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博 士 論 文

良率指標 S_{pk} 的抽樣性質與樣本資訊下估計良率的精準度

Sampling Properties of the Yield Index S_{pk} With Estimation Accuracy and Sample Size Information

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摘　　　　　要

Boyles 在 1994 年提出一個良率指標取名為 S_{pk}。該指標為常態製程提供一個精準正確的良率衡量。Lee *et al*.在 2002 年提出一個常態近似的方法來估計製程的 S_{pk} 值。在本論文，我們延伸擴展前人的成就，在三種不同的情況下考量運用良率指標的抽樣分配：(i)多重抽樣樣本下(ii)摺合逼近法的抽樣分配(iii)允收抽樣情況。我們在多重抽樣樣本下推導出良率指標的抽樣分配，並且發現針對相同的 S_{pk} 值，製程平均值落在規格上下限中心時良率指標估計量的變異程度最大。針對一些常用的指標需求，S_{pk} 的信賴下限已列成表。為了讓使用者更易於接受推導出來的常態逼近分配，我們檢查了真實的型 I 誤差並與一開始就設定好的顯著水準比較。我們計算了該常態逼近會收斂到與真值只有特定差異的需求樣本數。最後，一個可再充電鋰電池製程實例被呈現並說明從業者如何應用 S_{pk} 的信賴下限到多重抽樣的樣本。

接著，我們使用摺合逼近法來估計製程的 S_{pk} 值，並與常態逼近法做比較。比較的結果顯示摺合的方法的確比常態逼近法在估計製程良率及 S_{pk} 值上更為準確。根據摺合的方法，我們建構一個逐步且有效率的步驟來說明如何估計製程良率。我們亦研究了摺合方法的準確性，提供在特定檢定力需求下，以及在特定收斂需求下所需要的採集的樣本個數。本論文最後一個部份，我們考慮根據 S_{pk} 值來做允收與否的決策。允收抽樣計畫是一個提供買賣雙方對於檢驗貨品是否符合產品品質需求的決策法則。我們提出一個以 S_{pk} 為依據的計數值抽樣計畫來處理不良率為極小 PPM 的製程。我們根據同時解一對非線性的方程式，建立一個有效率的方法來決定所需抽樣的個數以及接受貨品與否的臨界值。根據我們設計出來的抽樣計畫，從業者可以決定檢驗貨批所需的數量和相對應的允收決策值。

關鍵字：製程能力，產品良率，多重抽樣，信賴下限，臨界值，檢定力，允收抽樣計畫，貨批檢驗，不良率

Sampling Properties of the Yield Index S_{pk} With Estimation Accuracy and Sample Size Information

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Abstract

The yield index S_{pk} proposed by Boyles (1994) provides an exact measure on the production yield of normal processes. Lee *et al.* (2002) considered a normal approximation for estimating S_{pk}. In the thesis, we extend the results and consider the sampling distribution of the yield index in three conditions (i) for multiple samples, (ii) in the convolution method, and (iii) for acceptance sampling. Under multiple samples, we derive the sampling distribution for the estimator $\hat{S}'_{pk}$ of S_{pk}, and observe that for the same S_{pk}, the variance of $\hat{S}'_{pk}$ would be largest when the process mean is on the center of specification limits. Lower bounds of S_{pk} are tabulated for some commonly used capability requirements. To assess the normally approximated distribution of $\hat{S}'_{pk}$, we also check out the actual type I error and compare with the preset significant level. We also compute the sample sizes required for the normal approximation to converge to the actual S_{pk} within a designated accuracy. Then, a real-world application of the one-cell rechargeable Li-ion battery packs is presented to illustrate how practitioners can apply the lower bounds to actual data collected in multiple samples.

Next, we consider a convolution approximation for estimating S_{pk}, and compare with the normal approximation. The comparison results show that the convolution method does provide a more accurate estimation to S_{pk} as well as the production yield than the normal approximation. An efficient step-by-step procedure based on the convolution method is developed to illustrate how to estimate the production yield. Also investigated is the accuracy of the convolution method which provides useful information about sample size required for designated power levels, and for convergence. Finally, we consider the acceptance determination based on the S_{pk} index. Acceptance sampling plans provide the vendor and the buyer decision rules for lot sentencing to meet their product quality needs. A variables sampling plan based on the index S_{pk} is proposed to handle processes requiring very low PPM fraction of defectives. We develop an effective method for obtaining the required sample sizes n and the critical acceptance value c_0 by solving simultaneously two nonlinear equations. Based on the designed sampling plan, the practitioners can determine the number of production items to be sampled for inspection and the corresponding critical acceptance value for lot sentencing.

