area of statistics, where Hotelling's T^2 is a popular statistic for monitoring the mean vector of a multivariate variable. But it has some drawbacks and the major one is that it does not directly detect the significant individual variable(s) when the aggregated T^2 statistic indicates that the process mean vector had changed.

There have been researches to deal with this major drawback of Hetelling's T^2 statistic, for example, Mason, Tracy, and Young (1995), Hawkins (1993), Murphy (1987), Doganaksoy, Faltin, and Tucker (1991), and so on. In this chapter, we propose a method, which is based on the likelihood principle and can determine the likelihood as to which variable or a set of variables that is most likely to have caused a shifted in the mean vector for the case with unknown population covariance. Several examples are used to illustrate the effectiveness of the proposed method. A possible further extension of the current chapter is to consider the case where subgroups contain more than one observation.

TABLES

Table 4.1 The Data of Switch Drums from Flury and Riedwyl (1988)

	X_1	X_2	X_3	X_4	X_5		X_1	X_2	X_3	X_4	X_5		X_1	X_2	X_3	X_4	X_5
1	20	8	13	9	7	18	19	10	13	11	7	35	16	10	15	11	8
2	20	12	17	12	11	19	17	9	13	9	6	36	18	6	12	8	6
3	15	7	9	7	2	20	19	11	15	11	9	37	19	14	16	16	13
4	16	12	14	13	8	21	17	8	13	9	7	38	17	12	16	13	12
5	20	12	16	13	9	22	17	11	14	11	8	39	18	13	16	14	12
6	16	10	15	10	8	23	18	7	13	8	7	40	17	8	13	10	10
7	16	10	12	10	5	24	19	11	15	11	9	41	15	8	11	10	8
8	20	11	15	11	8	25	16	9	11	9	6	42	22	9	13	9	8
9	18	10	14	12	9	26	15	11	16	12	9	43	18	10	13	12	9
10	19	11	14	11	7	27	17	10	16	11	11	44	17	13	15	14	12
11	21	7	11	7	4	28	20	9	12	10	7	45	16	10	12	10	8
12	17	10	12	10	6	29	18	13	15	14	10	46	20	12	17	12	12
13	19	10	14	10	8	30	20	12	15	14	10	47	20	11	15	12	11
14	18	12	14	12	7	31	16	10	12	10	6	48	19	11	14	11	8
15	15	10	12	12	7	32	20	11	15	13	9	49	20	11	14	12	11
16	14	10	12	11	6	33	17	11	13	12	7	50	18	11	14	12	10
17	21	10	14	10	7	34	18	11	13	13	8						

Table 4.2 Simulated Data of Switch Drums from Hawkins (1991)

	X_1	X_2	X_3	X_4	X_5		X_1	X_2	X_3	X_4	X_5
1	17.265	11.788	15.101	13.903	10.465	26	18.325	7.0040	12.773	8.1360	5.3260
2	17.384	6.9960	11.552	7.2530	6.6410	27	17.652	9.9300	13.904	9.7470	6.1050
3	16.517	10.277	11.724	13.013	9.1110	28	16.615	11.221	14.151	12.629	10.601
4	14.997	10.682	12.087	11.457	6.3200	29	14.606	8.5420	11.834	9.5870	5.7900
5	17.633	9.3480	12.672	10.475	5.4810	30	19.074	9.5500	13.044	11.688	9.4500
6	16.041	11.320	13.957	11.474	8.1760	31	22.449	10.093	16.306	11.806	11.239
7	15.339	10.384	12.313	9.4010	7.2520	32	18.401	11.856	13.608	10.832	7.7090
8	17.144	12.254	14.931	13.715	11.135	33	18.556	12.174	14.111	11.965	9.0740
9	20.351	10.028	14.271	11.124	8.9940	34	17.727	10.740	13.100	11.012	8.4290
10	19.586	11.083	15.019	12.126	9.4410	35	19.141	10.033	13.524	10.800	8.3830
11	20.153	13.100	16.231	13.628	8.7800	36	17.554	9.1320	11.563	10.554	6.5930
12	18.044	9.6990	11.807	11.655	7.5130	37	19.564	10.784	14.200	11.909	10.681
13	17.041	9.7480	13.576	9.3330	7.3160	38	20.985	10.191	15.129	11.301	9.0870
14	17.671	13.223	15.937	15.119	12.129	39	20.745	10.781	14.403	9.4690	7.4510
15	16.306	9.1400	13.239	10.982	8.9000	40	15.395	7.7940	10.602	8.8260	5.9330
16	15.977	9.9040	12.822	9.9100	7.1900	41	16.014	7.9280	11.872	7.1970	6.6490
17	18.517	11.401	16.883	13.162	12.861	42	15.220	12.697	15.201	13.891	10.849
18	16.591	12.875	14.542	13.787	7.9310	43	19.978	11.101	13.687	11.554	8.2840
19	17.576	10.686	13.072	11.257	5.9330	44	20.886	9.5780	13.724	10.914	11.075
20	17.225	8.9430	13.033	9.0880	6.1760	45	16.323	8.7110	12.084	8.5340	5.7150
21	19.234	11.575	15.192	11.809	11.418	46	16.120	10.997	14.497	12.918	11.921
22	19.379	10.421	13.095	11.898	7.8810	47	17.037	8.3620	13.290	10.097	9.4840
23	16.009	7.4780	10.291	7.2070	3.1600	48	13.065	11.625	14.923	12.589	12.446
24	15.944	10.086	14.438	10.652	6.9160	49	16.188	9.1400	13.284	10.991	9.1260
25	16.541	8.1970	12.520	9.5860	9.3040	50	22.047	10.824	14.796	10.872	9.2640

Table 4.3 Conditional Likelihood, $\ell'_j(H_{j0}|H_a)$, for Example 1 with p_1=1,2

$\ell'_j(H_{j0}|H_a)$ for p_1=1				
X_1	X_2	X_3	X_4	X_5
3.2946×10^{-3}	1.8504×10^{-1}	1.8549×10^{-1}	1.6491×10^{-1}	2.3213×10^{-2}

$\ell'_j(H_{j0}|H_a)$ for $p_1 = 2$				
(X_1, X_2)	(X_1, X_3)	(X_1, X_4)	(X_1, X_5)	(X_2, X_3)
2.8832×10^{-4}	6.8438×10^{-6}	1.8899×10^{-4}	4.0213×10^{-7}	6.5961×10^{-2}
(X_2, X_4)	(X_2, X_5)	(X_3, X_4)	(X_3, X_5)	(X_4, X_5)
8.3798×10^{-2}	7.0577×10^{-3}	7.0577×10^{-2}	6.1045×10^{-3}	4.2361×10^{-3}

Table 4.4 Conditional Likelihood, $\ell'_j(H_{j0}|H_a)$, for Example 2 with p_1= 1,2

$\ell'_j(H_{j0}|H_a)$ for p_1=1			
X_1	X_2	X_3	X_4
1.2731×10^{-2}	2.4014×10^{-2}	3.2872×10^{-3}	3.6412×10^{-2}

$\ell'_j(H_{j0}|H_a)$ for $p_1 = 2$			
(X_1, X_2)	(X_1, X_3)	(X_1, X_4)	(X_2, X_3)
6.1306×10^{-4}	2.0351×10^{-4}	1.7315×10^{-4}	5.8331×10^{-5}
(X_2, X_4)	(X_3, X_4)		
4.5075×10^{-4}	2.2421×10^{-7}		

Table 4.5 The Results of Example 1 for Scenario (1) of Case (A)

	Scenario (1) ($p_1 = 1$)	Number of Times Ranked First					Number of Times Ranked Second				
Δ'	$(2.5,0,0,0,0)\cdot D_\sigma$	X_1	X_2	X_3	X_4	X_5	X_1	X_2	X_3	X_4	X_5
$\Delta^{*'}$	$(\mathbf{2.5},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})\cdot D_\sigma$	330	84	67	14	5	50	131	187	89	43
	$(0,2.5,\mathbf{0},\mathbf{0},\mathbf{0})\cdot D_\sigma$	7	54	252	142	45	7	29	106	179	179
	$(\mathbf{2.5},\mathbf{0},\mathbf{0},2.5,\mathbf{0})\cdot D_\sigma$	91	136	193	50	30	59	184	104	26	127
	$(\mathbf{2.5},2.5,2.5,2.5,2.5)\cdot D_\sigma$	470	8	21	1	0	10	109	66	150	165

Table 4.6 The Results of Example 1 for Scenario (2) of Case (A)

	Scenario (1) ($p_1 = 1$)	Number of Times Ranked First					Number of Times Ranked Second				
Δ'	$(2.5,0,0,0,2.5)\cdot D_\sigma$	X_1	X_2	X_3	X_4	X_5	X_1	X_2	X_3	X_4	X_5
$\Delta^{*'}$	$(\mathbf{2.5},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	188	103	166	29	14	93	133	111	107	56
	$(0,\mathbf{0},\mathbf{0},2.5,0)\cdot D_\sigma$	16	184	236	38	26	26	223	185	31	35
	$(\mathbf{2.5},2.5,\mathbf{0},\mathbf{0},0)\cdot D_\sigma$	166	49	198	77	10	143	26	127	171	33
	$(\mathbf{2.5},\mathbf{0},2.5,\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	248	155	11	36	50	75	156	15	160	94
	$(\mathbf{2.5},2.5,2.5,2.5,\mathbf{2.5})\cdot D_\sigma$	292	4	4	4	196	178	13	3	16	290
	Scenario (2) ($p_1 = 2$)	Number of Times Ranked First									
Δ'	$(2.5,0,0,0,2.5)\cdot D_\sigma$	x_1,x_2	x_1,x_3	x_1,x_4	x_1,x_5	x_2,x_3	x_2,x_4	x_2,x_5	x_3,x_4	x_3,x_5	x_4,x_5
$\Delta^{*'}$	$(\mathbf{2.5},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	15	46	9	5	57	232	6	16	85	29
	$(0,\mathbf{0},\mathbf{0},2.5,0)\cdot D_\sigma$	11	8	1	23	433	0	9	13	2	0
	$(\mathbf{2.5},2.5,\mathbf{0},\mathbf{0},0)\cdot D_\sigma$	34	143	81	31	5	0	0	203	1	2
	$(\mathbf{2.5},\mathbf{0},2.5,\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	21	16	15	27	0	326	17	0	0	78
	$(\mathbf{2.5},2.5,2.5,2.5,\mathbf{2.5})\cdot D_\sigma$	0	4	1	451	1	31	3	3	0	6
	Scenario (2) ($p_1 = 2$)	Number of Times Ranked Second									
Δ'	$(2.5,0,0,0,2.5)\cdot D_\sigma$	x_1,x_2	x_1,x_3	x_1,x_4	x_1,x_5	x_2,x_3	x_2,x_4	x_2,x_5	x_3,x_4	x_3,x_5	x_4,x_5
$\Delta^{*'}$	$(\mathbf{2.5},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	41	62	21	10	116	66	28	27	79	50
	$(0,\mathbf{0},\mathbf{0},2.5,0)\cdot D_\sigma$	105	66	5	60	36	4	78	97	49	0
	$(\mathbf{2.5},2.5,\mathbf{0},\mathbf{0},0)\cdot D_\sigma$	27	185	116	36	10	0	0	122	4	0
	$(\mathbf{2.5},\mathbf{0},2.5,\mathbf{0},\mathbf{2.5})\cdot D_\sigma$	105	2	42	44	1	81	69	1	0	155
	$(\mathbf{2.5},2.5,2.5,2.5,\mathbf{2.5})\cdot D_\sigma$	36	92	170	15	2	98	33	10	10	34

Table 4.7 The Results of Example 1 for Scenario (3) of Case (A)

