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CHAOTIC VIBRATIONS OF THE ONE-DIMENSIONAL MIXED WAVE SYSTEM

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In this paper, we consider the initial-boundary value problem of the one-dimensional linear mixed wave equation $\omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0$ ($d \in \mathbb{R}$, $c > 0$) on an interval, where the boundary condition at the left endpoint is linear, pumping energy into the system, while the boundary condition at the right endpoint has odd-degree nonlinearity. This problem is said to be the one-dimensional mixed wave system. The solution of the one-dimensional mixed wave system corresponds to the iteration of an interval map h. Thus, the mixed wave system is said to be chaotic if the interval map h is chaotic in the sense of Li–Yorke. In this paper, we show that the mixed wave system is chaotic under some conditions.

Keywords: Chaotic vibrations; mixed wave system.

1. Introduction

In [Chen *et al.*, 1998b; Chen *et al.*, 2004; Huang & Feng, 2006], the one-dimensional wave equation is considered:

$$\omega_{tt} - \omega_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{1}$$

with the boundary conditions

$$\begin{cases} \omega_t(0,t) + \eta\omega_x(0,t) = 0, \quad \eta > 0, \\ \qquad\qquad \eta \neq 1, \quad t > 0, \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), \\ \qquad \alpha \in (0,1], \quad \beta > 0, \quad t > 0, \end{cases} \tag{2}$$

and the initial conditions

$$\begin{aligned} \omega(x,0) &= \varphi(x) \in C^1([0,1]), \\ \omega_t(x,0) &= \psi(x) \in C^0([0,1]). \end{aligned} \tag{3}$$

The boundary condition at the left endpoint $x = 0$ is linear, pumping energy into the system since $\eta > 0$ and the boundary condition at the right endpoint $x = 1$ is a van der Pol condition which is a well-known self-regulating mechanism in automatic control. And in [Chen *et al.*, 2002; Huang, 2003b], they consider the one-dimensional mixed wave equation

$$\begin{aligned} \omega_{xx} - v\omega_{tx} - \omega_{tt} = 0, \quad v > 0, \\ 0 < x < 1, \quad t > 0, \end{aligned} \tag{4}$$

with the boundary conditions

$$\omega_x(0,t) = 0, \quad t > 0, \tag{5}$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t),$$

$$\alpha \in \left(0, \frac{v + \sqrt{v^2+4}}{2}\right], \quad \beta > 0, \quad t > 0, \tag{6}$$

and the initial conditions

$$\begin{aligned} \omega(x,0) &= \varphi(x) \in C^1([0,1]), \\ \omega_t(x,0) &= \psi(x) \in C^0([0,1]). \end{aligned} \tag{7}$$

The boundary condition at the left endpoint $x = 0$ is the homogeneous Neumann condition and the boundary condition at the right endpoint $x = 1$ is the van der Pol condition.

In order to obtain more parameters, we consider the one-dimensional mixed wave equation

$$\omega_{tt} - d\omega_{tx} - c^2\omega_{xx} = 0, \quad d \in \mathbb{R}, \quad c > 0, \quad 0 < x < 1, \quad t > 0, \tag{8}$$

with the boundary conditions ($c_1 = (d + \sqrt{d^2 + 4c^2})/2$ and $c_2 = (-d + \sqrt{d^2 + 4c^2})/2$)

$$\omega_t(0,t) + \eta\omega_x(0,t) = 0, \quad \eta > 0, \quad \eta \neq c_2, \quad t > 0, \tag{9}$$

and

$$\omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^{2m+1}(1,t), \quad \alpha \in \left(0, \frac{1}{c_1}\right], \quad \beta > 0, \quad m \in \mathbb{N}, \quad t > 0. \tag{10}$$

And with the initial conditions

$$\omega(x,0) = \varphi(x) \in C^1([0,1]), \quad \omega_t(x,0) = \psi(x) \in C^0([0,1]). \tag{11}$$

Consider the case $d = 0$ and $c^2 = 1$ (i.e. $c_1 = c_2 = 1$) in Eq. (8), the equation is reduced to Eq. (1). In [Chen *et al.*, 1998b], they proved the 1D wave system Eqs. (1)–(3) is chaotic when the parameter η enters the region $[(3\sqrt{3} - 1 - \alpha)/(3\sqrt{3} + 1 + \alpha), 1) \cup (1, (3\sqrt{3} + 1 + \alpha)/(3\sqrt{3} - 1 - \alpha)]$ for any given $\alpha \in (0, 1]$, $\beta > 0$. In [Chen *et al.*, 2004; Huang & Feng, 2006], they characterized the dynamical behavior in terms of the growth of the total variation of the interval map. And they proved that for any given $\alpha \in (0, 1]$, there exist four constants $\underline{\eta_0}$, $\underline{\eta_H}$, $\overline{\eta_H}$ and $\overline{\eta_0}$ with $0 < \underline{\eta_0} < \underline{\eta_H} < 1 < \overline{\eta_H} < \overline{\eta_0} < \infty$ such that the total variation of the interval map remains bounded, is unbounded, is unbounded exponentially when the parameter η belongs to $(0, \underline{\eta_0}) \cup (\overline{\eta_0}, \infty)$, $(\underline{\eta_0}, \underline{\eta_H}) \cup (\overline{\eta_H}, \overline{\eta_0})$ and $(\underline{\eta_H}, 1) \cup (1, \overline{\eta_H})$, respectively. In particular, the last case corresponds to chaos in the 1D wave system Eqs. (1)–(3). In this paper, we show the 1D mixed wave system Eqs. (8)–(11) is chaotic when the parameter η satisfies either

