國 立 交 通 大 學
工業工程與管理學系碩士班

碩士論文

對稱與非對稱規格製程準確性之估計與檢定
Estimating and Testing Process Accuracy with Extension to Asymmetric Tolerances

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中華民國 九十四 年 六 月

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對稱與非對稱規格製程準確性之估計與檢定

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摘要

Pearn (1998)提出製程準確性指標C_a來衡量製程的集中程度(群集於中心之能力)。在本篇論文中，我們根據估計量$\hat{C}_a$之明確累積分配函數(CDF)推導$\hat{C}_a$估計量之統計特性。藉由臨界值、P 值和最低信賴區間值之計算，我們發展出檢測製程準確性之方法。除此之外，為了使工廠製程量測員更容易量測產品之製程準確性，我們進一步將衡量製程準確性指標C_a推廣到非對稱規格之製程，並找出非對稱規格製程準確性指標的估計值$\hat{C}_a''$。工廠製程量測員根據附錄裡的製程指標表格，就能快速地作出可信賴的決策與評估產品的製程品質是否需要馬上改善。在論文的最後，我們用三個實際例子來說明如何估計與檢定製程準確性。

關鍵字：非對稱規格、 臨界值、 製程準確性、 製程集中度

Estimating and Testing Process Accuracy with Extension to Asymmetric Tolerances

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Abstract

Pearn *et al*. (1998) introduced the process accuracy index C_a to measure the degree of process centering (the ability to cluster around the center). In this paper, we derive an explicit form for the cumulative distribution function of the estimator $\hat{C}_a$. Subsequently, the distributional and inferential properties of the estimated process accuracy index C_a are provided. Calculations of the critical values, *p*-values, and lower confidence bounds are developed for testing process accuracy. Further, a generalization of C_a for cases with asymmetric tolerances is proposed to measure the process accuracy. The distributional properties of the corresponding natural estimator are investigated. Based on the results practitioners can easily perform the process accuracy testing, and make reliable decisions on whether actions should be taken to improve the process quality. Three application examples are given to illustrate how we test process accuracy using the actual data collected from the factory.

Keywords: Asymmetric tolerances; Critical value; Process accuracy; Process centering.

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Notations

T : target value

LSL : the lower specification limits preset by the process engineers

USL : the upper specification limits preset by the process engineers

d : the half specification width

m : the midpoint between the upper and lower specification limits

μ : the population mean

σ : the population standard deviation

%NC : the fraction of Non-Conformities

n : the number of the sample size

C_a : the process accuracy index of the symmetric tolerances

C_a'' : the process accuracy index of the asymmetric tolerances

$\hat{C}_a$: the estimator of C_a

$\hat{C}_a''$: the estimator of C_a''

$\bar{X}$: the sample mean

S : the sample standard deviation

D_u : the specification width between USL and T in asymmetric tolerances

D_l : the specification width between T and LSL in asymmetric tolerances

d^* : the minimum of D_u and D_l

α : the risk of misjudging an incapable process as a capable one

1. Introduction

In recent years, process capability indices (PCIs) have received substantial research attention in quality assurance and statistical literatures as well (see Kotz and Lovelace (1998), Palmer and Tsui (1999), Kotz and Johnson (2002), Spiring *et al*. (2003) for more details). Those indices have become popular as unit-less measures on whether a process is capable of reproducing items meeting the quality requirement preset by the product designer. Based on analyzing the PCIs, a production department can trace and improve a poor process so that the quality level can be enhanced and the requirements of the customers can be satisfied.

The first, and the original, process capability index was C_p. The C_p index reflects product consistency by considering the overall process variability relative to the manufacturing tolerance, which is designed to provide an indirect measure of potential ability to meet requirements (see Juran (1974), Sullivan (1984, 1985) and Kane (1986)). The C_a index measures the degree of process centering (the ability to cluster around the center), which can be regarded as a process accuracy index (see Pearn *et al*. (1998)). The indices C_p and C_a are defined as the following:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_a = 1 - \frac{|\mu - m|}{d},$$

where LSL and USL are the lower and upper specification limit, μ is the process mean, σ is the process standard deviation, m is the mid-point between the upper and the lower specification limits, and d is the half length of the specification interval. In addition, that is $m = (USL + LSL)/2$, and $d = (USL - LSL)/2$. For processes with two-sided specification limits, the percentage of non-conforming items ($\%NC$) can be calculated as $1 - F(USL) + F(LSL)$, where $F(\cdot)$ is the cumulative distribution function of the process characteristic X. On the assumption of normality, $\%NC$ can be expressed in parts per million (PPM) as:

$$\%NC = \left[1 - \Phi\left(\frac{USL - \mu}{\sigma}\right) + \Phi\left(\frac{LSL - \mu}{\sigma}\right)\right] \times 10^6$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. If the process is perfectly centered at the specification range ($\mu = m$), then the $\%NC$ can be expressed as $2\Phi(-3C_p)$. For example, $C_p = 1.00$ corresponds to $\%NC =$ 2700 PPM, and $C_p = 1.33$ corresponds to $\%NC =$ 63 PPM. However, C_p does not refer to the mean of the process, it will not give an exact measure of percentage NC in the general case, i.e. $\mu \neq m$. Therefore, it provides a lower bound on $\%NC$ with $2\Phi(-3C_p)$. On the other hand, the index C_a provides a quantified measure of the ability to cluster around the center, which alerts the user if the process mean deviates from its midpoint. For example, $C_a = 1$ indicates that the process is perfectly centered ($\mu = m$), $C_a = 0$ indicates that the process mean μ is located at one of the specification limits. Thus, when $0 < C_a < 1$, the process mean is located between the mid-point and one of the specification limits. Obviously, if $C_a < 0$ then it indicates that μ fall outside the specification limits (i.e. $\mu > USL$ or $\mu < LSL$), the process is severely off-center and it needs an immediate troubleshooting. Table 1 displays various C_a values and the corresponding ranges of the departure magnitude of μ.

Table 1. C_a values and ranges of μ.

C_a value	Range of μ
$C_a = 1.00$	$\mu = M$
$0.75 < C_a < 1.00$	$0 < \lvert \mu - M \rvert < d/4$
$0.67 < C_a < 0.75$	$d/4 < \lvert \mu - M \rvert < d/3$
$0.50 < C_a < 0.67$	$d/3 < \lvert \mu - M \rvert < d/2$
$0.25 < C_a < 0.50$	$d/2 < \lvert \mu - M \rvert < 3d/4$
$0.00 < C_a < 0.25$	$3d/4 < \lvert \mu - M \rvert < d$
$C_a = 0.00$	$\mu = LSL$ or $\mu = USL$
$C_a < 0.00$	$\mu < LSL$ or $\mu > USL$

We remark here that process capability approach can be used only if the manufacturing process is under statistical control. If the process is out of control in the early stages of process capability analysis, it will be unreliable and meaningless to estimate process capability. Proper understanding and accurate estimation of the capability index is essential for the company to maintain a capable supplier. The usual practice of judging process capability by evaluating the point estimates of process capability indices, have flaws since there is no assessment of the sampling errors. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions, learning that capability measures must be reported in confidence intervals or via capability testing (Chou *et al*. (1990), Kushler and Hurley (1992), Kotz *et al*. (1993), Nagata and Nagahata (1994), Vännman and Kotz (1995), Tang *et al*. (1997), Hoffman (2001), Zimmer *et al*. (2001), Pearn and Lin (2002)). Critical values are usually used for making decisions in capability testing with designated Type I error α, the risk of misjudging an incapable process as a capable one. The *p*-values present the actual risk of misjudging an incapable process as a capable one. That is, if *p*-value $< \alpha$ then we reject the null hypothesis, and conclude that the process is capable with actual Type I error α.

Therefore, in order to assess process performance and make decisions in manufacturing capability testing, the exact cumulative distribution functions need to be derived in advance. In this paper, we first give a brief introduction on process capability indices. In Section 2, we derive an explicit form for the cumulative distribution function and probability density function of the estimator $\hat{C}_a$. Subsequently, some distributional and inferential properties of the estimated process accuracy index C_a are provided. The calculations of critical value, *p*-value and lower confidence bound are developed for testing process quality. Furthermore, several implementation issues and extensions of cases with asymmetric tolerances are discussed in Section 3, and discussed decision making rule in Section 4. Practitioners can use the proposed results to perform quality testing and determine the process can reproduce product items to meet the specified quality requirement. For illustrative purpose, three real-world applications are presented in Section 5. Finally, some concluding remarks are made in Section 6.

2. Estimation of Process Capability Accuracy Index C_a for Single Sample

To estimate the accuracy index C_a, Pearn *et al*. (1998) considered the natural estimator $\hat{C}_a$ as the following, where $\bar{X}=\sum_{i=1}^{n} x_i / n$ is the conventional estimator of the process mean μ, which may be obtained from a stable process.

$$\hat{C}_a = 1-\frac{\left|\bar{X}-m\right|}{d}.$$

The estimator can be alternatively written as:

$$\hat{C}_a = 1-\frac{|\bar{X}-m|}{d} = 1-\frac{\sigma}{d\sqrt{n}}\times\frac{\sqrt{n}\,|\bar{X}-m|}{\sigma} = 1-\frac{1}{3\sqrt{n}\,C_p}\times\frac{\sqrt{n}\,|\bar{X}-m|}{\sigma}.$$

on the assumption of normality, the statistic $\sqrt{n}\,|\bar{X}-m|/\sigma$ has a folded normal distribution as defined by Leone *et al*. (1961). Thus, the probability density function (PDF) of $\hat{C}_a$ can be expressed as:

$$f_{\hat{C}_a}(x) = 6C_p\sqrt{\frac{n}{2\pi}}\cosh\left[9nC_p^2(1-C_a)(1-x)\right]\exp\left\{\frac{-9nC_p^2\left[(1-x)^2+(1-C_a)^2\right]}{2}\right\},$$

for $-\infty < x \le 1$ and $\cosh(t)=\left[\exp(t)+\exp(-t)\right]/2$ is a Hyperbolic function. The first two moments of $\hat{C}_a$ therefore can be calculated as follows (Pearn *et al*. (1998)), and the variance of $\hat{C}_a$ also can be obtained by calculating $Var(\hat{C}_a)=E(\hat{C}_a^2)-E^2(\hat{C}_a)$.

$$E(\hat{C}_a) = C_a - \frac{1}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left(\frac{-\delta_1}{2}\right) + 2(1-C_a)\Phi\left(-\sqrt{\delta_1}\right),$$

$$E(\hat{C}_a^2) = C_a^2 - \frac{1}{9nC_p^2} - \frac{2}{3C_p}\sqrt{\frac{2}{n\pi}}\exp\left(\frac{-\delta}{2}\right) + 4(1-C_a)\Phi\left(-\sqrt{\delta}\right),$$

where $\delta = 9n(C_p - C_{pk})^2$. The estimator $\hat{C}_a$ is biased, but as the sample size approaches to infinity, the following three terms, $[2/(n\pi)]^{1/2}$, $\exp(-\delta/2)$, and $\Phi(-\sqrt{\delta})$ all converge to zero. Therefore, the estimator $\hat{C}_a$ is asymptotically unbiased. For the percentage bias to be less than five percent (i.e. $|E(\hat{C}_a) - C_a|/C_a \le 0.05$) it required $n > 30$, and for one percent (i.e. $|E(\hat{C}_a) - C_a|/C_a \le 0.01$) it requires $n > 650$.

On the assumption of normality, Pearn *et al*. (1998) showed that the natural estimator $\hat{C}_a$ of the process accuracy index C_a, is the maximum likelihood estimator (MLE), consistent, asymptotically efficient estimator and $\sqrt{n}(\hat{C}_a - C_a)$ converges to $N(0, 1/(9C_p^2))$ in distribution. Therefore, owing to $\sqrt{n}(\hat{C}_a - C_a)$ converges to $N(0,1/(9C_p^2))$, then $3\sqrt{n}\tilde{C}_p(\hat{C}_a - C_a)$ converges to $N(0,1)$ in distribution. An approximate $100(1-\alpha)\%$ confidence interval of C_a can be established as the following:

$$\left[\hat{C}_a - \frac{Z_{\alpha/2}}{3\sqrt{n}\tilde{C}_p}, \hat{C}_a + \frac{Z_{\alpha/2}}{3\sqrt{n}\tilde{C}_p}\right]$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quartile for the standard normal distribution $N(0,1)$, $\tilde{C}_p = b_{n-1}\hat{C}_p$, $b_{n-1} = (2/(n-1))^{1/2} \times \Gamma[(n-1)/2]/\Gamma[(n-2)/2]$, and $\Gamma(k) = \int_0^\infty t^{k-1}\mathrm{e}^{-t}\,dt$ is a gamma function. While a $100(1-\alpha)\%$ lower confidence limit on C_a can be constructed by using only the lower limit as $\hat{C}_a - z_\alpha/(3\sqrt{n}\,\tilde{C}_p)$.