	Scenario (1) ($p_1 = 1$)	Number of Times Ranked First					Number of Times Ranked Second				
Δ'	$(0,2.5,0,2.5,0)\cdot D_\sigma$	X_1	X_2	X_3	X_4	X_5	X_1	X_2	X_3	X_4	X_5
$\Delta^{*'}$	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	130	228	76	60	6	64	94	160	136	46
	$(2.5,0,\mathbf{0},0,\mathbf{0})\cdot D_\sigma$	18	25	343	16	98	12	22	98	30	338
	$(2.5,0,\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	60	18	248	116	58	36	19	102	99	244
	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},2.5)\cdot D_\sigma$	125	183	132	35	25	101	100	132	145	22
	$(2.5,\mathbf{2.5},2.5,\mathbf{2.5},2.5)\cdot D_\sigma$	11	317	18	147	7	6	153	7	318	16
	Scenario (3) ($p_1 = 2$)	Number of Times Ranked First									
Δ'	$(2.5,0,0,0,2.5)\cdot D_\sigma$	X_1,X_2	X_1,X_3	X_1,X_4	X_1,X_5	X_2,X_3	X_2,X_4	X_2,X_5	X_3,X_4	X_3,X_5	X_4,X_5
$\Delta^{*'}$	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	3	5	3	5	235	140	2	35	10	62
	$(2.5,0,\mathbf{0},0,\mathbf{0})\cdot D_\sigma$	0	5	0	5	0	59	0	5	426	0
	$(2.5,0,\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	5	9	6	8	0	0	1	98	282	91
	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},2.5)\cdot D_\sigma$	4	30	17	25	124	233	0	66	0	1
	$(2.5,\mathbf{2.5},2.5,\mathbf{2.5},2.5)\cdot D_\sigma$	1	0	2	0	0	479	3	0	15	0
	Scenario (3) ($p_1 = 2$)	Number of Times Ranked Second									
Δ'	$(2.5,0,0,0,2.5)\cdot D_\sigma$	X_1,X_2	X_1,X_3	X_1,X_4	X_1,X_5	X_2,X_3	X_2,X_4	X_2,X_5	X_3,X_4	X_3,X_5	X_4,X_5
$\Delta^{*'}$	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	16	15	8	8	100	162	23	48	17	103
	$(2.5,0,\mathbf{0},0,\mathbf{0})\cdot D_\sigma$	1	13	0	65	4	275	34	51	56	1
	$(2.5,0,\mathbf{0},\mathbf{2.5},\mathbf{0})\cdot D_\sigma$	3	38	14	11	0	1	4	189	118	122
	$(\mathbf{0},\mathbf{2.5},\mathbf{0},\mathbf{2.5},2.5)\cdot D_\sigma$	82	70	19	10	156	103	1	56	0	3
	$(2.5,\mathbf{2.5},2.5,\mathbf{2.5},2.5)\cdot D_\sigma$	103	1	109	0	8	16	133	15	112	3

Table 4.8 The Results of Example 1 for Scenario (4) of Case (A)

	Scenario (1) ($p_1=1$)	Number of Times Ranked First					Number of Times Ranked Second				
Δ'	(0,0,2.5,0,2.5)·D_σ	X_1	X_2	X_3	X_4	X_5	X_1	X_2	X_3	X_4	X_5
$\Delta^{*\prime}$	(**0**,**0**,**2.5**,**0**,**2.5**)·D_σ	124	88	258	20	10	94	157	91	82	76
	(**0**,**0**,0,2.5,0)·D_σ	205	247	21	20	7	230	155	30	55	30
	(2.5,**0**,**2.5**,**0**,0)·D_σ	10	110	311	46	23	13	229	90	161	7
	(2.5,**0**,**2.5**,**0**,**2.5**)·D_σ	8	78	332	32	50	6	241	56	100	97
	(2.5,2.5,**2.5**,2.5,**2.5**)·D_σ	7	2	372	0	119	8	3	114	3	372
	Scenario (4) ($p_1=2$)	Number of Times Ranked First									
Δ'	(2.5,0,0,0,2.5)·D_σ	X_1,X_2	X_1,X_3	X_1,X_4	X_1,X_5	X_2,X_3	X_2,X_4	X_2,X_5	X_3,X_4	X_3,X_5	X_4,X_5
$\Delta^{*\prime}$	(**0**,**0**,**2.5**,**0**,**2.5**)·D_σ	0	25	0	17	203	129	3	70	27	26
	(**0**,**0**,0,2.5,0)·D_σ	368	13	56	13	2	0	4	0	44	0
	(2.5,**0**,**2.5**,**0**,0)·D_σ	1	4	1	1	239	157	0	97	0	0
	(2.5,**0**,**2.5**,**0**,**2.5**)·D_σ	0	1	0	2	254	90	9	85	22	37
	(2.5,2.5,**2.5**,2.5,**2.5**)·D_σ	1	4	0	2	0	6	0	5	482	0
	Scenario (4) ($p_1=2$)	Number of Times Ranked Second									
Δ'	(2.5,0,0,0,2.5)·D_σ	X_1,X_2	X_1,X_3	X_1,X_4	X_1,X_5	X_2,X_3	X_2,X_4	X_2,X_5	X_3,X_4	X_3,X_5	X_4,X_5
$\Delta^{*\prime}$	(**0**,**0**,**2.5**,**0**,**2.5**)·D_σ	7	50	6	15	102	110	18	81	59	52
	(**0**,**0**,0,2.5,0)·D_σ	89	41	201	84	5	0	18	0	62	0
	(2.5,**0**,**2.5**,**0**,0)·D_σ	6	6	2	0	198	136	1	150	0	1
	(2.5,**0**,**2.5**,**0**,**2.5**)·D_σ	0	2	0	0	105	89	15	154	70	65
	(2.5,2.5,**2.5**,2.5,**2.5**)·D_σ	0	169	0	17	18	135	16	127	10	8

Table 4.9 The Conditional $T^2_{i.j}$ Values in MTY Decompositions for Flury and Riedwyl's data

$T^2_{i.j}$					
$T^2_{3.2}$	$T^2_{2.3}$	$T^2_{4.2}$	$T^2_{2.4}$	$T^2_{5.2}$	$T^2_{2.5}$
0.4701	1.364×10^{-4}	8.3046	8.6375	3.222×10^{-3}	0.2547
$T^2_{4.3}$	$T^2_{3.4}$	$T^2_{5.3}$	$T^2_{3.5}$	$T^2_{5.4}$	$T^2_{4.5}$
2.6164	3.4193	0.2299	0.9513	2.1044	2.023

Table 4.10 Conditional Likelihood, $\ell'_j(H_{j0}|H_a)$ for Flury and Riedwyl' data

$\ell'_j(H_{j0}\|H_a)$ for p_1=1				
X_1	X_2	X_3	X_4	X_5
3.1534×10^{-3}	1.8052×10^{-2}	1.3859×10^{-2}	1.9752×10^{-2}	1.5991×10^{-2}
$\ell'_j(H_{j0}\|H_a)$ for $p_1 = 2$				
(X_1, X_2)	(X_1, X_3)	(X_1, X_4)	(X_1, X_5)	(X_2, X_3)
7.7808×10^{-4}	9.2096×10^{-4}	5.5605×10^{-4}	6.7946×10^{-4}	5.0332×10^{-2}
(X_2, X_4)	(X_2, X_5)	(X_3, X_4)	(X_3, X_5)	(X_4, X_5)
5.5643×10^{-4}	4.7339×10^{-2}	9.5774×10^{-3}	4.0861×10^{-2}	1.8103×10^{-2}
$\ell'_j(H_{j0}\|H_a)$ for $p_1 = 3$				
(X_1, X_2, X_3)	(X_1, X_2, X_4)	(X_1, X_2, X_5)	(X_1, X_3, X_4)	(X_1, X_3, X_5)
3.2131×10^{-4}	4.2031×10^{-6}	1.9207×10^{-5}	1.4404×10^{-4}	3.0768×10^{-4}
(X_1, X_2, X_3)	(X_1, X_4, X_5)	(X_2, X_3, X_4)	(X_2, X_3, X_5)	(X_3, X_4, X_5)
1.4629×10^{-4}	1.4665×10^{-4}	1.4991×10^{-2}	3.9315×10^{-5}	3.5339×10^{-3}

Chapter 5

Summary, Conclusions, and Future Work

In this dissertation, we have presented and studied three subjects for monitoring multivariate process control.

In Chapter 2, we have proposed and studied a control chart based on the one-sided likelihood ratio test that is specifically designed for detecting dispersion decreases for multivariate processes. Both cases when the in-control covariance matrix $\mathbf{\Sigma}_0$ is known or unknown are considered and, for each case, the LRT statistic is derived. A comparative simulation study is conducted and shows that the proposed control chart indeed outperforms the existing two-sided-tests-based control charts in terms of the average run length, when process dispersion decreases. The applicability and effectiveness of the proposed control chart are demonstrated through a real example and two simulated examples.

In Chapter 3, we have proposed and studied a combined control chart constructed by combining the two one-sided likelihood-ratio-test-based control charts that are specifically designed for detecting dispersion increases and decreases respectively for multivariate processes. The chart for the increases part was proposed by Yen and Shiau (2008) and the decrease part is proposed in Chapter 2 and they are in essence a multivariate extension of the two-sided unequal tail test of univariate variance. Both cases when the in-control covariance matrix $\mathbf{\Sigma}_0$ is known or unknown are considered. It was shown that the control limit does not depend on $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$. Two real examples and two simulated examples are used to illustrate the applicability and effectiveness of our proposed combined chart.

The proposed control chart in Chapter 2 is Shewhart-type. It is well known that EWMA and CUSUM charts are more sensitive to small changes. An EWMA extension of the proposed chart will be reported in a follow-up paper. Furthermore, for another research issue, we also can conduct researches in the same line as that proposed in Yen and Shiau (2008) and in Chapter 2 but now dealing with individual observations instead of subgroups. The difficultly comes from the fact that we cannot obtain an estimator of the covariance matrix $\boldsymbol{\Sigma}$ from a single observation vector. Note that with a subgroup of size n, for the sample covariance matrix to be positive definite with probability one, we need $n > p$ (Dykstra, 1970). A common remedy for this is to borrow some strength from neighbors. More specifically, we can use the EWMA approach to obtain an estimate for the covariance matrix at time $t > p$. One can develop new LRT-like control charts respectively for monitoring (i) only increase, (ii) only decrease, and (iii) either directions in process dispersion by first using the EWMA approach to obtain estimators for the covariance matrix and then following similar approaches as we did in Yen and Shiau (2008), in Chapter 2, and in Chapter 3 for the subgroup case. Both cases when the in-control covariance matrix $\boldsymbol{\Sigma}_0$ is known or unknown can be considered. Therefore, this future work would cover more situations and have wider range for applications. The EWMA approach gives a legitimate estimator for the covariance matrix but causes a troublesome side-effect that the monitoring statistics are correlated at nearby time points, which makes computing ARL and finding the appropriate control limit a lot more difficult. It needs to find some better methods to speed up computation.

In chapter 4 of this dissertation, Multivariate SPC is an important application area of statistics, where Hotelling's T^2 is a popular statistic for monitoring the mean vector of a multivariate variable. But it has some drawbacks and the major one is that it does not directly detect the significant individual variable(s) when the aggregated T^2 statistic indicates that the process mean vector had changed.

When an out-of-control signal is given by a T^2 chart, we propose a method, which is based on the likelihood principle and can determine the likelihood as to which variable or a set of variables that is most likely to have caused a shift in the mean vector for the case of unknown population covariance. Our proposed method computes the conditional likelihood that an individual mean (or a group of means) is out of control, given $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ is rejected because of a significant T^2 value. By ranking these likelihoods, we can identify the mean that is mostly likely to be out-of-control. Several examples are used to illustrate the effectiveness of the proposed method.

Our method, which is a diagnostic tool, is different from most of the existing methods such as that in Mason, Tracy, and Young (1995), because their methods do not assume the rejection of $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ as given and this is why their critical values for their statistics are all based on certain central (instead of noncentral) distributions. That is, these methods basically assume that all the other individual means are in control when they are testing a particular univariate mean. Nevertheless, our method should be considered as a complementary tool, not a substitute, to the existing methods. Furthermore, a possible further extension of the current method is to consider the case where subgroups contain more than one observation.

Another slated issue is that of determining which parameters of the covariance matrix have actually changed when a control chart detects an out-of-control signal. Such a task is more complicated than that of the multivariate process mean due to the complexity of the covariance matrix. Unlike the case of the process mean with p parameters, there are a total of $p(p+1)/2$ parameters in the covariance matrix that could possibly change and trigger an out-of-control signal. It is of eminent importance to be able to further pinpoint which of these parameters are out of control. Therefore, developing a diagnostic technique could be a potential future research topic as well.

