Int. J. Production Economics 117 (2009) 338–359

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

An overview of theory and practice on process capability indices for quality assurance

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ARTICLE INFO

Article history:
Received 11 April 2007
Accepted 2 January 2009
Available online 10 December 2008

Keywords:
Expected relative loss
Fraction non-conforming
Process capability indices
Process consistency
Process relative departure
Quality assurance

ABSTRACT

Process capability indices (PCIs), C_p, C_a, C_{pk}, C_{pm}, and C_{pmk} have been developed in certain manufacturing industry as capability measures based on various criteria, including process consistency, process departure from a target, process yield, and process loss. It is noted in certain recent quality assurance and capability analysis works that the three indices, C_{pk}, C_{pm}, and C_{pmk} provide the same lower bounds on the process yield. In this paper, we investigate the behavior of the actual process yield, in terms of the number of non-conformities (in ppm), for processes with fixed index values of $C_{pk} = C_{pm} = C_{pmk}$, possessing different degrees of process centering. We also extend Johnson's [1992. The relationship of C_{pm} to squared error loss. Journal of Quality Technology 24, 211–215] result formulating the relationship between the expected relative squared loss and PCIs. Also a comparison analysis among PCIs is carried out based on various criteria. The result illustrates some advantages of using the index C_{pmk} over the indices C_{pk} and C_{pm} in measuring process capability (yield and loss), since C_{pmk} always provides a better protection for the customers. Additionally, several extensions and applications to real world problem are also discussed. The paper contains some material presented in the Kotz and Johnson [2002. Process capability indices—a review, 1992–2000. Journal of Quality Technology 34(1), 1–19] survey but from a different perspective. It also discusses the more recent developments during the years 2002–2006.

1. Introduction

Understanding the structure of a process and quantifying process performance no doubt are essential for successful quality improvement initiatives. Process capability analysis has become—in the course of some 20 years—an important and well-defined tool in applications of statistical process control (SPC) to a continuous improvement of quality and productivity. The relationship between the actual process performance and the specification limits (or tolerance) may be quantified using suitable process capability indices. Process capability indices (PCIs), in particular C_p, C_a, C_{pk}, C_{pm} and C_{pmk}, which provide numerical measures of whether or not a manufacturing process is capable to meet a predetermined level of production tolerance, have received substantial attention in research activities as well as an increased usage in process assessments and purchasing decisions during last two decades. By now (2006) there are several books (on different levels) cited in the references, which provide discussions of various PCIs. A number of authors have promoted the use of various process capability indices and examined (with a various degree of completeness) their properties.

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0925-5273/$ - see front matter
doi:10.1016/j.ijpe.2008.11.008

The first process capability index appearing in the engineering literature was presumably the simple "precision" index C_p (Juran, 1974; Sullivan, 1984, 1985; Kane, 1986). This index considers the overall process variability relative to the manufacturing tolerance as a measure of process precision (or product consistency).[1] Another index C_a, a function of the process mean and the specification limits, referred to as an "accuracy" index, is geared to measure the degree of process centering relative to the manufacturing tolerance (see, e.g., Pearn et al., 1998). This index is closely related to an earlier measure originally introduced in the Japanese literature (see Section 3). Formally:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_a = 1 - \frac{|\mu - m|}{d}, \tag{1}$$

where μ is the process mean, σ is the process standard deviation, USL and LSL are the upper and the lower specification limits, $d = (USL-LSL)/2$ is the half specification width related to the manufacturing tolerance and $m = (USL+LSL)/2$ is the midpoint between the upper and lower specification limits. Due to its simplicity, C_p cannot provide an assessment of process centering (targeting). The index C_{pk}, on the other hand, takes both the magnitude of process variance and the process departure from the midpoint m into consideration. It may be written as $C_{pk} = C_p \times C_a$ a product of the two basic indices C_p and C_a. The standard definition is

$$C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\} = \frac{d - |\mu - m|}{3\sigma}. \tag{1$'$}$$

As alluded above the index C_{pk} was developed because C_p does not adequately deal with cases where process mean μ is not centered (the mean does not equal to the midpoint m). However, C_{pk} by itself still cannot provide an adequate measure of process centering. That is, a large value of C_{pk} does not provide information about the location of the mean in the tolerance interval $USL-LSL$. The C_p and C_{pk} indices are appropriate measures of progress for quality improvement situations when reduction of variability is the guiding factor and process yield is the primary measure of a success. However, they are not related to the cost of failing to meet customers' requirement of the target. A well-known pioneer in the quality control, G. Taguchi, on the other hand, pays special attention on the loss in product's worth when one of product's characteristics deviates from the customers' ideal value T.

To take this factor into account, Hsiang and Taguchi (1985) introduced the index C_{pm}, which was also later proposed independently by Chan et al. (1988). The index is motivated by the idea of squared error loss and this loss-based process capability index C_{pm}, sometimes called the Taguchi index. The index is geared towards measuring the ability of a process to cluster around the target, and reflects the degrees of process targeting (centering). The index C_{pm} incorporates the variation of production items relative to the target value and the specification limits which are preset in a factory. The index C_{pm} is defined as

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\tau}, \tag{2}$$

where as above $USL-LSL$ is the allowable tolerance range of the process, $d = (USL-LSL)/2$ is the half-interval length, and $\tau = [\sigma^2 + (\mu - T)^2]^{1/2}$ is a measure of the average product deviation from the target value T. This index C_{pm} can also be expressed as a function of the two basic indices C_p and C_a, explicitly $C_{pm} = C_p/\{1 + [3C_p(1 - C_a)]^2\}^{1/2}$. The quantity $\tau^2 = E[(X - T)^2]$ combines two variation components: (i) variation relative to the process mean (σ^2) and (ii) deviation of the process mean from the target ($(\mu - T)^2$).

Pearn et al. (1992) proposed the process capability index C_{pmk}, which combines the features of the three earlier indices C_p, C_{pk} and C_{pm}. The index C_{pmk} (motivated by the structure of C_{pk} (1)$'$) alerts the user whenever the process variance increases and/or the process mean deviates from its target value. The index C_{pmk} has been referred to as the third-generation capability index, and is defined as

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{3}$$

Comparing the pair of indices (C_{pmk}, C_{pm}), analogously to (C_{pk}, C_p), we obtain the relation $C_{pmk} = C_{pm} \times C_a = (C_{pm} \times C_{pk})/C_p$. Consequently, C_{pmk} can be expressed as $C_{pmk} = C_pC_a/\{1 + [3C_p(1 - C_a)]^2\}^{1/2}$ in terms of the "elementary indices". More recently, Vännman (1995) has proposed a superstructure $C_p(u, v) = (d - u|\mu - m|)/\{3[\sigma^2 + v(\mu - T)^2]^{1/2}\}$ of capability indices for processes based on normal distribution, which includes C_p, C_{pk}, C_{pm} and C_{pmk} as particular cases. By setting $u, v = 0$ and 1, we obtain the four indices $C_p(0, 0) = C_p$, $C_p(1, 0) = C_{pk}$, $C_p(0, 1) = C_{pm}$, and $C_p(1, 1) = C_{pmk}$. These indices are effective tools for process capability analysis and quality assurance. Two basic process characteristics: the process location in relation to its target value, and the process spread (i.e. the overall process variation) are combined to determine formulas for these capability indices. The closer the process output is to the target value and the smaller is the process spread, the more capable the process is. The first feature (closeness to the target) is reflected in the denominator while the second one (the process spread) appears in the numerators of these four indices. In other words, the larger the value of a PCI, the more capable is the process. In this paper, all derivations are carried out assuming that the process is in a state of statistical control and the characteristics under investigation arise from a normal distribution. Moreover, the target value is taken to be the midpoint of the specification limits: $T = m$ (which is common in practical situation) unless stated otherwise.

During the last two decades many authors have promoted the use of various PCIs and examined them

[1] We have not been able to discover any publications on C_p between Juran (1974) and Sullivan (1984).

with a different degree of completeness. These contributions include (in the chronological order): Chan et al. (1988), Chou et al. (1990), Boyles (1991), Pearn et al. (1992), Kushler and Hurley (1992), Rodriguez (1992), Kotz and Johnson (1993), Vännman and Kotz (1995), Bothe (1997), Kotz and Lovelace (1998), Franklin (1999), Palmer and Tsui (1999), Wright (2000), Jessenberger and Weihs (2000), Pearn and Shu (2003), Vännman and Hubele (2003), Pearn and Wu (2005), Wu (2007) as well as references therein. Applications of these indices range over a great variety of situations and productions such as manufacturing of semiconductor products (Hoskins et al., 1988), head gimbals assembly for memory storage systems (Rado, 1989), jet-turbine engine components (Hubele et al., 1991), flip-chips and chip-on-board (Noguera and Nielsen, 1992), rubber edge (Pearn and Kotz, 1994), wood products (Lyth and Rabiej, 1995), aluminum electrolytic capacitors (Pearn and Chen, 1997a), audio-speaker drivers (Chen and Pearn, 1997), Pulux Surround (Pearn and Chang, 1998), liquid crystal display module (Chen and Pearn, 2002), and couplers and wavelength division multiplexers (Wu and Pearn, 2005a).

Kotz and Johnson (2002) provided a compact survey (with interpretations and comments) of some 170 publications on PCIs, during 1992–2000. Spiring et al. (2003) consolidated the research findings of process capability analysis and provide a bibliography of papers for the period 1990–2002. We shall attempt to describe, in an organized manner the interconnection between the PCIs described above and (i) the process yield, in an organized manner, (ii) the process loss, (iii) the process departure from target and (iv) process variability. This may clarify the role of the index C_{pmk} which is still the least understood by practitioners.

2. The PCIs and process consistency

The general idea behind a PCI is to compare what the process "should do" with what the process is "actually doing" (Kotz and Lovelace, 1998). The specification interval should reflect the bounds on usability of the product, so that controlling the process will result in a high-quality product. What the process is "actually doing" refers primarily to process variability (the lower the variability, the lower is the proportion of items that falls outside tolerance limits). Therefore, the quantity

$$Q = \left(\frac{\text{process spread}}{\text{specification interval}}\right) \times 100\% = \left(\frac{6\sigma}{USL - LSL}\right) \times 100\% = \left(\frac{3\sigma}{d}\right) \times 100\% \tag{4}$$

is used to quantify the percentage of the specification band utilized by the process. This quantity should be rendered as low as possible: the lower is the value of the ratio, the lower would be the proportion of the specification interval utilizing the process data. For example, the value 1 indicates that the process variability (or spread) utilizes the whole width of the specification interval (tolerance band). For an on-target normally distributed process, this would result in about 0.27% (2700 parts per million (ppm)) non-conforming units. (Equivalently the area outside the limits $\mu + 3\sigma$ and $\mu - 3\sigma$ of a normal $N(\mu, \sigma)$ distribution is 0.27%.) A value of 0.75 means that the process spread uses 75% of the tolerance band. In fact, $Q = 0.75$ is equivalent to $C_p = 1.33$ which implies the availability of 0.01% of non-conforming units. Thus, it is desirable to have a Q as small as possible. Indeed large values of Q (particularly those greater than 1.00) would not be acceptable since this indicates that the natural range of variation of the process does not fit within the tolerance band. The process spread relative to specification interval (tolerance band) for the normal distribution is illustrated graphically in Fig. 1.

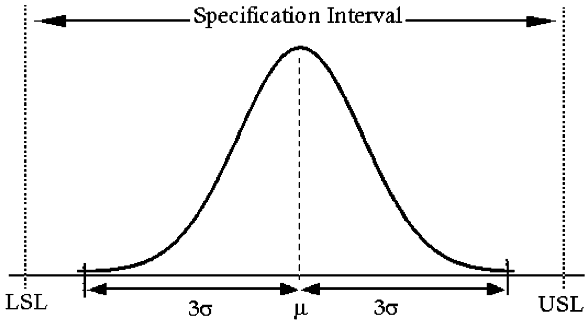

Fig. 1. Process spread with specification interval.

The ratio (4) can be rewritten as

$$\begin{aligned} Q &= (1/C_p) \times 100\% = (C_a/C_{pk}) \times 100\% \\ &= \{1/[C_{pm}(1+\xi^2)^{1/2}\} \times 100\% \\ &= \{C_a/[C_{pm}(1+\xi^2)^{1/2}\} \times 100\%, \end{aligned}$$

where $\xi = (\mu - T)/\sigma$. Therefore, when the process is centered (i.e. $\mu = T = m$ and hence $\xi = 0$, $C_a = 1.0$), all of the four indices provide the same bound on the process relative consistency ($Q \leqslant (1/C) \times 100\%$ with $C_p = C_{pk} = C_{pm} = C_{pmk} = C$).

3. The PCIs and process relative departures

As mentioned before, neither C_p nor C_{pk} indices alone are sufficient to tell us the whole story, since both indices have their individual drawbacks. By examining the relationship between them and using these indices as a pair, a substantial amount of information can be gleaned about the process (without the worrisome confounding of μ and σ in the C_{pk} case). The C_p and C_{pk} indices are specifically related by the process capability index k, i.e. $C_{pk} = C_p \times (1 - k)$, which was one of the original Japanese indices. This index is defined as

$$k = \frac{|\mu - m|}{d}. \left(\text{cf.} \quad c_a = 1 - \frac{|\mu - m|}{d}(1)\right). \tag{4'}$$

This index describes process capability in terms of departure of process mean μ from the center point m and provides a quantified measure of the degree of "off-centrality" of the process. For example, $k = 0$ indicates that the process is perfectly centered on target ($\mu = m$), $k = 1$ on the other hand shows that the process mean is located at one of the specification limits (far away from the center point). For $0<k<1$, the process mean is located somewhere between the target and one of the specification limits. If $k>1$ it shows that μ falls outside the specification limits (i.e. $\mu>USL$ or $\mu<LSL$), the process is severely off-centered and an immediate troubleshooting is necessary. The complement of k: $C_a=1-k$ measures the degree of process centering (the ability to cluster around the center), which was described before.