Subsequently, we now derive an explicit form for the cumulative distribution function (CDF) of the estimator $\hat{C}_a$ as follows. We first define $D = \sqrt{n}d/\sigma$, $Z' = \sqrt{n}(\bar{X} - m)/\sigma$ which is distributed as $N(\delta, 1)$, where $\delta = \sqrt{n}(\mu - m)/\sigma$, and $H = |Z'|$ which is a folded normal distribution. Then, the estimator $\hat{C}_a$ can be rewritten as $\hat{C}_a = 1 - H/D$. Thus,

$$F_{\hat{C}_a}(x) = P\left\{\hat{C}_a \le x\right\} = P\left\{1 - \frac{H}{D} \le x\right\} = 1 - P\left\{H \le D(1-x)\right\} = 1 - \int_0^{D(1-x)} f_H(h)dh \quad (1)$$

since $H = |Z'|$ is a folded normal distribution, we have

$$f_H(t) = \phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right), \quad \xi = (\mu - m)/\sigma. \quad (2)$$

Substituting Eq. (2) into Eq. (1) gives

$$F_{\hat{C}_a}(x) = 1 - \int_0^{D(1-x)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right) \right] dt, \text{ for } -\infty \le x \le 1. \tag{3}$$

By differential with respect to x gives the probability density function (PDF) of $\hat{C}_a$ as:

$$f_{\hat{C}_a}(x) = D\left[\phi\left(D(1-x) + \xi\sqrt{n}\right) + \phi\left(D(1-x) - \xi\sqrt{n}\right) \right], \text{ for } -\infty \le x \le 1.$$

Figures 1 and 2 display the PDF and CDF of $\hat{C}_a$ with C_a = 0.50, ξ = 1 for various sample sizes n = 25(25)100, respectively. Figures 3 and 4 display the PDF and CDF of $\hat{C}_a$ with C_a = 0.75, ξ = 1 for various sample sizes n = 25(25)100, respectively.

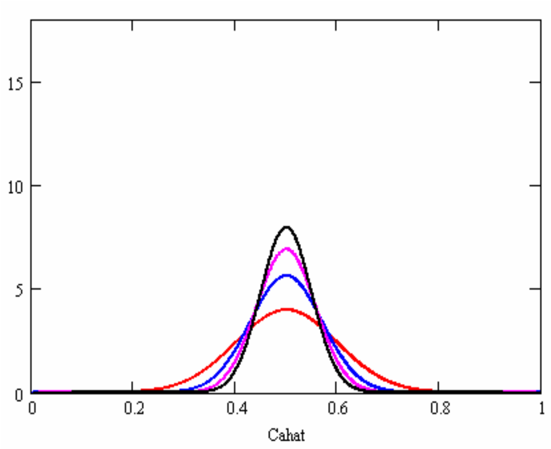

Figure 1. PDF plots of $\hat{C}_a$ with C_a = 0.50, ξ = 1 for n = 25(25)100 (from bottom to top)

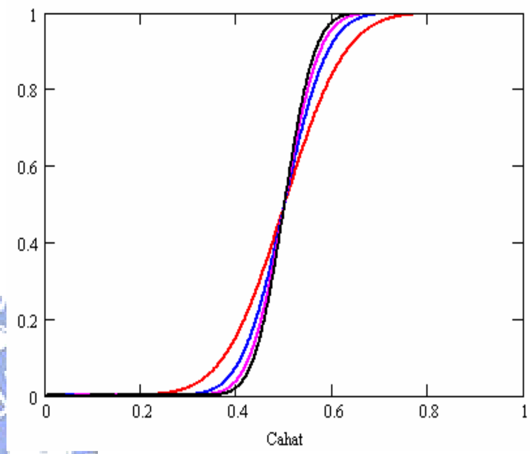

Figure 2. CDF plots of $\hat{C}_a$ with C_a = 0.50, ξ = 1 for n = 25(25)100 (when $\hat{C}_a$ >0.50, from bottom to top)

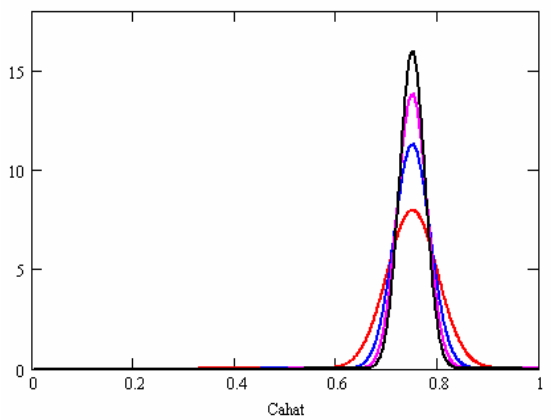

Figure 3. PDF plots of $\hat{C}_a$ with C_a = 0.75, ξ =1 for n = 25(25)100 (from bottom to top)

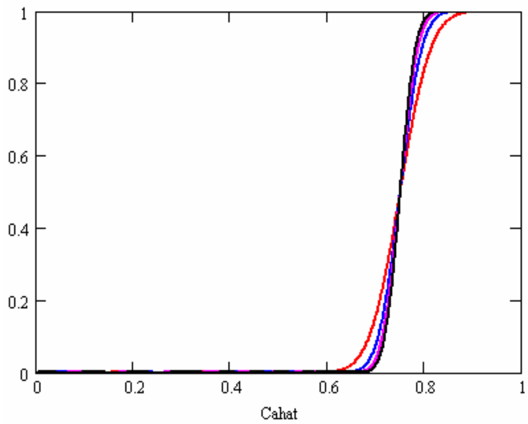

Figure 4. CDF plots of $\hat{C}_a$ with C_a = 0.75, ξ =1 for n = 25(25)100 (when $\hat{C}_a$ > 0.75, from bottom to top)

From these Figures, we can see that as the value of C_a increases, the spread of the distribution decreases. As the sample sizes n increase, the spread of the distribution decreases. Accordingly, in order to determine whether a given process capability meets the customers' demands and runs under the desired quality condition, the statistical hypothesis testing can be stated as follows:

H_0: $C_a \le C$ (process is inaccurate),

H_1: $C_a > C$ (process is accurate).

We will reject the null hypothesis H_0 ($C_a \le C$), when $\hat{C}_a > c_0$ with Type I error $\alpha(c_0) = \alpha$, the chance of incorrectly concluding an inaccurate process ($C_a \le C$) as accurate ($C_a > C$). Based on the CDF of $\hat{C}_a$ expressed in Eq. (3), given values of capability requirement C, parameter ξ, sample size n, and risk α, the critical value c_0 can be obtained by solving the equation $P(\hat{C}_a > c_0 \mid C_a = C) = \alpha$ using available numerical methods.

$$\int_0^{b\sqrt{n}(1-c_0)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right) \right] dt = \alpha . \tag{4}$$

where $b = d/\sigma$, $\xi = (\mu - m)/\sigma$. Given $C_a = C$, $b = d/\sigma$ can be expressed as $b = |\xi|/(1-C)$, the p-value corresponding to c^*, a specific value of $\hat{C}_a$ calculated from the sample data, is:

$$p-value = P(\hat{C}_a \ge c^* \mid C_a = C) = \int_0^{b\sqrt{n}(1-c^*)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right) \right] dt . \tag{5}$$

Furthermore, based on the CDF of $\hat{C}_a$ expressed in Eq. (3), the lower confidence limits which conveying critical information regarding the true process capability is also developed as below. In fact, given the sample of size n, the confidence level γ, the estimated value $\hat{C}_a$, and ξ, the lower confidence bounds C_a^L can be obtained using numerical integration technique with iterations, to solve the following Eq. (6) with $b = |\xi|/(1-C_a^L)$.

$$\int_0^{b\sqrt{n}\left(1-\hat{C}_a\right)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right) \right] dt = 1-\gamma . \tag{6}$$

Table 7. to Table 10. display the lower confidence bounds C_a^L of $\hat{C}_a$ = 0.75, for various parameter values, with α = 0.05, ξ = 1.0(0.1)2.0, and n = 10(10)100. Table 7. and Table 8. display the lower confidence bounds C_a^L of $\hat{C}_a$ = 0.50, for various parameter values, with α = 0.05, ξ = 1.0(0.1)2.0, and n = 10(10)100.

Accordingly, practitioners can use the proposed results to perform quality testing and determine the process can reproduce product items to meet the specified process precision requirement. Under the assumption of normality, we proved that the cumulative distribution function of $\hat{C}_a$ can be expressed in terms of a mixture of the normal distributions. Using the index C_a, the engineers can access the process performance and monitor the manufacturing processes on routine basis. However from the Eq. (2) to Eq. (6), the distribution characteristic parameter $\xi = (\mu - m)/\sigma$ is usually unknown, which has to be estimated in real applications, naturally by substituting μ and σ by the sample mean $\bar{X}$ and the sample variance $S^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 / (n-1)$. Such approach introduces additional sample errors from estimating ξ, and would be less reliable. Consequently, any decisions made would provide less quality assurance to the customers.

To eliminate the need for further estimating the characteristic parameter $\xi = (\mu - m)/\sigma$, we examine the sensitivity of critical value c_0 against the parameter ξ. Before testing the sensitivity of c_0 and ξ, we need to find the range of ξ. When C_a=0.25 and the $|\mu - m| = 3d/4$, then $\xi = 3d/4\sigma = 9/4C_p$. If the process is capable, the C_p will be required bigger than 1.33. Hence, we need to set $\xi \ge 3$ when C_a=0.25.

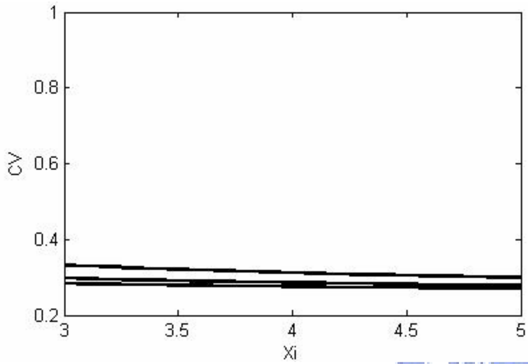

Figure 5. Plots of critical value vs $|\xi|$ = 3.0(0.1)5.0, for C_a=0.25, n = 25, 75 and 150, $\alpha = 0.05$ (from top to bottom).

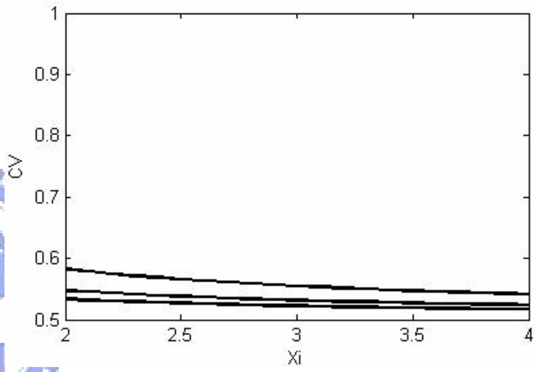

Figure 6. Plots of critical value vs $|\xi|$ = 2.0(0.1)4.0, for C_a=0.50, n = 25, 75 and 150, $\alpha = 0.05$ (from top to bottom).

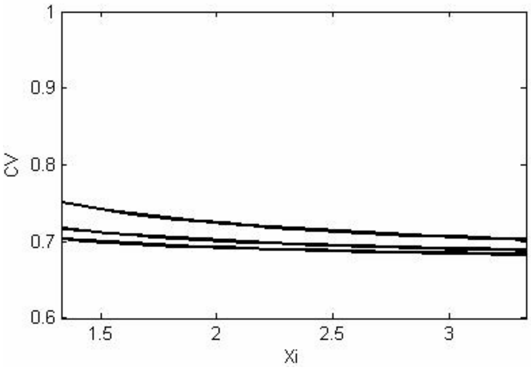

Figure 7. Plots of critical value vs $|\xi|$ = 1.33(0.1)3.33, for C_a=0.67, n = 25, 75 and 150, $\alpha = 0.05$ (from top to bottom).

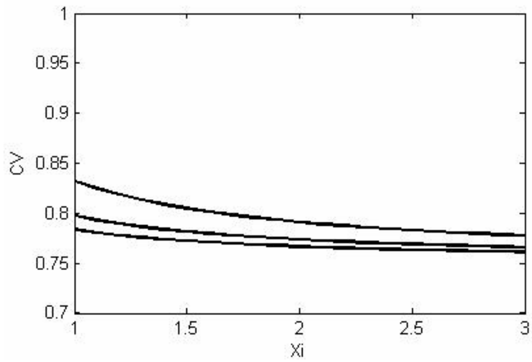

Figure 8. Plots of critical value vs $|\xi|$ = 1.0(0.1)3.0, for C_a=0.75, n = 25, 75 and 150, $\alpha = 0.05$ (from top to bottom).