Appendix A

A.1 Proof of Theorem 2.1

The likelihood function of n observations, $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$, is

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{X}_{tj}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_{tj}-\boldsymbol{\mu})\right\}. \tag{A.1.1}$$

Recall that $\boldsymbol{\Theta} = \boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}$. To maximize $L(\boldsymbol{\mu}, \boldsymbol{\Theta})$, we first note that $\hat{\boldsymbol{\mu}} \equiv \bar{\boldsymbol{X}}_t$ is the MLE of $\boldsymbol{\mu}$. Since $\boldsymbol{\Sigma}_0$ is known, rewrite the log likelihood function of (A.1.1) concentrated with respect to $\hat{\boldsymbol{\mu}}$ as

$$\ell(\hat{\boldsymbol{\mu}}, \boldsymbol{\Theta}) = -\frac{pn}{2}\log 2\pi - \frac{n}{2}\log|\boldsymbol{\Sigma}_0 - \boldsymbol{\Theta}| - \frac{n}{2}tr\Big[\boldsymbol{S}_t(\boldsymbol{\Sigma}_0 - \boldsymbol{\Theta})^{-1}\Big]. \tag{A.1.2}$$

Since $\boldsymbol{\Theta}$ is symmetric and positive semidefinite, from Theorem 4.14 of Schott (2005), there exists a nonsingular matrix $\boldsymbol{\Gamma}$ such that $\boldsymbol{\Theta} = \boldsymbol{\Gamma}\boldsymbol{D}_\varsigma\boldsymbol{\Gamma}'$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{\Gamma}\boldsymbol{\Gamma}'$, where $\boldsymbol{D}_\varsigma = diag(\varsigma_1, \cdots, \varsigma_p)$ with $\varsigma_1 \geq \cdots \geq \varsigma_k > \varsigma_{k+1} = \cdots = \varsigma_p = 0$ being the roots of $|\boldsymbol{\Theta} - \varsigma\boldsymbol{\Sigma}_0| = 0$, by the assumptions that $\boldsymbol{\Sigma}_0$ is positive definite and $rank(\boldsymbol{\Theta}) = k$. Let $\boldsymbol{D}_\delta = diag(\delta_1, \cdots, \delta_p)$ with $\delta_i > 0$, $i = 1, \cdots, p$, be the roots of $|\boldsymbol{\Sigma} - \delta\boldsymbol{\Sigma}_0| = 0$. Then $\boldsymbol{D}_\delta = \boldsymbol{I}_p - \boldsymbol{D}_\varsigma$ with $\delta_i = 1 - \varsigma_i$, $i = 1, \cdots, p$ and $0 < \delta_1 \leq \cdots \leq \delta_k < \delta_{k+1} = \cdots = \delta_p = 1$. Hence, $|\boldsymbol{\Sigma}_0 - \boldsymbol{\Theta}| = |\boldsymbol{\Gamma}\boldsymbol{\Gamma}' - \boldsymbol{\Gamma}\boldsymbol{D}_\varsigma\boldsymbol{\Gamma}'| = |\boldsymbol{\Gamma}\boldsymbol{D}_\delta\boldsymbol{\Gamma}'| = |\boldsymbol{\Sigma}_0||\boldsymbol{D}_\delta|$. Thus $\log|\boldsymbol{\Sigma}_0 - \boldsymbol{\Theta}| = \log|\boldsymbol{\Sigma}_{\boldsymbol{0}}| + \log|\boldsymbol{D}_\delta|$. Similarly, there exists a nonsingular matrix $\boldsymbol{Z}$ such that $\boldsymbol{S}_t = \boldsymbol{Z}\boldsymbol{D}_d\boldsymbol{Z}'$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{Z}\boldsymbol{Z}'$, where $\boldsymbol{D}_d = diag(d_1, \cdots, d_p)$ with $d_1 \geq \cdots \geq d_p$ being the roots of $|\boldsymbol{S}_t - d\boldsymbol{\Sigma}_0| = 0$. Then $tr[\boldsymbol{S}_t(\boldsymbol{\Sigma}_0 - \boldsymbol{\Theta})^{-1}] = tr[(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})\boldsymbol{D}_d(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})'\boldsymbol{D}_\delta^{-1}]$. Substituting these results into (A.1.2), we have the following log likelihood function concentrated with respect to $\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{X}}_t$:

$$\ell(\boldsymbol{D}_\delta, \boldsymbol{\Gamma}) = -\frac{pn}{2}\log 2\pi - \frac{n}{2}\log|\boldsymbol{\Sigma}_0| - \frac{n}{2}\log|\boldsymbol{D}_\delta| - \frac{n}{2}tr\Big[(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})\boldsymbol{D}_d(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})'\boldsymbol{D}_\delta^{-1}\Big]. \quad \text{(A.1.3)}$$

Since $\boldsymbol{\Gamma}\boldsymbol{\Gamma}' = \boldsymbol{Z}\boldsymbol{Z}'$, we have $\boldsymbol{I}_p = \boldsymbol{\Gamma}^{-1}\boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{\Gamma}'^{-1} = (\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})'$. Thus $\boldsymbol{\Gamma}^{-1}\boldsymbol{Z}$ is an orthogonal matrix. By the theorem of Von Neumann (1937) (see Appendix C), we obtain that $\min_{\boldsymbol{\Gamma}^{-1}\boldsymbol{Z}}\Big\{tr[(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})\boldsymbol{D}_d(\boldsymbol{\Gamma}^{-1}\boldsymbol{Z})'\boldsymbol{D}_\delta^{-1}]\Big\} = tr[\boldsymbol{D}_d\boldsymbol{D}_\delta^{-1}]$ and a minimizing value of $\boldsymbol{\Gamma}^{-1}\boldsymbol{Z}$ is $\boldsymbol{I}_p$. Therefore, by choosing $\boldsymbol{\Gamma} = \boldsymbol{Z}$, maximizing $\ell(\boldsymbol{D}_\delta, \boldsymbol{\Gamma})$ in (A.1.3) is reduced to maximizing

$$\begin{aligned}\ell(\boldsymbol{D}_\delta) &= -\frac{pn}{2}\log 2\pi - \frac{n}{2}\log|\boldsymbol{\Sigma}_0| - \frac{n}{2}\log\ |\boldsymbol{D}_\delta| - \frac{n}{2}tr\Big[\boldsymbol{D}_d\boldsymbol{D}_\delta^{-1}\Big] \\ &= -\frac{pn}{2}\log 2\pi - \frac{n}{2}\log|\boldsymbol{\Sigma}_0| - \frac{n}{2}\sum_{i=1}^{p}\{\log\delta_i + \frac{d_i}{\delta_i}\} \qquad \text{(A.1.4)}\end{aligned}$$

with respect to $\delta_1, \cdots, \delta_p$. Note that, for fixed d_i, $\log\delta_i + \frac{d_i}{\delta_i}$ reaches its minimum at $\delta_i = d_i$. Thus the maximum of (A.1.4) with respect to $\delta_1\cdots\delta_p$ over the region $0 < \delta_1 \le \cdots \le \delta_k < \delta_{k+1} = \cdots = \delta_p = 1$ occurs at

$$\begin{array}{lll} \delta_i = d_i\ , & if\ \ 0 < d_i < 1 & \Big\}\ \ for\ \ i = 1, \cdots, k, \\ \delta_i = 1\ \ , & if\ \ d_i \ge 1 & \\ \delta_i = 1\ \ , & & \ \ \ for\ \ i = k+1, \cdots, p. \end{array}$$

Let p^* be the number of $0 < d_i < 1$ and $k^* = \min(k, p^*)$. Then $\delta_i = d_i$, $i = 1, \cdots, k^*$, and $\delta_i = 1$, $i = k^* + 1, \cdots, p$.

Let $q^* = p - k^*$. Note that the maximum of (A.1.1) depends only on $\{d_i,\, i = 1, \cdots, k^*\}$ for given k. To simplify notation, denote $\max_{\boldsymbol{\mu},\boldsymbol{\Sigma}}\{L(\boldsymbol{\mu}, \boldsymbol{\Sigma})\}$ for given k by $L^*(k^*)$. Thus, after some simple algebra, the maximum likelihood function of (A.1.1) can be rewritten as

$$L^*(k^*) = (2\pi)^{-\frac{pn}{2}}e^{-\frac{pn}{2}}|\boldsymbol{\Sigma}_0|^{-\frac{n}{2}}\prod_{i=1}^{k^*}d_i^{-\frac{n}{2}}\prod_{i=k^*+1}^{p}\exp\Big[-\frac{n}{2}(d_i - 1)\Big].$$

It is trivial to show that $L^*(k^*)$ is nondecreasing in k^*. Let $k_0^* = \min(k_0, p^*)$ and $k_1^* = \min(k_1, p^*)$. We finally obtain that the LRT statistic for testing (2.2.1) is

$$\frac{\max_{H_0^*} L^*(k^*)}{\max_{H_1^*} L^*(k^*)} = \frac{L^*(k_0^*)}{L^*(k_1^*)} = \begin{cases} \prod_{i=k_0^*+1}^{k_1^*}\{d_i\exp[-(d_i - 1)]\}^{\frac{n}{2}} & ,\ for\ k_0^* < k_1^* \\ 1 & ,\ for\ k_0^* = k_1^* \end{cases}.$$

A.2 Proof of Theorem 2.2

When $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are unknown, the likelihood function of all observations, $\boldsymbol{X}_{11}, \cdots, \boldsymbol{X}_{mn}$, $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$, is

$$L(\boldsymbol{\mu}_0,\, \boldsymbol{\Sigma}_0,\, \boldsymbol{\mu},\, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{pmn}{2}} |\boldsymbol{\Sigma}_0|^{-\frac{mn}{2}} \exp\left[-\frac{1}{2}tr(\boldsymbol{A}{\boldsymbol{\Sigma}_0}^{-1}) - \frac{mn}{2}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0)'{\boldsymbol{\Sigma}_0}^{-1}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0)\right]$$
$$\times (2\pi)^{-\frac{pn}{2}} |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}tr(\boldsymbol{B}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})\right\}.$$

Thus, the log likelihood function is

$$\ell(\boldsymbol{\mu}_0,\, \boldsymbol{\Sigma}_0,\, \boldsymbol{\mu},\, \boldsymbol{\Sigma}) = -\frac{p(mn+n)}{2}\log 2\pi - \frac{mn}{2}\log|\boldsymbol{\Sigma}_0| - \frac{n}{2}\log|\boldsymbol{\Sigma}| - \frac{mn}{2}tr(\boldsymbol{S}_0\boldsymbol{\Sigma}_0^{-1})$$
$$-\frac{n}{2}tr(\boldsymbol{S}_t\boldsymbol{\Sigma}^{-1}) - \frac{mn}{2}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0) - \frac{n}{2}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})\,. \qquad \text{(A.2.1)}$$

We let $\boldsymbol{D}_\delta = diag(\delta_1, \cdots, \delta_p)$ with $0 < \delta_1 \leq \cdots \leq \delta_k < \delta_{k+1} = \cdots = \delta_p = 1$ being the roots of $|\boldsymbol{\Sigma} - \delta\boldsymbol{\Sigma}_0| = 0$. Again, by Theorem 4.14 of Schott (2005), there exists a nonsingular matrix $\boldsymbol{\Phi}$ such that $\boldsymbol{\Sigma} = \boldsymbol{\Phi}\boldsymbol{D}_\delta\boldsymbol{\Phi}'$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{\Phi}\boldsymbol{\Phi}'$. Recall that $\boldsymbol{S}_t = \boldsymbol{Y}\boldsymbol{D}_\beta\boldsymbol{Y}'$ and $\boldsymbol{S}_0 = \boldsymbol{Y}\boldsymbol{Y}'$. So, (A.2.1) becomes

$$\ell(\boldsymbol{\mu}_0,\, \boldsymbol{\mu},\, \boldsymbol{D}_\delta,\, \boldsymbol{\Phi}) = -\frac{1}{2}\{p(mn+n)\log 2\pi + mn\log|\boldsymbol{\Phi}\boldsymbol{\Phi}'| + n\log|\boldsymbol{\Phi}\boldsymbol{D}_\delta\boldsymbol{\Phi}'|$$
$$+mntr[\boldsymbol{Y}\boldsymbol{Y}'(\boldsymbol{\Phi}\boldsymbol{\Phi}')^{-1}] + ntr[\boldsymbol{Y}\boldsymbol{D}_\beta\boldsymbol{Y}'(\boldsymbol{\Phi}\boldsymbol{D}_\delta\boldsymbol{\Phi}')^{-1}]\Big\}$$
$$-\frac{mn}{2}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\bar{\bar{\boldsymbol{X}}} - \boldsymbol{\mu}_0) - \frac{n}{2}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{X}}_t - \boldsymbol{\mu})\,. \qquad \text{(A.2.2)}$$

To maximize $\ell(\boldsymbol{\mu}_0,\, \boldsymbol{\mu},\, \boldsymbol{D}_\delta,\, \boldsymbol{\Phi})$, we first note that $\hat{\boldsymbol{\mu}}_0 = \bar{\bar{\boldsymbol{X}}}$ and $\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{X}}_t$ are the MLEs of $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}$, respectively. Also,

$$tr[\boldsymbol{Y}\boldsymbol{Y}'(\boldsymbol{\Phi}\boldsymbol{\Phi}')^{-1}] = tr[\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{\Phi}'^{-1}] = tr[(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})\boldsymbol{D}_\beta^{-1}(\boldsymbol{\Phi}^{-1}Y\boldsymbol{D}_\beta^{\frac{1}{2}})'],$$

$$tr[\boldsymbol{Y}\boldsymbol{D}_\beta\boldsymbol{Y}'(\boldsymbol{\Phi}\boldsymbol{D}_\delta\boldsymbol{\Phi}')^{-1}] = tr[\boldsymbol{D}_\delta^{-1}(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})'].$$

By substituting the above results into (A.2.2), the log likelihood function concentrated with respect to $\hat{\boldsymbol{\mu}}_0 = \bar{\bar{\boldsymbol{X}}}$ and $\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{X}}_t$ is

$$\ell(\boldsymbol{D}_\delta, \boldsymbol{\Phi}) = -\frac{1}{2}\{p(mn+n)\log 2\pi + 2mn\log|\boldsymbol{\Phi}| + 2n\log|\boldsymbol{\Phi}| + n\log|\boldsymbol{D}_\delta|$$

$$+mntr\{(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})\boldsymbol{D}_\beta^{-1}(\boldsymbol{\Phi}^{-1}\boldsymbol{D}\boldsymbol{D}_\beta^{\frac{1}{2}})'\} + ntr\{\boldsymbol{D}_\delta^{-1}(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})(\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}})'\}\}. \quad \text{(A.2.3)}$$