The index C_{pm} takes the proximity of process mean μ from the target value T into account, thus being more sensitive to process "departures" than C_{pk}. Since the structure of C_{pm} is based on the average process loss relative to the manufacturing tolerance, it provides an upper bound on the average process loss. Furthermore, under the assumption that $T=m$, definition (2) can be rewritten as

$$C_{pm}=\frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-m)^2}}=\frac{C_p}{\sqrt{1+\xi^2}}, \tag{4a}$$

where now

$$\xi=\frac{\mu-m}{\sigma}. \tag{4b}$$

Chan et al. (1988) discussed this ratio and the sampling properties of an estimated C_{pm}, Boyles (1991) has provided an analysis of the index C_{pm} and its usefulness in measuring process centering. He observes that both C_{pk} and C_{pm} coincide with C_p when $\mu=m$ and decrease as μ departs from m. However, $C_{pk}<0$ for $\mu>USL$ or $\mu<LSL$, whereas C_{pm} of process with $|\mu-m|>0$ is strictly bounded above by the C_p value of a process with $\sigma=|\mu-m|$ (see Eqs. (4a) and (4b)). Consequently,

$$C_{pm}<\frac{USL-LSL}{6|\mu-m|}. \tag{5}$$

The index C_{pm} approaches to zero asymptotically as $|\mu-m|$ tends to infinity. On the other hand, while $C_{pk}=(d-|\mu-m|)/(3\sigma)$ (see Eq. (1)′) increases without bound for fixed μ as σ tends to zero, C_{pm} is bounded above by $C_{pm}<d/(3|\mu-m|)$ (recall that $d=(USL-LSL)/2$). The right-hand side of the equation is the limiting value of C_{pm} as σ tends to zero, and equals to C_p value of a process with $\sigma=|\mu-m|$. It follows from (5) that a necessary condition for $C_{pm}\geqslant 1$ is $|\mu-m|<d/3$.

Kotz and Johnson (1999) examined the relations between C_p, C_{pk} and C_{pm}. Roughly speaking, for a fixed k value, the value of C_{pm} is greater than C_{pk} for small values of C_p, but is less than C_{pk} for larger values of C_p. In fact

$$\frac{C_{pk}}{C_{pm}}=(1-k)\sqrt{1+\left(\frac{\mu-m}{\sigma}\right)^2}=(1-k)\sqrt{1+9C_p^2k^2}.$$

Thus, the relation $C_{pk}>(or<)C_{pm}$ is according to whether a$(1-k)(1+9C_p^2k^2)^{1/2}>(or<)1$, i.e. according to $C_p>(or<)$ $(1/3)(1-k)[(2-k)/k]^{1/2}$. Furthermore, the same authors (2002) noted that

$$\frac{C_{pk}}{C_{pm}}=\left(1-\frac{1}{3C_p}\left|\frac{\mu-m}{\sigma}\right|\right)\sqrt{1+\left(\frac{\mu-m}{\sigma}\right)^2}$$
$$=\left(1-\frac{1}{3C_p}|\xi|\right)\sqrt{1+\xi^2}<\left(1-\frac{1}{3C_p}|\xi|\right)\left(1+\frac{1}{2}\xi^2\right).$$

Hence the relation $C_{pk}<C_{pm}$ is certainly valid if

$\frac{1}{3}C_p|\xi|>\frac{1}{2}\xi^2$ or, equivalently, when $k<\frac{2}{9C_p^2}$.

3.1. In defense of the index C_{pmk}

Now the index $C_{pmk}=(d-|\mu-m|)/\{3[\sigma^2+(\mu-T)^2]^{1/2}\}$ is constructed by combining the yield-based index C_{pk} and the loss-based index C_{pm}, thus taking into account the process yield (meeting the manufacturing specifications) and the process loss (variation from the target). When the process mean μ departs from the target value T, the reduction of the value of C_{pmk} is more substantial than those of C_p, C_{pk}, and C_{pm}. Hence, the index C_{pmk} responds to the departure of the process mean μ from the target value T faster than the other three basic indices C_p, C_{pk}, and C_{pm}, while also being sensitive to the changes of the process variation. We note that a process meeting the capability requirement "$C_{pk}\geqslant C$" may not meet the requirement "$C_{pm}\geqslant C$" and vice versa. The discrepancy between these two indices is due to the fact that the C_{pk} index measures primarily the process yield, while C_{pm} focuses to a large extent mainly on the process loss. However, if the process meets the capability requirement "$C_{pmk}\geqslant C$", then a fortiori "$C_{pk}\geqslant C$" and "$C_{pm}\geqslant C$" are fulfilled (since $C_{pmk}\leqslant C_{pk}$ and $C_{pmk}\leqslant C_{pm}$). In fact, the definition of C_{pmk} given by (3) can be rewritten as $C_{pmk}=C_{pk}/\{1+[(\mu-T)/\sigma]^2\}^{1/2}$ or $C_{pmk}=(1-|\mu-m|/d)\times C_{pm}$ $(=(1-k)\times C_{pm})$. The index C_{pmk} is worse than C_{pk} being associated with a certain percentage of non-conforming product, however, one should not choose this index if process yield is the main interest. C_{pmk} (and usually C_{pm}) is much more sensitive than other capability indices to the deviation of the process mean relative to m. In fact when μ is equal to m, $C_{pmk}=C_{pk}$. If the mean of process moves away from m, then, C_{pmk} decreases more rapidly than C_{pk} does (although both are 0 when μ equals one of the specification limits). Conversely, when μ is brought closer to m, C_{pmk} increases much faster than C_{pk} does. The C_{pmk} has its maximum value when the process is centered. Viewing C_{pmk} as a mixture of C_{pk} and C_{pm}, C_{pmk} behaves "more like C_{pm}" if σ^2 is small, and "more like C_{pk}" if σ^2 is large (Jessenberger and Weihs, 2000).

In addition to the above advantages, C_{pmk} reveals the larger information about the location of the process mean. Given a C_{pk} index of 1.0, all we can say about μ is that it is somewhere between the LSL and the USL, i.e., $m-d<\mu<m+d$ or $k<1$, where as above d equals $(USL-LSL)/2$ (cf. Eqs. (1) and (1)′). As far as the C_{pm} index is concerned, it can be shown (Bothe, 1997) that the distance between μ and m must be less than $d/(3C_{pm})$. Therefore, given a C_{pm} index of 1.0, we know that $m-d/3<\mu<m+d/3$ or $k<1/3$. This is a narrower interval than the one obtained for C_{pk} equals to 1.0. For the C_{pmk} index, it can be shown that the distance between μ and m is less than $d/(3C_{pmk}+1)$. Consequently when C_{pmk} index of 1.0, it follows that $m-d/4<\mu<m+d/4$ or $k<1/4$, being a narrower interval than the one obtained for $C_{pm}=1$. Ranking of the three indices (C_{pk}, C_{pm}, C_{pmk})

from the most sensitive to the least sensitive with respect to the departures of the process mean from the target value we thus have: (i) C_{pmk}, (ii) C_{pm}, and (iii) C_{pk} (see also from Pearn and Kotz, 1994).

4. The PCIs and process yield

Process yield has been for some times the most common and standard criterion used in the manufacturing industries for judging process performance. Units are inspected according to specification limits placed on certain key product characteristics and are sorted into two categories: passed (conforming) (C) or rejected non-conforming. In the early days prior to mid eighties of the 20th century, fraction non-conforming for manufacturing processes were usually calculated by counting the number of non-conforming items in samples of 25 or 30 and then extrapolating the results. With the rapid advancement of the manufacturing technology, suppliers began to require their products be of a high quality involving very low fraction of non-conformities. The fraction of non-conformities is usually less than 0.01%, and often measured in ppm. The traditional methods of figuring out the fraction non-conforming are no longer applicable since any sample of a "reasonable size" may very likely contain no defective product items. Hence an alternative method of measuring the fraction of non-conforming is to use the capability indices, discussed in Section 3 all of which are functions of an item's specification limits (the tolerance range) and the variation of the process producing the item.

If a proportion of conforming items is the primary concern, a most natural measure is the proportion itself referred to the yield, which is defined as

$$\text{Yield} = \int_{LSL}^{USL} \mathrm{d}F(x),$$

where $F(x)$ is the cumulative distribution function of the measured random characteristic X, and USL and LSL are, as before, the upper and the lower specification limits, respectively. The use of the yield as a quality measure implies that each rejected unit costs the producer an additional amount (to scrap or repair) while each passed unit involves no additional cost.

It is often assumed (not always correctly) that the product characteristic, X, follows the normal distribution, $N(\mu, \sigma^2)$. In this case the fraction of non-conforming (%NC) may be expressed as

$$\%NC = 1 - P(LSL \leqslant X \leqslant USL) = P(X < LSL) + P(X > USL) = \Phi\left(\frac{LSL-\mu}{\sigma}\right) + 1 - \Phi\left(\frac{USL-\mu}{\sigma}\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function (c.d.f.) of the standard normal distribution $N(0,1)$ (see Fig. 1). Since $USL = m + d$ and $LSL = m - d$, we have

$$\%NC = \Phi\left(\frac{m-d-\mu}{\sigma}\right) + 1 - \Phi\left(\frac{m+d-\mu}{\sigma}\right) = \Phi\left(-\frac{d+\mu-m}{d}\cdot\frac{d}{\sigma}\right) + \Phi\left(-\frac{d-\mu+m}{d}\cdot\frac{d}{\sigma}\right).$$

Therefore,

$$P(NC) = \Phi\left[-\frac{(1+\delta)}{\gamma}\right] + \Phi\left[-\frac{(1-\delta)}{\gamma}\right]$$

(we identify here proportion with probability) where $\delta = (\mu - m)/d$ and $\gamma = \sigma/d$. Furthermore, since %NC is an even function of δ, %NC may be rewritten as

$$\%NC = \Phi\left(-\frac{1+|\delta|}{\gamma}\right) + \Phi\left(-\frac{1-|\delta|}{\gamma}\right). \tag{6}$$

Noting that $C_a = 1 - |\delta|$ and $C_p = 1/(3\gamma)$ (Eq. (1)), the above expression for %NC also can be expressed as

$$\%NC = \Phi[-3C_pC_a] + \Phi[-3C_p(2 - C_a)].$$

4.1. Yield assurance based on C_{pk}

The index C_{pk} has been regarded as a yield-based index. It provides bounds on the process yield, $2\Phi(3C_{pk}) - 1 \leqslant Yield < \Phi(3C_{pk})$, for a normally distributed process (Boyles, 1991). For a C_{pk} at level of 1, one would expect that not more than 2700 ppm fall outside the specification limits (fraction of defectives). At a C_{pk} level of 1.33, the defect rate drops to 66 ppm. To achieve less than 0.544 ppm defect rate, a C_{pk} level of 1.67 is required. At a C_{pk} level of 2.0, the likelihood of a defective part drops to the minuscule 2 parts per billion (ppb). Note a drastic decrease in the fraction of defectives as C_{pk} increases from 1 to $1\frac{1}{3}$, say.

This bound may be established by noting that for a process with fixed C_{pk} the number of non-conformities (product items falling outside of the specification interval [LSL, USL]) is bounded but the actual number of non-conformities will vary depending upon the location of the process mean and the magnitude of the process variation. First we rewrite the definition of the index C_{pk} in terms of standardized parameters $\delta = (\mu - m)/d$ and $\gamma = \sigma/d$.

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma} = \frac{1 - |(\mu - m)/d|}{3(\sigma/d)} = \frac{1-|\delta|}{3\gamma} = \begin{cases} \dfrac{1+\delta}{3\gamma} & \text{for } LSL \leqslant \mu \leqslant m \\ \dfrac{1-\delta}{3\gamma} & \text{for } m < \mu \leqslant USL. \end{cases} \tag{7}$$

For a positive C_{pk}, $C_{pk} > 0$ (a natural situation), the exact expected fraction of non-conforming formula for %NC can be expressed as a function of C_{pk} and C_a or equivalently that of C_{pk} and C_p as follows:

$$\%NC = \Phi[-3C_{pk}] + \Phi[-3C_{pk}(2 - C_a)/C_a] \tag{8}$$

and

$$\%NC = \Phi[-3C_{pk}] + \Phi[-3(2C_p - C_{pk})]. \tag{9}$$

It follows from (8) that when the process mean μ is located within the specification limits, i.e. $0 < C_a \leqslant 1$ or

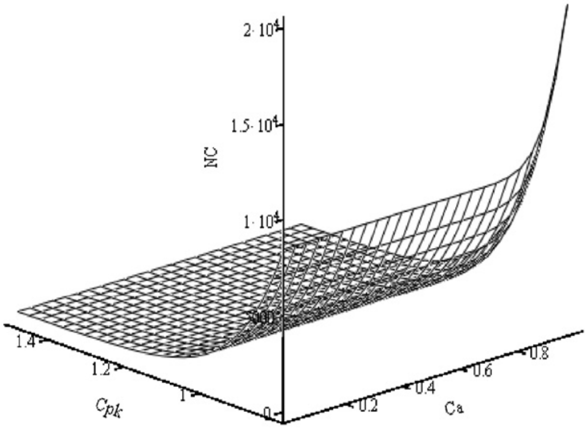

Fig. 2. Surface plot of NC for $0.8 \leqslant C_{pk} \leqslant 1.5$ and $0 \leqslant C_a \leqslant 1$.