Figure 5 to 8 plot the curves of the critical value c_0 versus the parameter ξ, n = 25, 75 and 150 with Type I error α = 0.05, for four levels of C_a = 0.25, 0.50, 0.67 and

0.75 in all cases with accuracy up to 10^{-4}, respectively. From these Figures, the results indicate that as the value of n increasing, the critical value c_0 is closed to C_a.

3. Estimation of Process Accuracy index C_a for Asymmetric Tolerances

A process is said to have a symmetric tolerance if the target value T is set to be the mid-point of the specification interval, i.e. $T = m = (USL + LSL)/2$. Most research in quality assurance literature has focus on cases in which the manufacturing tolerance is symmetric. Examples include Kane (1986), Chan *et al*. (1988), Boyles (1991), Pearn *et al*. (1992), Vännman (1995), Vännman and Kotz (1995), Spiring (1997), Hoffman (2001), Zimmer *et al*. (2001), Vännman and Hubele (2003) and many others. Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances ($T \neq m$) often occur in the manufacturing industry. From the customer's point of view, asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other (see Boyles (1994), Vännman (1997), and Wu and Tang (1998)). Usually they are not related to the shape of the supplier's process distribution. However, asymmetric tolerances can also arise in situations where the tolerances are symmetric to begin with, but the process distribution is skewed or follows a non-normal distribution. Dealing with this, the data have been transformed to achieve approximate normality, as shown by Chou *et al*. (1998) who have used Johnson curves to transform non-normal process data. Moreover, these indices presented above, are designed to monitor the performance for only normal and near-normal processes with symmetric tolerances, which are shown to be inappropriate for cases with asymmetric tolerances (Boyles (1994), Pearn and Chen (1998)). Unfortunately, there has been comparatively little research published on cases with asymmetric tolerances. Exceptions are Choi and Owen (1990), Boyles (1994), Vännman (1997), Chen (1998), Pearn and Chen (1998), Pearn *et al*. (1999), Chen *et al*. (1999), and Jessenberger and Weihs (2000).

To overcome these drawbacks, we modify C_a index for the asymmetric cases ($USL - T \neq T - LSL$), denoted here as C_a''.

$$C_a'' = 1 - \frac{A^*}{d^*},$$

where $A^* = Max\{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$, $d^* = min\{D_u, D_l\}$, $D_u = USL - T$, $D_l = T - LSL$. Obviously, if $T = m$ (symmetric tolerance), then $d^* = D_u = D_l = d$, $A^* = |\mu - m|$ and C_a'' reduces to the original index C_a. The factor A^* ensures that the new generalization C_a'' obtains its maximal value at $\mu = T$ (process is on-target) regardless of whether the tolerances are symmetric ($T = m$) or asymmetric ($T \neq m$).

Figure 3 displays the plots of C_a'' for processes with $10 \le \mu \le 50$ where $(LSL, T, USL) = (10, 40, 50)$ is an asymmetric tolerance. $C_a'' = 0$ can be verified when the process mean is on the specification limit ($\mu = LSL$ or $\mu = USL$). On the other hand, $C_a'' > 0$ when $LSL < \mu < USL$. Thus, given two processes E and F with $\mu_E > T$ and $\mu_F < T$, satisfying $(\mu_E - T)/D_u = (T - \mu_F)/D_l$ (i.e., processes E and F have equal departure ratio), the C_a'' values given to processes E and F are the same. For example, consider processes E and F with $\mu_E = 45 > T$ and $\mu_F = 25 < T$. Clearly, the corresponding departure ratios are 1/2 for both processes E and F (i.e. (45 – 40)/10 = (40 – 25)/30 = 1/2). We have $C_a'' = 0.50$ for both processes E and F (see Figure 3). In addition, the index C_a'' decreases when mean μ shifts away from target T in either direction. In fact, C_a'' decreases faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. This is an advantage since the index would respond faster to the shift towards "the wrong side" of T than towards the middle of the specification interval.

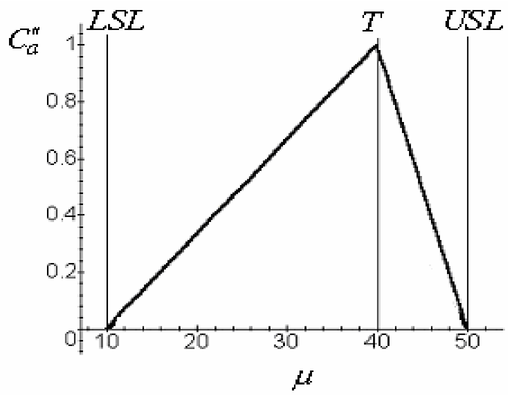

Figure 9. Plots of C_a'' values for processes with $10 \le \mu \le 50$ under $(LSL, T, USL) = (10, 40, 50)$.

To estimate the generalization C_a'', we consider the natural estimator $\hat{C}_a''$ defined as:

$$\hat{C}_a'' = 1 - \frac{\hat{A}^*}{d^*},$$

where $\hat{A}^* = Max\left\{d^*\left(\bar{X} - T\right)/D_u, d^*\left(T - \bar{X}\right)/D_l\right\}$ and $\bar{X} = \sum_{i=1}^{n} x_i/n$. We now define $D^* = n^{1/2}\left(d^*/\sigma\right)$, $Z = n^{1/2}\left(\bar{X} - T\right)/\sigma$, $Y = \left[Max\left\{\left(d^*/D_u\right)Z, -\left(d^*/D_l\right)Z\right\}\right]^2$ $\lambda = \delta^2$, and $\delta = n^{1/2}\left(\mu - T\right)/\sigma$. Then, the estimator $\hat{C}_a''$ can be rewritten as:

$$\hat{C}_a'' = 1 - \frac{\sqrt{Y}}{D^*}.$$

Under the assumption of normality, Z is distributed as the normal distribution $N(\delta, 1)$ with mean δ and variance 1. We note that the statistic Z^2 follows a non-central chi-square distribution with one degree of freedom and non-centrality parameter $\lambda = \delta^2$. Chen (1998) defined the distribution of Y as a weighted non-central chi-square distribution with one degree of freedom and non-centrality parameter λ under the assumption of normality. Chen (1998) also derived the probability density function of Y as:

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\ \Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2}\ f_{Y_j}\left(y/d_i^2\right) \right\},\quad y>0. \tag{7}$$

where $\lambda = \delta^2$, $\delta = n^{1/2}(\mu - T)/\sigma$, $h_j(\lambda) = (2\lambda)^{j/2}/(j!)$, $d_1 = d^*/D_l$, $d_2 = d^*/D_u$ and Y_j is distributed as χ^2_{1+j}. Thus, the cumulative distribution function of $\hat{C}''_a$ can be derived as follows:

$$F_{\hat{C}''_a}(x) = P\left\{\hat{C}''_a \le x\right\} = P\left\{1 - \frac{\sqrt{Y}}{D^*} \le x\right\} = 1 - P\left(Y \le \left[D(1-x)\right]^2\right)$$

$$= 1 - \int_0^{[D^*(1-x)]^2} f_Y(y)dy \text{, for } x<1. \tag{8}$$

by substituting Eq. (7) into Eq. (8) gives

$$F_{\hat{C}''_a}(x) = 1 - \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-x)]^2} f_{Y_j}\left(y/d_i^2\right)dy \right\}\text{, for } x<1. \tag{9}$$

Changing the variable with $t = y/d_i^2$, then $dy = d_i^2\,dt$ for $i = 1, 2$, and the cumulative distribution function of $\hat{C}''_a$ can be rewritten as Eq. (10).

$$F_{\hat{C}''_a}(x) = \begin{cases} 1 - \dfrac{e^{-\lambda/2}}{2\sqrt{\pi}} \displaystyle\sum_{j=0}^{\infty} h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} (-1)^{ij} \int_0^{[D^*(1-x)/d_i]^2} f_{Y_j}(t)dt, & x<1 \\ 1, & x \ge 1 \end{cases} \tag{10}$$

Taking the derivative of the cumulative distribution function of $\hat{C}''_a$ in x, the probability density function of $\hat{C}''_a$ will be obtained as expressed in Eq. (11).

$$f_{\hat{C}''_a}(x) = \frac{e^{-\lambda/2}}{\sqrt{\pi}} \sum_{j=0}^{\infty} h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} (-1)^{ij} f_{Y_j}\left([(1-x)D^*/d_i]^2\right)\left[(1-x)(D^*/d_i)^2\right],$$

$$x<1. \tag{11}$$

If the manufacturing tolerance is symmetric ($T = m$), then $d^* = D_u = D_l = d$, $d_1 = d_2 = 1$, and the cumulative distribution function of $\hat{C}''_a$ in Eq. (9) reduces to:

$$F_{\hat{C}''_a}(x) = \begin{cases} 1 - \displaystyle\sum_{\ell=0}^{\infty} P_\ell(\lambda) \int_0^{[D(1-x)]^2} f_{Y_{2\ell}}(t)dt, & x<1 \\ 1, & x \ge 1 \end{cases} \tag{12}$$

and the corresponding probability density function is:

$$f_{\hat{C}_a''}(x) = \sum_{\ell=0}^{\infty} P_\ell(\lambda) \frac{D^{2\ell+1}(1-x)^{2\ell}}{\Gamma\left((2\ell+1)/2\right) 2^{(2\ell-1)/2}} e^{-[D(1-x)]^2/2}, \quad x < 1 \tag{13}$$

where $D = n^{1/2}(d/\sigma)$ and $P_\ell(\lambda) = e^{-\lambda/2}(\lambda/2)^\ell/(\ell!)$.

As the same as symmetric tolerances, we also display the CDF and PDF of $\hat{C}_a''$. For an asymmetric case ($D_l : D_u = 6:4$), depending to the Eq. (11), Figures 10 and 11 display the PDF and CDF of $\hat{C}_a''$ with $C_a'' = 0.75$, $\xi = 0.5$ for various sample sizes $n = 10$, 20 and 50, respectively. Figures 12 and 13 display the PDF and CDF of $\hat{C}_a''$ with $C_a'' = 0.50$, $\xi = 1.0$ for various sample sizes $n = 10$, 20 and 50, respectively. From these figures, we observe that as n increases the spread decreases and so does the skewness. We also observe that the estimated index $\hat{C}_a''$ is approximately unbiased for sample size $n > 50$.

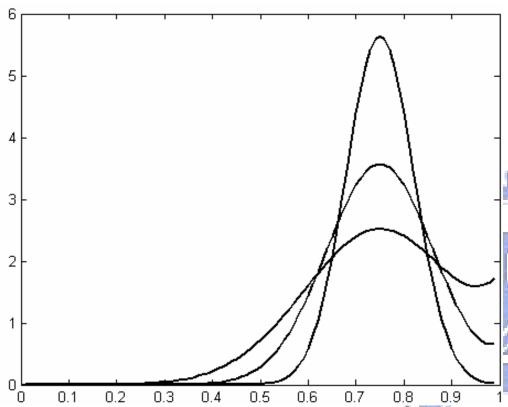

Figure 10. The PDF plots of $\hat{C}_a''$ with $C_a = 0.75$, $D_l : D_u = 6:4$, $\xi = 1.0$ for n = 10, 20, and 50 (from bottom to top).

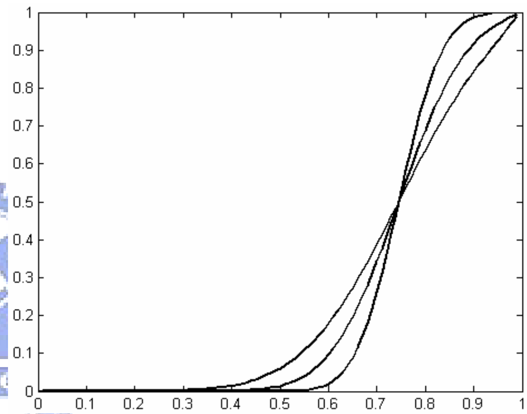

Figure 11. The CDF plots of $\hat{C}_a''$ with $C_a''=0.75$, $D_l : D_u = 6:4$, $\xi = 0.5$ for n = 10, 20, and 50 (when $\hat{C}_a'' > 0.75$, from bottom to top).

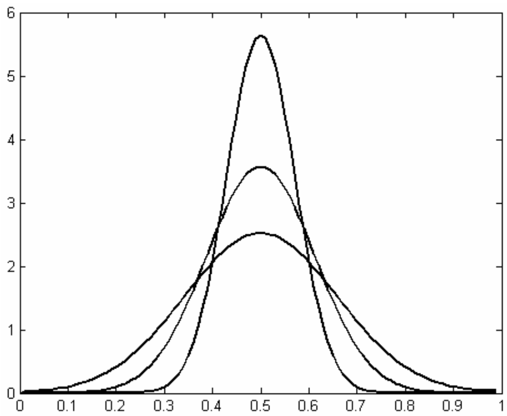

Figure 12. The PDF plots of $\hat{C}_a''$ with $C_a'' = 0.50$, $D_l : D_u = 6:4$, $\xi = 1.0$ for n = 10, 20, and 50 (from bottom to top).