Next, we want to find $\boldsymbol{D}_\delta$ and $\boldsymbol{\Phi}$ that maximize (A.2.3). By Singular Value Decomposition, there exist orthogonal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ and a diagonal matrix $\boldsymbol{D}_r = diag(r_1, \cdots, r_p)$ such that

$$\boldsymbol{\Phi}^{-1}\boldsymbol{Y}\boldsymbol{D}_\beta^{\frac{1}{2}} = \boldsymbol{U}\boldsymbol{D}_r\boldsymbol{V}. \quad \text{(A.2.4)}$$

By substituting (A.2.4) into (A.2.3), (A.2.3) becomes

$$\begin{aligned}\ell(\boldsymbol{D}_\delta, \boldsymbol{D}_r, \boldsymbol{U}, \boldsymbol{V}) &= -\frac{1}{2}\{p(mn+n)\log 2\pi + 2mn\log|\boldsymbol{\Phi}| + 2n\log|\boldsymbol{\Phi}| + n\log|\boldsymbol{D}_\delta| \\ &\quad +mn\,tr[(\boldsymbol{U}\boldsymbol{D}_r\boldsymbol{V})\boldsymbol{D}_\beta^{-1}(\boldsymbol{U}\boldsymbol{D}_r\boldsymbol{V})'] + n\,tr[\boldsymbol{D}_\delta^{-1}(\boldsymbol{U}\boldsymbol{D}_r\boldsymbol{V})(\boldsymbol{U}\boldsymbol{D}_r\boldsymbol{V})']\Big\} \\ &= -\frac{p(mn+n)\log 2\pi}{2} - \frac{mn+n}{2}\log|\boldsymbol{D}_\beta| - (mn+n)\log|\boldsymbol{Y}| + \frac{mn+n}{2}\log|\boldsymbol{D}_r|^2 \\ &\quad -\frac{n}{2}\log|\boldsymbol{D}_\delta| - \frac{mn}{2}tr[\boldsymbol{D}_r^2\boldsymbol{V}\boldsymbol{D}_\beta^{-1}\boldsymbol{V}'] - \frac{n}{2}tr[\boldsymbol{D}_\delta^{-1}\boldsymbol{U}\boldsymbol{D}_r^2\boldsymbol{U}']\}. \quad \text{(A.2.5)}\end{aligned}$$

In order to maximize (A.2.5), we first fix the diagonal matrices $\boldsymbol{D}_\delta$ and $\boldsymbol{D}_r^2$ and find the $\boldsymbol{U}$ and $\boldsymbol{V}$ that maximize $\ell(\boldsymbol{D}_\delta, \boldsymbol{D}_r^2, \boldsymbol{U}, \boldsymbol{V})$. Again, by the theorem of Von Neuman (1937), we have $\min_{\boldsymbol{V}}\{tr[\boldsymbol{D}_r^2\boldsymbol{V}\boldsymbol{D}_\beta^{-1}\boldsymbol{V}']\} = tr[\boldsymbol{D}_r^2\boldsymbol{D}_\beta^{-1}]$ and $\min_{\boldsymbol{U}}\{tr[\boldsymbol{D}_\delta^{-1}\boldsymbol{U}\boldsymbol{D}_r^2\boldsymbol{U}']\} = tr[\boldsymbol{D}_\delta^{-1}\boldsymbol{D}_r^2]$, and $\boldsymbol{U} = \boldsymbol{V} = \boldsymbol{I}$ are one of the minimizers. With $\boldsymbol{U} = \boldsymbol{I}$ and $\boldsymbol{V} = \boldsymbol{I}$, the concentrated likelihood function becomes

$$\begin{aligned}\ell(\boldsymbol{D}_\delta, \boldsymbol{D}_r) &= C + \frac{mn+n}{2}\log|\boldsymbol{D}_r|^2 - \frac{n}{2}\log|\boldsymbol{D}_\delta| - \frac{mn}{2}tr[\boldsymbol{D}_r^2\boldsymbol{D}_\beta^{-1}] - \frac{n}{2}tr[\boldsymbol{D}_\delta^{-1}\boldsymbol{D}_r^2] \\ &= C + \frac{1}{2}\sum_{i=1}^{p}\left[(mn+n)\log r_i^2 - n\log\delta_i - mn\frac{r_i^2}{\beta_i} - n\frac{r_i^2}{\delta_i}\right] \\ &= C + \frac{1}{2}\sum_{i=1}^{p}\left[(mn+n)\log r_i^2 - (\frac{mn}{\beta_i} + \frac{n}{\delta_i})r_i^2 - n\log\delta_i\right], \quad \text{(A.2.6)}\end{aligned}$$

where $C = -\frac{p(mn+n)\log 2\pi}{2} - \frac{mn+n}{2}\log|\boldsymbol{D}_\beta| - (mn+n)\log|\boldsymbol{Y}|$.

Note that $\{r_i\}_{i=1}^p$ depends on $\{\delta_i\}_{i=1}^p$. For fixed δ_i, $i = 1, 2, \cdots, p$, it can be easily shown that the maximum of (A.2.6) occurs at $r_i^2 = (mn+n)(\frac{mn}{\beta_i} + \frac{n}{\delta_i})^{-1}$. Then the concentrated likelihood function becomes

$$\ell(\boldsymbol{D}_\delta) = C - \frac{1}{2}\sum_{i=1}^{p}\Big[n\log\delta_i + (mn+n)\log(\frac{mn}{\beta_i} + \frac{n}{\delta_i}) + (mn+n) - (mn+n)\log(mn+n)\Big]. \tag{A.2.7}$$

It is then obvious that the maximum of (A.2.7) with respect to $\delta_1, \cdots, \delta_p$ over the region $0 < \delta_1 \leq \cdots \leq \delta_k < \delta_{k+1} = \cdots = \delta_p = 1$ occurs at

$$\begin{array}{ll} \left.\begin{array}{ll} \delta_i = \beta_i \ , & if \ \ 0 < \beta_i < 1 \\ \delta_i = 1 \ \ , & if \ \ \beta_i \geq 1 \end{array}\right\} & for \ \ i = 1, \cdots, k, \\ \delta_i = 1 \ \ , & for \ \ i = k+1, \cdots, p. \end{array}$$

Let $p^\star$ be the number of $0 < \beta_i < 1$ and $k^\star = \min(k, p^\star)$. Then $\delta_i = \beta_i$, $i = 1, \cdots, k^\star$, and $\delta_i = 1$, $i = k^\star + 1, \cdots, p$.

Let $q^\star = p - k^\star$. Then the maximum of (A.2.7) is

$$C - \frac{(mn+n)p}{2} - \frac{(q^\star)(mn+n)\log(mn+n)}{2} + \frac{(mn+n)}{2}\sum_{i=1}^{p}\log\beta_i$$
$$-\frac{n}{2}\sum_{i=1}^{k^\star}\log\beta_i - \frac{(mn+n)}{2}\sum_{i=k^\star+1}^{p}\log(n\beta_i + mn).$$

Denote this by $\ell(k^\star)$ and the corresponding maximum likelihood function by $L^*(k^\star)$. Then

$$L^*(k^\star) = C^*(mn+n)^{-\frac{q^\star(mn+n)}{2}}\prod_{i=1}^{k^\star}\beta_i^{-\frac{n}{2}}\prod_{i=k^\star+1}^{p}(n\beta_i + mn)^{-\frac{mn+n}{2}},$$

where $C^* = (2\pi)^{-\frac{(mn+n)p}{2}} e^{-\frac{(mn+n)p}{2}} |\boldsymbol{Y}|^{-(mn+n)}$.

It is trivial to show that $L^*(k^\star)$ is nondecreasing in $k^\star$. Let $k_0^\star = \min(k_0, p^\star)$ and $k_1^\star = \min(k_1, p^\star)$. Then the LRT statistic for testing (2.2.1) is

$$\frac{\max\limits_{H_0^*} L^*(k^\star)}{\max\limits_{H_1^*} L^*(k^\star)} = \frac{L^*(k_0^\star)}{L^*(k_1^\star)} = \begin{cases} \prod\limits_{i=k_0^\star+1}^{k_1^\star}\Big[\frac{\beta_i^w}{(w\beta_i + 1 - w)}\Big]^{\frac{mn+n}{2}} & , \ if \ \ k_0^\star < k_1^\star \\ 1 & , \ if \ \ k_0^\star = k_1^\star \end{cases},$$

where $w = \frac{1}{m+1}$.

A.3 A Theorem of Von Neumann(1937)

Theorem C.1 (Von Neumann) *For* $\boldsymbol{Q}$ *orthogonal and* $\boldsymbol{D}_s$ *and* $\boldsymbol{D}_t$ *diagonal* ($s_1 \geq \cdots \geq s_p > 0$, $t_1 \geq \cdots \geq t_p > 0$),

$$\min_{\boldsymbol{Q}} tr(\boldsymbol{D}_s^{-1}\boldsymbol{Q}\boldsymbol{D}_t\boldsymbol{Q}') = tr(\boldsymbol{D}_s^{-1}\boldsymbol{D}_t)$$

and a minimizing value of $\boldsymbol{Q}$ *is* $\boldsymbol{Q} = \boldsymbol{I}$.

Appendix B

B.1 Distributions of T_1^2 and $T_2^2|\{T_1^2 = t_1^2\}$

From the partition of S in (4.3.4) we have (Srivastava and Khatri (1979, p. 8))

$$S^{-1} = \begin{pmatrix} S_{11}^{-1} + S_{11}^{-1} S_{12} S_{22.1}^{-1} S_{21} S_{11}^{-1} & -S_{11}^{-1} S_{12} S_{22.1}^{-1} \\ -S_{22.1}^{-1} S_{21} S_{11}^{-1} & S_{22.1}^{-1} \end{pmatrix}.$$

Then, T^2 can be written as

$$\begin{aligned} T^2 &= \frac{N}{N+1}(X_t - \bar{X})' S^{-1} (X_t - \bar{X}) \\ &= \frac{N}{N+1} \begin{pmatrix} X_t^{(1)} - \bar{X}^{(1)} \\ X_t^{(2)} - \bar{X}^{(2)} \end{pmatrix}' \begin{bmatrix} S_{11}^{-1} + S_{11}^{-1} S_{12} S_{22.1}^{-1} S_{21} S_{11}^{-1} & -S_{11}^{-1} S_{12} S_{22.1}^{-1} \\ -S_{22.1}^{-1} S_{21} S_{11}^{-1} & S_{22.1}^{-1} \end{bmatrix} \begin{pmatrix} X_t^{(1)} - \bar{X}^{(1)} \\ X_t^{(2)} - \bar{X}^{(2)} \end{pmatrix} \\ &= \frac{N}{N+1}(X_t^{(1)} - \bar{X}^{(1)})' S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)}) \\ &+ \frac{N}{N+1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21} S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)})]' S_{22.1}^{-1} [(X_t^{(2)} - \bar{X}^{(2)}) - S_{21} S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)})] \\ &= T_1^2 + T_2^2, \text{ say} \end{aligned}$$

where

$$T_1^2 = \frac{N}{N+1}(X_t^{(1)} - \bar{X}^{(1)})' S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)}),$$

and

$$T_2^2 = \frac{N}{N+1}[(X_t^{(2)} - \bar{X}^{(2)}) - S_{21} S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)})]' S_{22.1}^{-1} [(X_t^{(2)} - \bar{X}^{(2)}) - S_{21} S_{11}^{-1} (X_t^{(1)} - \bar{X}^{(1)})].$$

Then, from Anderson (2003, p 143), T_1^2 follows $\frac{(N-1)p_1}{N-p_1} F_{p_1, N-p_1, \lambda_1}$ with noncentrality

$$\lambda_1 = \frac{N}{N+1}(\mu^{(1)} - \mu_0^{(1)})' \Sigma_{11}^{-1} (\mu^{(1)} - \mu_0^{(1)}).$$

Next we will find the distribution of $T_2^2|\{T_1^2 = t_1^2\}$. First, since (Theorem 3.3.9 of Gupta and Nagar (2000, p. 94))

$$S_{21}|\{S_{11} = s_{11}\} \sim N_{q_1,p_1}(\Sigma_{21}\Sigma_{11}^{-1}s_{11}, \frac{1}{N-1}\Sigma_{22.1} \otimes s_{11})$$

we have

$$S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})|\left\{(X_t^{(1)}, \bar{X}^{(1)}, S_{11}) = (x_t^{(1)}, \bar{x}^{(1)}, s_{11})\right\}$$
$$\sim N_{q_1}(\Sigma_{21}\Sigma_{11}^{-1}s_{11}s_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)}), \frac{\Sigma_{22.1}}{N-1}\frac{N+1}{N}\ t_1^2), \tag{B.1.1}$$

with $t_1^2 = \frac{N}{N+1}(x_t^{(1)} - \bar{x}^{(1)})' s_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)})$. Furthermore, since$X_t$,$\bar{X}$, and$S$ are independent, from (B.1.1) and the fact that $X_t^{(2)} - \bar{X}^{(2)}|\left\{(X_t^{(1)}, \bar{X}^{(1)}) = (x_t^{(1)}, \bar{x}^{(1)})\right\} \sim N_{q_1}(\mu^{(2)} - \mu_0^{(2)} - \Sigma_{21}\Sigma_{11}^{-1}(x_t^{(1)} - \bar{x}^{(1)} - (\mu^{(1)} - \mu_0^{(1)})), \frac{N+1}{N}\Sigma_{22.1})$ we have

$$(X_t^{(2)} - \bar{X}^{(2)}) - S_{21}S_{11}^{-1}(X_t^{(1)} - \bar{X}^{(1)})|\left\{(X_t^{(1)}, \bar{X}^{(1)}, S_{11}) = (x_t^{(1)}, \bar{x}^{(1)}, s_{11})\right\}$$
$$\sim N_{q_1}(\mu^{(2)} - \mu_0^{(2)} - \Sigma_{21}\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)}), \frac{N+1}{N}(1 + \frac{t_1^2}{N-1})\Sigma_{22.1}). \tag{B.1.2}$$

