Appendix A

Given a two-D curve α parameterized in arc length s, the unit vector T and curvature κ of α are define as

$$\frac{d\alpha}{ds} \equiv T \quad , \tag{A.1}$$

and

$$\frac{d^2\alpha}{ds^2} \equiv \kappa N \quad , \tag{A.2}$$

where N indicates the unit normal vector.

Since

$$\frac{d\alpha}{ds} \cdot \frac{d\alpha}{ds} = 1 , \tag{A.3}$$

by differentiate (A.3) we have

$$\frac{d\alpha}{ds} \cdot \frac{d^2\alpha}{ds^2} = 0 . \tag{A.4}$$

Therefore, T and N are orthonormal vectors.

If α is parameterized in any parameter t, that is $\alpha(t) = [x(t) \quad y(t)]$, $\frac{d\alpha}{ds}$ and $\frac{d^2\alpha}{ds^2}$ can be expressed as

$$\frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = \left|\frac{d\alpha}{dt}\right| T , \tag{A.5}$$

and

$$\frac{d^2\alpha}{dt^2} = \frac{d\left|\frac{d\alpha}{dt}\right|}{dt} T + \left|\frac{d\alpha}{dt}\right| \frac{dT}{ds}\frac{ds}{dt} = \frac{d\left|\frac{d\alpha}{dt}\right|}{dt} T + \kappa \left|\frac{d\alpha}{dt}\right|^2 N \quad .\text{(A.6)}$$

The cross product of $\frac{d\alpha}{ds}$ and $\frac{d^2\alpha}{ds^2}$ can be expressed as

$$\frac{d\alpha}{dt} \times \frac{d^2\alpha}{dt^2} = 0 + \kappa \left|\frac{d\alpha}{dt}\right|^3 T \times N . \tag{A.7}$$

Using (A.7), we can calculate κ as

$$\kappa = \frac{\left|\frac{d\alpha}{dt} \times \frac{d^2\alpha}{dt^2}\right|}{\left|\frac{d\alpha}{dt}\right|^3} = \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{3/2}} . \tag{A.8}$$

Appendix B

Given a point p on surface S, the tangent plane of p is indicated as T(S). The curvature of p along different directions can be calculated using the first fundamental form and second fundamental form in differential geometry. Assume a curve α lies on S, that is

$$\alpha(s) = S(u(s), v(s)) . \tag{B.1}$$

The unit tangent vector T can be obtained by

$$T = \frac{d\alpha(s)}{ds} = \frac{\partial S}{\partial u}\frac{du}{ds} + \frac{\partial S}{\partial v}\frac{dv}{ds} . \tag{B.2}$$

The first fundamental form in differential geometry is expressed as

$$I_1 \equiv T \cdot T = \frac{d\alpha(s)}{ds} \cdot \frac{d\alpha(s)}{ds} = (\frac{\partial S}{\partial u})^2 (\frac{du}{ds})^2 + 2\frac{\partial S}{\partial u}\frac{\partial S}{\partial v}\frac{du}{ds}\frac{dv}{ds} + (\frac{\partial S}{\partial v})^2 (\frac{dv}{ds})^2 . \tag{B.3}$$

The first fundamental form I_1 measure length of curves, angles of tangent vector, and areas without referring back to the neighbor space R_3 [39]. Let E, F, and G, express the coefficient of the quadric form,

$$E \equiv (\frac{\partial S}{\partial u})^2 , \tag{B.4}$$

$$F \equiv \frac{\partial S}{\partial u}\frac{\partial S}{\partial v} , \tag{B.5}$$

and

$$G \equiv (\frac{\partial S}{\partial v})^2 . \tag{B.6}$$

The unit normal vector of α can be calculated by

$$N = \frac{\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}}{\left| \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \right|} . \tag{B.7}$$