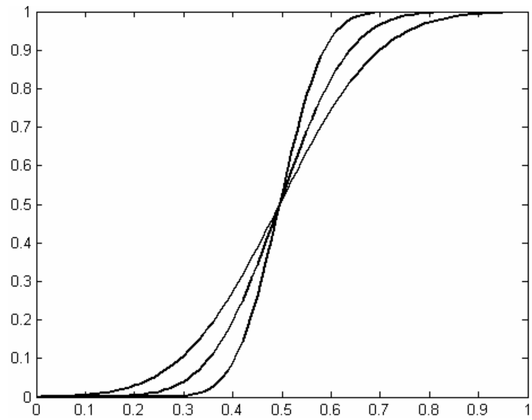

Figure 13. The CDF plots of $\hat{C}_a''$ with $C_a'' = 0.50$, $D_l : D_u = 6:4$, $\xi = 1.0$ for n = 10, 20, and 50 (when $\hat{C}_a'' > 0.50$, from bottom to top).

Moreover, the r-th moment of $\hat{C}''_a$ may be obtained as :

$$E\left(\hat{C}''_a\right)^r = \sum_{j=0}^{r} \frac{r!}{j!(r-j)!}\left(\frac{-\sigma}{\sqrt{n}}\right)^{r-j} E\left(Max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right)^{r-j}. \tag{14}$$

To obtain the expected value and variance of $\hat{C}''_a$, we first need to calculate the following (see Pearn and Chen (1998)):

$$E\left(Max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right)$$

$$= \left(\frac{1}{D_u} + \frac{1}{D_l}\right)\frac{e^{-\lambda/2}}{\sqrt{2\pi}} + Max\left\{\frac{\delta}{D_u}, \frac{-\delta}{D_l}\right\}\left[1-2\Phi(-|\delta|)\right] + \left(\frac{\delta}{D_u} - \frac{\delta}{D_l}\right)\Phi(-|\delta|), \tag{15}$$

$$E\left(Max^2\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{D_u^2} + \frac{1}{D_l^2}\right) + \left(\frac{\lambda}{D_u^2} + \frac{\lambda}{D_l^2}\right)\Phi(-|\delta|) + \left(\frac{1}{D_u^2} - \frac{1}{D_l^2}\right)\left\{\frac{\delta e^{-\lambda/2}}{\sqrt{2\pi}} + \frac{\delta}{2|\delta|}\left[1-2\Phi(-|\delta|)\right]\right\}$$

$$+ Max^2\left\{\frac{\delta}{D_u}, \frac{-\delta}{D_l}\right\}\left[1-2\Phi(-|\delta|)\right]. \tag{16}$$

hence, we have:

$$E\left(\hat{C}''_a\right) = 1 - \frac{\sigma}{\sqrt{n}} E\left(Max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right), \quad Var\left(\hat{C}''_a\right) = E\left(\hat{C}''_a\right)^2 - E^2\left(\hat{C}''_a\right),$$

where $E\left(\hat{C}''_a\right)^2 = 1 - 2\frac{\sigma}{\sqrt{n}} E\left(Max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right) + \frac{\sigma^2}{n} E\left(Max^2\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right)$.

noting that $\frac{\sigma}{\sqrt{n}} Max\left\{\frac{\delta}{D_u}, \frac{-\delta}{D_l}\right\} = 1 - C''_a$, we have

$$E\left(\hat{C}''_a\right) = C''_a - \frac{1}{3C_p^*}\left(\frac{d^*}{D_u} + \frac{d^*}{D_l}\right)\frac{e^{-\lambda/2}}{\sqrt{2n\pi}} + 2(1-C''_a)\Phi(-|\delta|) - \frac{\sigma}{\sqrt{n}}\left(\frac{\delta}{D_u} - \frac{\delta}{D_l}\right)\Phi(-|\delta|),$$

$$E\left(\hat{C}''_a\right)^2 = (C''_a)^2 + \frac{1}{2n(3C_p^*)^2}\left(\left(\frac{d^*}{D_u}\right)^2 + \left(\frac{d^*}{D_l}\right)^2\right) - \frac{2}{3C_p^*}\left(\frac{d^*}{D_u} + \frac{d^*}{D_l}\right)\frac{e^{-\lambda/2}}{\sqrt{2n\pi}}$$

$$+4(1-C''_a)\Phi\left(-|\delta|\right)+\left\{\frac{1}{n(3C_p^*)^2}\left(\left(\frac{d^*}{D_u}\right)^2+\left(\frac{d^*}{D_l}\right)^2\right)-2(1-C''_a)^2\right\}\Phi\left(-|\delta|\right)$$

$$+\frac{1}{n(3C_p^*)^2}\left(\left(\frac{d^*}{D_u}\right)^2-\left(\frac{d^*}{D_l}\right)^2\right)\left\{\frac{\sigma e^{-\lambda/2}}{\sqrt{2\pi}}+\frac{\delta}{2|\delta|}\left[1-2\Phi\left(-|\delta|\right)\right]\right\}$$

$$-2\frac{\sigma}{\sqrt{n}}\left(\frac{\delta}{D_u}-\frac{\delta}{D_l}\right)\Phi\left(-|\delta|\right).$$

where $C_p^*=d^*/3\sigma$. Additionally, if the manufacturing tolerance is symmetric ($T=m$), then $E\left(\hat{C}_a''\right)$, $E\left(\hat{C}_a''\right)^2$ and $Var\left(\hat{C}_a''\right)$ can be reduced to the results obtained in Pearn *et al.* (1998). We note that the estimator $\hat{C}''_a$ is biased. The magnitude of the bias is $B\left(\hat{C}''_a\right)=E\left(\hat{C}''_a\right)-C''_a$. The mean square error can be expressed as $MSE\left(\hat{C}''_a\right) =Var\left(\hat{C}''_a\right)+B^2\left(\hat{C}''_a\right)$. To investigate the behavior of the estimator $\hat{C}''_a$, the bias and the mean square error are calculated for various values of $\xi=\left(\mu-T\right)/\sigma$, $b=d^*/\sigma$, $d_l=d/D_l$, $d_u=d/D_u$, and sample size *n*.

Form Table 3, 4, 5 and 6(in Appendix), we observe that as the sample size *n* increases, both the bias and the mean square error of $\hat{C}''_a$ decrease. Figure 15 displays the plot of the *MSE* of $\hat{C}''_a$ (vs. *n*) with $\xi=$ 0.5, 0 and 1.0 (from bottom to top) for fixed *b* = 2 and $D_l:D_u=6:4$. Figure 16 displays the plot of the *MSE* of $\hat{C}''_a$ (vs. *n*) with b = 2, 3, 4 and 5 (from top to bottom) for fixed $\xi=0.5$. By the way, we observe that as the value of b increases, the *MSE* of $\hat{C}''_a$ decreases. Proper sample sizes for capability estimation are essential. The smaller the sample size is, the higher the value of $\hat{C}''_a$ is required to justify the true process capability.

4. Decision Making Rule

Using the index C_a'', the engineers can access the process performance and monitor the manufacturing processes on routine basis. To obtain a decision making rule we consider a testing hypothesis with the null hypothesis $C''_a\le C$ (the process is inaccurate) and the alternative hypothesis $C''_a>C$ (the process accurate). The null hypothesis will be rejected if $\hat{C}''_a>c_0$, where the constant c_0, called the critical value, is determined so that the significance level of the test is α, i.e., $\mathrm{P}\left(\hat{C}_a''>c_0\,|\,C_a''=C\right)=\alpha$. The decision making rule to be used is then that, for given values of risk α and sample size n, the process will be considered accurate if $\hat{C}''_a>c_0$ and inaccurate if $\hat{C}''_a\le c_0$.

We note that by setting $\xi=\left(\mu-T\right)/\sigma$ and $b=d^*/\sigma$, the index C_a'' can be rewritten as $C''_a=\left[b+\xi/\max\left\{1,r\right\}\right]/b$ for $\xi<0$ and $C''_a=\left[b-\xi\min\left\{1,r\right\}\right]/b$ for

$\xi \geq 0$ where $r = D_l / D_u$. Hence, the value of C_a'' can be calculated given values of ξ, b, and r. For example, if (ξ, b, r) = (-1, 3, 3/2) then C_a'' =[3+(-1)/max{1,3/2}]/3 = 0.7778. If $C_a'' = C$, we have $b = (-1)\left[\xi / \max\{1, r\}\right] / (1 - C)$ for $\xi < 0$ and $b = \left[\xi \min\{1, r\}\right] / (1 - C)$ for $\xi \geq 0$. In addition, since $D^* = n^{1/2}\left(d^* / \sigma\right)$ and $b = d^* / \sigma$ then $B^2 = nb^2$. Therefore, if $C_a'' = C$ then

$$D^{*2} = \begin{cases} n\left(\left[\xi / \max\{1, r\}\right] / (1 - C)\right)^2, & \xi < 0 \\ n\left(\left[\xi \min\{1, r\}\right] / (1 - C)\right)^2, & \xi \geq 0. \end{cases} \tag{17}$$

We can use the central chi-square distribution and the normal distribution to find the critical value c_0 satisfying $\mathrm{P}\left(\hat{C}_a'' > c_0 \mid C_a'' = C\right) = \alpha$, i.e., $1 - F_{\hat{C}_a}(c_0) = \alpha$ given $C_a'' = C$. We note that c_0 is larger than zero in general, hence we can find c_0 by Eq. (10).

$$\mathrm{P}\left(\hat{C}_a'' > c_0 \mid C_a'' = C\right)$$

$$= \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda) \Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-c_\alpha)]^2} f_{Y_j}\left(y / d_i^2\right) dy \right\} = \alpha \tag{18}$$

where $\lambda = \delta^2$, $\delta = \sqrt{n}(\mu - T)/\sigma$, $h_j(\lambda) = (2\lambda)^{j/2} / (j!)$, $d_1 = d^* / D_l$, $d_2 = d^* / D_u$. We point out that if $T = m$ (symmetric tolerances) then $\hat{C}_a''$ reduces to $\hat{C}_a$. Table 11 (in Appendix) displays the critical values c_0 for d/σ = 2, 3, 4 and 5 with sample sizes n =10, 25(25)150, $|\xi|$ = 0.5(0.5)1.5 and α = 0.01 and 0.05 for $T = m$. Table 12 (in Appendix) displays the critical values c_0 for d/σ = 2, 3, 4 and 5 with sample sizes n =10, 25, 50 and 75, $|\xi|$ = 0.5(0.5)1.5 and α = 0.01 and 0.05 for $T \neq m$ ($D_l : D_u = 6:4$).

To test if the process meets the precision (quality) requirement, we first determine the value of C and α-risk. Since both the process parameters μ and σ are unknown, then parameter $\xi = (\mu - T)/\sigma$ is also unknown. But we can estimate ξ by calculating the value $\hat{\xi} = \left(\bar{X} - T\right)/S$ from the sample. If the estimated value $\hat{C}_a''$ is larger then the critical value c_0 $(C_a'' > c_0)$, then we conclude that the process meets the precision capability requirement $(C_a'' > C)$. Otherwise, we do not have sufficient information to conclude that the process meets the present precision requirement. In this case, we would believe that $C_a'' \leq C$ (the process is inaccurate). In order to make the practitioners easily to perform the process accuracy testing, we show the Table 13 to Table 28 (In Appendix) for critical value of the asymmetric cases ($D_l : D_u = 7:3$, $D_l : D_u = 6:4$, $D_l : D_u = 4:6$ and $D_l : D_u = 3:7$) with C_a'' = 0.75, 0.67, 0.50 and 0.25, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(1.0)1.0$.

We also can calculate the *p*-value, i.e. the probability that $\hat{C}_a''$ exceed the observed estimated index given the values of *C*, $\xi = (\mu - T)/\sigma$, and sample size *n*, and then compare this probability with the significance level α. If the estimated index value is c^*, given the values of C, ξ and sample size n, then the p-value can be calculated as:

$$p\text{-value} = \mathrm{P}\left(\hat{C}_a'' > c^* \mid C_a'' = C\right)$$

$$= \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda) \Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-c^*)]^2} f_{Y_j}\left(y/d_i^2\right) dy \right\}, \tag{19}$$

If the p-value is small than the α-risk, than we conclude that the process meets the precision requirement $\left(C_a'' > C\right)$. Otherwise, we do not have sufficient information to conclude that the process meets the present precision requirement. In this case, we would believe that $C_a'' \le C$ (the process is inaccurate).