From (B.1.2) and the fact that $S_{22.1} \sim W_{q_1}(N - p_1 - 1, \frac{1}{N-1}\Sigma_{22.1})$ and is independent of (S_{21}, S_{11}), then, from Anderson (2003, p. 143), the conditional distribution of $T_2^{2(*)} \equiv \frac{T_2^2}{(1+t_1^2/(N-1))}$, given $(X_t^{(1)}, \bar{X}^{(1)}, S_{11}) = (x_t^{(1)}, \bar{x}^{(1)}, s_{11})$, is a noncentral $\frac{(N-1)q_1}{(N-1-p_1)-q_1+1}F_{q_1,N-p,\lambda_2} = \frac{(N-1)q_1}{N-p}F_{q_1,N-p,\lambda_2}$ distribution with noncentrality

$$\lambda_2 = \frac{N}{N+1}\frac{[(\mu^{(2)} - \mu_0^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)})]'\Sigma_{22.1}^{-1}[(\mu^{(2)} - \mu_0^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(\mu^{(1)} - \mu_0^{(1)})]}{(1 + t_1^2/(N-1))}$$
$$= \frac{\lambda - \lambda_1}{(1 + t_1^2/(N-1))}. \tag{B.1.3}$$

Since this conditional distribution depends on $(x_t^{(1)}, \bar{x}^{(1)}, s_{11})$ but through t_1^2, it is also the conditional of T_2^2, given t_1^2, so we obtain the results in Theorem 1.

B.2 Approximation of Expectation in (4.3.10)

If $F \sim F_{v_1,v_2,\tau}$, then its p.d.f is $p(f) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\tau}{2}}(\frac{\tau}{2})^\beta}{B(\frac{v_2}{2},\frac{v_1}{2}+\beta)\cdot\beta!}(\frac{v_1}{v_2})^{\frac{v_1}{2}+\beta}(\frac{v_2}{v_2+v_1f})^{\frac{v_1+v_2}{2}+\beta}f^{\frac{v_1}{2}-1+\beta}$, and from *Theorem* 1(*ii*) that $\frac{(N-p)T_2^{2(*)}}{(N-1)q_1} \sim F_{q_1,N-p,\lambda_2}$, we have

$$f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2 - t_1^2}{(1 + t_1^2/(N-1))}|x_t^{(1)}, \bar{x}^{(1)}, s_{11})$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda_2}{2}}(\frac{\lambda_2}{2})^{\beta}}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\cdot\beta!}(\frac{q_1}{N-p})^{\frac{q_1}{2}+\beta}\left(\frac{N-p}{N-p+q_1\frac{(N-p)(t^2-t_1^2)}{q_1(N-1)(1+t_1^2/(N-1))}}\right)^{\frac{q_1+N-p}{2}+\beta}$$

$$\times(\frac{(N-p)(t^2-t_1^2)}{q_1(N-1)(1+t_1^2/(N-1))})^{\frac{q_1}{2}-1+\beta}\cdot\frac{(N-p)}{(N-1)q_1}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda_2}{2}}(\frac{\lambda_2}{2})^{\beta}(\frac{q_1}{N-p})^{\frac{q_1}{2}+\beta}(\frac{N-p}{q_1})^{\frac{q_1}{2}+\beta}}{(N-1)B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\cdot\beta!}(1+\frac{(t^2-t_1^2)}{(N-1)(1+t_1^2/(N-1))})^{-(\frac{q_1+N-p}{2}+\beta)}$$

$$\times(\frac{(t^2-t_1^2)}{(N-1)(1+t_1^2/(N-1))})^{\frac{q_1}{2}-1+\beta}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda_2}{2}}(\frac{\lambda_2}{2})^{\beta}}{(N-1)B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\cdot\beta!}(1+\frac{(t^2-t_1^2)}{(N-1)(1+t_1^2/(N-1))})^{-(\frac{N-p_1}{2}+\beta)}$$

$$\times(\frac{(t^2-t_1^2)}{(N-1)(1+t_1^2/(N-1))})^{\frac{q_1}{2}-1+\beta}$$

$$= \frac{e^{-\frac{1}{2}\lambda_2}}{(N-1)}\sum_{\beta=0}^{\infty}\frac{(\frac{\lambda_2}{2})^{\beta}\Big[\frac{t^2-t_1^2}{(N-1)(1+t_1^2/(N-1))}\Big]^{\frac{q_1}{2}-1+\beta}}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\beta!\Big[1+\frac{t^2-t_1^2}{(N-1)(1+t_1^2/(N-1))}\Big]^{\frac{N-p_1}{2}+\beta}}$$

Let $g_{q_1,N-p}(t_1^2) = f_{T_2^{2(*)}|(X_t^{(1)},\bar{X}^{(1)},S_{11})}(\frac{t^2-t_1^2}{(1+t_1^2/(N-1))}|x_t^{(1)},\bar{x}^{(1)},s_{11})\cdot\frac{1}{(1+t_1^2/(N-1))}$, then

$$g_{q_1,N-p}(t_1^2) = \frac{e^{-\frac{1}{2}\lambda_2}}{(N-1)}\sum_{\beta=0}^{\infty}\frac{(\frac{\lambda_2}{2})^{\beta}\Big[\frac{t^2-t_1^2}{(N-1)(1+t_1^2/(N-1))}\Big]^{\frac{q_1}{2}-1+\beta}}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\beta!\Big[1+\frac{t^2-t_1^2}{(N-1)(1+t_1^2/(N-1))}\Big]^{\frac{N-p_1}{2}+\beta}}\cdot\frac{1}{(1+t_1^2/(N-1))}$$

$$= \frac{e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)}}}{(N-1)}\sum_{\beta=0}^{\infty}\frac{(\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)})^{\beta}\Big[\frac{t^2-t_1^2}{N-1+t_1^2}\Big]^{\frac{q_1}{2}-1+\beta}}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\beta!\Big[\frac{N-1+t^2}{N-1+t_1^2}\Big]^{\frac{N-p_1}{2}+\beta}}\cdot\frac{N-1}{(N-1)+t_1^2}$$

$$= e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{((N-1)+t_1^2)}}\sum_{\beta=0}^{\infty}\frac{(\frac{(N-1)(\lambda-\lambda_1)}{2})^{\beta}}{(N-1+t^2)^{\frac{N-p_1}{2}+\beta}B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\beta!}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{(N-1+t_1^2)^{-\frac{N-p}{2}+\beta}}$$

$$= e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)}}\frac{(N-1+t_1^2)^{\frac{N}{2}}}{(N-1+t^2)^{\frac{N}{2}}}\sum_{\beta=0}^{\infty}\frac{(\frac{(N-1)(\lambda-\lambda_1)}{2})^{\beta}(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)\beta!}\frac{(N-1+t_1^2)^{\frac{-p}{2}-\beta}}{(N-1+t^2)^{\frac{-p_1}{2}+\beta}} \quad \text{(B.2.1)}$$

Next, we will find the asymptotic expansion for (B.2.1). First, we give the following Lemma:

Lemma 1 *If* $g(x) = \sum_{k=1}^{m}\alpha_k x^{-k}$ $(1 \le m \le \infty)$, *then* $\exp(g(x)) = \sum_{j=0}^{\infty}\beta_j(m)x^{-j}$, *where* $\beta_j's$ *satisfy the following recursive relation:*

$$\beta_0(m) = 1, \quad \beta_j(m) = \frac{1}{j}\sum_{k=0}^{\min(k,m)} k\alpha_k\beta_{j-k}(m)x^{-j}, j = 1,2,...$$

Proof. Note $g'(x) = (-k)\sum_{k=1}^{m} \alpha_k x^{-(k+1)}$, *so*

$$\frac{d}{dx}(\exp(g(x))) = \frac{d}{dx}(\sum_{j=0}^{\infty} \beta_j(m) x^{-j}),$$

or $g'(x)\exp(g(x)) = \sum_{j=0}^{\infty}(-j)\beta_j(m)x^{-(j+1)}$, *or*

$$\left(\sum_{j=0}^{\infty}\sum_{k=1}^{m} k\alpha_k\beta_j(m)x^{-(j+k+1)}\right) = \sum_{j=0}^{\infty} j\beta_j(m)x^{-(j+1)}.$$

By comparing the powers, we prove the lemma with, for example, $\beta_0(m) = 1$,$\beta_1(m) = \alpha_1$, $\beta_2(m) = 2\alpha_2 + \alpha_1^2$,$\beta_3(m) = 3\alpha_3 + 4\alpha_1\alpha_2 + \alpha_1^3$.

The following is the asymptotic expansion for log of gamma function (Barnes (1988, p64) or Anderson (2003, p318)):

$$\log\Gamma(x+h) = \log\sqrt{2\pi} + (x+h-\frac{1}{2})\log(x) - x - \sum_{r=1}^{m}(-1)^r\frac{B_{r+1}(h)}{r(r+1)x^r} + R_{m+1}(x), \quad \text{(B.2.2)}$$

where $R_{m+1}(x) = O(x^{-(m+1)})$ as $|x| \to \infty$ (i.e., $\left|\frac{R_{m+1}(x)}{x^{-(m+1)}}\right|$ is bounded as $|x| \to \infty$), and $B_r(h)$ is the Bernoulli polynomial of degree r and order unity defined by $\frac{re^{h\tau}}{e^{\tau}-1} = \sum_{r=0}^{m}\frac{\tau^r}{r!}B_r(h)$. The first three polynomial are ($B_0(h) = 1$)

$$B_1(h) = h - \frac{1}{2}, \quad B_2(h) = h^2 - h + \frac{1}{6}, \quad B_3(h) = h^3 - \frac{3}{2}h^2 + \frac{1}{2}h. \quad \text{(B.2.3)}$$

Now, $\frac{1}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)} = \frac{\Gamma(\frac{N-p}{2}+\frac{q_1}{2}+\beta)}{\Gamma(\frac{N-p}{2})\Gamma(\frac{q_1}{2}+\beta)}$ and from (B.2.2):

$$\log\left(\Gamma(\frac{N}{2}+h)\right) = \log\sqrt{2\pi} + (\frac{N}{2}+h-\frac{1}{2})\log(\frac{N}{2}) - \frac{N}{2} - \sum_{r=1}^{m}\frac{(-1)^r}{r(r+1)}B_{r+1}(h)(\frac{N}{2})^{-r} + R_{m+1}(N), \quad \text{(B.2.4)}$$

where $R_{m+1}(N) = O(N^{-(m+1)})$, we can twice apply the result in (B.2.4) to obtain

$$\log\left(\Gamma(\frac{N}{2}+\frac{q_1-p}{2}+\beta)\right)$$

$$= \log\sqrt{2\pi} + (\frac{N}{2}+\frac{q_1-p}{2}+\beta-\frac{1}{2})\log(\frac{N}{2}) - \frac{N}{2} - \sum_{r=1}^{m}\frac{(-2)^r}{r(r+1)}B_{r+1}(\frac{q_1-p}{2}+\beta)(\frac{1}{N})^r + R_{m+1}(N), \quad \text{(B.2.5)}$$

and

$$\log\left(\Gamma(\frac{N}{2}+\frac{-p}{2})\right)$$

$$=\log\sqrt{2\pi}+(\frac{N}{2}-\frac{p}{2}-\frac{1}{2})\log(\frac{N}{2})-\frac{N}{2}-\sum_{r=1}^{m}\frac{(-2)^r}{r(r+1)}B_{r+1}(\frac{-p}{2})(\frac{1}{N})^r+R_{m+1}(N). \quad \text{(B.2.6)}$$

Hence,

$$\log\left(\frac{\Gamma(\frac{N}{2}-\frac{p}{2}+\frac{q_1}{2}+\beta)}{\Gamma(\frac{N}{2}-\frac{p}{2})}\right)$$

$$=(\frac{q_1}{2}+\beta)\log(\frac{N}{2})+\sum_{r=1}^{m}\frac{(-2)^r}{r(r+1)}\left(B_{r+1}(\frac{-p}{2})-B_{r+1}(\frac{q_1-p}{2}+\beta)\right)(\frac{1}{N})^r+R_{m+1}(N).$$