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2$$

or

$$c_2 < \eta \leq \frac{c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}$$

for any given parameters c_1, c_2, α, β, m satisfying the inequality

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0,$$

and when the parameter η satisfies either

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \leq \eta < c_2 \quad \text{or} \quad \eta > c_2$$

for any given parameters c_1, c_2, α, β, m satisfying the inequality

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) \leq 0.$$

Consider the case $d = -v$ and $c^2 = 1$ (i.e. $c_1 = (-v + \sqrt{v^2+4})/2$ and $c_2 = (v + \sqrt{v^2+4})/2$) in Eq. (8), the equation is reduced to Eq. (4). In [Chen *et al.*, 2002], they proved the 1D mixed wave system Eqs. (4)–(7) is chaotic when the parameters (v, α) enter a certain subregion of $S = \{(v,\alpha) \in \mathbb{R}^2 \mid 0 < v < \infty, 0 < \alpha \leq (v + \sqrt{v^2+4})/2\}$. In [Huang, 2003b], they proved there exist three subregions S_1^0, S_1^1 and S_2 of S such that the growth of the total variation of the interval map remains bounded, is unbounded, is unbounded exponentially when the parameters (v, α) belong to S_1^0, S_1^1 and S_2, respectively. In particular, the last case corresponds to chaos in the 1D mixed wave system Eqs. (4)–(7). In this paper, we consider the boundary condition at the left endpoint is Eq. (9) which is different from Eq. (5). We show the 1D mixed wave system Eqs. (8)–(11) is chaotic for any given $c_1 \leq 1/\alpha$ if the parameters η, c_2, α, β, m satisfy

$$\eta > c_2 \quad \text{and}$$

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \geq 0.$$

And the 1D mixed wave system Eqs. (8)–(11) is chaotic for sufficiently small c_1 if the parameters η, c_2, α, β, m satisfy some conditions or for sufficiently large c_1 if the parameters η, c_2, α, β, m satisfy some other conditions. Notice that the initial conditions in the system have some properties which we will discuss later.

The general solution of Eq. (8) is

$$\omega(x,t) = u(c_1 t + x) + v(c_2 t - x), \tag{12}$$

where u, v are arbitrary C^2-function. Substituting Eq. (12) into Eqs. (9) and (10) we have

$$v'(c_2 t) = -\frac{c_1+\eta}{c_2-\eta} u'(c_1 t), \quad t > 0,$$

and

$$\begin{aligned}&\beta(c_1 u'(c_1 t+1) + c_2 v'(c_2 t - 1))^{2m+1}\\ &\quad + \left(\frac{1}{c_1} - \alpha\right)(c_1 u'(c_1 t + 1) + c_2 v'(c_2 t - 1))\\ &\quad - \left(1 + \frac{c_2}{c_1}\right) v'(c_2 t - 1) = 0, \quad t > 0.\end{aligned}$$

And by using the substitution

$$z(c_1 t) = \begin{cases} v'\left(\dfrac{c_2}{c_1}\left(c_1 t - \dfrac{c_1}{c_2}\right)\right), & 0 \le t \le \dfrac{1}{c_2}, \\ \dfrac{\eta + c_1}{\eta - c_2} u'\left(c_1 t - \dfrac{c_1}{c_2}\right), & t > \dfrac{1}{c_2}, \end{cases} \tag{13}$$

we have the difference equation

$$\begin{aligned}&\beta\left(c_1 \frac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) + c_2 z(\tau)\right)^{2m+1} + \left(\frac{1}{c_1} - \alpha\right)\\ &\quad \times \left(c_1 \frac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) + c_2 z(\tau)\right)\\ &\quad - \left(1 + \frac{c_2}{c_1}\right) z(\tau) = 0,\end{aligned} \tag{14}$$

where $\tau = c_1 t$, $\Delta = 1 + (c_1/c_2)$. The initial condition of Eq. (14) is

$$z(c_1 t) = \begin{cases} \dfrac{\psi(1 - c_2 t) - c_1 \varphi'(1 - c_2 t)}{c_1 + c_2}, \\ \qquad 0 \le t \le \dfrac{1}{c_2}. \\ \dfrac{\eta + c_1}{\eta - c_2} \dfrac{\psi\left(c_1 t - \dfrac{c_1}{c_2}\right) + c_2 \varphi'\left(c_1 t - \dfrac{c_1}{c_2}\right)}{c_1 + c_2}, \\ \qquad \dfrac{1}{c_2} < t \le \dfrac{1}{c_1} + \dfrac{1}{c_2}. \end{cases} \tag{15}$$

Remark 1.1. In this paper, we assume that the initial values $\varphi(0)$ and $\psi(0)$ are chosen such that $z(\tau)$ is continuous on $[0, 1 + (c_1/c_2)]$ and satisfy the compatibility condition

$$\begin{aligned}&\beta\left(c_1 \frac{\eta - c_2}{\eta + c_1} z(\Delta) + c_2 z(0)\right)^{2m+1} + \left(\frac{1}{c_1} - \alpha\right)\\ &\quad \times \left(c_1 \frac{\eta - c_2}{\eta + c_1} z(\Delta) + c_2 z(0)\right) - \left(1 + \frac{c_2}{c_1}\right) z(0) = 0.\end{aligned}$$

Definition 1.2. In this paper, we denote the range of $z(\tau)$ on $[0, \Delta]$ to be the compact interval Λ, i.e. $\Lambda = z([0, \Delta])$.