Furthermore, based on the CDF of $\hat{C}_a''$ expressed in Eq. (9), the lower confidence bounds which conveying critical information regarding the true process capability level γ, the estimated value $\hat{C}_a''$ and ξ, the lower confidence bounds $C_a^{L''}$ can be obtained using numerical integration technique with iterations, to solve the following Eq. (9) with $b = |\xi| \Big/ \left(1 - C_a^{L''}\right)$.

$$\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda) \Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-\hat{C}_a'')]^2} f_{Y_j}\left(y/d_i^2\right) dy \right\} = 1 - \gamma \tag{20}$$

Table 29-32 (In Appendix) display the lower confidence bounds for the asymmetric cases ($D_l : D_u = 7:3$, $D_l : D_u = 6:4$, $D_l : D_u = 4:6$ and $D_l : D_u = 3:7$) with $\hat{C}_a''$ = 0.75, 0.67, 0.50 and 0.25, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $\xi = 0.1(1.0)1.0$.

Accordingly, practitioners can use the proposed results to perform quality testing and determine the process can reproduce product items to meet the specified process precision requirement.

5. Illustrative Examples

In the following three examples, we use the critical value, p-value and lower confidence bounds to test if the process is accurate.

5.1 Example I of Testing Plastic Process for Precision

In the following, we consider an example of manufacturing the plastic taken from the factory located on the manufacture-based Industrial Park, Taiwan. The plastics company sells its product to other manufacturers to be used in making such consumer products as automobile bumpers and pens. One of the quality characteristics of plastic is the melt flow (g/min) at $200^0 F$. This measures the viscosity of the liquid. The plastic is poured into molds to make razors, pens, disposable lighters, and many other familiar products. The melt flow indicates how the plastic will fill the molds.

The plastics company monitors the process that manufactures the plastic by regularly measuring the melt flow. The company that purchases the plastic is also concerned about the melt flow of the material it is purchasing and so it checks a sample of the raw material. By analyzing the data, the company determines whether the plastic is ready for sale. If the plastic does not have the correct melt flow, then air bubbles may form when the plastic is poured into a mold, causing under-filling or over-filling of the mold.

With active efficient quality control effort and effective improvement plan implemented, the manufacturing process of plastic is justified to be in statistical control and follows a near-normal distribution by checking the histogram, normal probability plot, and Shapiro-Wilk test. The plastic is considered to be acceptable if the melt flow falls within the manufacturing specification limits. That is, the melt flow must fall between $9.6\ g/\min$ and $10.4\ g/\min$, i.e. $(LSL, T, USL) =$ (9.6, 10, 10.4). Since process accuracy is an important criterion set to $C_a \geq 0.75$ to ensure good process centering. A sample of $n =$ 100 is taken from the factory, with the calculated $\hat{C}_a =$ 0.837. The two process characteristic parameters, $\hat{\xi} = \left|\bar{X} - m\right|/S$, and d/S, are also calculated from the sample, to find the corresponding critical value $c_0 =$ 0.791 for $\alpha =$ 0.05 (the risk of wrongly concluding an inaccurate process as accurate). Since $\hat{C}_a >$ 0.791, the process is concluded satisfying the precision requirement. Thus, with 95% confidence we believe $C_a \geq 0.75$ (or $|\mu - m| < d/4$), and no further accuracy improvement action needs to be taken at this time. Subsequently, we calculate the lower confidence bounds $C_a^L = 0.8049 > C_a$ and conclude that the true value of the process accuracy index C_a is no less than 0.8049 with 95% level of confidence.

5.2 Example II of Testing the Laser Marking on IC Packages

Integrated circuit (IC) packages are used for encapsulating the chips and connections against environmental, electrical and electromagnetic effects, and for establishing thermal path from the semi-conductor junction to the environment and electrical connection from the chip to the printed circuit board (PCB). In mask marking of IC packages, the marking area is determined by the output power of the laser, the cross-section of the beam and the properties of the material being marked. Marked area of 75 mm^2 at 10 J/pulse with pulse duration of about 0.69 ms has been reported for uncoated plastic packages. For the coated plastic packages (e.g. marker ink or varnish), the marked area can reach 600 mm^2 per pulse with pulse energy ranging from 0.1 J to 0.75 J and a pulse repetition frequency of 30 Hz. For mask marking, a minimum character height of about 0.5 mm has been reported.

The quality of a mark is assessed by its legibility characteristics such as mark contrast, mark with, mark depth, spattering, and microcracks. The characteristics are usually evaluated using complementary techniques such as optical microscopy, ultrosonics microscopy, electron microscopy, surface roughness measurement, and contrast evaluation devices. The acceptance of level of each of these characteristics generally depends on the manufacturer's requirements. Mark width refers to the width of the line segment that forms a character. With the mask image marking, the mark width in the characters is essentially determined by the mask geometry and the lens imaging quality. It can be as small as a few micro-meters, which can only be read under a microscope. In beam deflected marking, the line width is mainly determined by the focused beam spot size, which varies between 20-100 mm.

The IC packages company monitors the process that laser marking on IC packages by measuring the mark width. According to the customer's requirement, this company considers the following normally distributed process with asymmetric tolerances LSL = 20 mm, T = 26.5 mm and USL = 32 mm for the mark width and the α-risk = 0.05. We calculate that $D_l = T - LSL =$ 6.5, $D_u = USL - T =$ 5.5, $d^* = \min\{D_l, D_u\} =$ 5.5. The sample of n = 100, the sample mean $\bar{X}$ = 27.35 and the sample standard deviation S = 2.0. We can calculate $\hat{A}^* = Max\left\{d^*\left(\bar{X} - T\right)/D_u, d^*\left(T - \bar{X}\right)/D_l\right\}$ = 0.85, $\hat{\xi} = \left(\bar{X} - T\right)/S$ = 0.425 and $\hat{C}_a''$ = 0.845. To test if the process meets the capability requirement, we define a process with $C_a'' \geq 0.75$ is accurate. We find the corresponding critical value c_0 = 0.791 for α = 0.05. Since $\hat{C}_a''$ > 0.791, the process is concluded satisfying the precision requirement. Thus, with 95% confidence we believe $C_a'' \geq 0.75$

We find the corresponding p-value is 0.0532 from calculating Eq. (19). Because the α-risk = 0.05 is small than 0.0532, we do not have sufficient information to conclude that the process meets the present precision requirement. Subsequently, the lower confidence bounds $C_a^{L''}$ = 0.593, so we can conclude that the true value of the process accuracy index C_a'' is no less than 0.593 with 95% level of confidence.

5.3 Example III of Testing the Length of Steel Meter Sticks

The iron and steel industry manufactures many kinds of product such as plate products, steel stick and wire rod products, hot-rolled products, cold-rolled products and laminated products. Then, the steel manufacturing process is composed of raw material preparation, ironmaking, steelmaking, and rolling. For the production of the steel stick mill, depicted in Figure 14, begins at the heating of billets in the reheating furnace, and then the billets rolled through rough mill, intermediate mill, finish mill and 3-roller type reducing and sizing mill to the products size. If for coil stick products then formed by pouring reel to become coil stick products. If for straight sticks then cut by flying shear, cooled in a cooling bed and sheared, compacted to bundles.

For a steel factory producing steel meter sticks, the upper specification limit, *USL* for the length of the steel meter sticks is set to 1001 mm, the lower specification limit, *LSL* of the length of the steel meter sticks is set to 999 mm and the target value is set to T = 1000 mm. Hence, a meter stick outside these specification limit is considered as nonconforming, or defective. We have collected a total of 100 observations and subtracted 1000 mm from our data to obtain a deviation from target measure. These results are shown in Table 2. After calculating from the sample data, we obtain the sample mean $\bar{X} = 0.1495$, and the sample standard deviation $S = 0.3603$.

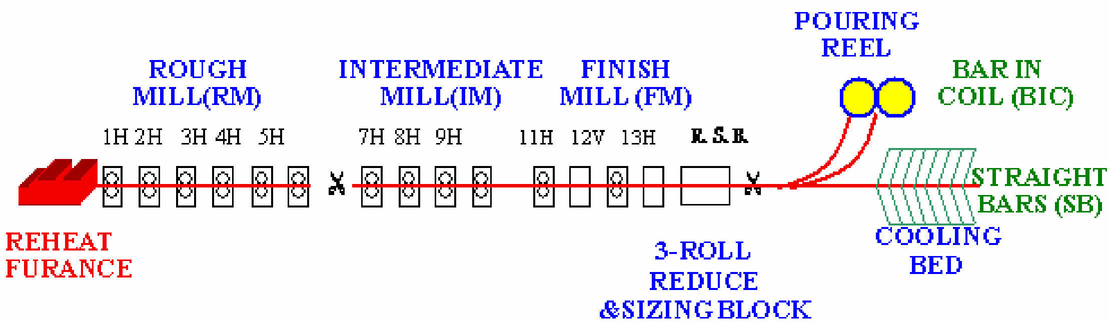

Figure 14. The manufacturing flow chart of the steel stick

Table 2. The sample data of the steel meter sticks deviation from target measure

0.14	-0.09	0.26	-0.06	0.32	-0.14	-0.06	-0.74	-0.44	0.27
0.49	-0.26	0.22	-0.06	0.18	-0.21	-0.08	-0.06	-0.45	0.11
-0.54	0.04	0.32	-0.08	-0.03	0.45	-0.12	0.09	0.20	0.28
-0.04	-0.52	-0.30	0.25	-0.05	0.18	0.31	-0.11	-0.12	0.03
0.28	-0.05	0.63	0.20	-0.15	-0.04	0.46	-0.19	-0.02	-0.25
0.59	0.28	-0.51	0.34	0.50	0.38	0.25	0.48	0.69	0.73
0.46	0.29	0.49	-0.62	0.27	0.60	-0.42	0.47	0.38	0.19
0.84	0.44	0.12	0.46	-0.41	0.29	0.51	0.28	0.62	0.38
0.44	0.56	0.69	0.63	0.42	0.22	0.35	0.32	0.50	0.59
-0.39	0.54	0.48	0.39	0.60	-0.45	-0.45	0.16	-0.67	0.20

Since process accuracy is an important criterion set to $C_a \geq 0.75$ to ensure good process centering. The two process characteristic parameters, $\hat{\xi} = |\bar{X} - m|/S$ =0.4149 and $d/S = 2.775$, and the estimator of process accuracy index, $\hat{C}_a = 0.8505$, are calculated from the sample. Hence, we find the corresponding critical value c_0 = 0.8491 for $\alpha = 0.05$. Since $\hat{C}_a$ >0.8491, the process is considered satisfying the accuracy requirement. Furthermore, following the calculation of *p*-value =

$0.0477 < \alpha = 0.05$ and the lower confidence bounds C_a^L = 0.7524, we have sufficient information to conclude that this process is accurate and the true value of the process accuracy index C_a is no less than 0.7524 with 95% level of confidence.

6. Conclusions

The idea of measuring process capability is not new, but the use of indices to communicate information about processes and the specified requirements has become widespread. In this paper, to measure the degree of process centering (the ability to cluster around the center), the process accuracy index C_a is investigated. An explicit form for the cumulative distribution function of the estimator $\hat{C}_a$ is derived. Subsequently, the distributional and inferential properties of the estimated process accuracy index C_a are provided. The calculations of critical value, *p*-value and lower confidence bound are developed for testing process quality. Furthermore, a new generalization of C_a for cases with asymmetric tolerances is proposed to measure the process accuracy, and the distributional properties of the new estimator are investigated. The results obtained are useful for the practitioners in performing the process accuracy testing, and decisions making rule therefore can be obtained on whether actions should be taken to improve the process quality. For illustrative purpose, three application examples are given to show how to test process accuracy using the actual data collected from the factory.