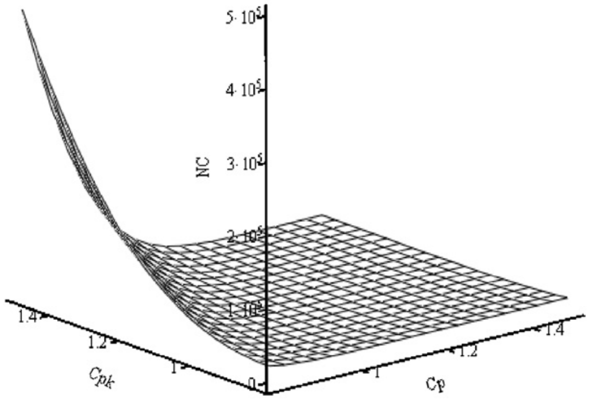

Fig 4. Surface plot of NC for $0.8 \leqslant C_{pk} \leqslant 1.5$ and $0.8 \leqslant C_p \leqslant 1.5$.

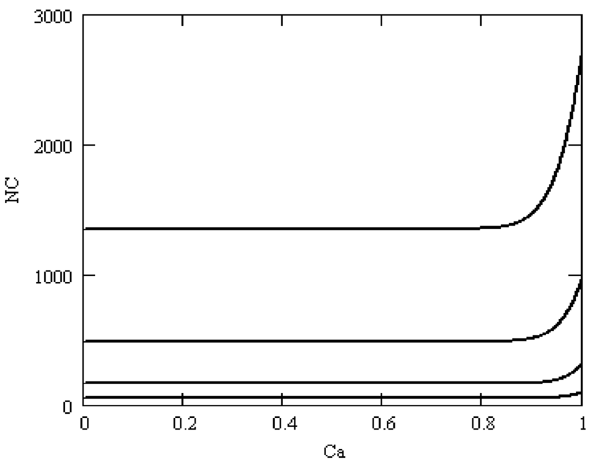

Fig 3. NC plots for $C_{pk} = 1.0(0.1)1.3$ with $0 < C_a \leqslant 1$ (from top to bottom).

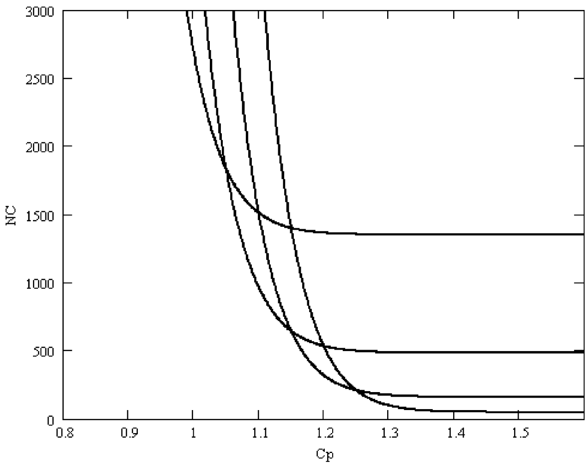

Fig. 5. NC plots for $C_{pk} = 1.0(0.1)1.3$ with $0.8 \leqslant C_p \leqslant 1.5$ (from top to bottom).

$C_{pk} > 0$, we have the bounds on $\%NC$ with $\Phi(-3C_{pk}) < \%NC \leqslant 2\Phi(-3C_{pk})$ since the standard normal c.d.f. $\Phi(\cdot)$ is an increasing function. That is equivalent to the bounds on the process yield $2\Phi(3C_{pk}) - 1 \leqslant Yield < \Phi(3C_{pk})$. For $C_a = 1.0$, the process is perfectly centered ($\mu = m$). For $C_a = 0$, the process mean is at one of the specification limits ($\mu = USL$ or $\mu = LSL$) (cf. the beginning of Section 3). For processes with fixed C_{pk}, the number of non-conformities attains its maximum for a centered process ($C_a = 1.0$), and $\%NC$ is reduced when the process mean departs from the center (namely C_a decreases). Fig. 2 displays the surface plot of the actual number of the non-conformities (in ppm) for $0.8 \leqslant C_{pk} \leqslant 1.5$ and $0 < C_a \leqslant 1$. Fig. 3 plots the actual number of the non-conformities (in ppm) for $C_{pk} =$ 1.0, 1.1, 1.2 and 1.3, with $0 < C_a \leqslant 1$. Fig. 4 displays the surface plot of the actual number of the non-conformities (in ppm) for $0.8 \leqslant C_{pk} \leqslant 1.5$ and $0.8 \leqslant C_p \leqslant 1.5$. Fig. 5 plots the actual number of the non-conformities (in ppm) for $C_{pk} = 1.0$, 1.1, 1.2, and 1.3, with $0.8 \leqslant C_p \leqslant 1.5$. Note that for $C_{pk} \geqslant 1.3$, the curves in Figs. 3 and 5 are almost indistinguishable.

4.2. Yield assurance based on C_{pm}

In the case when $T = m$, the definition of C_{pm} index (2) can be rewritten as a function of the standardized parameters δ and γ, as follows:

$$C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - m)^2}} = \frac{1}{3\sqrt{\gamma^2 + \delta^2}}.$$

Hence,

$$\gamma^2 = \frac{1}{(3C_{pm})^2} - \delta^2 = \left(\frac{1}{3C_{pm}} + \delta\right)\left(\frac{1}{3C_{pm}} - \delta\right),$$

or, equivalently,

$$\gamma = \sqrt{\left(\frac{1}{3C_{pm}} + \delta\right)\left(\frac{1}{3C_{pm}} - \delta\right)} = \sqrt{\left(\frac{1}{3C_{pm}} + |\delta|\right)\left(\frac{1}{3C_{pm}} - |\delta|\right)}$$

holds for $0 \leqslant |\delta| \leqslant 1/(3C_{pm})$, i.e., for $1 - 1/(3C_{pm}) \leqslant C_a \leqslant 1$. We thus obtain the following explicit relationship between the exact expected proportion of NC and the indices

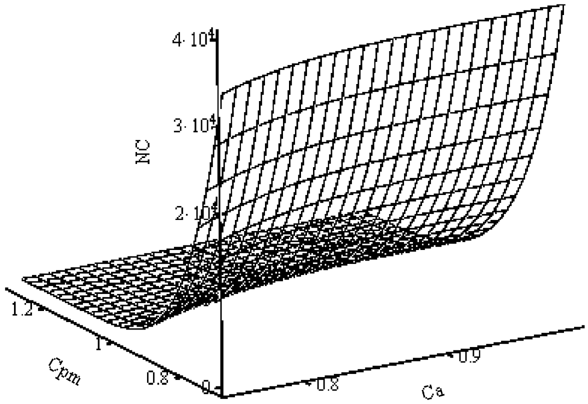

Fig. 6. Surface plot of NC for $0.7 \leqslant C_{pm} \leqslant 1.3$ and $0.75 \leqslant C_a \leqslant 1$.

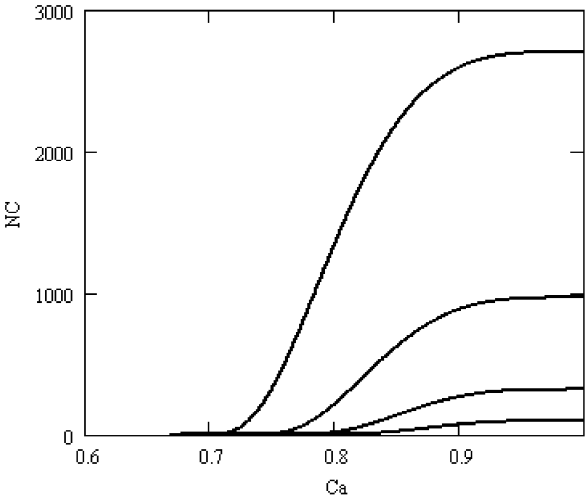

Fig. 7. NC for $C_{pm} = 1.0(0.1)1.3$, with $0.6 \leqslant C_a \leqslant 1$ (from top to bottom in the plot).

C_{pm} and C_a valid for $1 - 1/(3C_{pm}) \leqslant C_a \leqslant 1$:

$$\%NC = \Phi\left[-\frac{2 - C_a}{\sqrt{\frac{1}{(3C_{pm})^2} - (1 - C_a)^2}}\right] + \Phi\left[-\frac{C_a}{\sqrt{\frac{1}{(3C_{pm})^2} - (1 - C_a)^2}}\right]. \qquad (10)$$

Relation (10) shows that for a perfectly centered process (i.e. $C_a = 1$), the fraction of non-conforming has an upper bound with $\%NC \leqslant 2\Phi(-3C_{pm})$. Fig. 6 displays the surface plot of the actual number of the non-conformities (in ppm) for $0.7 \leqslant C_{pm} \leqslant 1.3$ and $0.75 \leqslant C_a \leqslant 1$. Fig. 7 plots the actual number of non-conformities (in ppm) for $C_{pm} = 1.0$, 1.1, 1.2, and 1.3 (from top to bottom in the plot), with $0.6 \leqslant C_a \leqslant 1$. We note that for $C_{pm} > 1.3$, the curves become close to each other.

4.3. *Yield assurance based on C_{pmk}*

Using a similar technique for deriving the formula of the exact expected proportion of non-conforming, and the relation $C_{pmk} = C_{pm} \times C_a$, we obtain the following exact expected proportion of non-conforming in terms of C_{pmk} and C_a as

$$\%NC = \Phi\left[-\frac{2 - C_a}{\sqrt{(C_a/3C_{pmk})^2 - (1 - C_a)^2}}\right] + \Phi\left[-\frac{C_a}{\sqrt{(C_a/3C_{pmk})^2 - (1 - C_a)^2}}\right]. \qquad (11)$$

Similarly, for $C_a = 1$, the fraction of non-conforming has an upper bound with $\%NC \leqslant 2\Phi(-3C_{pmk})$. Fig. 8 displays the surface plot of the actual number of non-conformities (in ppm) for $0.5 \leqslant C_{pmk} \leqslant 1.3$ and $0.8 \leqslant C_a \leqslant 1$. Fig. 9

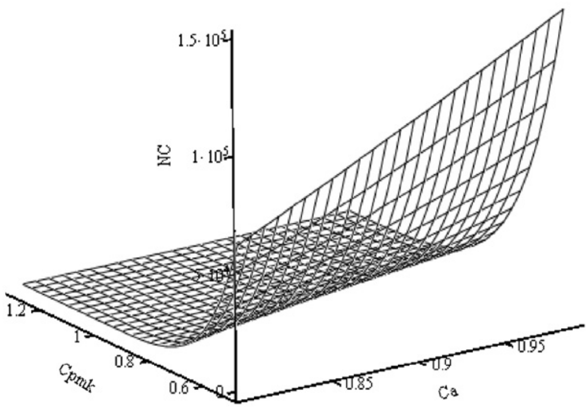

Fig. 8. Surface plot of NC for $0.5 \leqslant C_{pmk} \leqslant 1.3$ and $0.8 \leqslant C_a \leqslant 1$.

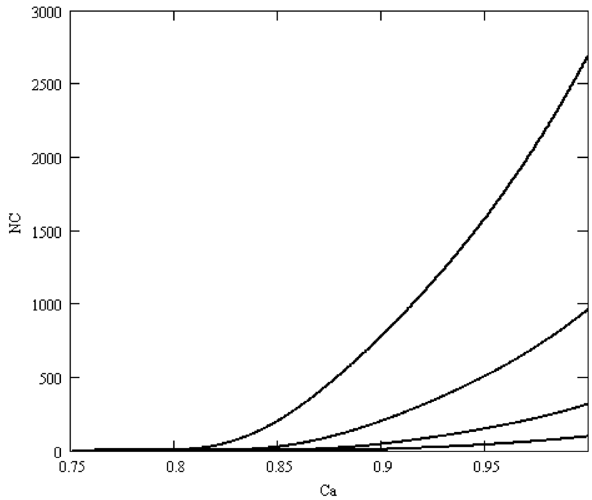

Fig. 9. NC for $C_{pmk} = 1.0(0.1)1.3$, with $0.75 \leqslant C_a \leqslant 1$ (from top to bottom in the plot).

Table 1
Bounds on %NC and C_a for $C_{pk} = C_{pm} = C_{pmk} = C$, respectively.

C	C_{pk}		C_{pm}		C_{pmk}	
	Bounds on NC (ppm)	Bounds on C_a	Bounds on NC (ppm)	Bounds on C_a	Bounds on NC (ppm)	Bounds on C_a
1.00	2699.796	$0 \leqslant C_a \leqslant 1$	2699.796	$0.667 \leqslant C_a \leqslant 1$	2699.796	$0.750 \leqslant C_a \leqslant 1$
1.33	66.334	$0 \leqslant C_a \leqslant 1$	66.334	$0.750 \leqslant C_a \leqslant 1$	66.334	$0.800 \leqslant C_a \leqslant 1$
1.50	6.795	$0 \leqslant C_a \leqslant 1$	6.795	$0.778 \leqslant C_a \leqslant 1$	6.795	$0.818 \leqslant C_a \leqslant 1$
1.67	0.554	$0 \leqslant C_a \leqslant 1$	0.554	$0.800 \leqslant C_a \leqslant 1$	0.554	$0.833 \leqslant C_a \leqslant 1$
2.00	0.002	$0 \leqslant C_a \leqslant 1$	0.002	$0.833 \leqslant C_a \leqslant 1$	0.002	$0.857 \leqslant C_a \leqslant 1$

plots the actual number of non-conformities (in ppm) for $C_{pmk} = 1.0$, 1.1, 1.2, and 1.3 (from top to bottom), with $0.75 \leqslant C_a \leqslant 1$. We note that for $C_{pmk} > 1.3$, the corresponding curves are almost indistinguishable.