The dependence of $z(\tau + \Delta)$ on $z(\tau)$ is given implicitly by a function f_λ. To see this we define

$$F(x, v) = \beta x^{2m+1} + \left(\frac{1}{c_1} - \alpha\right) x - \left(1 + \frac{c_2}{c_1}\right) v,$$

which is strictly increasing on x if v is to be fixed. Hence, for each v_0 there exists a function $g(v_0) = x_0$ such that $F(g(v_0), v_0) = 0$. Thus, we have

$$g(z(\tau)) = c_1 \frac{\eta - c_2}{\eta + c_1} z(\tau + \Delta) + c_2 z(\tau), \quad \tau > 0,$$

which implies that

$$z(\tau + \Delta) = \frac{\eta + c_1}{c_1 (\eta - c_2)} (g(z(\tau)) - c_2 z(\tau)), \quad \tau > 0.$$

Therefore, we have the definition as below.

Definition 1.3. We denote $f_\lambda(z(\tau)) = z(\tau + \Delta)$ to be the function satisfying Eq. (14) for all $\tau \ge 0$, where $\lambda = (\eta, c_1, c_2, \alpha, \beta, m)$.

Since

$$f_\lambda(z(\tau)) = z(\tau + \Delta) \quad \text{for all } \tau > 0,$$

we can use the map f_λ and the interval Λ to generate $z(\tau)$ for all $\tau > 0$. And the corresponding solution of the mixed wave system Eqs. (8)–(11) is calculated via the formulae

$$\begin{aligned}\omega(x, t) &= \int_{\frac{1}{c_2}}^{t + \frac{x}{c_1} + \frac{1}{c_2}} c_1 \frac{\eta - c_2}{\eta + c_1} z(c_1 \tau) d\tau\\ &\quad + \int_0^{t - \frac{x}{c_2} + \frac{1}{c_2}} c_2 z(c_1 \tau) d\tau.\end{aligned}$$

Definition 1.4. The mixed wave system is said to be chaotic if the interval map f_λ is chaotic in the sense of Li–Yorke.

2. The Chaotic Region of the Mixed Wave System

In this section, we want to show the chaotic region of the mixed wave system Eqs. (8)–(11). First, we consider the equation

$$H(x,y) = \beta\left(c_1\frac{\eta - c_2}{\eta + c_1}y + c_2x\right)^{2m+1} + \left(\frac{1}{c_1} - \alpha\right) \times \left(c_1\frac{\eta - c_2}{\eta + c_1}y + c_2x\right) - \left(1 + \frac{c_2}{c_1}\right)x = 0, \tag{16}$$

where η, c_1, c_2, β are positive ($\eta \neq c_2$), $0 < \alpha \leq 1/c_1$ and $m \in \mathbb{N}$.

Definition 2.1. We denote that

$$v_c = \frac{c_1}{c_1 + c_2}\left[\frac{1 + \alpha c_2}{c_2(2m+1)} + \frac{1}{c_1} - \alpha\right] \times \sqrt[2m]{\frac{1 + \alpha c_2}{(2m+1)\beta c_2}}$$

in the following lemmas.

Lemma 2.2. *Let $y = h(x)$ be the unique function which satisfies Eq. (16) where η, c_1, c_2, α, β, m are to be fixed. Then the function h is odd and h has local extrema at*

$$\left(v_c, \frac{2m}{2m+1}\frac{1 + \alpha c_2}{c_1 + c_2}\frac{\eta + c_1}{\eta - c_2}\sqrt[2m]{\frac{1 + \alpha c_2}{(2m+1)\beta c_2}}\right)$$

and

$$\left(-v_c, -\frac{2m}{2m+1}\frac{1 + \alpha c_2}{c_1 + c_2}\frac{\eta + c_1}{\eta - c_2}\sqrt[2m]{\frac{1 + \alpha c_2}{(2m+1)\beta c_2}}\right).$$

Furthermore, the function h is strictly monotonic on $(-\infty, -v_c)$, $(-v_c, v_c)$ and (v_c, ∞).