Keywords: Process capability, production yield, multiple samples, lower confidence bound, critical value, power of test, acceptance sampling plan, lot sentencing, fraction of defectives

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Chapter 1
Introduction

1.1. Process Capability Indices

Production yield, for a long time, has been a standard criterion used in the manufacturing industry as a common measure on process performance, and defined as the percentage of processed product unit that falls within the manufacturing specification limits. For product units falling out of the manufacturing tolerance, additional cost would be incurred to the factory for scrapping or repairing the product. All passed product units, which incur no additional cost to the factory, are equally accepted by the producer. Numerous process capability indices (PCI) have been proposed to the manufacturing industry, to provide numerical measures on the production yield as well as process performance. Those indices, such as C_p, C_{pk}, C_{pm}, C_{pmk}, and S_{pk}, establish the relationship between the actual process performance and the manufacturing specifications, which have been the focus of the recent research in statistical and quality assurance literatures. The explicit forms of the indices are defined as follows:

$$C_a = 1 - \frac{|\mu - \mathrm{m}|}{\mathrm{d}}, \quad C_p = \frac{USL - LSL}{6\sigma}, \quad C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\} = C_a C_p,$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \quad C_{pmk} = \min\left\{\frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}, \text{ and}$$

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\},$$

where *USL* and *LSL* are the upper and lower specification limits, respectively, μ is the process mean, σ is the process standard deviation, m = (*USL* + *LSL*)/2 is the center of the specification limits, d = (*USL* – *LSL*)/2 is the half length of the specification limits, *T* is the target value, $\Phi(\cdot)$ is the cumulatively distribution function (CDF) of the standard normal variable, and $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$.

1.2. Literatures Review

The index C_p measures the overall process variation relative to the specification tolerance, therefore only reflects the process precision (the product consistency) (see Juran [20]; Kane [21]). Owing to the simplicity of the design, C_p can not reflect the tendency of process centering. In order to reflect the deviations of process mean from the target value, several indices similar in nature to C_p, such as C_{pk}, C_{pm}, C_{pmk}, have been proposed. Those indices attempt to take into consideration the magnitude of process variance as well as process location. The C_{pk} index was developed because the C_p index can not adequately deal with cases that process mean is not centered. However, a large value of C_{pk} does not really say anything about the location of the mean in the tolerance interval. The C_{pk} index has been regarded as a yield-based index since it provides bounds on production yield for a normally distributed process. The C_p and C_{pk} indices are appropriate measures of progress for quality improvement paradigms in which reduction of variability is the guiding principle and production yield is the primary measure of success.

Taguchi, on the other hand, emphasizes the loss in a product's worth, rather than the production yield, when one of its characteristics departs from the target value. Hsiang and Taguchi [16] introduced the index C_{pm}, which was also proposed independently by Chan *et al.* [4]. The C_{pm} index is related to the idea of squared error loss, $loss(X) = (X - T)^2$, and has been called the Taguchi index. The C_{pm} index incorporates the process variation with respect to the target value with the manufacturing specifications preset in the factory, which reflects the degree of process targeting. Chan *et al.* [4] also discussed the sampling properties of the natural estimator of C_{pm}. Boyles [2] provided a definitive analysis of C_{pm} and its usefulness in measuring process targeting. Pearn and Shu [38] provided explicit formulas with efficient algorithms to obtain the lower confidence bound of C_{pm} using the maximum likelihood estimator (MLE) of C_{pm}. Pearn *et al.* [39] developed a two-phase supplier selection procedure based on the C_{pm} index providing useful information about sample size required for a designated selection power.

Pearn *et al.* [35] proposed a third-generation capability index called C_{pmk}, which is constructed by combining the merits of the three indices C_p, C_{pk}, and C_{pm}. The index C_{pmk} alters the user either the process variance increases or the process mean deviates from its target value. The C_{pmk} index responds to the departure of the process mean from the target value T faster than the other three indices C_p, C_{pk}, and C_{pm}, while it remains sensitive to the changes of process variation. Vännman and Kotz [49] obtained the distribution of the estimated $C_p(u, v)$ for cases with on-center target. By taking $u = 1$ and $v = 1$, the distribution of $C_p(1, 1) = C_{pmk}$ is obtained. Chen and Hsu [7] proposed the asymptotic sampling distribution of C_{pmk}, and showed that the estimated C_{pmk} is consistent, asymptotically unbiased estimator of C_{pmk} and is asymptotically normal while the fourth moment of the characteristic X is finite. Wright [51] derived an explicit but rather complicated expression of the probability density function (PDF) of the estimated C_{pmk}. Pearn and Lin [37] alternatively expressed the CDF and PDF of the estimated C_{pmk} in terms of a mixture of the chi-square distribution and normal distribution. The CDF form of the estimated C_{pmk} obtained by Pearn and Lin [37] considerably simplify the complexity for analyzing the statistical properties of the estimated C_{pmk}.

1.3. Motivation of Research

Process yield is the most common and standard criteria used in the manufacturing industry for measuring process performance. The indices C_{pm} and C_{pmk} are designed to emphasize the loss in a product's worth when one of its characteristics departs from the target value T. The index C_p can provide yield estimation only for on-center processes, which can be expressed as:

$$Yield = 2\Phi(C_p) - 1,$$

and for processes with departure mean, the process yield would be less than $2\Phi(C_p) - 1$. The index C_{pk} can provide interval estimation on the process yield (Boyles [2]), that is

$$2\Phi(3C_{pk}) - 1 \le Yield \le \Phi(3C_{pk}).$$

Only the yield index S_{pk} provides an exact measure on the production yield.