Appling *Lemma B.1* and we can obtain

$$\frac{\Gamma(\frac{N}{2}-\frac{p}{2}+\frac{q_1}{2}+\beta)}{\Gamma(\frac{N}{2}-\frac{p}{2})}=(\frac{N}{2})^{(\frac{q_1}{2}+\beta)}\cdot\exp\left(\sum_{r=1}^{m}\alpha_r(\frac{1}{N})^r+R_{m+1}(N)\right),$$

$$=(\frac{N}{2})^{(\frac{q_1}{2}+\beta)}\cdot\left(\sum_{j=0}^{m}\beta_j(m)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right), \quad \text{(B.2.7)}$$

where $\alpha_r=\alpha_r(p,q_1,\beta)=\frac{(-2)^r}{r(r+1)}\left(B_{r+1}(\frac{-p}{2})-B_{r+1}(\frac{q_1-p}{2}+\beta)\right)$ and $\beta_j(m)=\beta_j(m;p,q_1,\beta)$.
We can use the result in (B.2.3) to$\beta_j(m)$ and the first three terms are

$$\beta_0(m)=1,$$

$$\beta_1(m)=\alpha_1=\frac{(-2)}{(1+1)}\left(B_2(\frac{-p}{2})-B_2(\frac{q_1-p}{2}+\beta)\right)$$

$$=-\Big[\Big((\frac{-p}{2})^2-(\frac{-p}{2})+\frac{1}{6}\Big)-\Big((\frac{q_1-p}{2}+\beta)^2-(\frac{q_1-p}{2}+\beta)+\frac{1}{6}\Big)\Big]$$

$$=(\frac{q_1}{2}+\beta)(\frac{q_1}{2}-p+\beta-1),$$

$$\beta_2(m)=2\alpha_2+\alpha_1^2,$$

where

$$\alpha_2=\frac{(-2)^2}{2(2+1)}\left(B_3(\frac{-p}{2})-B_3(\frac{q_1-p}{2}+\beta)\right)$$

$$=\frac{2}{3}\Big[\Big((\frac{-p}{2})^3-\frac{3}{2}(\frac{-p}{2})^2+\frac{1}{2}(\frac{-p}{2})\Big)-\Big((\frac{q_1-p}{2}+\beta)^3-\frac{3}{2}(\frac{q_1-p}{2}+\beta)^2+\frac{1}{2}(\frac{q_1-p}{2}+\beta)\Big)\Big]$$

$$= \frac{2}{3}\Big[-(\frac{q_1}{2}+\beta)\left((\frac{-p}{2})^2+(\frac{-p}{2})(\frac{q_1-p}{2}+\beta)+(\frac{q_1-p}{2}+\beta)^2\right)+\frac{3}{2}(\frac{q_1}{2}+\beta)(\frac{q_1}{2}-p+\beta)-\frac{1}{2}(\frac{q_1}{2}+\beta)\Big]$$

$$= (\frac{q_1}{2}+\beta)\Big[-\frac{2}{3}\left(\frac{p^2}{4}+(\frac{q_1-p}{2}+\beta)(\frac{q_1}{2}-p+\beta)\right)+(\frac{q_1}{2}-p+\beta)-\frac{1}{3}\Big]$$

$$= (\frac{q_1}{2}+\beta)\left(\frac{1}{3}(\frac{q_1}{2}-p+\beta)(3-q_1+p-2\beta)-\frac{p^2}{6}-\frac{1}{3}\right).$$

So, from (B.2.7), we can obtain

$$\frac{1}{B(\frac{N-p}{2},\frac{q_1}{2}+\beta)} = \frac{1}{\Gamma(\frac{q_1}{2}+\beta)}\cdot(\frac{N}{2})^{(\frac{q_1}{2}+\beta)}\cdot\left(\sum_{j=0}^{m}\beta_j(m)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right) \tag{B.2.8}$$

Next, to find the asymptotic expansion for $(1-\frac{a}{N})^{bN}$ in $\frac{1}{N}$, first we note

$$\log\left((1-\frac{a}{N})^{bN}\right) = (bN)\log(1-\frac{a}{N}).$$

Let $x=\frac{a}{N}$, so we are dealing with $\log(1-x)$. A Taylor series expansion for $\log(1-x)$at $x_0=0$ is $\log(1-x)=-\sum_{k=1}^{\infty}(\frac{1}{k})x^k$. Hence,

$$\log\left((1-\frac{a}{N})^{bN}\right) = (bN)\left(-\sum_{k=1}^{\infty}(\frac{1}{k})(\frac{a}{N})^k\right) = -(bN)\left(\sum_{k=1}^{\infty}(\frac{a^k}{k})(\frac{1}{N})^k\right)$$

$$= \sum_{j=0}^{\infty}(\frac{-ba^{j+1}}{j+1})(\frac{1}{N})^j = -ba+\sum_{j=1}^{\infty}(\frac{-ba^{j+1}}{j+1})(\frac{1}{N})^j.$$

Applying *Lemma B.1* and finally,

$$(1-\frac{a}{N})^{bN} = e^{-ba}\sum_{j=0}^{\infty}\tilde{\beta}_j(m;a,b)(\frac{1}{N})^j = e^{-ba}\left(\sum_{j=0}^{m}\tilde{\beta}_j(m;a,b)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right), \tag{B.2.9}$$

where $\tilde{\beta}_0(m;a,b)=1$, $\tilde{\beta}_1(m;a,b)=(\frac{-ba^{1+1}}{1+1})=\frac{-ba^2}{2}$, and $\tilde{\beta}_2(m;a,b)=2(\frac{-ba^{2+1}}{2+1})+(\frac{-ba^2}{2})^2$ $=\frac{-2ba^3}{3}+\frac{b^2a^4}{4}$.

For (B.2.1), we will twice apply the result of (B.2.9):

$$\frac{1}{(N-1+t^2)^{\frac{N}{2}}} = (N-1+t^2)^{-\frac{N}{2}} = N^{-\frac{N}{2}}(1-\frac{1-t^2}{N})^{-\frac{N}{2}} = N^{-\frac{N}{2}}e^{\frac{1-t^2}{2}}\sum_{j=0}^{\infty}\tilde{\beta}_j(m;1-t^2,-\frac{1}{2})(\frac{1}{N})^j$$

$$= N^{-\frac{N}{2}}e^{\frac{1-t^2}{2}}\left(\sum_{j=0}^{m}\tilde{\beta}_j(m;1-t^2,-\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right), \tag{B.2.10}$$

where $\tilde{\beta}_0(m;1-t^2,-\frac{1}{2})=1$, $\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})=\frac{(1-t^2)^2}{4}$, $\tilde{\beta}_2(m;1-t^2,-\frac{1}{2})=\frac{(1-t^2)^3}{3}+\frac{(1-t^2)^4}{16}$, and

$$(N-1+t_1^2)^{\frac{N}{2}}=N^{\frac{N}{2}}(1-\frac{1-t_1^2}{N})^{\frac{N}{2}}=N^{\frac{N}{2}}e^{-\frac{1-t_1^2}{2}}\sum_{j=0}^{\infty}\tilde{\tilde{\beta}}_j(m;1-t_1^2,\frac{1}{2})(\frac{1}{N})^j$$
$$=N^{\frac{N}{2}}e^{-\frac{1-t_1^2}{2}}\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;1-t_1^2,\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right), \quad \text{(B.2.11)}$$

with $\tilde{\tilde{\beta}}_0(m;1-t_1^2,\frac{1}{2})=1$,$\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})=\frac{-(1-t_1^2)^2}{4}$, and $\tilde{\tilde{\beta}}_2(m;1-t_1^2,\frac{1}{2})=\frac{-(1-t_1^2)^3}{3}+\frac{(1-t_1^2)^4}{16}$. Hence, from (B.2.8), (B.2.10), and (B.2.11), we can rewrite (B.2.1) as

$$e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)}}\sum_{\beta=0}^{\infty}\frac{(\frac{(N-1)(\lambda-\lambda_1)}{2})^{\beta}(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\frac{(N-1+t_1^2)^{\frac{-p}{2}-\beta}}{(N-1+t^2)^{\frac{-p_1}{2}+\beta}}$$
$$\times(\frac{N}{2})^{(\frac{q_1}{2}+\beta)}\cdot\left(\sum_{j=0}^{m}\beta_j(m;p,q_1,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)e^{\frac{t_1^2-t^2}{2}}$$
$$\times N^{-\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\beta}_j(m;1-t^2,-\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$
$$\times N^{\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;1-t_1^2,\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right). \quad \text{(B.2.12)}$$

At the moment, one can check that the powers of N in the last expansion is $\beta+(\frac{p_1}{2}-\beta)+(-\frac{p}{2}-\beta)+(\frac{q_1}{2}+\beta)+(-\frac{N}{2})+\frac{N}{2}=\frac{p_1+q_1}{2}-\frac{p}{2}=0$, as expected.

That means, the last term in the equation (B.2.1) is independent of N after we group the powers of N.

Now,

$$\left(\frac{(N-1)(\lambda-\lambda_1)}{2}\right)^{\beta}=\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}N^{\beta}(1-\frac{1}{N})^{\beta}$$
$$=\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}N^{\beta}\sum_{k=0}^{\beta}C_k^{\beta}(-\frac{1}{N})^k, \quad \text{(B.2.13)}$$

and we apply *Lemma B.1* and obtain

$$(N-1+t^2)^{\frac{p_1}{2}-\beta}=N^{\frac{p_1}{2}-\beta}(1-\frac{1-t^2}{N})^{\frac{p_1}{2}-\beta}$$
$$=N^{\frac{p_1}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m;t^2,p_1,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right) \quad \text{(B.2.14)}$$

$(g(N) = (\frac{p_1}{2} - \beta)\log(1 - \frac{1-t^2}{N}) = (\frac{p_1}{2} - \beta)\left(-\sum_{k=1}^{\infty}(\frac{1}{k})(\frac{1-t^2}{N})^k\right) = \sum_{k=1}^{\infty}\left(\frac{-(1-t^2)^k}{k}(\frac{p_1}{2} - \beta)\right)(\frac{1}{N})^k)$ where $\tilde{\tilde{\tilde{\beta}}}_0(m; t^2, p_1, \beta) = 1$, $\tilde{\tilde{\tilde{\beta}}}_1(m; t^2, p_1, \beta) = -(1-t^2)(\frac{p_1}{2} - \beta)$, and $\tilde{\tilde{\tilde{\beta}}}_2(m; t^2, p_1, \beta) = 2\frac{-(1-t^2)^2}{2}(\frac{p_1}{2} - \beta) + (-(1-t^2)(\frac{p_1}{2} - \beta))^2 = (1-t^2)^2(\frac{p_1}{2} - \beta)(\frac{p_1}{2} - \beta - 1)$.

And

$$
\begin{aligned}
(N-1+t_1^2)^{\frac{-p}{2}-\beta} &= N^{\frac{-p}{2}-\beta}(1 - \frac{1-t_1^2}{N})^{\frac{-p}{2}-\beta} \\
&= N^{\frac{-p}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m; t_1^2, p, \beta)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \qquad \text{(B.2.15)}
\end{aligned}
$$

$(g(N) = (\frac{-p}{2} - \beta)\log(1 - \frac{1-t_1^2}{N}) = (\frac{-p}{2} - \beta)\left(-\sum_{k=1}^{\infty}(\frac{1}{k})(\frac{1-t_1^2}{N})^k\right) = \sum_{k=1}^{\infty}\left(\frac{(1-t_1^2)^k}{k}(\frac{p}{2} + \beta)\right)(\frac{1}{N})^k)$ where $\tilde{\tilde{\tilde{\beta}}}_0(m; t_1^2, p_1, \beta) = 1$, $\tilde{\tilde{\tilde{\beta}}}_1(m; t_1^2, p_1, \beta) = (1-t_1^2)(\frac{p}{2} + \beta)$, and $\tilde{\tilde{\tilde{\beta}}}_2(m; t_1^2, p, \beta) = 2\frac{(1-t_1^2)^2}{2}(\frac{p}{2} + \beta) + ((1-t_1^2)(\frac{p}{2} + \beta))^2 = (1-t_1^2)^2(\frac{p}{2} + \beta)(\frac{p}{2} + \beta + 1)$

Finally, substituting (B.2.13), (B.2.14), and (B.2.15) into (B.2.12) and then (B.2.1) becomes

$$
\begin{aligned}
& e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)}} \sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!} \cdot e^{\frac{t_1^2-t^2}{2}} \cdot \left(\frac{\lambda-\lambda_1}{2}\right)^{\beta} \cdot N^{\beta}\left(\sum_{k=0}^{\beta}C_k^{\beta}(-\frac{1}{N})^k\right) \\
& \times(\frac{N}{2})^{(\frac{q_1}{2}+\beta)} \cdot \left(\sum_{j=0}^{m}\beta_j(m; p, q_1, \beta)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \\
& \times N^{-\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\beta}_j(m; 1-t^2, -\frac{1}{2})(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \\
& \times N^{\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m; 1-t_1^2, \frac{1}{2})(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \\
& \times N^{\frac{p_1}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m; t^2, p_1, \beta)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \\
& \times N^{\frac{-p}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m; t_1^2, p, \beta)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \qquad \text{(B.2.16)}
\end{aligned}
$$

where there terms in the power of N are canceled.