Proof. Since $H(-x, h(-x)) = H(-x, -h(x)) = 0$, we can see h is odd. Then use

$$\frac{d}{dx}H(x,y) = (2m+1)\beta\left(c_1\frac{\eta - c_2}{\eta + c_1}y + c_2x\right)^{2m} \times \left(c_1\frac{\eta - c_2}{\eta + c_1}y' + c_2\right) + \left(\frac{1}{c_1} - \alpha\right) \times \left(c_1\frac{\eta - c_2}{\eta + c_1}y' + c_2\right) - \left(1 + \frac{c_2}{c_1}\right) = 0$$

and carry out the computations. We have the results. ■

Lemma 2.3. *Suppose the parameters η, c_1, c_2, α, β, m are to be fixed in Eq. (16). Then the function h intersects the x-axis at the points*

$$\left(-\frac{1}{c_2}\sqrt[2m]{\frac{1 + \alpha c_2}{\beta c_2}}, 0\right), \quad (0, 0), \quad \left(\frac{1}{c_2}\sqrt[2m]{\frac{1 + \alpha c_2}{\beta c_2}}, 0\right).$$

Proof. Straightforward verification by computing

$$H(x, 0) = \beta(0 + c_2x)^{2m+1} + \left(\frac{1}{c_1} - \alpha\right)(0 + c_2x) - \left(1 + \frac{c_2}{c_1}\right)x = 0,$$

we have $x(\beta c_2^{2m+1}x^{2m} - \alpha c_2 - 1) = 0$ which implies $x = 0, \pm(1/c_2)\sqrt[2m]{(1 + \alpha c_2)/(\beta c_2)}$. ■

Lemma 2.4. *Suppose the parameters η, c_1, c_2, α, β, m are to be fixed in Eq. (16). Then the function h intersects the line $y = x$ at the points*

$$\left(-\frac{\eta + c_1}{\eta(c_1 + c_2)}\sqrt[2m]{\frac{1 + \alpha\eta}{\beta\eta}}, -\frac{\eta + c_1}{\eta(c_1 + c_2)}\sqrt[2m]{\frac{1 + \alpha\eta}{\beta\eta}}\right), \quad (0, 0),$$

and

$$\left(\frac{\eta + c_1}{\eta\,(c_1 + c_2)}\sqrt[2m]{\frac{1 + \alpha\eta}{\beta\eta}}, \frac{\eta + c_1}{\eta\,(c_1 + c_2)}\sqrt[2m]{\frac{1 + \alpha\eta}{\beta\eta}}\right).$$

Proof. Straightforward verification by computing $H(x, x) = 0$. ■

Definition 2.5. We denote that

$$B = \frac{\eta + c_1}{|2c_1c_2 + (c_2 - c_1)\eta|} \times \sqrt[2m]{\frac{2\eta + 2\alpha c_1c_2 + (c_1 - c_2) + \alpha\eta\,(c_2 - c_1)}{\beta[2c_1c_2 + (c_2 - c_1)\eta]}}$$

in the following lemmas.

Lemma 2.6. *Suppose the parameters η, c_1, c_2, α, β, m are to be fixed in Eq. (16) with*

$$\eta > c_2 \quad \text{and} \quad 2c_1c_2 + (c_2 - c_1)\eta \neq 0.$$

Then the function h intersects the line $y = -x$ at the points

$$(-B, B), \quad (0, 0), \quad (B, -B),$$

if (i) $c_1 \le c_2$ *or if* (ii) $c_1 > c_2$ *and* $2c_1c_2 + (c_2 - c_1)\eta > 0$ *or if* (iii) $c_1 > c_2$ *and* $2c_1c_2 + (c_2 - c_1)\eta < 0$ *and* $(2\eta + c_1 - c_2)/(2c_1c_2 + (c_2 - c_1)\eta) > -\alpha$.

Furthermore, if the parameters are not in these three cases then the function h *intersects the line* $y = -x$ *only at the point* $(0, 0)$.

Proof. Straightforward verification by computing $H(x, -x) = 0$ and the three cases imply that

$$\frac{2\eta + 2\alpha c_1 c_2 + (c_1 - c_2) + \alpha\eta\,(c_2 - c_1)}{\beta[2c_1c_2 + (c_2 - c_1)\eta]}$$

is positive. Otherwise, $(2\eta + 2\alpha c_1c_2 + (c_1 - c_2) + \alpha\eta(c_2 - c_1))/(\beta[2c_1c_2 + (c_2 - c_1)\eta])$ is zero or negative. ■

Remark 2.7. It is easy to see the case when $\eta > c_2$ and $2c_1c_2 + (c_2 - c_1)\eta = 0$ that implies the function h intersects the line $y = -x$ only at the point $(0, 0)$.

Lemma 2.8. *Suppose the parameters* η, c_1, c_2, α, β, m *are to be fixed in Eq.* (*16*).

(i) *If* $0 < \eta < c_2$ *and*

$$M = -\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \le \frac{\eta+c_1}{\eta\,(c_1+c_2)}\sqrt[2m]{\frac{1+\alpha\eta}{\beta\eta}},$$

then the iterates of every point in the set

$$U \equiv \left(-\infty, -\frac{\eta+c_1}{\eta\,(c_1+c_2)}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}\right) \cup \left(\frac{\eta+c_1}{\eta\,(c_1+c_2)}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}, \infty\right)$$

escape to $\pm\infty$, *while those of any point in* $\mathbb{R}\backslash\overline{U}$ *are attracted to the bounded invariant interval* $I \equiv [-M, M]$ *of* h.

(ii) *If* $\eta > c_2$ *and* h *intersects the line* $y = -x$ *at three points and*

$$M = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \le B,$$

then the iterates of every point in the set $U \equiv (-\infty, -B) \cup (B, \infty)$ *escape to* $\pm\infty$, *while those of any point in* $\mathbb{R}\backslash\overline{U}$ *are attracted to the bounded invariant interval* $I \equiv [-M, M]$ *of* h.