For reader's convenience we summarize in Table 1 the formulas for %NC in the various cases discussed above.

4.4. Yield comparison among PCIs

For a normally distributed process, the C_{pk} index provides a lower bound on the process yield, Yield$\geqslant 2\Phi(3C_{pk}) - 1$, or %$NC \leqslant 2\Phi(-3C_{pk})$ for $LSL \leqslant \mu \leqslant USL$. Furthermore, based on the C_{pm} index, Ruczinski (1996) obtained a lower bound on the process yield as Yield$\geqslant 2\Phi(3C_{pm}) - 1$, or %$NC \leqslant 2\Phi(-3C_{pm})$ for $C_{pm} > \sqrt{3}/3$. The bound, however, has never been analytically justified for quality assurance purposes based on the C_{pmk} index. It is not clear therefore whether C_{pmk} is related to the process yield, since the relationship between C_{pmk} and the process yield (or proportion of non-conforming) has not been available. Recently, Pearn and Lin (2005) provided a mathematical derivation of upper bound formula for C_{pmk} on process yield, in terms of the number of non-conformities (in ppm) as

$$0 \leqslant \%NC \leqslant 2\Phi(3C_{pmk}) \quad \text{for } C_{pmk} \geqslant \sqrt{2}/3.$$

Based on the yield analysis among capability indices C_{pk}, C_{pm}, and C_{pmk}, the result illustrates that the three indices provide the same lower bounds on process yield for normally distributed processes, that is, Yield$\geqslant 2\Phi(3C_{pk}) - 1 = 2\Phi(3C_{pm}) - 1$. For example, if it is given that $C_{pk} = 1.00$ we have the information on the process yield only through the upper bound %$NC \leqslant 2699.796$ ppm and no information on C_0. However, if $C_{pm} = 1.00$ we have the information on the process yield through the upper bound %$NC \leqslant 2699.796$ ppm and the process centering measure $0.667 \leqslant C_a \leqslant 1.00$. Finally for $C_{pmk} = 1.00$ we have the same upper bound on process yield %$NC \leqslant 2699.796$ ppm and a narrower process centering measure $0.750 \leqslant C_a \leqslant 1.00$. Figs. 10 and 11 plot the actual number of the non-conformities (in ppm) for $C_{pk} = C_{pm} = C_{pmk} = 1.00$ and 1.50, with the bound of $0 < C_a \leqslant 1$ for C_{pk}, with the bound of $1 - 1/(3C_{pm}) \leqslant C_a \leqslant 1$ for C_{pm}, and the bound of $1 - 1/(1 + 3C_{pmk}) \leqslant C_a \leqslant 1$ for C_{pmk}. These results indicate on an advantage of using index C_{pmk} over the indices C_{pk} and C_{pm} when (together) measuring the process yield. Indeed C_{pmk} provides a better protection for the customers in terms of the quality yield of the products. Table 1 displays the bounds on %NC and C_a for $C_{pk} = C_{pm} = C_{pmk} = C$, respectively (for $C = 1(\frac{1}{3})2$).

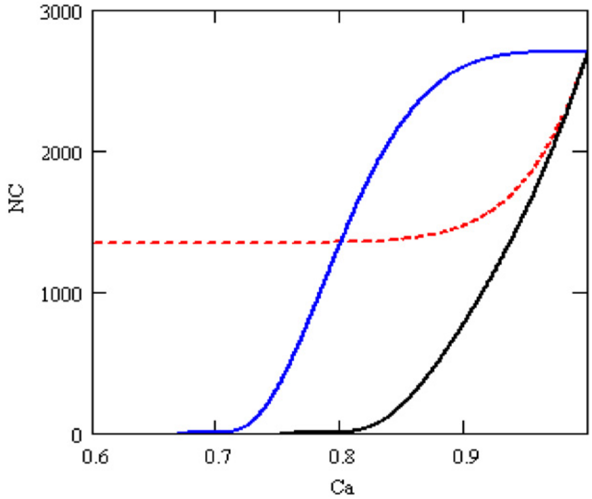

Fig. 10. The actual number of NC curves for $C_{pk} = 1.0$ (top), $C_{pm} = 1.0$ (dash), and $C_{pmk} = 1.0$ (bottom) for various allowed values of C_a.

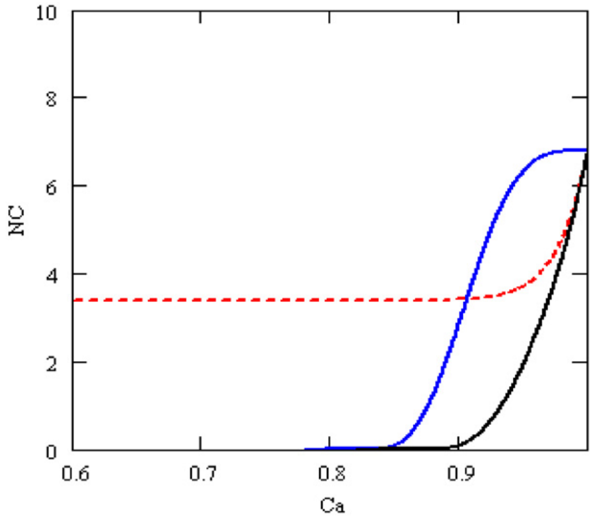

Fig. 11. The actual number of NC curves for $C_{pk} = 1.5$ (top), $C_{pm} = 1.5$ (dash), and $C_{pmk} = 1.0$ (bottom) for various allowed values of C_a.

In certain manufacturing industries, reducing the fraction of non-conformities or the expected proportion of non-conforming items is of primarily concern and a guiding principle for quality improvement. In such cases, keeping the process centered (on-target) may not be a good strategy for maintaining adequate process capability since the number of non-conformities reaches its maximum when the process is centered (i.e. $C_a = 1.0$), and the %NC reduces when the process mean departs from the center (i.e. C_a decreases) for a fixed C_{pk}, C_{pm} or C_{pmk} (see Figs. 10 and 11). For other manufacturing industries, a reduction of deviation from the target value serves as a guiding principle (e.g. Taguchi's quality philosophy, or certain modern quality improvement theories). In such cases, the efforts should not be focused entirely on reducing the fraction of the non-conformities. Here keeping a process centered (on-target) would be considered satisfactory. Note that if μ happens to be far away from the target T (the corresponding C_a is small) then the process would not be viewed as capable even if σ is so small so that the %NC is small. (Recall the expressions for C_a and %NC.)

$$\%NC = \Phi\left[-\frac{2-C_a}{\sqrt{\left(\frac{C_a}{3C_{pmk}}\right)^2-(1-C_a)^2}}\right] + \Phi\left[-\frac{C_a}{\sqrt{\left(\frac{C_a}{3C_{pmk}}\right)^2-(1-C_a)^2}}\right]. \quad (11)$$

5. The PCIs and the expected relative loss

A disadvantage of an yield measure is that it does not distinguish at all among the products which fall inside of the specification limits. Customers, however, do observe the unit-to-unit differences of this characteristic, especially if the variance is large and/or the mean is offset from the target. With the increased importance of clustering around the target (rather than conforming to specification limits) and utilization of loss functions an alternative approach to PCIs is being developed. Instead of numbers or fraction of non-conforming, various economic/production costs (or losses) offer opportunities for developing an improved assessment, monitoring, and comparisons of process capability. Hsiang and Taguchi (1985) have presented an approach to quality improvement in which reduction of deviation from the target value serves as the guiding principle. According to this approach any measured value x of a product characteristic X in general results in a loss to the consumer.

Proceeding along these lines, it was observed that the loss for each lot is often not necessarily the same, even though the lots have the same fraction defectives. Hence, to implement the Taguchi's loss criterion, the loss caused by the deviation from its target value is expressed as a quadratic function with respect to the difference between the actual value and the target value to distinguish between the products by increasing the penalty as the departure from the target increases. The squared loss function for the product characteristic X in symmetric case can be expressed as

$$Loss(X) = w(X-T)^2,$$

where as above T denotes the target value of X and w is a positive constant. (The choice of a quadratic function may be viewed to be somewhat arbitrary and is no doubt influenced by over of three hundred years tradition of using mean square errors, etc., in statistical applications augmented by certain optimal properties.) This implies that the loss is zero when the process outcome is on target and is positive for any deviation from the target. The expected loss can be evaluated as

$$E[Loss(X)] = w\int_{-\infty}^{\infty}(x-T)^2\,dF(x) = w[(\mu-T)^2+\sigma^2],$$

where $F(\cdot)$ is the underlying c.d.f. of X.

A disadvantage of the expected loss lies in a difficulty of setting a standard for the proposed index since it increases from zero to infinity. To overcome this drawback, Johnson (1992) has defined the worth of the product $W(X)$, which can be expressed as a function of X with

$$W(X) = W_T - w(X-T)^2,$$

where W_T denotes the worth of the product when X is precisely on target. Defining Δ to be the distance of X from the target T at which the worth of the product is zero, we obtain $0 = W_T - w\Delta^2$ or $\Delta^2 = W_T/w$. If the loss at the specification limits (either USL or LSL) is A_0 and the distance from the specification limits to the target T is d, then $A_0 = Loss(USL) = Loss(LSL) = Loss(T \pm d) = wd^2$. From the above we have that $A_0/W_T = d^2/\Delta^2$ and the expected relative squared loss, say L_e, can be rewritten as

$$L_e = \frac{E[Loss(X)]}{\Delta^2} = \frac{E[Loss(X)]}{d^2}\left(\frac{A_0}{W_T}\right), \quad (12)$$

which provides a unitless measure of process performance in terms of the loss value of the product for industrial applications. The distributional and statistical properties of estimators of the loss index L_e have been investigated in Johnson (1992) and Pearn et al. (2004a).

5.1. Loss assurance based on C_{pk}

Below we shall extend (without details) the derivation of the relationship between the expected relative squared loss and C_{pk} by rewriting the index C_{pk} in the form

$$C_{pk} = \frac{2dC_a}{6\sqrt{[\sigma^2+(\mu-T)^2]-(\mu-T)^2}} = \frac{1}{3}\sqrt{\frac{A_0C_a^2}{W_TL_e - A_0(1-C_a)^2}}. \quad (13)$$

Consequently, the expected relative squared losses based on the C_{pk} index, denoted by $L_{C_{pk}}$ can be expressed as

$$L_{C_{pk}} = \frac{A_0}{W_T}(1-C_a)^2 + \frac{1}{9}\frac{A_0C_a^2}{W_TC_{pk}^2}. \quad (14)$$

By taking the derivative of $L_{C_{pk}}$ with respect to C_a we arrive at the following equation:

$$\frac{\partial L_{C_{pk}}}{\partial C_a} = \frac{2A_0}{W_T}\left[\left(\frac{9C_{pk}^2+1}{9C_{pk}^2}\right)C_a - 1\right].$$

Thus, the expected relative loss with C_{pk}, $L_{C_{pk}}$, has the minimum loss $(A_0/W_T)/(1+9C_{pk}^2)$ when $C_a = 9C_{pk}^2/(1+9C_{pk}^2)$. If the index $C_a > 9C_{pk}^2/(1+9C_{pk}^2)$, then $L_{C_{pk}}$ decreases as C_a increases. Conversely, $L_{C_{pk}}$ increases as C_a decreases if $C_a < 9C_{pk}^2/(1+9C_{pk}^2)$. For example, at a level of $C_{pk} = 1.00$, the $L_{C_{pk}}$ has the minimum loss $A_0/(10W_T)$ at $C_a = 0.9$. Figs. 12 and 13 display the surface plot and the contour plot of the loss $L_{C_{pk}}$ in terms of A_0/W_T for $0.5 \leqslant C_{pk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$, respectively.

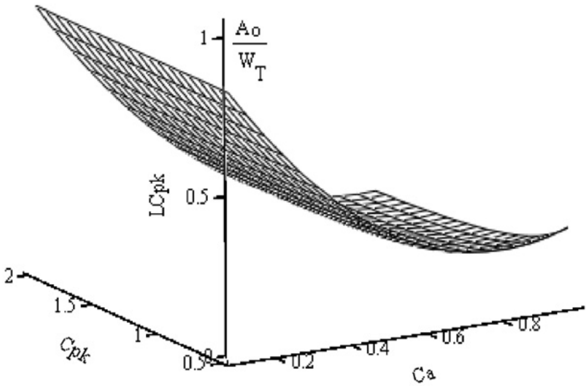

Fig. 12. The surface plot of $L_{C_{pk}}$ for $0.5 \leqslant C_{pk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

5.2. Loss assurance based on C_{pm}

From the definition of expected relative squared loss L_e (12), the relationship between L_e and C_{pm} index can easily be derived as

$$C_{pm} = \frac{2d}{6\Delta\sqrt{E[Loss(X)]}} = \frac{1}{3}\sqrt{\frac{A_0}{W_T L_e}}. \tag{15}$$

(Recall that $A_0 = Loss(LSL)$; see also the introduction to Section 5.)