Last, the only thing is the first factor in (B.2.1), namely, $e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{(N-1+t_1^2)}}$. Now,

$$
-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{N-1+t_1^2} = -\frac{(\lambda-\lambda_1)}{2}(N-1)(N-1+t_1^2)^{-1}
$$

$$= -\frac{(\lambda-\lambda_1)}{2}(1-\frac{1}{N})(1-\frac{1-t_1^2}{N})^{-1} = -\frac{(\lambda-\lambda_1)}{2}(1-\frac{1}{N})\left(\sum_{j=0}^{\infty}\left(\frac{1-t_1^2}{N}\right)^j\right)$$

$$= -\frac{(\lambda-\lambda_1)}{2}\left(\sum_{j=0}^{\infty}\left(\frac{1-t_1^2}{N}\right)^j\right) + \frac{(\lambda-\lambda_1)}{2}(\frac{1}{N})\left(\sum_{j=0}^{\infty}\left(\frac{1-t_1^2}{N}\right)^j\right)$$

$$= \sum_{j=0}^{\infty}\left(-\frac{(\lambda-\lambda_1)(1-t_1^2)^j}{2}\right)\left(\frac{1}{N}\right)^j + \sum_{j=0}^{\infty}\left(\frac{(\lambda-\lambda_1)(1-t_1^2)^j}{2}\right)\left(\frac{1}{N}\right)^{j+1}$$

$$= -\frac{(\lambda-\lambda_1)}{2} - \sum_{j=1}^{\infty}\frac{(\lambda-\lambda_1)(1-t_1^2)^j}{2}\left(\frac{1}{N}\right)^j + \sum_{j=1}^{\infty}\left(\frac{(\lambda-\lambda_1)(1-t_1^2)^{j-1}}{2}\right)\left(\frac{1}{N}\right)^j$$

$$= -\frac{(\lambda-\lambda_1)}{2} + \sum_{j=1}^{\infty}\left(\frac{(\lambda-\lambda_1)(1-t_1^2)^{j-1}}{2} - \frac{(\lambda-\lambda_1)(1-t_1^2)^j}{2}\right)\left(\frac{1}{N}\right)^j$$

$$= -\frac{(\lambda-\lambda_1)}{2} + \sum_{j=1}^{\infty}\frac{(\lambda-\lambda_1)(1-t_1^2)^{j-1}(1-(1-t_1^2))}{2}\left(\frac{1}{N}\right)^j$$

$$= -\frac{(\lambda-\lambda_1)}{2} + \sum_{j=1}^{\infty}\frac{(\lambda-\lambda_1)t_1^2(1-t_1^2)^{j-1}}{2}\left(\frac{1}{N}\right)^j \qquad \text{(B.2.17)}$$

Hence, from (B.2.17) and *Lemma B.1*,

$$e^{-\frac{1}{2}\frac{(N-1)(\lambda-\lambda_1)}{N-1+t_1^2}} = e^{-\frac{(\lambda-\lambda_1)}{2}} \cdot e^{\sum_{j=1}^{\infty}\frac{(\lambda-\lambda_1)t_1^2(1-t_1^2)^{j-1}}{2}(\frac{1}{N})^j}$$

$$= e^{-\frac{(\lambda-\lambda_1)}{2}} \cdot \left(\sum_{j=0}^{m}\beta_j^*(m;t_1^2,\lambda,\lambda_1)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right), \qquad \text{(B.2.18)}$$

where $\beta_0^*(m;t_1^2,\lambda,\lambda_1) = 1$, $\beta_1^*(m;t_1^2,\lambda,\lambda_1) = \frac{(\lambda-\lambda_1)t_1^2(1-t_1^2)^{1-1}}{2} = \frac{(\lambda-\lambda_1)t_1^2}{2}$ and $\beta_2^*(m;t_1^2,\lambda,\lambda_1)$ $= 2\frac{(\lambda-\lambda_1)t_1^2(1-t_1^2)^{2-1}}{2} + \left(\frac{(\lambda-\lambda_1)t_1^2}{2}\right)^2 = \frac{(\lambda-\lambda_1)t_1^2(1-t_1^2)}{2} + \left(\frac{(\lambda-\lambda_1)t_1^2}{2}\right)^2$

Finally, from (B.2.16) and (B.2.18), we can obtain

(B.2.1)$= e^{-\frac{(\lambda-\lambda_1)}{2}} \cdot \left(\sum_{j=0}^{m}\beta_j^*(m;t_1^2,\lambda,\lambda_1)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right) \cdot$

$$\times \sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!} \cdot e^{\frac{t_1^2-t^2}{2}} \cdot \left(\frac{\lambda-\lambda_1}{2}\right)^\beta \cdot N^\beta\left(\sum_{k=0}^{\beta}C_k^\beta(-\frac{1}{N})^k\right)$$

$$\times(\frac{N}{2})^{(\frac{q_1}{2}+\beta)} \cdot \left(\sum_{j=0}^{m}\beta_j(m;p,q_1,\beta)(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right)$$

$$\times N^{-\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\beta}_j(m;1-t^2,-\frac{1}{2})(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right)$$

$$\times N^{\frac{N}{2}}\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;1-t_1^2,\frac{1}{2})(\frac{1}{N})^j + O_{m+1}(\frac{1}{N})\right)$$

$$\times N^{\frac{p_1}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;t^2,p_1,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\times N^{\frac{-p}{2}-\beta}\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m;t_1^2,p,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$= e^{-\frac{(\lambda-\lambda_1)}{2}}\cdot\left(\sum_{j=0}^{m}\beta_j^*(m;t_1^2,\lambda,\lambda_1)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\times\sum_{\beta=0}^{\infty}\left\{\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot e^{\frac{t_1^2-t^2}{2}}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}\left(\sum_{k=0}^{\beta}C_k^{\beta}(-\frac{1}{N})^k\right)\right.$$

$$\times 2^{-(\frac{q_1}{2}+\beta)}\cdot\left(\sum_{j=0}^{m}\beta_j(m;p,q_1,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\times\left(\sum_{j=0}^{m}\tilde{\beta}_j(m;1-t^2,-\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\times\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;1-t_1^2,\frac{1}{2})(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\times\left(\sum_{j=0}^{m}\tilde{\tilde{\beta}}_j(m;t^2,p_1,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)$$

$$\left.\times\left(\sum_{j=0}^{m}\tilde{\tilde{\tilde{\beta}}}_j(m;t_1^2,p,\beta)(\frac{1}{N})^j+O_{m+1}(\frac{1}{N})\right)\right\} \tag{B.2.19}$$

In (B.2.19) we may group terms with the same powers of $(\frac{1}{N})$, and take only the first 3 terms (at most), that is, in $(\frac{1}{N})^0$, $(\frac{1}{N})^1$, $(\frac{1}{N})^2$. Hence, (B.2.19) becomes

$$f_{asymp}(t^2,t_1^2|x_t^{(1)},\bar{x}^{(1)},s_{11}) = (\frac{1}{N})^0 f_{asymp}^{(0)} + (\frac{1}{N})f_{asymp}^{(1)} + (\frac{1}{N})^2 f_{asymp}^{(2)} + O_3(\frac{1}{N}) \tag{B.2.20}$$

where

$$f_{asymp}^{(0)} = e^{-\frac{(\lambda-\lambda_1)}{2}+\frac{t_1^2-t^2}{2}}\left(\sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}2^{-(\frac{q_1}{2}+\beta)}\right), \tag{B.2.21}$$

$$f_{asymp}^{(1)} = e^{-\frac{(\lambda-\lambda_1)}{2}+\frac{t_1^2-t^2}{2}}\sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}2^{-(\frac{q_1}{2}+\beta)}$$

$$\times\left(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^{\beta}+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})\right.$$

$$\left.+\tilde{\tilde{\beta}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\right)$$

$$= e^{-\frac{(\lambda-\lambda_1)}{2}+\frac{t_1^2-t^2}{2}}\sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}2^{-(\frac{q_1}{2}+\beta)}.$$

$$\times\Bigg(\frac{(\lambda-\lambda_1)t_1^2}{2}-\beta+(\frac{q_1}{2}+\beta)(\frac{q_1}{2}-p+\beta-1)+\frac{(1-t^2)^2}{4}-\frac{(1-t_1^2)^2}{4}$$
$$-(1-t^2)(\frac{p_1}{2}-\beta)+(1-t_1^2)(\frac{p}{2}+\beta)\Bigg)$$
$$=e^{-\frac{(\lambda-\lambda_1)}{2}+\frac{t_1^2-t^2}{2}}\sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}2^{-(\frac{q_1}{2}+\beta)}$$
$$\times\Bigg(\frac{(\lambda-\lambda_1)t_1^2}{2}-\frac{(1-t_1^2)(1-t_1^2-2p)}{4}+\frac{(1-t^2)(1-t^2-2p_1)}{4}$$
$$+\frac{q_1}{2}(\frac{q_1}{2}-p-1)-(t^2+t_1^2+p_1-\beta)\beta\Bigg) \qquad \text{(B.2.22)}$$

$$f_{asymp}^{(2)}=e^{-\frac{(\lambda-\lambda_1)}{2}+\frac{t_1^2-t^2}{2}}\sum_{\beta=0}^{\infty}\frac{(t^2-t_1^2)^{\frac{q_1}{2}-1+\beta}}{\Gamma(\frac{q_1}{2}+\beta)\beta!}\cdot\left(\frac{\lambda-\lambda_1}{2}\right)^{\beta}2^{-(\frac{q_1}{2}+\beta)}$$
$$\times\Bigg\{\Bigg(\beta_2^*(m;t_1^2,\lambda,\lambda_1)+C_2^{\beta}+\beta_2(m)+\tilde{\beta}_2(m;1-t^2,-\frac{1}{2})+\tilde{\tilde{\beta}}_2(m;1-t_1^2,\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_2(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_2(m;t_1^2,p,\beta)\Bigg)$$
$$+\beta_1^*(m;t_1^2,\lambda,\lambda_1)\Bigg(-C_1^{\beta}+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Bigg)$$
$$+(-C_1^{\beta})\Bigg(\beta_1^*(m;t_1^2,\lambda,\lambda_1)+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Bigg)$$
$$+\beta_1(m)\Bigg(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^{\beta}+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Bigg)$$
$$+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})\Bigg(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^{\beta}+\beta_1(m)+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Bigg)$$
$$+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})\Bigg(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^{\beta}+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Bigg)$$

$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)\Big(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^\beta+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})$$
$$+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Big)$$
$$+\tilde{\tilde{\tilde{\beta}}}_1(m;t_1^2,p,\beta)\Big(\beta_1^*(m;t_1^2,\lambda,\lambda_1)-C_1^\beta+\beta_1(m)+\tilde{\beta}_1(m;1-t^2,-\frac{1}{2})$$
$$+\tilde{\tilde{\beta}}_1(m;1-t_1^2,\frac{1}{2})+\tilde{\tilde{\tilde{\beta}}}_1(m;t^2,p_1,\beta)\Big)\Big\} \tag{B.2.23}$$

Finally, we integrate (B.2.20) with respect to $\bar{X}^{(1)}$ and S_{11},so the expectation in (4.3.10) is

$$\int\int_{s_{11}>0,\bar{x}^{(1)},t_1^2\le t^2} f_{asymp}(t^2,t_1^2|x_t^{(1)},\bar{x}^{(1)},s_{11})\cdot f_{\bar{X}^{(1)}}(\bar{x}^{(1)})\cdot f_{S_{11}}(s_{11})d\bar{x}^{(1)}ds_{11}$$
$$=\int\int_{s_{11}>0,\bar{x}^{(1)},t_1^2\le t^2}\Big((\frac{1}{N})^0 f_{asymp}^{(0)}+(\frac{1}{N})f_{asymp}^{(1)}+(\frac{1}{N})^2 f_{asymp}^{(2)}+O_3(\frac{1}{N})\Big)\cdot f_{\bar{X}^{(1)}}(\bar{x}^{(1)})\cdot f_{S_{11}}(s_{11})d\bar{x}^{(1)}ds_{11} \tag{B.2.24}$$

so the proof is completed.