(iii) *If* $\eta > c_2$ *and* h *intersects the line* $y = -x$ *only at* $(0, 0)$, *then the iterates of every point in* $\mathbb{R}$ *are attracted to the bounded invariant interval* $I \equiv [-M, M]$ *of* h *where*

$$M = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}.$$

Proof. The results of (i) and (ii) follow easily from the above lemmas and other piecewise monotonic properties of h, as can be directly confirmed by graphical analysis. We omit the details (see Figs. 1 and 2).

(iii) If $\eta > c_2$ and h intersects the line $y = -x$ only at $(0, 0)$, then $|h(x)| < |x|$ for all $x \in (-\infty, -(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}) \cup ((1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}, \infty)$. Thus, $|h^n(x)|$ is strictly decreasing on $(-\infty, -(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}) \cup ((1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}, \infty)$. Hence, the iterates of every point in $\mathbb{R}$ are attracted to the bounded invariant interval $I \equiv [-M, M]$ of h (see Fig. 3). ■

Lemma 2.9. *Suppose the parameters* η, c_1, c_2, α, β, m *are to be fixed in Eq.* (*16*) *and satisfy the inequality*

$$\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\left|\frac{\eta+c_1}{\eta-c_2}\right|\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}} \ge \frac{1}{c_2}\sqrt[2m]{\frac{1+\alpha c_2}{\beta c_2}}, \quad (17)$$

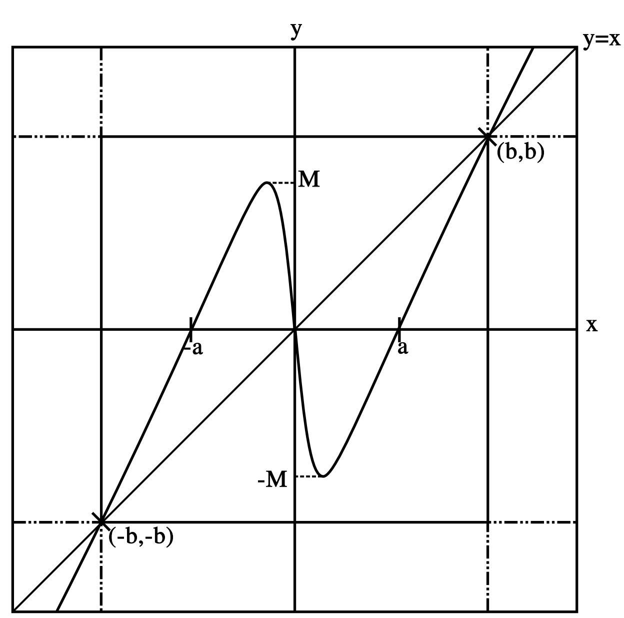

Fig. 1. $M < b$.

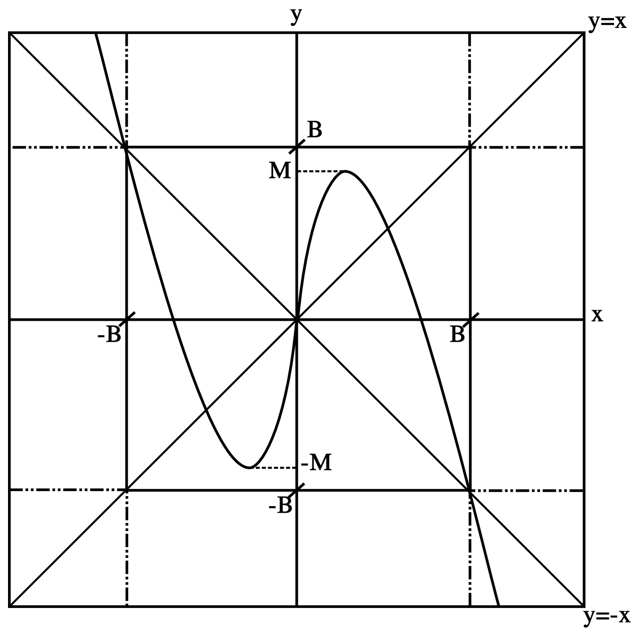

Fig. 2. $M < B$.

then the interval map h is chaotic in the sense of Li–Yorke if the domain of h contains the interval $[-(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}, (1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}]$.

Proof

(i) If $\eta > c_2$, then

$$h(v_c) = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}$$

is the local maximum. Since h is strictly increasing on $[0, v_c]$ and $h(v_c) \geq (1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}$, there exists one unique point $p_1 \in (0, v_c]$ such that $h(p_1) = (1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}$. Similarly, there exists one unique point $p_2 \in (0, p_1)$ such that $h(p_2) = p_1$. Hence, we have

$$0 = h^3(p_2) < p_2 < h(p_2) < h^2(p_2) \text{ (see Fig. 4).}$$

Thus, h has points of all periods implying chaos [Li & Yorke, 1975].