Thus, the expected relative squared losses based on the C_{pm} index, $L_{C_{pm}}$, can be rewritten as

$$L_{C_{pm}} = \frac{1}{9}\frac{A_0}{W_T C_{pm}^2} \quad \text{(c.f. (14))}. \tag{16}$$

From the expression of $L_{C_{pm}}$ given by (16), we note that C_{pm} has the property of being the so-called a larger-the-better index. Thus, small values of C_{pm} may be due to a high expected loss resulting in a poorer process capability. In addition, $L_{C_{pm}}$ is a constant $A_0/(9W_T C_{pm}^2)$ for all values of C_a. Figs. 14 and 15 display the surface and the contour plots of the expected relative loss $L_{C_{pm}}$ for $0.5 \leqslant C_{pm} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$, respectively.

5.3. Loss assurance based on C_{pmk}

We shall briefly comment on the abilities of the index C_{pmk} to provide a loss assurance. The relationship between the expected relative squared loss L_e and C_{pmk} can be expressed directly as follows:

$$C_{pmk} = C_{pm} \times C_a = \frac{1}{3}\sqrt{\frac{A_0 C_a^2}{W_T L_e}} \quad \text{(c.f. (15))}. \tag{17}$$

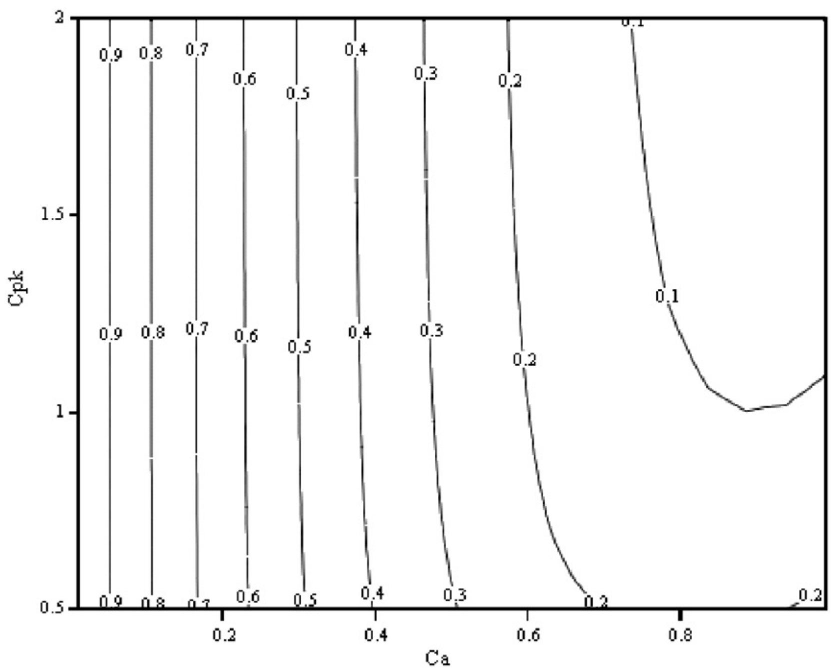

Fig. 13. The contour plot of $L_{C_{pk}}$ for $0.5 \leqslant C_{pk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

Hence the expected relative squared losses based on C_{pmk}, $L_{C_{pmk}}$, can be rewritten as

$$L_{C_{pmk}} = \frac{1}{9}\frac{A_0 C_a^2}{W_T C_{pmk}^2}. \tag{18}$$

Expression (18) shows that $L_{C_{pmk}}$ increases as C_a increases and reaches its maximum at $C_a = 1.0$. For example, at level of $C_{pmk} = 1.00$, $L_{C_{pmk}}$ has the maximum loss $A_0/(9W_T)$ obtained at $C_a = 1.0$. Figs. 16 and 17 display the surface and the contour plots of the expected relative quadratic loss $L_{C_{pmk}}$ for $0.5 \leqslant C_{pmk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$, respectively.

5.4. Loss comparison among PCIs

We shall now compare the expected relative squared losses of PCIs given in (14), (16) and (18). The following features of L_e'S for $e = C_p$, C_{pk}, and C_{pmk} are worth noting: (i) $L_{C_{pm}}$ remains a constant $A_0/(9W_T C_{pm}^2)$ for all values of C_a, (ii) $L_{C_{pk}}$ attains a minimum value when $C_a = 9C_{pk}^2/(1+9C_{pk}^2)$ and (iii) $L_{C_{pmk}}$ increases as C_a increases and reaches its maximum value at $C_a = 1.0$.

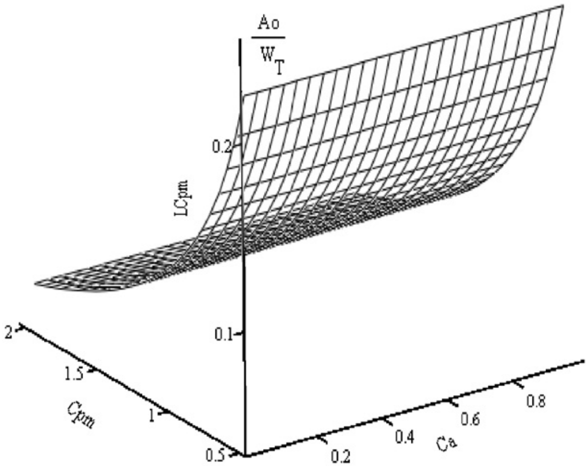

Fig. 14. The surface plot of $L_{C_{pm}}$ for $0.5 \leqslant C_{pm} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

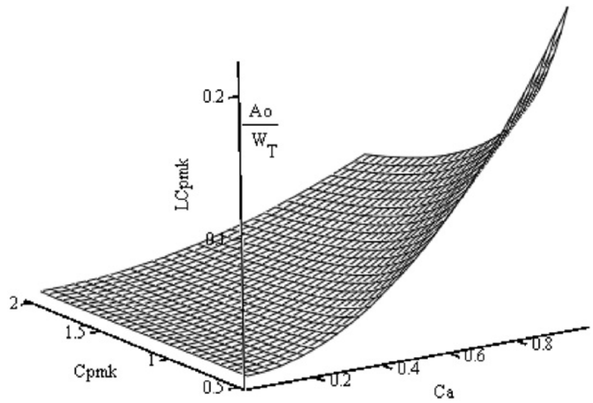

Fig. 16. The surface plot of $L_{C_{pmk}}$ for $0.5 \leqslant C_{pmk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

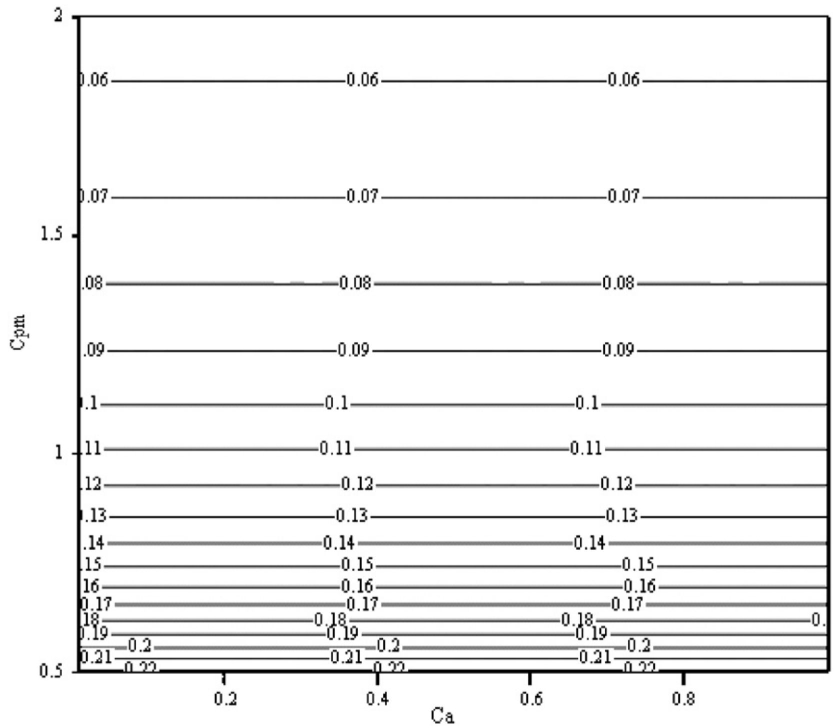

Fig. 15. The contour plot of $L_{C_{pm}}$ for $0.5 \leqslant C_{pm} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

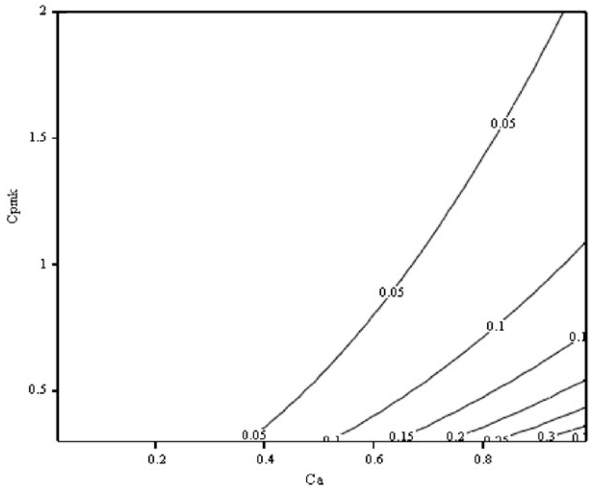

Fig. 17. The contour plot of $L_{C_{pmk}}$ for $0.5 \leqslant C_{pmk} \leqslant 2.0$ and $0 \leqslant C_a \leqslant 1$.

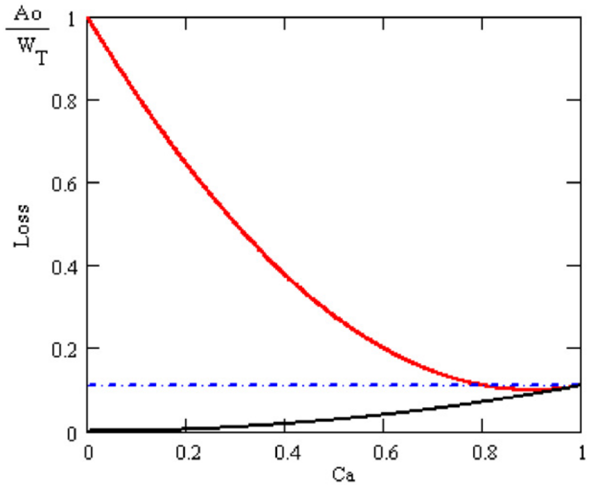

Fig. 18. The expected loss function curves for $C_{pk} = 1.0$ (top), $C_{pm} = 1.0$ (dash), and $C_{pmk} = 1.0$ (bottom) with various C_a.

Suppose now that the four capability indices values of C_p, C_{pk}, C_{pm} and C_{pmk} are all set to be equal to C (a positive constant). Then the expected relative squared losses of the three indices (C_{pk}, C_{pm} and C_{pmk}) have the same value $A_0/(9W_TC^2)$ when the process is centered (i.e. $C_a = 1.0$). Moreover, under specified capability indices values $C_{pk} = C_{pm} = C_{pmk} = C$, the ratios of the expected relative squared losses for these PCIs satisfy:

$$\frac{L_{C_{pmk}}}{L_{C_{pm}}} = C_a^2, \tag{19}$$

$$\frac{L_{C_{pk}}}{L_{C_{pmk}}} = 1 + 9C^2(1 - C_a)^2/C_a^2, \tag{20}$$

$$\frac{L_{C_{pk}}}{L_{C_{pm}}} = C_a^2 + 9C^2(1 - C_a)^2. \tag{21}$$

(Note that these three ratios become 1 for $C_a = 1$.)

For the ratio of the expected relative squared losses between C_{pmk} and C_{pm} (Eq. (19)), one concludes that $L_{C_{pmk}} \leqslant L_{C_{pm}}$ since of the value of C_a is between 0 and 1 for $LSL \leqslant \mu \leqslant USL$. For the ratio of the expected relative squared losses between C_{pk} and C_{pmk} (Eq. (20)) since $L_{C_{pk}}/L_{C_{pmk}} > 1$ in all cases except for $C_a = 1$, we obtain that $L_{C_{pmk}} \leqslant L_{C_{pk}}$ for $0 \leqslant C_a \leqslant 1$. Finally the Expression (21) shows that the ratio of the expected relative squared loss between C_{pk} and C_{pm} satisfies

$$L_{C_{pm}} \leqslant L_{C_{pk}} \quad \text{for } C_a \leqslant \frac{9C^2 - 1}{9C^2 + 1} \quad \text{and}$$
$$L_{C_{pm}} > L_{C_{pk}} \quad \text{for } C_a > \frac{9C^2 - 1}{9C^2 + 1},$$

C being the common value.

These results illustrate the advantage of using the index C_{pmk} over the indices C_{pk} and C_{pm} when measuring squared process loss, since C_{pmk} indeed provides a better protection to the customers in terms of the quality loss of

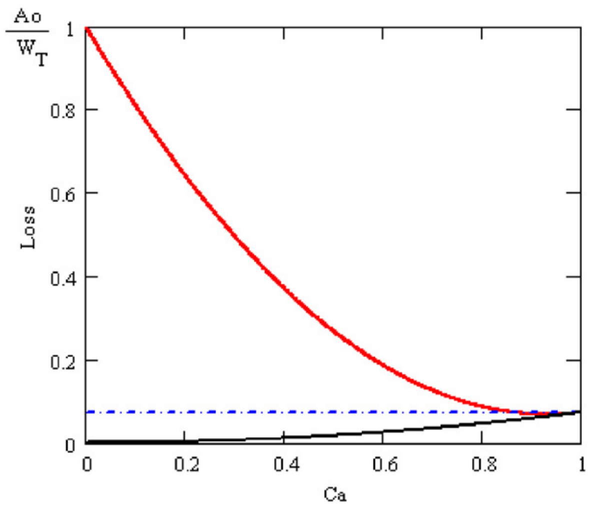

Fig. 19. The expected loss function curves for $C_{pk} = 1.5$ (top), $C_{pm} = 1.5$ (dash), and $C_{pmk} = 1.5$ (bottom) with various C_a.

the products. Figs. 18 and 19 plot the expected relative losses with $C_{pk} = C_{pm} = C_{pmk} = 1.00$ and 1.50 for $0 \leqslant C_a \leqslant 1$.