B.3 Distribution of T in (4.4.2)

Let $\Sigma^{\frac{1}{2}}=\begin{bmatrix}\Sigma_{11}^{\frac{1}{2}} & 0\\ \Sigma_{21}\Sigma_{11}^{-\frac{1}{2}} & \Sigma_{22.1}^{\frac{1}{2}}\end{bmatrix}$, where $\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}'}=\Sigma$. Then the inverse of $\Sigma^{\frac{1}{2}}$ is

$$(\Sigma^{-\frac{1}{2}})'=\begin{bmatrix}(\Sigma_{11}^{-\frac{1}{2}})' & 0\\ -(\Sigma_{22.1}^{-\frac{1}{2}})'\Sigma_{21}\Sigma_{11}^{-1} & (\Sigma_{22.1}^{-\frac{1}{2}})'\end{bmatrix},$$

where $(\Sigma^{-\frac{1}{2}})(\Sigma^{-\frac{1}{2}})'=\Sigma^{-1}$. And let $Z=(\Sigma^{-\frac{1}{2}})'\Big[\sqrt{\frac{N}{N+1}}\left(X_t-\bar{X}\right)\Big]$ and $S^*=(\Sigma^{-\frac{1}{2}})'S\Sigma^{-\frac{1}{2}}=\left((\Sigma^{-\frac{1}{2}})'S^{\frac{1}{2}}\right)\left((\Sigma^{-\frac{1}{2}})'S^{\frac{1}{2}}\right)'=S^{*\frac{1}{2}}(S^{*\frac{1}{2}})'$, where $S^{*\frac{1}{2}}=(\Sigma^{-\frac{1}{2}})'S^{\frac{1}{2}}$ and $(S^{*-\frac{1}{2}})'=(S^{-\frac{1}{2}})'\Sigma^{\frac{1}{2}}$. Then $Z\sim N_p(\mu^{(Z)},(\Sigma^{-\frac{1}{2}})'\Sigma(\Sigma^{-\frac{1}{2}})=I_p)$, where $\mu^*=\sqrt{\frac{N}{N+1}}(\Sigma^{-\frac{1}{2}})'(\mu-\mu_0)$, and $S^*\sim W_p(N-1,(N-1)^{-1}\Sigma^{-\frac{1}{2}}\Sigma\Sigma'^{-\frac{1}{2}}=(N-1)^{-1}I_p)$ are independent. The joint density of Z and $S^*$is

$$f_{Z,S^*}(Z,S^*)=f_Z(Z)\cdot f_{S^*}(S^*)$$
$$=(2\pi)^{-\frac{p}{2}}\exp(-\frac{1}{2}(Z-\mu^*)'(Z-\mu^*))\times(2^{\frac{(N-1)p}{2}}\Gamma_p(\frac{N-1}{2}))^{-1}|S^*|^{\frac{N-1-p-1}{2}}$$
$$|\frac{I_p}{N-1}|^{-\frac{N-1}{2}}etr(-\frac{1}{2}(N-1)S^*).$$

Transforming $T = (S^{*-1/2})'Z$, with $J(Z \to T) = |S^*|^{1/2}$, and$Z = S^{*1/2}T$, we obtain the joint density of T and S^* as

$$f_{T,S^*}(T, S^*) = f_{Z,S^*}(S^{*\frac{1}{2}}T, S^*) \cdot J(Z \to T) = f_Z(S^{*\frac{1}{2}}T) \cdot f_{S^*}(S^*) \cdot |S^*|^{1/2}$$

$$= (2\pi)^{-\frac{p}{2}} \exp(-\frac{1}{2}(S^{*\frac{1}{2}}T - \mu^*)'(S^{*\frac{1}{2}}T - \mu^*)) \times \ (2^{\frac{(N-1)p}{2}}\Gamma_p(\frac{N-1}{2}))^{-1}$$

$$\times |S^*|^{\frac{N-1-p-1}{2}} |\frac{I_p}{N-1}|^{-\frac{N-1}{2}} etr(-\frac{1}{2}(N-1)S^*)|S^*|^{\frac{1}{2}}$$

$$= (2\pi)^{-\frac{p}{2}} \exp(-\frac{1}{2}[T'S^*T + \mu^{*'}\mu^* - 2(S^{*\frac{1}{2}}T)'\mu^*]) \cdot (2^{\frac{(N-1)p}{2}}\Gamma_p(\frac{N-1}{2}))^{-1}$$

$$\times |S^*|^{\frac{N-1-p}{2}} |\frac{I_p}{N-1}|^{-\frac{N-1}{2}} etr(-\frac{1}{2}(N-1)S^*)$$

$$= (2\pi)^{-\frac{p}{2}} \cdot (2^{\frac{(N-1)p}{2}}\Gamma_p(\frac{N-1}{2}))^{-1} \cdot |\frac{I_p}{N-1}|^{-\frac{N-1}{2}} \cdot etr\{-\frac{1}{2}\mu^{*'}\mu^*\}\cdot$$

$$|S^*|^{\frac{N-1-p}{2}} \cdot etr\{-\frac{1}{2}(TT' + (N-1)I_p)S^* + \mu^*(S^{*\frac{1}{2}}T)'\}. \quad \text{(B.3.1)}$$

To find the marginal density of T, we integrate (C.1) with respect to S^*. Let $U = \frac{1}{2}(TT' + (N-1)I_p)^{\frac{1}{2}'}S^*(TT'+(N-1)I_p)^{\frac{1}{2}}$, then $J(S^* \to T) = |TT'+(N-1)I_p|^{-\frac{p+1}{2}}$ (see, for example, Gupta and Nagar (2000), p. 13), $S^* = 2(TT' + (N-1)I_p)^{-\frac{1}{2}}U(TT' + (N-1)I_p)^{-\frac{1}{2}'}$, and $(S^{*\frac{1}{2}})' = \sqrt{2}(TT' + (N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}$. And we can write

$$\int_{S^*>0} |S^*|^{\frac{N-1-p}{2}} \cdot etr\{-\frac{1}{2}(TT' + (N-1)I_p)S^* + \mu^*(S^{*\frac{1}{2}}T)'\}dS^*$$

$$= \int_{U>0} \Big\{|2(TT' + (N-1)I_p)^{-\frac{1}{2}}U(TT' + (N-1)I_p)^{-\frac{1}{2}'}|^{\frac{N-1-p}{2}} \cdot$$

$$etr\{-U + \sqrt{2}\mu^{(Z)}T'(TT' + (N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}\} \cdot \left|\frac{1}{2}(TT' + (N-1)I_p)\right|^{-\frac{p+1}{2}}\Big\}dU$$

$$= 2^{\frac{Np}{2}}|TT' + (N-1)I_p|^{-\frac{N}{2}}\int_{U>0} |U|^{\frac{N-1-p}{2}} \cdot etr\{-U + \sqrt{2}\mu^*T'(TT' + (N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}\}dU. \quad \text{(B.3.2)}$$

Substituting (C.2) into (C.1) and using multivariate gamma function (Gupta and Nagar (2000), p. 19), $\Gamma_p(a) = \pi^{-\frac{p(p-1)}{4}}\prod_{i=1}^{p}\Gamma(a - \frac{i-1}{2})$, where $Re(a) > \frac{1}{2}(p-1)$, we obtain

$$f_T(T) = (2\pi)^{-\frac{p}{2}} \cdot (2^{\frac{(N-1)p}{2}}\Gamma_p(\frac{N-1}{2}))^{-1} \cdot |\frac{I_p}{N-1}|^{-\frac{N-1}{2}} \cdot etr\{-\frac{1}{2}\mu^{*'}\mu^*\}$$

$$\times\, 2^{\frac{Np}{2}}|TT'+(N-1)I_p|^{-\frac{N}{2}}\int_{U>0}|U|^{\frac{N-1-p}{2}}\cdot etr\{-U+\sqrt{2}\mu^*T'(TT'+(N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}\}dU$$

$$=\frac{\pi^{-\frac{p}{2}}\Gamma_p(\frac{N}{2})}{\Gamma_p(\frac{N-1}{2})}\cdot|(N-1)I_p|^{\frac{N-1}{2}}|(N-1)I_p|^{-\frac{N}{2}}(1+(N-1)^{-1}T'T)^{-\frac{N}{2}}$$

$$\times\, etr\{-\frac{1}{2}\mu^{*'}\mu^*\}\cdot\Gamma_p(\frac{N}{2})^{-1}\cdot\int_{U>0}|U|^{\frac{N-1-p}{2}}\cdot etr\{-U+\sqrt{2}\mu^*T'(TT'+(N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}\}dU$$

$$=\frac{\pi^{-\frac{p}{2}}\cdot\Gamma(\frac{(N-p)+p}{2})}{\Gamma(\frac{N-p}{2})}(N-1)^{-\frac{p}{2}}(1+(N-1)^{-1}T'T)^{-\frac{N}{2}}\cdot I_T(\mu-\mu_0,T) \tag{B.3.3}$$

where

$$I_T(\mu-\mu_0,T)=etr\{-\frac{1}{2}\frac{N}{N+1}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\}\cdot\Gamma_p(\frac{N}{2})^{-1}$$

$$\times\int_{U>0}|U|^{\frac{N-1-p}{2}}etr\{-U+\sqrt{\frac{2N}{N+1}}(\Sigma^{-\frac{1}{2}})'(\mu-\mu_0)T'(TT'+(N-1)I_p)^{-\frac{1}{2}}U^{\frac{1}{2}'}\}dU.$$

Thus, if$\mu-\mu_0=0$, then $I_T(0,T)=\Gamma_p(\frac{N}{2})^{-1}\cdot\int_{U>0}|U|^{\frac{N-1-p}{2}}etr\{-U\}dU=1$ since $\int_{U>0}|U|^{\frac{N-1-p}{2}}\cdot etr\{-U\}dU=\Gamma_p(\frac{N}{2})$ is a multivariate gamma function, and we obtain the central case. Hence we complete the proof.

B.4 Marginal Distribution of T_1 and T_2 in (4.4.3)

First, recall that $T_1=\sqrt{\frac{N}{N+1}}(S_{11}^{-\frac{1}{2}})'(X_t^{(1)}-\bar{X}^{(1)})$ and

$$T_2=\sqrt{\frac{N}{N+1}}(S_{22.1}^{-\frac{1}{2}})'\left((X_t^{(2)}-\bar{X}^{(2)})-S_{21}S_{11}^{-1}(X_t^{(1)}-\bar{X}^{(2)})\right)$$

are partitions of T with dimensions p_1 and q_1, respectively. When $\mu-\mu_0=0$, from (4.4.5), we have

$$f_T(T)=\frac{\pi^{-\frac{p}{2}}\cdot\Gamma(\frac{N}{2})}{\Gamma(\frac{N-p}{2})}\cdot(N-1)^{-\frac{p}{2}}\cdot(1+(N-1)^{-1}T_1'T_1+(N-1)^{-1}T_2'T_2)^{-\frac{N}{2}}$$

$$=\frac{\pi^{-\frac{p_1}{2}}\cdot\Gamma(\frac{N}{2})(N-1)^{-\frac{p_1}{2}}}{\Gamma(\frac{N-p_1}{2})}(1+(N-1)^{-1}T_1'T_1)^{-\frac{N}{2}}$$

$$\times\frac{\pi^{-\frac{p_2}{2}}\cdot\Gamma(\frac{N-p_1}{2})(N-1)^{-\frac{p_2}{2}}}{\Gamma(\frac{N-p}{2})}\frac{(1+(N-1)^{-1}T_1'T_1+(N-1)^{-1}T_2'T_2)^{-\frac{N}{2}}}{(1+(N-1)^{-1}T_1'T_1)^{-\frac{N}{2}}}$$

$$=\frac{\pi^{-\frac{p_1}{2}}\cdot\Gamma(\frac{N}{2})(N-1)^{-\frac{p_1}{2}}}{\Gamma(\frac{N-p_1}{2})}(1+(N-1)^{-1}T_1'T_1)^{-\frac{N}{2}}$$

$$\times \frac{\pi^{-\frac{q_1}{2}} \cdot \Gamma(\frac{N-p_1}{2})(N-1)^{-\frac{q_1}{2}}}{\Gamma(\frac{N-p}{2})}(1 + \frac{(N-1)^{-1}T_2'T_2}{1+(N-1)^{-1}T_1'T_1})^{-\frac{N}{2}}$$

$$= f_{T_1}(T_1) \cdot f_{T_2|T_1}(T_2|T_1), \text{ say} \qquad \text{(B.4.1)}$$

where $f_{T_1}(T_1)$ is the marginal density of T_1 and $f_{T_2|T_1}(T_2|T_1)$ is the conditional density of T_2 given T_1. Similarly, by simply interchanging T_1 and T_2, we can obtain

$$f_T(T) = f_{T_2}(T_2) \cdot f_{T_1|T_2}(T_1|T_2), \qquad \text{(B.4.2)}$$

where $f_{T_2}(T_2) = \frac{\pi^{-\frac{q_1}{2}} \cdot \Gamma(\frac{N-p+q_1}{2})(N-1)^{-\frac{q_1}{2}}}{\Gamma(\frac{N-p}{2})}(1+(N-1)^{-1}T_2'T_2)^{-\frac{N-p+q_1}{2}}$ is the marginal density of T_2 and $f_{T_1|T_2}(T_1|T_2) = \frac{\pi^{-\frac{p_1}{2}} \cdot \Gamma(\frac{N}{2})(N-1)^{-\frac{p_1}{2}}}{\Gamma(\frac{N-p+q_1}{2})}(1+(N-1)^{-1}T_2'T_2)^{-\frac{p_1}{2}}(1+\frac{(N-1)^{-1}T_1'T_1}{1+(N-1)^{-1}T_2'T_2})^{-\frac{N}{2}}$ is the conditional density of T_1 given T_2.

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