(ii) If $0 < \eta < c_2$, then

$$h(v_c) = \frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2}\sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}$$

is the local minimum and

$$h(-v_c) = -\frac{2m}{2m+1}\frac{1+\alpha c_2}{c_1+c_2}\frac{\eta+c_1}{\eta-c_2} \times \sqrt[2m]{\frac{1+\alpha c_2}{(2m+1)\beta c_2}}$$

is the local maximum. Since h is strictly decreasing on $[-v_c, v_c]$ and $h(v_c) \leq -(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}$, there exist one unique point $p_1 \in (0, v_c]$ such that $h(p_1) = -(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}$. And since h is odd, there exists one unique point $p_2 \in (-p_1, 0)$ such that $h(p_2) = p_1$. Similarly, there exists one unique point $p_3 \in (0, -p_2)$ such that $h(p_3) = p_2$

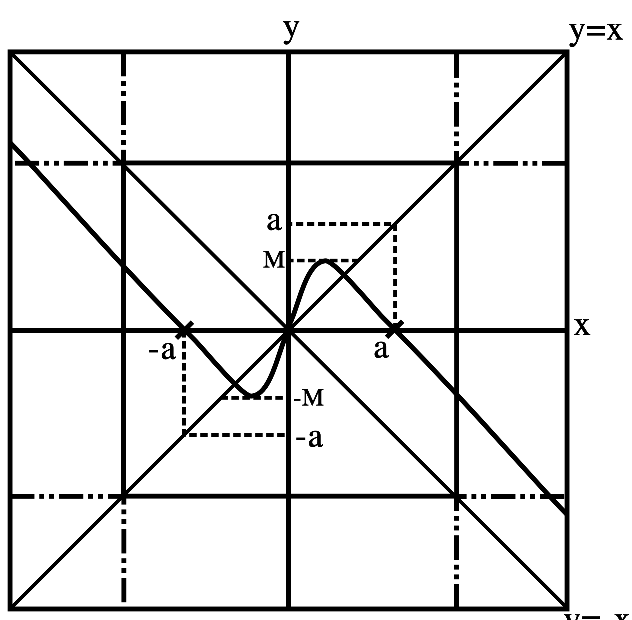

Fig. 3. The function h intersects the line $y = -x$ only at the point $(0, 0)$.

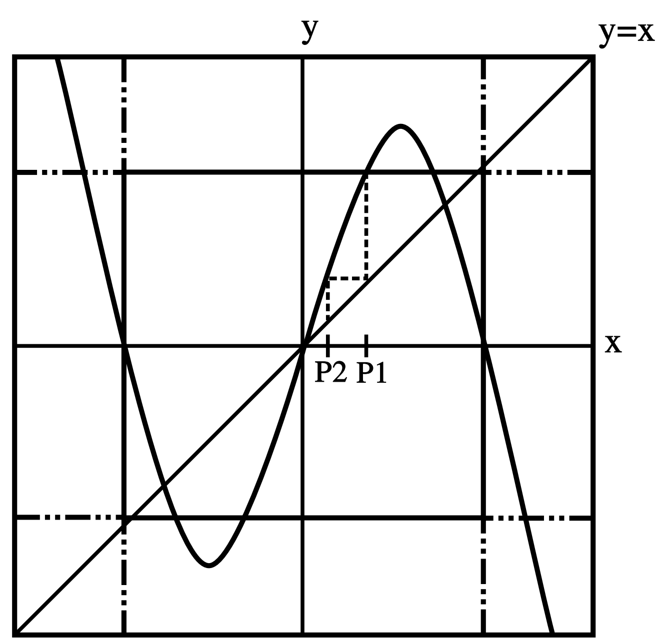

Fig. 4. $p_2 < p_1$.

and then there exists one unique point $p_4 \in (-p_3, 0)$ such that $h(p_4) = p_3$. Then there exists one unique point $p_5 \in (0, -p_4)$ such that $h(p_5) = p_4$. Hence, we have

$$0 = h^6(p_5) < p_5 < h^2(p_5) < h^4(p_5) \text{ (see Fig. 5).}$$

Thus, $g = h^2$ has points of all periods implying chaos [Li & Yorke, 1975]. Therefore, h is chaotic in the sense of Li–Yorke. ■

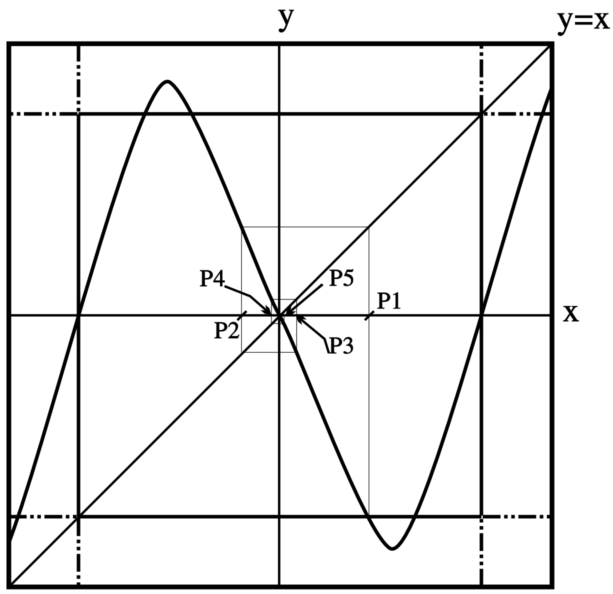

Fig. 5. $p_5 < p_3 < p_1$.