The use of various loss functions in quality assurance settings is becoming more widespread as the Taguchi's approach becomes more prominent. Spiring (1993), Sun et al. (1996), and Spiring and Yeung (1998) have developed a class of loss functions that provide practitioners with a wide range of choices that can be used in depicting loss due to departures from the process target. Research efforts related to PCIs and the loss properties would appear to offer opportunities that could potentially reduce (but not yet eliminate!) practitioners', managers', and researchers' concerns and discrepancies in the area of process capability. As it was already alluded above, theoretical statisticians and economists have for many years used the squared error loss function when making decisions or evaluating decision rules. English and Taylor

(1993) investigated the loss imparted to society by examining the expected losses arising in non-normal populations. Gupta and Kotz (1997) attempted to relate the relative loss to a modified C_{pm} index, which they refer to as C_{pq}. However, for the most part there has been little research effort devoted to the area of loss and loss functions along the lines of assessing process capability. This may be due in part to criticisms of quadratic/squared error losses. Criticism of the quadratic loss function is widespread in the literature and includes statistical decision analysts (Box and Tiao, 1992; Berger, 1985) quality assurance practitioners and researchers (Tribus and Szonyi, 1989; Leon and Wu, 1992), for the reasons of its failure to provide a quantifiable maximum loss (i.e., unbounded loss) and because the sizes of losses are severe for extreme deviations from the target. Pearn et al. (1992) and other researchers also pointed out that the squared error loss function is necessarily very often chosen due to the simplicity of mathematical derivations rather for its success in depicting actual process losses.

6. Extensions and applications

The theory and methodology of PCIs have been successfully applied to real world problems including cases with asymmetric tolerances, data collected as multiple subsamples, tool wear, gauge measurement error, supplier selection, multi-process product, multiple quality characteristics and so on.

6.1. Extensions to asymmetric tolerances

A process is said to have a symmetric tolerance if the target value T is set to be the mid-point of the specification interval [LSL, USL], i.e. $T=(USL+LSL)/2$. In the manufacturing industry cases with asymmetric tolerances ($T \neq m$) often occur. From the customer's point of view, asymmetric tolerances indicate that deviations from the target that are less tolerable in one direction than the other (see e.g. Boyles, 1994; Vännman, 1997; Wu and Tang, 1998). Nevertheless, asymmetric tolerances can also arise in those situations where the tolerances are symmetric to start with, but the process distribution is skewed following a non-normal distribution. To deal with this situation the data are usually transformed to achieve approximate normality. A prominent example of this approach is Chou et al. (1998) who have used the well-known, Johnson (1949) curves to transform the non-normal process data. Other research focused on cases with asymmetric tolerances include Choi and Owen (1990), Boyles (1994), Vännman (1997), Chen (1998), Pearn et al. (1999a, b), Chen et al. (1999), and more recent Jessenberger and Weihs (2000), Pearn et al. (2006) and Chang and Wu (2008).

Several generalizations of C_{pk} including C_{pk}^{*}, C'_{pk} have been proposed to handle processes with asymmetric tolerances (see Kane, 1986; Franklin and Wasserman, 1992; Kushler and Hurley, 1992 for details). Unfortunately, these generalizations understate or overstate the process capability, depending on the position in many cases of μ relative to T. To remedy the situation, Pearn and Chen (1998) proposed the index C''_{pk}—another generalization of C_{pk}—for processes with asymmetric tolerances. The motivation for the new index C''_{pk} is based on the general criteria stipulated by Boyles (1994), Choi and Owen (1990) and Pearn et al. (1992) when a $d^{*}=D_u=D_l=d$ analyzing and comparing the existing capability indices dealing with (a) process yield; (b) process centering; (c) other process characteristics. The generalization C''_{pk} (the Pearn–Chen index) is formally defined as

$$C''_{pk}=\frac{d^{*}-A^{*}}{3\sigma},$$

where $A^{*}=\max\{d^{*}(\mu-T)/D_u, d^{*}(T-\mu)/D_l\}$, $d^{*}=\min D_u, D_l\}$, $D_u=USL-T$ and $D_l=T-LSL$. Note that $d^{*}(\mu-T)/D_u=[\min\{D_u,D_l\}(\mu-T)]/(USL-T)$. Obviously, whenever $T=m$ (a symmetric tolerance), $A^{*}=|\mu-m|$ and C''_{pk} reduces to the index C_{pk}. The index C''_{pk} attains the maximal values at $\mu=T$, regardless of whether the preset specification tolerances are symmetric or not. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric (with respective to T). The index C''_{pk} takes into account the asymmetry of the loss function. Thus, given two processes E and F with $\mu_E>T$ and $\mu_F<T$, satisfying $(\mu_E-T)/D_u=(T-\mu_F)/D_l$ (i.e., processes E and F have equal "departure ratio"), the C''_{pk} values for processes E and F are the same provided $\sigma_E=\sigma_F$. In addition, the index C''_{pk} decreases when mean μ shifts away from target T in either direction. Actually, C''_{pk} decreases faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. This is an advantage since the index would respond faster to the shift towards "the wrong side" of T other than towards the middle of the specification interval. Pearn and Chen (1998) also provide a thorough comparison among the three indices, C_{pk}, C'_{pk} and C''_{pk}. The estimation of this index C''_{pk}, PDF, CDF of its estimator $\hat{C}''_{pk}$, and a decision making procedure for C''_{pk} can be found in Pearn and Chen (1998) and Pearn et al. (1999b).

For cases with asymmetric tolerances, generalizations of C_{pm} and C_{pmk} can be developed along the similar lines. Chen et al. (1999) and Pearn et al. (1999a) considered extensions of C_{pm} and C_{pmk} to handle a process with asymmetric tolerances. Under the normally distributed assumption, the explicit forms of the PDF and the CDF of the estimated index $\hat{C}''_{pm}$ and $\hat{C}''_{pmk}$ with asymmetric tolerances are derived.

6.2. Extensions to multiple subsamples

The results obtained so far regarding the statistical properties of the estimated capability indices were based on a single sample. However, a common practice in process control is to estimate the PCIs by using the past "in-control data" from subsamples, especially, when a daily-based or a weekly-based production control plan is implemented for monitoring process stability. To use estimators based on several small subsamples and then interpret the results as if they were based on a single sample may generate incorrect conclusions, and vice versa. In order to use the past in-control data from subsamples to provide decisions regarding process capability, the distribution of the estimated capability index based on multiple subsamples should be considered.

When using subsamples, Kirmani et al. (1991) have investigated the distribution of estimators of C_p based on the sample standard deviations of the subsamples. Li et al. (1990) have investigated the distribution of estimators of C_p and C_{pk} based on the ranges of the subsamples. Vännman and Hubele (2003) considered the indices in the super structure class defined by $C_p(u,v)$ and derived the distribution of the estimators of $C_p(u,v)$, in the case when the estimators of the process parameters μ and σ are based on subsamples. Consider the case when the characteristic of the process is normally distributed and we have h subsamples, where the sample size of the ith subsample being n_i. For each i, $i=1,2,\ldots,h$, let x_{ij}, $j=1,2,\ldots,n_i$, be a random sample from a normal distribution with mean μ and variance σ^2 measuring the characteristic under consideration. Assume that the process is monitored using a $\bar{X}$-chart together with a S-chart. For each subsample let $\bar{x}_i$ and s_i^2 denote the sample mean and sample variance, respectively, of the ith sample and let N denote the total number of observations, namely,

$$\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}, \quad s_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2 \quad \text{and} \quad N = \sum_{i=1}^{h} n_i.$$

Let $N_1 = \sum_{i=1}^{h}(n_i-1) = N-h$. When all the subsamples are of the same size n, $N = hn$ and $N_1 = h(n-1)$. As an estimator of μ and σ^2, we use the overall sample mean and the pooled sample variance, respectively. These are the unbiased estimators, i.e.

$$\hat{\mu} = \bar{\bar{x}} = \frac{1}{N}\sum_{j=1}^{n_i} n_i\bar{x}_i, \quad \hat{\sigma}^2 = s_p^2 = \frac{1}{N_1}\sum_{j=1}^{n_i}(n_i-1)s_i^2.$$

For the C_{pk} index, the natural estimator based on multiple samples can be expressed as

$$\hat{C}_{pk}^M = \min\left\{\frac{USL-\bar{\bar{x}}}{3s_p}, \frac{\bar{\bar{x}}-LSL}{3s_p}\right\} = \frac{d-|\bar{\bar{x}}-m|}{3s_p}.$$

Using the techniques available for cases with a single sample, the CDF of $\hat{C}_{pk}^M$ can be derived. Consequently, the critical values, lower confidence bounds, and the manufacturing capability calculations also can be carried out. For cases with multiple subsamples, several estimators of C_{pm} and C_{pmk} can be derived using similar technique (see Vännman and Hubele, 2003; Wu and Pearn, 2005b; Wu, 2008 for more details). Hubele and Vännman (2004) considered the pooled and un-pooled estimators of the variance from subsamples, and provide the sampling distributions of the corresponding estimators of C_{pm}. The un-pooled variance estimator is equivalent to the usual "overall" or "long-term" variance estimator, whereas the pooled variance estimator is based on a control chart relating "within" and "short-term" variance estimator. Namely, when the process has undergone a change in variation, the un-pooled estimator captures all of the variation, whereas the pooled one captures only the component of within subsamples variation (see, e.g., Cryer and Ryan, 1990; Hubele and Vännman, 2004).

6.3. Extensions to tool wear problem

In the 21st century, manufacturing systems are geared towards meeting the challenges of a quality-based competition. Process capability studies and analyses have become critical issues in process control; indeed a number of guidelines are available for process capability assessment. Moreover, certain conditions such as normally distributed output, statistical independence of observed values and the existence of only random variation (resulted from chance causes) ought to be stipulated for this assessment. These conditions may not be fully satisfied in a practical set-up and some departures are quite likely to occur. Tool wear, naturally, constitutes a dominant and inseparable component of variability in many "machining" processes, and hence it represents a systematic assignable cause. Process capability assessment in such cases may turn out to be a bit tricky since the standard procedure may not always provide accurate results.

Observe that a process capability analysis is valid only when the process under investigation is free of any special or assignable causes (i.e., being in-control). A process is said to have a "tool wear problem" when a variation due to a certain systematic cause is present. There are, in fact, two areas of interest when studying one process: process stability and process capability. It is important to have clear guideless about control before developing the plan for a tool wear process. Specifically is the intent of our plan to detect changes in the process or is our goal just to monitor the tool? An action to be taken in an out-of-control situation should be determined by the intent of the plan. Statistical process studies and the ongoing control can be quite complicated in case that we are dealing with machine processes possessing a tool wear. Indeed such a wear is a fact, and it is essential for processes that exhibit tool wear to be controlled, to maintain high part quality and to maximize the tool life. In its simplest and most common form, tool wear data tend to have an upward or a downward slope over time. To determine this trend, a best-fit line to the data ought to be generated. For standard control charts, the grand average and the control limits are usually horizontal. In contrast, when a tool wear is present, the control limits will be parallel to the tool wear slope. Once control has been assessed, the capability of the process can be then determined.

Some investigators attempt to remove the variability associated with the systematic cause. For example, Yang and Hancock (1990) recommended that in computing the basic C_p index, an unbiased estimator of σ can be obtained to be $\sigma/(1-\rho)^{1/2}$, where ρ is defined as the average correlation factor. Some other authors make a general assumption of linear degradation in the tool. For instance, Quesenberry (1988) suggested that tool wear can be modeled over an interval of tool life by a regression model and assumes that the tool wear rate is either known or a good estimate of it is available, and that the process mean can be adjusted after each batch without an error. However, the procedure of model-building does not appear to be either easy or directly applicable to realistic conditions. Long and De Coste (1988) approach is to remove first the linearity by regressing on the means of the subgroups and then to determine the process capability. These authors discussed the techniques for obtaining the best-fit line for the data, calculating the control limits, comparing the slopes to determine different tools, and finally calculating the capability of the process. These techniques are based on the assumption that tools are "consistent" within their tool groups. As the data are recorded over several tools, the subgroup averages are plotted over time. A best-fit line is then determined using the methodology of the standard linear regression analysis.

Evidently when systematic assignable causes are present and tolerated, the overall variation of the process (σ^2) is then composed of the variation due to random causes (σ_r^2) and the variation due to assignable causes (σ_a^2), that is, $\sigma^2 = \sigma_r^2 + \sigma_a^2$. The traditional PCI measures disregard the portions of the overall variation, (in the presence of tool wear) that are due to assignable causes. Hence any estimates of the process capability will confound the true capability with these two sources. In order to get a true measure of a process capability, any variation due to an assignable cause must be removed from the measure of a process capability. However, the above approaches tacitly assume a static process capability over a cycle. By allowing the process capability to be dynamic within a cycle, as well as from a cycle to cycle, one could circumvent some of the problems encountered. Spiring (1989, 1991) viewed this as a dynamic process which is in a constant change. In this dynamic model, the capability of the process may vary, possibly even in a predictable manner. Spiring has devised a modification of C_{pm} index for this dynamic process under the influence of systematic assignable causes. In this scenario the goal is to maintain a minimum level of capability at all times. As a result, the capability will be cyclical in nature and its period being defined by the frequency of the process/tolling adjustments. Even when the assignable cause variation is not systematic, as is in the case with tool or die wear, one ought to be able to deal with random fluctuations of the process mean over time. Quite often in practice, deviations from the target value are due to assignable causes, which are easy to pinpoint such as shift-to-shift changes, differences in the raw material batches, environmental factors, etc.