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Appendix

Table 3 The values of C''_a, $B\left(\hat{C}''_a\right)$ and $MSE\left(\hat{C}''_a\right)$ for b = 2, ξ = -1.0(0.5)1.0, $D_l : D_u = 6:4$, and n = 10(10)50

n	$\xi=-1.0$		$\xi=-0.5$		$\xi=0.0$		$\xi=0.5$		$\xi=1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	-0.0001	0.0245	-0.0064	0.0128	-0.1051	0.0250	-0.0064	0.0055	-0.0001	0.0109
20	-0.0000	0.0125	-0.0008	0.0102	-0.0743	0.0125	-0.0008	0.0044	-0.0000	0.0056
30	-0.0000	0.0083	-0.0001	0.0078	-0.0607	0.0083	-0.0001	0.0034	-0.0000	0.0037
40	-0.0000	0.0062	-0.0000	0.0061	-0.0526	0.0065	0.0000	0.0027	-0.0000	0.0028
50	-0.0000	0.0050	-0.0000	0.0050	-0.0470	0.0050	0.0000	0.0022	-0.0000	0.0022
C_a''	0.5000		0.7500		1.0000		0.8333		0.6667	

Table 4. The values of C''_a, $B\left(\hat{C}''_a\right)$ and $MSE\left(\hat{C}''_a\right)$ for b = 3, ξ = -1.0(0.5)1.0, $D_l : D_u = 6:4$, and n = 10(10)50

n	$\xi=-1.0$		$\xi=-0.5$		$\xi=0.0$		$\xi=0.5$		$\xi=1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0000	0.0109	-0.0043	0.0061	-0.0701	0.0128	-0.0043	0.0029	0.0000	0.0048
20	0.0000	0.0056	-0.0005	0.0046	-0.0496	0.0064	-0.0005	0.0020	0.0000	0.0025
30	0.0000	0.0037	-0.0001	0.0035	-0.0405	0.0043	-0.0001	0.0015	0.0000	0.0016
40	0.0000	0.0028	0.0000	0.0027	-0.0350	0.0032	0.0000	0.0012	0.0000	0.0012
50	0.0000	0.0022	0.0000	0.0022	-0.0313	0.0026	0.0000	0.0010	0.0000	0.0010
C_a''	0.6667		0.8333		1.0000		0.8889		0.7778	

Table 5. The values of C''_a, $B\left(\hat{C}''_a\right)$ and $MSE\left(\hat{C}''_a\right)$ for b = 4, ξ = -1.0(0.5)1.0, $D_l : D_u = 6:4$, and n = 10(10)50

n	$\xi=-1.0$		$\xi=-0.5$		$\xi=0.0$		$\xi=0.5$		$\xi=1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0000	0.0061	-0.0032	0.0036	-0.0526	0.0076	-0.0032	0.0018	0.0000	0.0027
20	0.0000	0.0031	-0.0004	0.0026	-0.0372	0.0038	-0.0004	0.0012	0.0000	0.0014
30	0.0000	0.0021	-0.0001	0.0020	-0.0303	0.0025	-0.0001	0.0009	0.0000	0.0009
40	0.0000	0.0016	0.0000	0.0015	-0.0263	0.0019	0.0000	0.0007	0.0000	0.0007
50	0.0000	0.0012	0.0000	0.0012	-0.0235	0.0015	0.0000	0.0006	0.0000	0.0006
C_a''	0.7500		0.8750		1.0000		0.9167		0.8333	

Table 6. The values of C_a'', $B\left(\hat{C}_a''\right)$ and $MSE\left(\hat{C}_a''\right)$ for b = 5, ξ = -1.0(0.5)1.0, $D_l : D_u = 6:4$, and n = 10(10)50

n	$\xi = -1.0$		$\xi = -0.5$		$\xi = 0.0$		$\xi = 0.5$		$\xi = 1.0$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.0000	0.0039	-0.0026	0.0023	-0.0421	0.0051	-0.0026	0.0012	0.0000	0.0017
20	0.0000	0.0020	-0.0003	0.0017	-0.0297	0.0025	-0.0003	0.0008	0.0000	0.0009
30	0.0000	0.0013	-0.0001	0.0013	-0.0243	0.0017	-0.0001	0.0006	0.0000	0.0006
40	0.0000	0.0010	0.0000	0.0010	-0.0210	0.0013	0.0000	0.0004	0.0000	0.0004
50	0.0000	0.0008	0.0000	0.0008	-0.0188	0.0010	0.0000	0.0004	0.0000	0.0004
C_a''	0.8000		0.9000		1.0000		0.9333		0.8667	

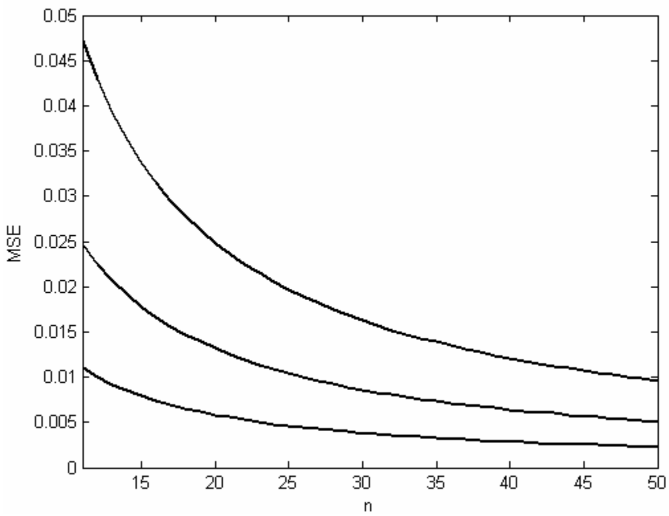

Figure 15. MSE plot of $\hat{C}_a''$ (versus n) for d^*/σ = 2, $D_l : D_u = 6:4$, with ξ=1.0, 0, and 0.5 (from bottom to top).

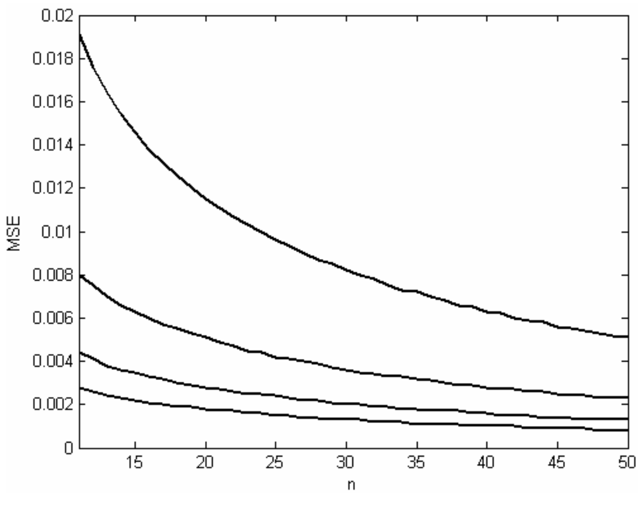

Figure 16. MSE plot of $\hat{C}_a''$ (versus n) for ξ=0.5, $D_l : D_u = 6:4$, with d^*/σ = 5, 4, 3 and 2 (from bottom to top).

Table 7. Lower confidence bounds C_a^L of $\hat{C}_a = 0.75$ for various parameter values, with $\alpha = 0.05$, $\xi = 1.0{:}0.1{:}2.0$, and $n = 10{:}10{:}100$.

ξ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
n=10	0.479	0.526	0.559	0.583	0.602	0.617	0.630	0.640	0.649	0.656	0.662
n=20	0.605	0.625	0.640	0.651	0.661	0.669	0.675	0.681	0.686	0.690	0.694
n=30	0643	0.656	0.667	0.675	0.682	0.688	0.692	0.696	0.700	0.703	0.706
n=40	0.662	0.673	0.681	0.688	0.693	0.698	0.702	0.705	0.708	0.710	0.713
n=50	0.674	0.683	0.690	0.696	0.700	0.704	0.708	0.710	0.713	0.715	0.717
n=60	0.683	0.690	0.696	0.701	0.705	0.709	0.712	0.714	0.717	0.719	0.720
n=70	0.689	0.696	0.701	0.706	0.709	0.712	0.715	0.717	0.719	0.721	0.723
n=80	0.694	0.700	0.705	0.709	0.712	0.715	0.718	0.720	0.722	0.723	0.725
n=90	0.698	0.703	0.708	0.712	0.715	0.717	0.720	0.722	0.723	0.725	0.726
n=100	0.701	0.706	0.710	0.714	0.717	0.719	0.721	0.723	0.725	0.726	0.728

Table 8. Lower confidence bounds C_a^L of $\hat{C}_a = 0.75$ for various parameter values, with $\alpha = 0.01$, $\xi = 1.0{:}0.1{:}2.0$, and $n = 10{:}10{:}100$.

ξ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
n=10	0.100	0.246	0.354	0.424	0.473	0.510	0.537	0.559	0.577	0.592	0.605
n=20	0.479	0.526	0.559	0.583	0.602	0.617	0.630	0.640	0.649	0.656	0.662
n=30	0.566	0.593	0.613	0.629	0.641	0.651	0.660	0.667	0.673	0.678	0.683
n=40	0.605	0.625	0.640	0.651	0.661	0.669	0.675	0.681	0.686	0.690	0.694
n=50	0.628	0.643	0.656	0.665	0.673	0.680	0.685	0.690	0.694	0.698	0.701
n=60	0.643	0.656	0.667	0.675	0.682	0.688	0.692	0.696	0.700	0.703	0.706
n=70	0.654	0.666	0.675	0.682	0.688	0.693	0.698	0.701	0.704	0.707	0.710
n=80	0.662	0.673	0.681	0.688	0.693	0.698	0.702	0.705	0.708	0.710	0.713
n=90	0.669	0.678	0.686	0.692	0.697	0.701	0.705	0.708	0.711	0.713	0.715
n=100	0.674	0.683	0.690	0.696	0.700	0.704	0.708	0.710	0.713	0.715	0.717

Table 9. Lower confidence bounds C_a^L of $\hat{C}_a = 0.50$ for various parameter values, with $\alpha = 0.05$, $\xi = 2.0{:}0.1{:}3.0$, and $n = 10{:}10{:}100$.

ξ	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
n=10	0.324	0.335	0.345	0.354	0.362	0.369	0.375	0.381	0.386	0.391	0.395
n=20	0.387	0.394	0.400	0.405	0.410	0.414	0.418	0.421	0.424	0.427	0.430
n=30	0.412	0.417	0.421	0.425	0.429	0.432	0.435	0.438	0.440	0.442	0.444
n=40	0.425	0.429	0.433	0.436	0.439	0.442	0.445	0.447	0.449	0.451	0.453
n=50	0.434	0.438	0.441	0.444	0.446	0.449	0.451	0.453	0.455	0.456	0.458
n=60	0.441	0.444	0.447	0.449	0.452	0.454	0.456	0.457	0.459	0.461	0.462
n=70	0.446	0.448	0.451	0.453	0.455	0.457	0.459	0.461	0.462	0.464	0.465
n=80	0.449	0.452	0.454	0.457	0.459	0.460	0.462	0.464	0.465	0.466	0.467
n=90	0.453	0.455	0.457	0.459	0.461	0.463	0.464	0.466	0.467	0.468	0.469
n=100	0.455	0.458	0.460	0.462	0.463	0.465	0.466	0.468	0.469	0.470	0.471

Table 10. Lower confidence bounds C_a^L of $\hat{C}_a = 0.50$ for various parameter values, with $\alpha = 0.01$, $\xi = 2.0{:}0.1{:}3.0$, and $n = 10{:}10{:}100$.

ξ	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
n=10	0.209	0.231	0.249	0.265	0.279	0.292	0.303	0.313	0.322	0.330	0.338
n=20	0.324	0.335	0.345	0.354	0.362	0.369	0.375	0.381	0.386	0.391	0.395
n=30	0.365	0.373	0.380	0.387	0.393	0.398	0.402	0.407	0.411	0.414	0.418
n=40	0.387	0.394	0.400	0.405	0.410	0.414	0.418	0.421	0.424	0.427	0.430
n=50	0.402	0.407	0.412	0.417	0.421	0.424	0.428	0.431	0.434	0.436	0.439
n=60	0.412	0.417	0.421	0.425	0.429	0.432	0.435	0.438	0.440	0.442	0.444
n=70	0.419	0.424	0.428	0.431	0.435	0.438	0.440	0.443	0.445	0.447	0.449
n=80	0.425	0.429	0.433	0.436	0.439	0.442	0.445	0.447	0.449	0.451	0.453
n=90	0.430	0.434	0.437	0.440	0.443	0.446	0.448	0.450	0.452	0.454	0.456
n=100	0.434	0.438	0.441	0.444	0.446	0.449	0.451	0.453	0.455	0.456	0.458

Table 11. Critical value c_0 of C_a'' for various parameter values, with $D_l = D_u = d$, α = 0.01, 0.05 and n = 10, 25(25)150.