The curvature of α along the surface normal direction, normal curvature κ_n, can be expressed as

$$\kappa_n = \frac{d^2\alpha(s)}{ds^2} \cdot N = \frac{d(\frac{d\alpha(s)}{ds} \cdot N)}{ds} - \frac{d\alpha(s)}{ds} \cdot \frac{dN}{ds} . \tag{B.8}$$

Let $\frac{dN(s)}{ds} = dN(\frac{d\alpha}{ds}) = AT$. Now the normal curvature κ_n can be expressed as

$$\kappa_n = \frac{d(T \cdot N)}{ds} - \frac{d\alpha(s)}{ds} \cdot dN(\frac{d\alpha(s)}{ds}) = 0 - T \cdot dN(T) = -T \cdot AT \equiv I_2 , \tag{B.9}$$

where I_2 is the second fundamental form. By using matrix A, the normal curvature along different directions can be obtained. The second fundamental can be expressed as

$$\begin{aligned} I_2 &= -\frac{d\alpha(s)}{ds} \cdot \frac{dN}{ds} = -(\frac{\partial S}{\partial u}\frac{du}{ds} + \frac{\partial S}{\partial v}\frac{dv}{ds}) \cdot (\frac{\partial N}{\partial u}\frac{du}{ds} + \frac{\partial N}{\partial v}\frac{dv}{ds}) \\ &= -\{(\frac{\partial S}{\partial u} \cdot \frac{\partial N}{\partial u})(\frac{du}{ds})^2 + 2(\frac{\partial S}{\partial u} \cdot \frac{\partial N}{\partial v})\frac{du}{ds}\frac{dv}{ds} + (\frac{\partial S}{\partial v} \cdot \frac{\partial N}{\partial v})(\frac{dv}{ds})^2\} . \\ &= e(\frac{du}{ds})^2 + 2f\frac{du}{ds}\frac{dv}{ds} + g(\frac{dv}{ds})^2 \end{aligned} \tag{B.10}$$

Since $dN(\frac{d\alpha}{ds})$ can be expressed as

$$\begin{aligned} dN(\frac{d\alpha}{ds}) &= A\frac{d\alpha}{ds} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \frac{\partial S}{\partial u}\frac{du}{ds} \\ \frac{\partial S}{\partial v}\frac{dv}{ds} \end{bmatrix} = (a_{11}\frac{\partial S}{\partial u} + a_{21}\frac{\partial S}{\partial v})\frac{du}{ds} + (a_{12}\frac{\partial S}{\partial u} + a_{22}\frac{\partial S}{\partial v})\frac{dv}{ds} , \\ &= \frac{dN}{ds} = \frac{\partial N}{\partial u}\frac{du}{ds} + \frac{\partial N}{\partial v}\frac{dv}{ds} \end{aligned} \tag{B.11}$$

we have

$$\begin{aligned} \frac{\partial N}{\partial u} &= (a_{11}\frac{\partial S}{\partial u} + a_{12}\frac{\partial S}{\partial v}) \\ \frac{\partial N}{\partial v} &= (a_{21}\frac{\partial S}{\partial u} + a_{22}\frac{\partial S}{\partial v}) \end{aligned} . \tag{B.12}$$

By using (B.4-6) and (B.10-12), we have

$$-e = \frac{\partial S}{\partial u} \cdot \frac{\partial N}{\partial u} = a_{11}E + a_{22}F , \tag{B.13}$$

$$-f = \frac{\partial S}{\partial u} \cdot \frac{\partial N}{\partial v} = a_{12}E + a_{22}F , \tag{B.14}$$

$$-f = \frac{\partial S}{\partial v} \cdot \frac{\partial N}{\partial u} = a_{11}E + a_{21}F , \tag{B.15}$$

and

$$-g = \frac{\partial S}{\partial v} \cdot \frac{\partial N}{\partial v} = a_{12}F + a_{22}G . \tag{B.16}$$

By rearranging the elements, we have

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} , \tag{B.17}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} fF - eG & gF - fG \\ eF - fE & fF - gE \end{pmatrix}. \tag{B.18}$$