The measure of process capability for dynamic process proposed by Spiring (1991) is

$$C_{pm} = \frac{\min\{USL - T, T - LSL\}}{3\sqrt{\sigma_{rt}^2 + (\mu_t - T)^2}},$$

where USL, LSL and T as above but μ_t represents the mean and σ_{rt}^2—the variation (due to random causes only) of the process at time period t. As we have already remarked the actual value of μ_t and σ_{rt}^2 are seldom known, and in order to get an assessment of process capability these values ought to be estimated. These estimators will have to incorporate various existing sources of variation. Monitoring process's capability will thus require obtaining the value of C_{pm} or its suitable estimate at various times t over each cycle during the lifetime of the tool.

It thus follows that the proposed sampling scheme is similar to procedures used in monitoring a process for control charting. The general format is to gather k subgroups of size n from each cycle (e.g., the period from t_0 to t_1) over the lifetime of the tool. The value of k will be unique to each process and, in fact, may change from cycle to cycle within the process. On the other hand, sample size of less than five (i.e., $n<5$) are cautioned against, while larger samples (e.g., $n>25$, see Spiring, 1991) may also pose problems. The optimal sample size for assessing process capability in the presence of systematic assignable causes will thus vary for each process under consideration.

Assuming that the effect of the tool deterioration is linear over the sampling window, estimates of C_{pm} are available which would not involve contribution of the assignable causes. Typically such an estimator is of the form:

$$\hat{C}_{pm} = \frac{\min\{USL - T, T - LSL\}}{3\sqrt{MSE_t + \dfrac{n}{n-1}(\bar{X}_t - T)^2}}.$$

This measure of process capability considers only the proximity to the target value T and the variation associated with random causes as the linear effect of the tool wear is effectively removed by using

$$MSE_t = \frac{\sum_{I=1}^{N}(x_{t_{a_i}} - \hat{x}_{t_{a_i}})^2}{n-2}$$

at sequentially selected points (i.e., $t_{a_1}, t_{a_2}, t_{a_3}, \ldots$) rather than the standard estimator S^2. The MSE_t is the mean square error associated with the regression equation $x_{a_i} = \alpha_a + \beta t_{a_i} + \varepsilon_{a_i}$ and where t_{a_i} is the sequence number of the sampling unit and $\varepsilon_{a_i} \sim N(0,1)$. The coefficient β represents the linear change in the tool wear given a unit change in time/production. The method was proposed by Spiring (1991) who have suggested that the problem can be tackled by viewing the process capability as ordinal sequential dynamic rather than static process. This entails calculating a new index and constant monitoring it as the process advances. When the index reaches a preset minimum value, the processing is terminated and resetting/replacement is carried out.

6.4. Extensions to gauge measurement error

The inevitable variations in process measurements come from two sources: the manufacturing process and the gauge. Gauge capability reflects the gauge's precision, or lack of variation, which is not the same as calibration (the latter assures the gauge's accuracy). As it was emphasized on a numerous occasions, process capability measures the ability of a manufacturing process to meet preassigned specifications. Nowadays, many customers use process capability to judge supplier's ability to deliver quality products. Suppliers need to be aware of how gauges affect various process capability estimates.

The gauge capability consists of two parts: repeatability and reproducibility. Repeatability is the gauge's experimental or random error. Namely, when measuring the same specimen several times, the gauge will never provide exactly the same measurement. Reproducibility is the (quite annoying) inability of several inspectors or gauges to arrive at the same measurement value from a given specimen i.e. the variability due to different operators using the gauge (or different time periods, different environments, in general different conditions). To summarize we have

$$\sigma^2_{\text{Measurement error}} = \sigma^2_{\text{Gauge}} = \sigma^2_{\text{repeatability}} + \sigma^2_{\text{reproducibility}}$$

Estimates for $\sigma^2_{\text{repeatability}}$ and $\sigma^2_{\text{reproducibility}}$ come from a gauge study, or a GR&R (gauge repeatability and reproducibility) study. Barrentine (1991), Levinson (1995), Montgomery (2001, 2005), and Burdick et al. (2003), among others, describe various procedures for gauge studies.

Gauge capability is a gauge's ability to repeat and reproduce measurements. Its measurement is the percentage of tolerance consumed by (gauge) capability (PTCC). Montgomery (2001) referred to it as the precision-to-tolerance (or P/T) ratio. It is the ratio of the gauge's variation to the specification width; its smaller values are evidently preferable. Denoting the gauge's standard deviation as σ_{Gauge}, we have

$$\text{PTCC} = \frac{6\sigma_{\text{Gauge}}}{USL - LSL} \times 100\%.$$

Some authors and practitioners prefer to use the coefficient 5.15 instead of 6 (see e.g. Barrentine, 1991; Levinson, 1995). This formula above uses 6σ as a natural tolerance width for the gauge based on the normal distribution assumptions.

The gauge capability has a significant effect on process capability measurements. An inaccurate measurement system can thwart the benefits of improvement endeavors and results in poor quality. Analyzing process capability without considering gauge capability may often lead to unreliable decisions. It could cause a serious loss to producers if gauge capability is ignored in process capability estimation and testing. On the other hand, improving the gauge measurements and properly trained operators can reduce the measurement errors. Since measurement errors unfortunately cannot be avoided, using appropriate confidence coefficients and power becomes necessary. However, the real world is that no measurement is free from an error or uncertainty even if it is carried out with the aid of the most sophisticated and precise measuring devices. Any variation in the measurement process has a direct impact on the ability arrive at an execute sound judgment about the manufacturing process. Analyzing the effects of measurement errors on PCIs, Levinson (1995) and Mittag (1997) developed definitive techniques for quantifying the percentage error in process capability indices estimation in the presence of measurement errors.

Common approaches to GR&R studies, such as the Range method (see Montgomery and Runger, 1993a) and the ANOVA method (see Mandel, 1972; Montgomery and Runger, 1993b) assume that the distribution of the measurement errors is normal with a mean error of zero. Let the measurement errors be described by a random variable $M_e \sim N(0, \sigma^2_{M_e})$; Montgomery and Runger (1993b) determined the gauge capability λ using of the formula:

$$\lambda = \frac{6\sigma_{M_e}}{USL - LSL} \times 100\%.$$

For a measurement system to be deemed acceptable, the variability in the measurements due to this system ought to be less than a predetermined percentage of the engineering tolerance. Some guidelines for gauge acceptance have been developed by the Automotive Industry Action Group (AIAG, 2002).

Let $X \sim N(\mu, \sigma^2)$ be the relevant quality characteristic of a manufacturing process and consider this process capability in a measurement error system. Due to the measurement errors, the observed random variable $Y \sim N(\mu_Y = \mu,\ \sigma_Y^2 = \sigma^2 + \sigma^2_{M_e})$ is measured under the assumption that X and M_e (the measurement error) are stochastically independent (instead of measuring the actual variable X). The empirical capability index C^Y_{pk} will be obtained after substituting σ_Y for σ. The relationship between the true process capability $C_{pk} = \min\{(USL - \mu)/3\sigma, (\mu - LSL)/3\sigma\}$ and the empirical process capability C^Y_{pk} can be expressed as

$$\frac{C^Y_{pk}}{C_{pk}} = \frac{1}{\sqrt{1 + \lambda^2 C_p^2}}.$$

Since the variation of the observed data is larger than the variation of the original one, the denominator of the index C_{pk} becomes larger, and the true capability of the process will be understated if calculations of process capability index are based on the empirical data represented by Y. Suppose that the empirical data (the observed measurements contaminated by errors) $\{Y_i, i = 1, 2, \ldots, n\}$ are available, then the natural estimator $\hat{C}^Y_{pk}$ is

$$\hat{C}^Y_{pk} = \frac{d - |\bar{Y} - m|}{3S_Y},$$

which is obtained by replacing the process mean μ and the process standard deviation σ by their conventional estimators $\bar{Y} = \sum_{i=1}^n Y_i/n$ and $S_Y = [\sum_{i=1}^n (Y_i - \bar{Y})/(n-1)]^{1/2}$ from a "bonafide" stable process. When estimating the capability, the estimator $\hat{C}^Y_{pk}$ in the case of contaminated data, substantially underestimates the true capability in the presence of measurement errors. Consequently, if a statistical test is used to determine whether the process meets the capability requirement, the power of the test would drastically decrease.

In fact the discussions in Pearn and Liao (2005) indicated that the true process capability would be inappropriately underestimated if $\hat{C}^Y_{pk}$ is used. The probability that $\hat{C}^Y_{pk}$ is greater than c_0 would be less than a preassigned when using $\hat{C}_{pk}$. Thus, when estimating C_{pk} the α-risk using $\hat{C}^Y_{pk}$ is less than the α-risk of using $\hat{C}_{pk}$. The power of the test based on $\hat{C}^Y_{pk}$ is then also smaller than that based on $\hat{C}_{pk}$. Namely, the α-risk and the power of the test decrease with the measurement error. Since the lower confidence bound is underestimated and the power becomes small, the producers cannot firmly state that their processes meet the capability requirement even if their processes are indeed sufficiently capable. Adequate and even superior product units could be incorrectly rejected in this case. To help the situation, Pearn and Liao (2005) derived the CDF of $\hat{C}_{pk}^{Y}$ and considered the adjustment of the confidence bounds and of the critical values to provide a better capability assessment. Suppose that the required confidence coefficient is θ, then an adjusted confidence interval of C_{pk} with a lower confidence bound L^* can be obtained which improves the accuracy of capability assessment.

6.5. Applications to supplier selection

In an initial stage of production setting, the decision maker necessarily faces the problem of selecting the best manufacturing supplier out of several available candidates. There are many factors, such as quality, cost, service and so on, that ought to be taken into account when selecting the best supplier. Several selection rules have been proposed for selecting the means or the variances in the classical analysis of variance (ANOVA) (see e.g. Gibbons et al., 1977; Gupta and Panchapakesan, 1979; Gupta and Huang, 1981 for more details). The vast majority all of the selection rules are based on the ordered sample variances. A common drawback of these selection rules is that the information available in sample data cannot be efficiently used.

As it was mentioned in an earlier section of this paper, PCIs provide common quantitative measures of the manufacturing capability and production quality to be used by both producer and supplier as guidelines when signing a contract. Purchasing personnel could use the PCI to decide whether to accept or reject the products provided by suppliers. There are two common methods that are available to determine the better suppliers' PCI. Simply carry out a 100% inspection to calculate separately the PCI for each supplier, the suppliers can then be compared according to their respective true PCI values; this approach is, however, quite expensive and time-consuming and is nowadays rarely used in practice. The second method involves sampling implementation, and statistical testing is then applied to assess suppliers' process capabilities.

Tseng and Wu (1991) considered the problem of selecting the best manufacturing process from k available processes based on the (primitive) "precision" capability index C_p and proposed a modified likelihood ratio (MLR) selection rule. Details related to theoretical derivations of the MLR selection rule are given in their paper. Some tables of the sample size and of the critical values for selecting the best manufacturing have also been computed by controlling the probability of a correct selection (CS) and the error probability based on the proposed MLR selection rule. Furthermore, when the product quality characteristic data for each process follows a non-normal symmetric distribution, a simulation study has been carried out to examine the robustness of the selection rule. The results indicate that the proposed MLR selection rule is quite insensitive to a number of non-normal symmetric process distributions (including logistic and uniform).

Chou (1994) developed an approximate method for selecting a better supplier based on the one-sided capability indices C_{pu} and C_{pl} for equal sample sizes. Hubele et al. (2005) developed a Wald statistic for testing the equality of g C_{pu} (or C_{pl}) indices where $g \geqslant 2$ and there is no restriction on sample sizes drawn from the g processes. Based on the C_{pm} index a somewhat mathematically cumbersome approximation method has been developed by Huang and Lee (1995) for selecting a subset of processes containing the best supplier from a given set of processes. The method essentially compares the average loss of a group of candidate processes, and selects a subset of these processes with a small process loss $\tau^2 = E[(X-T)^2]$, which, at a certain level of confidence, contains the best process. Since the specification limits are usually fixed and determined in advance, searching for the largest C_{pm} is equivalent to searching for the smallest τ^2. The selection rule of Huang and Lee (1995) is that one retains the population i in the selected subset if and only if, $\tau_i^2 \leqslant c \times \min_{1 \leqslant j \leqslant g, j \neq i} \tau_j^2$ where the value of c is determined by a function of parameters, which can be in turn determined by calculations from the obtained samples. Pearn et al. (2004b) have investigated the accuracy of this selection method for the cases with two candidate processes. Additionally, a two-phase selection procedure was developed to select a superior supplier and further to examine the magnitude of the difference between the two suppliers. Chen and Chen (2004) developed an approximate confidence interval for the ratio C_{pm1}/C_{pm2} for selecting the better of two suppliers based on Boyles (1991) investigation. The performance was compared with three well-known Bootstrap confidence intervals (SB, PB, BCPB) using simulation. The results showed that the confidence intervals based on Boyles' (1991) and the SB method are better than those based on PB and BCPB methods. Additionally, they suggested that a sample size greater than 30 is necessary for the interval based on Boyles' method and greater than 50 for the interval based on SB method.