$\|\mu - T\|/\sigma$		0.5			1.0			1.5		
n	$\frac{d}{\sigma}$	C_a''	c_0		C_a''	c_0		C_a''	c_0	
			α=0.01	α=0.05		α=0.01	α=0.05		α=0.01	α=0.05
10	2	0.750	0.993	0.965	0.500	0.868	0.760	0.250	0.618	0.510
	3	0.833	0.995	0.977	0.667	0.912	0.840	0.500	0.745	0.673
	4	0.875	0.997	0.983	0.750	0.934	0.880	0.625	0.809	0.755
	5	0.900	0.997	0.986	0.800	0.947	0.904	0.700	0.847	0.804
25	2	0.750	0.973	0.914	0.500	0.733	0.664	0.250	0.483	0.414
	3	0.833	0.982	0.943	0.667	0.822	0.777	0.500	0.655	0.610
	4	0.875	0.987	0.957	0.750	0.866	0.832	0.625	0.741	0.707
	5	0.900	0.989	0.966	0.800	0.893	0.866	0.700	0.793	0.766
50	2	0.750	0.914	0.866	0.500	0.664	0.616	0.250	0.414	0.366
	3	0.833	0.943	0.911	0.667	0.777	0.744	0.500	0.610	0.578
	4	0.875	0.957	0.933	0.750	0.832	0.808	0.625	0.707	0.683
	5	0.900	0.966	0.947	0.800	0.866	0.847	0.700	0.766	0.747
75	2	0.750	0.884	0.845	0.500	0.634	0.595	0.250	0.384	0.345
	3	0.833	0.923	0.896	0.667	0.756	0.730	0.500	0.590	0.563
	4	0.875	0.942	0.922	0.750	0.817	0.797	0.625	0.692	0.672
	5	0.900	0.954	0.938	0.800	0.854	0.838	0.700	0.754	0.738
100	2	0.750	0.866	0.832	0.500	0.616	0.582	0.250	0.366	0.332
	3	0.833	0.911	0.888	0.667	0.744	0.722	0.500	0.578	0.555
	4	0.875	0.933	0.916	0.750	0.808	0.791	0.625	0.683	0.666
	5	0.900	0.947	0.933	0.800	0.847	0.833	0.700	0.747	0.733
125	2	0.750	0.854	0.824	0.500	0.604	0.574	0.250	0.354	0.324
	3	0.833	0.902	0.882	0.667	0.736	0.716	0.500	0.569	0.549
	4	0.875	0.927	0.912	0.750	0.802	0.787	0.625	0.677	0.662
	5	0.900	0.942	0.929	0.800	0.842	0.829	0.700	0.742	0.729
150	2	0.750	0.845	0.817	0.500	0.595	0.567	0.250	0.345	0.317
	3	0.833	0.896	0.878	0.667	0.730	0.712	0.500	0.563	0.545
	4	0.875	0.922	0.909	0.750	0.797	0.784	0.625	0.672	0.659
	5	0.900	0.938	0.927	0.800	0.838	0.827	0.700	0.738	0.727

Table 12. Critical value c_0 of C_a'' for various parameter values, with $D_l : D_u = 6:4$, α = 0.01, 0.05 and n = 10, 25, 50 and 75.

$\lvert\mu - T\rvert/\sigma$		0.5			1.0			1.5		
n	$\frac{d^*}{\sigma}$	C_a''	c_0		C_a''	c_0		C_a''	c_0	
			α =0.01	α =0.05		α =0.01	α =0.05		α =0.01	α =0.05
10	2	0.750	0.989	0.951	0.500	0.863	0.759	0.250	0.617	0.510
	3	0.833	0.993	0.967	0.667	0.909	0.839	0.500	0.745	0.673
	4	0.875	0.994	0.975	0.750	0.931	0.879	0.625	0.808	0.755
	5	0.900	0.995	0.980	0.800	0.945	0.903	0.700	0.847	0.804
25	2	0.750	0.965	0.909	0.500	0.732	0.664	0.250	0.482	0.414
	3	0.833	0.977	0.939	0.667	0.821	0.776	0.500	0.655	0.609
	4	0.875	0.982	0.954	0.750	0.866	0.832	0.625	0.741	0.707
	5	0.900	0.986	0.963	0.800	0.893	0.865	0.700	0.793	0.765
50	2	0.750	0.914	0.866	0.500	0.665	0.616	0.250	0.414	0.366
	3	0.833	0.942	0.910	0.667	0.776	0.744	0.500	0.609	0.577
	4	0.875	0.957	0.933	0.750	0.832	0.808	0.625	0.707	0.683
	5	0.900	0.965	0.946	0.800	0.866	0.846	0.700	0.765	0.746
75	2	0.750	0.884	0.844	0.500	0.634	0.595	0.250	0.387	0.345
	3	0.833	0.922	0.896	0.667	0.756	0.730	0.500	0.591	0.563
	4	0.875	0.942	0.922	0.750	0.817	0.797	0.625	0.693	0.672
	5	0.900	0.953	0.937	0.800	0.853	0.838	0.700	0.754	0.738

Table 13. Critical Values c_0 for various parameter values, C_a''=0.75, $D_l : D_u = 7:3$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.993	0.968	0.995	0.976	0.996	0.979	0.996	0.981	0.996	0.982
0.2	0.996	0.981	0.996	0.983	0.996	0.983	0.996	0.982	0.996	0.979
0.3	0.996	0.983	0.996	0.981	0.995	0.973	0.993	0.960	0.990	0.943
0.4	0.996	0.982	0.994	0.967	0.989	0.937	0.976	0.912	0.955	0.895
0.5	0.995	0.975	0.988	0.933	0.962	0.900	0.933	0.880	0.914	0.866
0.6	0.993	0.960	0.966	0.903	0.926	0.875	0.903	0.858	0.887	0.846
0.7	0.988	0.935	0.935	0.881	0.901	0.857	0.881	0.842	0.867	0.833
0.8	0.976	0.912	0.912	0.864	0.882	0.843	0.864	0.831	0.852	0.822
09	0.954	0.894	0.894	0.852	0.867	0.833	0.852	0.822	0.841	0.814
1.0	0.933	0.880	0.880	0.841	0.856	0.825	0.841	0.815	0.832	0.808

Table 14. Critical Values c_0 for various parameter values, C_a''=0.67, $D_l : D_u = 7:3$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.991	0.958	0.993	0.968	0.994	0.973	0.995	0.975	0.995	0.976
0.2	0.995	0.975	0.995	0.978	0.995	0.978	0.995	0.976	0.995	0.973
0.3	0.995	0.978	0.995	0.975	0.993	0.965	0.991	0.948	0.987	0.925
0.4	0.995	0.976	0.992	0.957	0.985	0.917	0.968	0.884	0.941	0.861
0.5	0.994	0.968	0.984	0.912	0.949	0.868	0.912	0.841	0.887	0.823
0.6	0.991	0.948	0.955	0.872	0.903	0.835	0.872	0.813	0.850	0.797
0.7	0.985	0.914	0.915	0.843	0.870	0.811	0.843	0.792	0.825	0.779
0.8	0.968	0.884	0.884	0.821	0.845	0.793	0.821	0.777	0.805	0.765
09	0.939	0.860	0.860	0.804	0.825	0.780	0.804	0.765	0.790	0.755
1.0	0.912	0.841	0.841	0.791	0.810	0.769	0.791	0.755	0.778	0.746

Table 15. Critical Values c_0 for various parameter values, C_a''=0.50, $D_l : D_u = 7:3$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.987	0.936	0.990	0.952	0.992	0.959	0.992	0.962	0.993	0.965
0.2	0.992	0.962	0.993	0.967	0.993	0.967	0.993	0.964	0.992	0.959
0.3	0.993	0.967	0.993	0.962	0.990	0.947	0.987	0.921	0.981	0.886
0.4	0.993	0.964	0.989	0.934	0.978	0.874	0.952	0.825	0.911	0.790
0.5	0.991	0.951	0.976	0.867	0.924	0.800	0.867	0.760	0.829	0.732
0.6	0.987	0.921	0.932	0.806	0.853	0.750	0.806	0.716	0.774	0.693
0.7	0.977	0.871	0.871	0.762	0.803	0.714	0.762	0.685	0.734	0.666
0.8	0.952	0.825	0.825	0.729	0.765	0.687	0.729	0.662	0.705	0.645
09	0.908	0.788	0.789	0.704	0.735	0.666	0.704	0.644	0.682	0.629
1.0	0.867	0.760	0.760	0.683	0.712	0.650	0.683	0.630	0.664	0.616

Table 16. Critical Values c_0 for various parameter values, C_a''=0.25, $D_l : D_u = 7:3$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.981	0.905	0.986	0.929	0.988	0.939	0.989	0.944	0.989	0.947
0.2	0.989	0.944	0.990	0.951	0.990	0.950	0.989	0.946	0.988	0.939
0.3	0.990	0.951	0.989	0.943	0.986	0.921	0.980	0.882	0.972	0.829
0.4	0.989	0.946	0.983	0.902	0.967	0.812	0.928	0.737	0.866	0.686
0.5	0.987	0.927	0.964	0.801	0.886	0.700	0.801	0.640	0.743	0.598
0.6	0.980	0.882	0.898	0.709	0.780	0.625	0.709	0.575	0.661	0.540
0.7	0.966	0.806	0.807	0.644	0.705	0.571	0.644	0.528	0.602	0.499
0.8	0.928	0.737	0.737	0.594	0.648	0.531	0.594	0.493	0.558	0.468
09	0.862	0.683	0.683	0.556	0.603	0.500	0.556	0.466	0.524	0.444
1.0	0.801	0.640	0.640	0.525	0.568	0.475	0.525	0.445	0.497	0.424

Table 17. Critical Values c_0 for various parameter values, C_a''=0.75, $D_l : D_u = 6:4$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.991	0.958	0.993	0.968	0.994	0.973	0.995	0.975	0.995	0.976
0.2	0.995	0.975	0.995	0.978	0.995	0.978	0.995	0.977	0.995	0.974
0.3	0.995	0.978	0.995	0.976	0.994	0.968	0.991	0.956	0.988	0.941
0.4	0.995	0.977	0.992	0.962	0.986	0.936	0.973	0.912	0.955	0.895
0.5	0.994	0.970	0.985	0.932	0.961	0.900	0.933	0.880	0.914	0.866
0.6	0.991	0.956	0.964	0.903	0.926	0.875	0.903	0.858	0.887	0.846
0.7	0.986	0.934	0.935	0.881	0.901	0.857	0.881	0.842	0.867	0.833
0.8	0.973	0.912	0.912	0.864	0.882	0.843	0.864	0.831	0.852	0.822
09	0.953	0.894	0.894	0.852	0.867	0.833	0.852	0.822	0.841	0.814
1.0	0.933	0.880	0.880	0.841	0.856	0.825	0.841	0.815	0.832	0.808

Table 18. Critical Values c_0 for various parameter values, C_a''=0.67, $D_l : D_u = 6:4$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.989	0.944	0.991	0.958	0.992	0.964	0.993	0.967	0.993	0.969
0.2	0.993	0.967	0.994	0.971	0.994	0.971	0.994	0.969	0.993	0.966
0.3	0.994	0.972	0.993	0.968	0.992	0.958	0.989	0.942	0.984	0.922
0.4	0.994	0.969	0.990	0.950	0.982	0.915	0.965	0.884	0.940	0.861
0.5	0.992	0.961	0.980	0.911	0.948	0.868	0.912	0.841	0.887	0.823
0.6	0.989	0.942	0.953	0.872	0.903	0.835	0.872	0.813	0.850	0.797
0.7	0.981	0.913	0.915	0.843	0.870	0.811	0.843	0.792	0.825	0.779
0.8	0.965	0.884	0.884	0.821	0.845	0.793	0.821	0.777	0.805	0.765
09	0.939	0.860	0.860	0.804	0.825	0.780	0.804	0.765	0.790	0.755
1.0	0.912	0.841	0.841	0.791	0.810	0.769	0.791	0.755	0.778	0.746

Table 19. Critical Values c_0 for various parameter values, C_a''=0.50, $D_l : D_u = 6:4$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.983	0.916	0.987	0.937	0.989	0.946	0.990	0.951	0.990	0.953
0.2	0.990	0.951	0.991	0.957	0.991	0.957	0.991	0.954	0.990	0.949
0.3	0.991	0.957	0.990	0.952	0.988	0.937	0.983	0.913	0.976	0.883
0.4	0.991	0.954	0.985	0.924	0.973	0.872	0.947	0.824	0.910	0.790
0.5	0.988	0.941	0.971	0.865	0.922	0.800	0.867	0.760	0.829	0.732
0.6	0.983	0.913	0.929	0.806	0.853	0.750	0.806	0.716	0.774	0.693
0.7	0.972	0.869	0.871	0.762	0.803	0.714	0.762	0.685	0.735	0.666
0.8	0.947	0.824	0.825	0.729	0.765	0.687	0.729	0.662	0.705	0.645
09	0.907	0.788	0.789	0.704	0.735	0.666	0.704	0.644	0.682	0.629
1.0	0.867	0.760	0.760	0.683	0.712	0.650	0.683	0.630	0.664	0.616

Table 20. Critical Values c_0 for various parameter values, C_a''=0.25, $D_l : D_u = 6:4$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.975	0.874	0.981	0.906	0.984	0.919	0.985	0.926	0.986	0.930
0.2	0.985	0.926	0.987	0.936	0.987	0.935	0.986	0.931	0.985	0.924
0.3	0.987	0.936	0.986	0.928	0.982	0.905	0.975	0.869	0.964	0.824
0.4	0.986	0.931	0.978	0.887	0.960	0.809	0.921	0.737	0.865	0.686
0.5	0.983	0.911	0.956	0.798	0.883	0.700	0.801	0.640	0.743	0.598
0.6	0.975	0.869	0.894	0.709	0.780	0.625	0.709	0.575	0.661	0.540
0.7	0.958	0.803	0.807	0.644	0.705	0.571	0.644	0.528	0.602	0.499
0.8	0.921	0.737	0.737	0.594	0.648	0.531	0.594	0.493	0.558	0.468
09	0.861	0.683	0.683	0.556	0.603	0.500	0.556	0.466	0.524	0.443
1.0	0.801	0.640	0.640	0.525	0.568	0.475	0.525	0.445	0.496	0.424