Now we want to show the chaotic region of η when c_1, c_2, α, β, m are to be fixed. There are two different cases as follows.

Proposition 2.10. *Suppose the parameters c_1, c_2, α, β, m are to be fixed in Eq. (16) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0.$$

Then the inequality (17) holds if and only if η satisfies either

$$0 < \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2$$

or

$$c_2 < \eta \le \frac{c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}.$$

Proof

(i) If $\eta > c_2$, then the inequality (17) is equivalent to

$$c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)] \ge \eta[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)].$$

And since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0,$$

the inequality (17) is equivalent to

$$c_2 < \eta \le \frac{c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}.$$

(ii) If $\eta < c_2$, then the inequality (17) is equivalent to

$$\eta[2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)] \ge c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)].$$

Furthermore, the inequality (17) is equivalent to

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2.$$

And since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2) \ge \sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1\left(1+\frac{c_2}{c_1}\right) > 0,$$

we have

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} > 0.$$

By (i) and (ii), the inequality (17) holds if and only if η satisfies either

$$0 < \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2$$

or

$$c_2 < \eta \le \frac{c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)}.$$

■

Proposition 2.11. *Suppose the parameters c_1, c_2, α, β, m are to be fixed in Eq. (16) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) \le 0.$$

Then the inequality (17) holds if and only if η satisfies either

$$\eta > c_2 \quad \text{or} \quad 0 < \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2.$$

Proof. If $\eta > c_2$, then the inequality (17) is equivalent to

$$c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)] \ge \eta[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)].$$

Since

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) \le 0,$$

we can conclude the inequality (17) always holds. Thus, the inequality (17) holds if and only if η satisfies either

$$\eta > c_2 \quad \text{or} \quad 0 < \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2.$$

■

Now we want to show the chaotic region of c_1 when η, c_2, α, β, m are to be fixed. There are three cases as follows.

Proposition 2.12. *Suppose the parameters η, c_2, α, β, m are to be fixed in Eq. (16) and satisfy the inequality*

$$\eta > c_2 \quad \text{and}$$

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \ge 0,$$

then the inequality (17) holds for any $c_1 \le 1/\alpha$.

Proof. If $\eta > c_2$, then the inequality (17) is equivalent to

$$c_1[2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)] \ge c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)].$$

Since

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \ge 0,$$

we have

$$\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2) < 0.$$

Thus, the inequality (17) holds for any $c_1 \le 1/\alpha$. ■

Proposition 2.13. *Suppose the parameters η, c_2, α, β, m are to be fixed in Eq. (16) and satisfy the inequality*

$$\eta > c_2 \quad \text{and}$$

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) < 0.$$

If

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)} > 0$$

and if c_1 satisfies

$$c_1 \le \min\left\{\frac{1}{\alpha}, \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)}\right\},$$

then the inequality (17) holds.

Proof. If $\eta > c_2$, then the inequality (17) is equivalent to

$$c_1[2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)] \geq c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)].$$

Since

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) < 0,$$

then the inequality (17) is equivalent to

$$c_1 \leq \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)}.$$

And since

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)} > 0,$$

the inequality (17) holds if

$$c_1 \leq \min\left\{\frac{1}{\alpha}, \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2)}\right\}.$$ ■

Proposition 2.14. *Suppose the parameters η, c_2, α, β, m are to be fixed in Eq.* (*16*) *and satisfy the inequality*

$$\eta < c_2 \quad \textit{and} \quad 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) > 0.$$

Then the inequality (*17*) *holds if c_1 satisfies*

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)} \leq c_1 \leq \frac{1}{\alpha}.$$

Proof. If $\eta < c_2$, then the inequality (17) is equivalent to

$$c_1[2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)] \geq c_2[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2)].$$

Since

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) > 0,$$

then the inequality (17) is equivalent to

$$c_1 \geq \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)}.$$

Thus, the inequality (17) holds if

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)} \leq c_1 \leq \frac{1}{\alpha}.$$ ■

3. Main Results

Definition 3.1. We say the mixed wave system Eqs. (8)–(11) has initial conditions of type I if the initial conditions satisfy Remark (1.1) and the union of the range of

$$F_0(x) \equiv \frac{\psi(x) - c_1\varphi'(x)}{c_1 + c_2}$$

on $[0, 1]$ and the range of

$$F_1(x) \equiv \frac{\eta + c_1}{\eta - c_2}\frac{\psi(x) + c_2\varphi'(x)}{c_1 + c_2}$$

on $[0, 1]$ contains the interval $I \equiv [-(1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}, (1/c_2)\sqrt[2m]{(1+\alpha c_2)/\beta c_2}]$, i.e. $I \subseteq \Lambda$ (see Definition (1.2)).

Remark 3.2. In the following theorems, for any given c and d we can compute that $c_1 = (d + \sqrt{d^2 + 4c^2})/2$ and $c_2 = (-d +$

$\sqrt{d^2+4c^2})/2$. Conversely, for any given c_1 and c_2 we can compute that $d = c_1 - c_2$ and $c = \sqrt{c_1 c_2}$.