6.6. Applications to multi-process performance analysis chart

Ever since Shewhart introduced his pioneering control charts in 1932, it has become a common practice for practitioners to use various control charts for monitoring different processes on a routine basis. As an example, when dealing with a variable data, the control chart technique usually employs a chart (such as a $\bar{X}$ chart) to monitor the process center and a chart (such as an R chart or an S chart) to monitor the process spread. These charts are easy to comprehend, and they effectively communicate the critical process information without even employing words and on formulas. Unfortunately, they are applicable only for a single process (one process at a time). Using them in multi-process environment could be a clumsy and time-consuming task for supervisors or shop engineers since it may require to analyze each individual chart in order to evaluate the overall status of a shop process control activities.

We know that PCIs measure the ability of a process to (re)produce products that meet certain specifications. We also have seen that capability of a process, there are mainly two characteristics of importance involved in an analysis of capability process: the process location in relation to its target value and the process spread. Recall also that the closer the process output is to the target value and the smaller is the process spread, the more capable will the process be. However, the fact that PCIs combine information about closeness to target with the process spread, and expresses the capability of a process by a single number, may in some instances become viewed as one of their major drawbacks. When a process is found to be "non-capable", the operator might well be interested in knowing whether the non-capability is caused because the process output is substantially off target or because the process spread is too large, or is a result of a combination of these two factors. To circumvent this defect a number of researchers in the last 15 years were suggesting to use different graphical methods to ascertain the improvement initiatives aimed at devising more capable processes (see, e.g. Gabel, 1990; Boyles, 1991; Tang et al., 1997; Deleryd and Vannman, 1999 among others).

A multi-process performance analysis chart (MPPAC), originally proposed by Singhal (1990), evaluates the performance of a multi-process product with symmetric bilateral specifications, determine priorities among multiple processes to achieve capability improvement and indicates whether reducing the variability or the departure of the process mean should be the main task for improvement. While C_{pu} and C_{pl} present the X-axis and Y-axis, respectively, in a MPPAC, C_p is the average of C_{pu} and C_{pl}, namely, $C_p = (C_{pu} + C_{pl})/2$. Moreover, the C_{pk} MPPAC provides an efficient route to process improvement by comparing the locations on the chart of processes before and after an improvement effort has been carried out. Singhal (1991) also provided a MPPAC with several well-defined capability zones by using the process capability indices C_p and C_{pk} for grouping processes in a multiple process environment into several performance categories on a single chart. This is indeed quite useful when process performance is measured in terms of capability indices. Different capability zones describe a status of each process, which is easy to interpret and assist in grouping the processes into performance categories to implement of quality improvement operations.

The American giant corporation Motorola Inc. introduced their very popular six sigma ($6\text{–}\sigma$) program which is equivalent to a defect rate of 3.4 ppm. This program corresponds to a C_p value of 2.0 or more and a C_{pk} value of 1.5 or more. All this is accomplished under an implicit normality assumption. In practical applications, when a product has several models with required different specifications which are required to be monitored and controlled, it may be difficult or time-consuming to carry out these factory control activities. As we have seen a MPPAC not only evaluates the performance of a multi-process product with symmetric bilateral specifications but also sets the priorities among multiple processes for capability improvement and indicate whether reducing the variability or adjusting the departure of the process mean should be the focus of our improvement operations. This renders a MPPAC to be an efficient tool for communicating between the product designer, manufacturers, quality engineers and among (often numerous) management departments.

Subsequently Pearn and Chen (1997b) proposed a modification of the C_{pk} MPPAC combining the third generation process capability indices, C_{pm} or C_{pmk}, in attempt to identify the problems which cause processes' failings to center around the target value. Furthermore, combining Singhal's MPPAC with asymmetric process capability index C_{pa}, and using unilateral characteristics, Chen et al. (2001) introduced more recently a process capability analysis chart (PCAC) to evaluate the process potential and performance for an entire product which may be composed of "the-smaller-the-better", "the-larger-the-better", symmetric and asymmetric specifications. We have already noted that the process yield of a multi-process product is lower than of any individual process yields. Similarly, when the entire product capability is preset to satisfy the required level, the individual process capabilities should exceed the preset standard for the entire product. Therefore, the overall process capability is recognized as "capable" if all the individual process capability indices are located within the process capability zone. Conversely, processes must be upgraded when some of the process capability indices are outside this zone. It is straightforward to distinguish process performance relative to the locations of the PCIs on a modified C_{pk} MPPAC. Hence, the modified C_{pk} MPPAC not only distinguishes between process capabilities, but also reveals the degree of quality accuracy for multi-process products. This renders the modified C_{pk} MPPAC as an effective and efficient tool for evaluating multi-process products, composed of various unilateral and bilateral specifications.

6.7. Extensions to multiple characteristics

As we have emphasized on several occasions in this book a PCI is a numerical summary that compares the behavior of a product (or processes characteristics) related to engineering specifications. Its convenience is due to compressing complex information about the process into a single number. At present, numerous customers request their suppliers to record capability indices for product characteristics on a regular basis. In a majority of companies it became a key index to evaluate product quality. A large number of quality engineers and statisticians have proposed methodologies for assessing product/process quality. However, the bulk of the studies associated with analyzing the quality and efficiency of a process are so far limited to a discussion of one single quality specification.

By now in most processes, the products possess multiple quality characteristics one. Multiple characteristics processes are so common that our studies to capability indices cannot be restricted to the univariate domain. The multivariate relationship among the quality characteristics may or may not be reflected in the engineering specifications. For instance, *USL*'s and *LSL*'s may be given separately for each quality characteristic. In two-dimension cases, these tolerance ranges compose a rectangular tolerance region. In higher dimensions, they form a hypercube. For more complex engineering specifications, the tolerance region could be quite intrinsic.

For processes with multiple characteristics, Bothe (1992) proposed a simple measurement of a tolerance region by taking the minimum of the measure for each single characteristic. Consider a v-characteristic process with v yield measures (percentage of conformities) P_1, $P_2,\ldots$, and P_v. The overall process yield is measured as $P = \min\{P_1, P_2,\ldots,P_v\}$. However, this approach does not reflect accurately the real situation. Suppose the process has five characteristics ($v = 5$) with equal characteristic yield measures $P_1 = P_2 = P_3 = P_4 = P_5 =$ in the amount of 99.73%. Using the approach considered by Bothe (1992), the overall process yield is calculated as $P = \min\{P_1, P_2, P_3, P_4, P_5\} = 99.73\%$ (or equivalently 2700 ppm of non-conformities). Assuming that the five characteristics are mutually independent, the actual overall process yield should be calculated as $P = P_1 \times P_2 \times \cdots \times P_5 = 98.66\%$ (or 134273 ppm of non-conformities), which is significantly less than the one calculated by Bothe (1992) method.

When these variables are related characteristics, the analysis ought to be based on a multivariate statistical technique. Chan et al. (1991), Taam et al. (1993), Pearn et al. (1992), Chen (1994), Karl et al. (1994), Shariari et al. (1995), Boyles (1996), Wang and Du (2000), Wang et al. (2000) and others have developed and presented multivariate capability indices for assessing capability. Wang and Chen (1998) and Wang and Du (2000) proposed multivariate extensions for C_p, C_{pk}, C_{pm} and C_{pmk} based on the technique of principal component analysis, which transforms number of original related measurement variables into a set of uncorrected linear functions. A comparison of three novel multivariate methodologies for assessing capability is illustrated (and their usefulness is discussed) in Wang et al. (2000).

At present, the research of multivariate PCIs is still very limited in comparison to the research of the univariate PCIs. A current problem for multivariate capability indices is that there is no consistency regarding a methodology for evaluating the capability. In addition, it is also quite difficult to obtain the relevant statistical properties needed for a more detailed inference for multivariate PCIs. Consequently there still exist essential difficulties in trying to assess the capability of a multivariate system by means of a single value. Obviously, further investigations in this field are needed and there are no guaranteed that a universal success be reached.

7. Concluding remarks: statistical properties and general observations

In view of the globalization trends capability indices are becoming more and more powerful standard tools for quality reporting through out the world, particularly, at the management level. Proper understanding and their accurate estimation are essential for a modern company to maintain the status of a capable supplier. Most supplier certification manuals include a discussion of process capability analysis and describe the recommended procedure for computing PCIs. Analyzing these capability indices, the production department is able to trace and improve inadequate processes in order to meet customers' needs. On the surface mathematical expression of these capability indices are easy to understand and seem to be straightforward to apply. However, in practice even the process mean μ and the process variance σ^2 are usually unknown. In order to calculate the index value, sample data must be collected and a degree of uncertainty would no doubt be introduced into capability assessments due to inevitable sampling errors. The approach by just looking at the calculated values of the estimated indices and making conclusions as to the given process is capable, is therefore highly unreliably approved. As the use of the capability indices grows, users are becoming more and more educated and sensitive to the impact of the estimators and their sampling distributions, and discovering that capability measures ought to be reported at least via confidence intervals or via capability testing. Statistical properties of the estimators of these indices under various process conditions have been investigated quite extensively, in the last 15 years. Zhang et al. (1990), Li et al. (1990), Pearn et al. (1992), Kushler and Hurley (1992), Kotz et al. (1993), Nagata and Nagahata (1994), Chen and Hsu (1995), Vännman and Kotz (1995), Tang et al. (1997), Zimmer and Hubele (1997), Vännman (1997), Pearn et al. (1998), Wright (1998, 2000), Hoffman (2001), Zimmer et al. (2001), Pearn and Lin (2002, 2004), Vännman and Hubele (2003) and Pearn and Shu (2003) are only a small part of works in the literature dealing with these problems.

We have to stress that the indices described above are designed to monitor the performance for solely normal and near-normal processes with symmetric tolerances, and are shown to be inappropriate for the majority of cases with asymmetric tolerances. For normal distributions, these PCI estimators based on the statistics $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$ are quite stable and reliable. However, for non-normal distributions, they become highly unstable since the distribution of the sample variance s^2 is quite sensitive to the departures from normality. Somerville and Montgomery (1996) presented an extensive study illustrating how poorly the normality-based capability indices may perform as a predictor of process fallout when the process is non-normally distributed. When normality-based capability indices are used to deal with non-normal process data, the values of the capability indices may be incorrect and might even misrepresent the actual product quality. Therefore normality-based process capability indices such as C_p, C_{pk}, C_{pm} and C_{pmk} are inappropriate to measure processes for non-normal distributions.

For non-normal distributions of X, Clements (1989) suggested that "6σ" in the expression for C_p be replaced by the length of the interval between the upper and lower 0.135 percentage points (corresponding to 6σ in a normal case) of the given distribution and considered fitting a distribution for X by means of the Pearson system of distributions to obtain the required percentiles. Pearn et al. (1992) suggested replacing "6σ" in the denominator of C_p by "6θ", where θ is chosen so that the "capability" is not affected to a large extent by the shape of the distribution at hand. English and Taylor (1993) examined the effect of the non-normality assumption on PCIs and concluded that C_{pk} is more sensitive to departures from normality than C_p. Kotz and Johnson (1993) provided a survey of earlier works on the properties of PCIs and their estimators when the distribution is non-normal. Johnson et al. (1994) introduced a "flexible" PCI which takes into account possible differences in variability above and below the target value. Vännman (1995) proposed a new family of indices $C_p(u, v)$ (mentioned earlier), parameterized by $(u,\ v)$ that includes many other indices as its special cases. Deleryd (1996) investigated the suitable u and v values of $C_p(u, v)$ when the process distribution is skewed. It is recommended that $C_p(1,1)$, which is equivalent to C_{pmk}, is most suited to handle non-normality of PCIs. Pearn and Chen (1997a) considered a generalization of $C_p(u, v)$, called $C_{Np}(u, v)$, which can be applied to processes with arbitrary distributions. Castagliola (1996) introduced a non-normal PCI calculation method by estimating the proportion of non-conforming items for the Burr distribution. A new index C_S, proposed by Wright (1995) incorporates an additional skewness correction factor in the denominator of C_{pmk}. Shore (1998) developed a new approach to analyzing non-normal quality data and used it for the process capability analysis. Chang et al. (2002) proposed a heuristic weighted standard deviation method to adjust the value of PCIs according to the degree of skewness, considering separately the standard deviations above and below the process mean. Tang and Than (1999) reviewed several methods and presented a comprehensive evaluation and comparison of their ability to handle non-normality of the original data.

It should be kept in mind that PCIs can be used only after it has been established that the manufacturing process is indeed under statistical control. For applications where routine-based data collection plans are implemented, a common practice of process control is to estimate the process capability by analyzing the past "in control" data. To estimate σ we typically use either the sample standard deviation or the sample range. Control charts can be utilized as a monitoring device or a logbook to show the effect of changes in the process performance. A process may be in control but not necessarily operating at an acceptable level. Hence, management intervention will be required either to improve the process capability, or to change the manufacturing requirements to ensure that the products meet at least the minimum acceptable level. If the process is out of control in the early stages of process capability analysis, it will be unreliable and meaningless to estimate process capability. In such situations the first step should be to find and eliminate the assignable causes of variability which would bring the process in-control.

On the whole, capability indices are very appealing. Like many powerful tools, the PCIs can sometimes inflict substantial damage if used incorrectly. Due to rapid additions to the literature of PCIs periodically these advances should periodically be monitored. We trust that this paper fulfils this obligation for the first six years of the 21st century. When appropriately calculated, these indices will provide a lot of vital information concerning the manner that the current output of a process satisfies customer requirements. On the other hand, incorrectly applied and/or interpreted, these indices can generate an abundance of misinformation that may cause practitioners to carry out incorrect decisions.

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