Table 21. Critical Values c_0 for various parameter values, C_a'' =0.75, $D_l : D_u = 4:6$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.987	0.937	0.990	0.953	0.992	0.960	0.992	0.964	0.993	0.966
0.2	0.992	0.964	0.993	0.969	0.993	0.969	0.993	0.968	0.992	0.965
0.3	0.993	0.969	0.993	0.966	0.991	0.959	0.988	0.948	0.984	0.936
0.4	0.993	0.968	0.989	0.953	0.982	0.932	0.969	0.911	0.953	0.895
0.5	0.991	0.961	0.981	0.929	0.958	0.899	0.933	0.880	0.914	0.866
0.6	0.988	0.948	0.961	0.902	0.926	0.875	0.903	0.858	0.887	0.846
0.7	0.981	0.931	0.935	0.881	0.901	0.857	0.881	0.842	0.867	0.833
0.8	0.969	0.911	0.912	0.864	0.882	0.843	0.864	0.831	0.852	0.822
09	0.952	0.894	0.894	0.852	0.868	0.833	0.852	0.822	0.841	0.814
1.0	0.933	0.880	0.880	0.841	0.856	0.825	0.841	0.815	0.832	0.808

Table 22. Critical Values c_0 for various parameter values, C_a'' =0.67, $D_l : D_u = 4:6$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.983	0.917	0.987	0.939	0.989	0.947	0.990	0.952	0.991	0.955
0.2	0.990	0.952	0.991	0.959	0.991	0.959	0.991	0.957	0.990	0.954
0.3	0.991	0.959	0.990	0.956	0.988	0.946	0.984	0.932	0.979	0.916
0.4	0.991	0.957	0.986	0.939	0.976	0.910	0.959	0.883	0.938	0.861
0.5	0.989	0.949	0.974	0.907	0.945	0.867	0.912	0.841	0.887	0.823
0.6	0.984	0.932	0.949	0.871	0.903	0.835	0.872	0.813	0.850	0.797
0.7	0.975	0.909	0.914	0.843	0.870	0.811	0.843	0.792	0.825	0.779
0.8	0.959	0.883	0.884	0.821	0.845	0.793	0.821	0.777	0.805	0.765
09	0.936	0.860	0.860	0.804	0.825	0.780	0.804	0.765	0.790	0.755
1.0	0.912	0.841	0.841	0.791	0.810	0.769	0.791	0.755	0.778	0.746

Table 23. Critical Values c_0 for various parameter values, C_a''=0.50, $D_l : D_u = 4:6$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.975	0.875	0.981	0.907	0.984	0.920	0.985	0.928	0.986	0.932
0.2	0.985	0.928	0.987	0.938	0.987	0.938	0.986	0.936	0.985	0.931
0.3	0.987	0.938	0.986	0.933	0.982	0.918	0.976	0.897	0.968	0.873
0.4	0.986	0.936	0.979	0.907	0.964	0.865	0.939	0.823	0.906	0.790
0.5	0.983	0.922	0.962	0.859	0.917	0.799	0.867	0.760	0.828	0.732
0.6	0.976	0.897	0.923	0.805	0.853	0.750	0.806	0.716	0.774	0.693
0.7	0.963	0.862	0.870	0.762	0.803	0.714	0.762	0.685	0.735	0.666
0.8	0.939	0.823	0.825	0.729	0.765	0.687	0.729	0.662	0.705	0.645
09	0.904	0.788	0.789	0.704	0.736	0.666	0.704	0.644	0.682	0.629
1.0	0.867	0.760	0.760	0.683	0.712	0.650	0.683	0.630	0.664	0.616

Table 24. Critical Values c_0 for various parameter values, C_a''=0.25, $D_l : D_u = 4:6$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.962	0.813	0.972	0.861	0.976	0.881	0.978	0.892	0.979	0.898
0.2	0.978	0.892	0.981	0.907	0.981	0.908	0.980	0.904	0.978	0.896
0.3	0.981	0.908	0.979	0.900	0.973	0.878	0.965	0.846	0.952	0.810
0.4	0.980	0.904	0.969	0.861	0.946	0.797	0.908	0.735	0.859	0.685
0.5	0.975	0.884	0.943	0.789	0.875	0.699	0.800	0.640	0.743	0.598
0.6	0.965	0.846	0.885	0.708	0.780	0.625	0.709	0.575	0.661	0.540
0.7	0.945	0.793	0.806	0.644	0.705	0.571	0.644	0.528	0.602	0.499
0.8	0.908	0.735	0.737	0.594	0.648	0.531	0.594	0.493	0.558	0.468
09	0.856	0.683	0.683	0.556	0.604	0.500	0.556	0.466	0.524	0.443
1.0	0.800	0.640	0.640	0.525	0.568	0.475	0.525	0.445	0.496	0.424

Table 25. Critical Values c_0 for various parameter values, C_a''=0.75, $D_l : D_u = 3:7$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.985	0.927	0.989	0.946	0.990	0.954	0.991	0.958	0.992	0.961
0.2	0.991	0.958	0.992	0.964	0.992	0.965	0.992	0.963	0.991	0.961
0.3	0.992	0.964	0.992	0.962	0.990	0.955	0.986	0.945	0.982	0.934
0.4	0.992	0.963	0.988	0.950	0.980	0.930	0.967	0.910	0.952	0.895
0.5	0.990	0.957	0.979	0.927	0.957	0.899	0.933	0.879	0.914	0.866
0.6	0.986	0.945	0.960	0.902	0.926	0.875	0.903	0.858	0.887	0.846
0.7	0.979	0.928	0.935	0.881	0.901	0.857	0.881	0.842	0.867	0.833
0.8	0.967	0.910	0.912	0.864	0.882	0.843	0.864	0.831	0.852	0.822
09	0.951	0.894	0.894	0.852	0.868	0.833	0.852	0.822	0.841	0.814
1.0	0.933	0.879	0.880	0.841	0.856	0.825	0.841	0.815	0.832	0.808

Table 26. Critical Values c_0 for various parameter values, C_a''=0.67, $D_l : D_u = 3:7$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.980	0.904	0.985	0.929	0.987	0.939	0.988	0.945	0.989	0.948
0.2	0.988	0.945	0.990	0.953	0.990	0.954	0.989	0.952	0.989	0.949
0.3	0.990	0.953	0.989	0.950	0.986	0.941	0.982	0.927	0.976	0.913
0.4	0.989	0.952	0.984	0.934	0.974	0.907	0.957	0.882	0.936	0.861
0.5	0.987	0.943	0.972	0.904	0.943	0.867	0.911	0.841	0.887	0.823
0.6	0.982	0.927	0.947	0.871	0.903	0.835	0.872	0.813	0.850	0.797
0.7	0.973	0.906	0.914	0.843	0.870	0.811	0.843	0.792	0.825	0.779
0.8	0.957	0.882	0.884	0.821	0.845	0.793	0.821	0.777	0.805	0.765
09	0.935	0.860	0.860	0.804	0.825	0.780	0.804	0.765	0.790	0.755
1.0	0.911	0.841	0.841	0.791	0.810	0.769	0.791	0.755	0.778	0.746

Table 27. Critical Values c_0 for various parameter values, C_a''=0.50, $D_l : D_u = 3:7$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\|\mu - T\|}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.970	0.855	0.978	0.892	0.981	0.908	0.983	0.916	0.984	0.922
0.2	0.983	0.916	0.985	0.929	0.985	0.930	0.984	0.927	0.983	0.922
0.3	0.985	0.929	0.984	0.925	0.980	0.910	0.973	0.890	0.964	0.868
0.4	0.984	0.927	0.976	0.900	0.960	0.860	0.935	0.821	0.904	0.790
0.5	0.981	0.914	0.958	0.855	0.914	0.799	0.866	0.759	0.828	0.732
0.6	0.973	0.890	0.920	0.804	0.853	0.750	0.806	0.716	0.774	0.693
0.7	0.959	0.857	0.870	0.762	0.803	0.714	0.762	0.685	0.735	0.666
0.8	0.935	0.821	0.825	0.729	0.765	0.687	0.729	0.662	0.705	0.645
09	0.902	0.788	0.789	0.704	0.736	0.666	0.704	0.644	0.682	0.629
1.0	0.866	0.759	0.760	0.683	0.712	0.650	0.683	0.630	0.664	0.616

Table 28. Critical Values c_0 for various parameter values, C_a''=0.25, $D_l : D_u = 3:7$, $\alpha = 0.01, 0.05$, $n = 10(10)50$, and $\xi = 0.1(0.1)1.0$.

$\frac{\|\mu - T\|}{\sigma}$	n = 10		n = 20		n = 30		n = 40		n = 50	
	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05	α = 0.01	α = 0.05
0.1	0.956	0.782	0.967	0.839	0.972	0.862	0.974	0.875	0.976	0.883
0.2	0.974	0.875	0.978	0.893	0.978	0.895	0.977	0.891	0.975	0.884
0.3	0.978	0.894	0.976	0.888	0.970	0.866	0.960	0.836	0.946	0.802
0.4	0.977	0.891	0.965	0.850	0.941	0.790	0.903	0.732	0.856	0.685
0.5	0.971	0.872	0.937	0.783	0.871	0.698	0.799	0.639	0.743	0.598
0.6	0.960	0.836	0.880	0.707	0.780	0.625	0.709	0.575	0.661	0.540
0.7	0.939	0.786	0.805	0.643	0.705	0.571	0.644	0.528	0.602	0.499
0.8	0.903	0.732	0.737	0.594	0.648	0.531	0.594	0.493	0.558	0.468
09	0.853	0.682	0.683	0.556	0.604	0.500	0.556	0.466	0.524	0.443
1.0	0.799	0.639	0.640	0.525	0.568	0.475	0.525	0.445	0.496	0.424

Table 29. Lower Confidence Bounds $C_a^{L''}$ for various parameter values, $\hat{C}_a''$=0.75, $D_l : D_u = 7:3$, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $\xi = 0.6(0.1)1.0$.

n	$\xi = 0.6$		$\xi = 0.7$		$\xi = 0.8$		$\xi = 0.9$		$\xi = 1.0$	
	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05
20	0.025	0.354	0.027	0.474	0.286	0.538	0.408	0.578	0.479	0.605
30	0.145	0.500	0.365	0.563	0.468	0.600	0.527	0.625	0.566	0.643
40	0.354	0.559	0.474	0.603	0.538	0.630	0.578	0.648	0.605	0.663
50	0.447	0.592	0.529	0.626	0.576	0.648	0.606	0.663	0.628	0.675

Table 30. Lower Confidence Bounds $C_a^{L''}$ for various parameter values, $\hat{C}_a''$=0.75, $D_l : D_u = 6:4$, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $\xi = 0.6(0.1)1.0$.

n	$\xi = 0.6$		$\xi = 0.7$		$\xi = 0.8$		$\xi = 0.9$		$\xi = 1.0$	
	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05
20	0.025	0.355	0.027	0.474	0.286	0.538	0.408	0.578	0.479	0.605
30	0.144	0.500	0.365	0.563	0.467	0.600	0.527	0.625	0.566	0.643
40	0.354	0.559	0.474	0.603	0.538	0.630	0.578	0.649	0.605	0.663
50	0.447	0.592	0.529	0.626	0.576	0.648	0.606	0.663	0.628	0.675

Table 31. Lower Confidence Bounds $C_a^{L''}$ for various parameter values, $\hat{C}_a''$=0.75, $D_l : D_u = 4:6$, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $\xi = 0.6(0.1)1.0$.

n	$\xi = 0.6$		$\xi = 0.7$		$\xi = 0.8$		$\xi = 0.9$		$\xi = 1.0$	
	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05
20	0.027	0.357	0.033	0.474	0.286	0.538	0.408	0.578	0.479	0.605
30	0.146	0.500	0.365	0.563	0.467	0.600	0.527	0.625	0.566	0.643
40	0.354	0.559	0.474	0.603	0.538	0.630	0.578	0.649	0.605	0.663
50	0.447	0.592	0.529	0.626	0.576	0.648	0.606	0.663	0.628	0.675

Table 32. Lower Confidence Bounds $C_a^{L''}$ for various parameter values, $\hat{C}_a''$=0.75, $D_l : D_u = 3:7$, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $\xi = 0.6(0.1)1.0$.

n	$\xi = 0.6$		$\xi = 0.7$		$\xi = 0.8$		$\xi = 0.9$		$\xi = 1.0$	
	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05	$\alpha =$ 0.01	$\alpha =$ 0.05
20	0.027	0.360	0.038	0.474	0.286	0.538	0.408	0.578	0.479	0.605
30	0.148	0.500	0.365	0.563	0.467	0.600	0.527	0.625	0.566	0.643
40	0.354	0.559	0.474	0.603	0.538	0.630	0.578	0.649	0.605	0.663
50	0.447	0.592	0.529	0.626	0.576	0.648	0.606	0.663	0.628	0.675