Theorem 3.3. *Suppose the parameters c, d, α, β, m are to be fixed in the mixed wave system Eqs. (8)–(11) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) > 0.$$

If the mixed wave system has initial conditions of type I and if η satisfies either

$$0 < \frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2$$

or

$$c_2 < \eta \le \frac{c_2[2mc_1(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)]}{\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2)},$$

then the mixed wave system is chaotic.

Proof. The result follows easily from Lemma 2.9 and Proposition 2.10. ■

Example 3.4. Consider the wave system Eqs. (1)–(3)

$$\begin{cases} \omega_{tt} - \omega_{xx} = 0, \quad 0 < x < 1, \quad t > 0. \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, \quad \eta > 0, \quad \eta \ne 1, \quad t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), \\ \qquad \alpha \in (0,1], \quad \beta > 0, \quad t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), \\ \qquad \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose the parameters α, β are to be fixed and the wave system has initial conditions of type I, where

$$\mathrm{I} = \left[-\sqrt{\frac{1+\alpha}{\beta}}, \sqrt{\frac{1+\alpha}{\beta}}\right].$$

If η satisfies either

$$1 < \eta \le \frac{3\sqrt{3}+1+\alpha}{3\sqrt{3}-1-\alpha} \quad \text{or} \quad \frac{3\sqrt{3}-1-\alpha}{3\sqrt{3}+1+\alpha} \le \eta < 1,$$

then the wave system is chaotic. In [Chen *et al.*, 1998b], they showed the same result as above.

Theorem 3.5. *Suppose the parameters c, d, α, β, m are to be fixed in the mixed wave system Eqs. (8)–(11) and satisfy the inequality*

$$\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_2(1+\alpha c_2) \le 0.$$

If the mixed wave system has initial conditions of type I and if η satisfies either

$$\eta > c_2 \quad \textit{or}$$

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_1+c_2) - 2mc_1(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) + \sqrt[2m]{2m+1}(2m+1)(c_1+c_2)} \le \eta < c_2,$$

then the mixed wave system is chaotic.

Proof. The result follows easily from Lemma 2.9 and Proposition 2.11. ■

Example 3.6. Consider the mixed wave system

$$\begin{cases} \omega_{tt} + 2\omega_{tx} - 3\omega_{xx} = 0, \quad 0 < x < 1, \quad t > 0. \\ \omega_t(0,t) + \eta\omega_x(0,t) = 0, \quad \eta > 0, \quad \eta \ne 3, \quad t > 0. \\ \omega_x(1,t) = \alpha\omega_t(1,t) - \beta\omega_t^3(1,t), \\ \qquad \alpha \in \left[\frac{2\sqrt{3}-1}{3}, 1\right], \quad \beta > 0, \quad t > 0. \\ \omega(x,0) = \varphi(x) \in C^1([0,1]), \\ \qquad \omega_t(x,0) = \psi(x) \in C^0([0,1]). \end{cases}$$

Suppose the parameters α, β are to be fixed and the system has initial conditions of type I, where

$$\mathrm{I} = \left[-\frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}, \frac{1}{3}\sqrt{\frac{1+3\alpha}{3\beta}}\right].$$

If η satisfies either

$$\eta > 3 \quad \text{or} \quad \frac{6\sqrt{3}-1-3\alpha}{2\sqrt{3}+1+3\alpha} \le \eta < 3,$$

then the mixed wave system is chaotic.

Theorem 3.7. *Suppose the parameters η, c_2, α, β, m are to be fixed in the mixed wave system Eqs. (8)–(11) and satisfy the inequality*

$$\eta > c_2 \quad \textit{and}$$

$$2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) \ge 0.$$

If the mixed wave system has initial conditions of type I, then the mixed wave system is chaotic for any $c_1 \le 1/\alpha$.

Proof. The result follows easily from Lemma 2.9 and Proposition 2.12. ■

Theorem 3.8. *Suppose the parameters η, c_2, α, β, m are to be fixed in the mixed wave system Eqs. (8)–(11) and satisfy the inequality*

$$\eta > c_2 \quad and \quad 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(\eta - c_2) < 0.$$

If the mixed wave system has initial conditions of type I and if

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2)} > 0,$$

then for any c_1 satisfying

$$c_1 \le \min\left\{\frac{1}{\alpha}, \frac{c_2[\sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)\,(\eta - c_2)}\right\}$$

the mixed wave system is chaotic.

Proof. The result follows easily from Lemma 2.9 and Proposition 2.13. ■

Theorem 3.9. *Suppose the parameters η, c_2, α, β, m are to be fixed in the mixed wave system Eqs. (8)–(11) and satisfy the inequality*

$$\eta < c_2 \quad and \quad 2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) > 0.$$

If the mixed wave system has initial conditions of type I and for any c_1 that satisfies

$$\frac{c_2[\sqrt[2m]{2m+1}(2m+1)(c_2 - \eta) - 2m\eta(1+\alpha c_2)]}{2mc_2(1+\alpha c_2) - \sqrt[2m]{2m+1}(2m+1)(c_2 - \eta)} \le c_1 \le \frac{1}{\alpha},$$

then the mixed wave system is chaotic.

Proof. The result follows easily from Lemma 2.9 and Proposition 2.14. ■

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