The organization of this dissertation is as follows. In chapter 2, we consider to estimate the process yield based on the S_{pk} index for multiple samples. In chapeter3, we propose the convolution method for estimating the process yield. In chapter 4, product acceptance determination based on the S_{pk} index is developed. In the final chapter, we make some conclusions for evaluating process performance based on the S_{pk} index.

Chapter 2

Estimating Process Yield Based on S_{pk} for Multiple Samples

2.1. Introduction

Production yield is defined as the percentage of processed product units passing the inspection. That is, the product characteristic must fall within the manufacturing tolerance. For processes with two-sided manufacturing specifications, the process yield can be calculated as $Yield = F(USL) - F(LSL)$, where $F(\cdot)$ is the cumulative distribution function of the process characteristic. If the process characteristic follows the normal distribution, then the process yield can be alternatively expressed as $Yield = \Phi[(USL\text{-}\mu)/\sigma]\text{-}\Phi[(LSL\text{-}\mu)/\sigma]$. Take C_{pk} for example, if C_{pk} = c, then the process yield would be in the range of $2\Phi(3c) - 1$ and $\Phi(3c)$, i.e. $2\Phi(3C_{pk}) - 1 \le Yield \le \Phi(3C_{pk})$ (Boyles [2]). To overcome this shortcoming, Boyles [3] proposed the yield index called S_{pk}. There is a one-to-one relationship between S_{pk} and the process yield, $Yield = 2\Phi(3S_{pk}) - 1$.

Most of the results obtained regarding the statistical properties of estimated capability indices are based on one single sample. However, a common practice in process control is to estimate the process capability indices by using past "in-control data" from multiple samples, particularly, when a daily-based or weekly-based production control plan is implemented for monitoring process stability. To use estimators based on several small multiple samples and interpret the results as if they were based on a single sample may result in incorrect conclusions. In order to use past in-control data from multiple samples to make decisions regarding process capability, the distribution of the estimated capability index based on multiple samples should be taken into account. When using multiple samples, Kirmani *et al.* [22] have investigated the distribution of estimators of based on the sample standard deviations of the multiple samples. Li *et al.* [27] have investigated the distribution of estimators of C_p and C_{pk} based on the ranges of the multiple samples. Vännman and Hubele [48] considered the indices in the class defined by $C_p(u,v)$ and derived the distribution of the estimators of $C_p(u,v)$, when the estimators of the process parameters μ and σ are based on multiple samples.

In Chapter 2, we investigate the behavior of an estimator of S_{pk} for multiple samples. In the second section (section 2.2.), we compare the yield index S_{pk} with the most commonly used index C_{pk}, and review some results of S_{pk} under single sample. In the third section (section 2.3.), we derive the sampling distribution for the estimator of S_{pk} under multiple samples and result in a normal approximation distribution. In the fourth section (section 2.4.), we find that the spread of $\hat{S}'_{pk}$ would be largest when the process mean is on the center of specification limits for the same S_{pk}, so we calculate the lower bounds of S_{pk} from our deriving distribution of $\hat{S}'_{pk}$ based on the situation with the largest variance for conservative. In the fifth section (section 2.5.), we show the accuracy of our normal-approximated distribution of $\hat{S}'_{pk}$ by displaying the histograms of lower bounds and the actual type I errors. Finally (in section 2.6.), we give an application example to describe how to use the lower bounds as listed in our tables.

2.2. The Yield Index S_{pk}

We consider a group of five processes as printed in Figure 2-1. For these processes, USL = 36.0, LSL = 24.0, and mean μ = 30.0, 30.5, 31.0, 31.5, 32.0, standard deviation σ = 2.0, 11/6, 5/3, 1.5, 4/3, respectively (from process A to E). The C_{pk} value and calculated yield of theses five processes are all the same as in Table 2-1(a), and the S_{pk} value and its calculated yield in Table 2-1(b). The 'Actual Yield' in Table 2-1(a) and 2-1(b) is defined by $\Phi((USL-\mu)/\sigma)-\Phi((LSL-\mu)/\sigma)$. We can see that for these five processes the calculated yield of C_{pk} can only guarantee the lower bound yield, however the calculated yield of S_{pk} value can truly reveal the actual yield of each process.

For single sample, Lee *et al.* [26] have derived the distribution of an estimator of S_{pk}. The estimator is defined as

$$\hat{S}_{pk} = S_{pk} + \frac{W}{6\sqrt{n}\phi(3S_{pk})} + O_p\left(\frac{1}{n}\right)$$

where W is normally distributed with a mean of zero and a variance of a^2+b^2,

$$a = \frac{1}{\sqrt{2}}\left\{\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\},\quad b = \phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right),$$

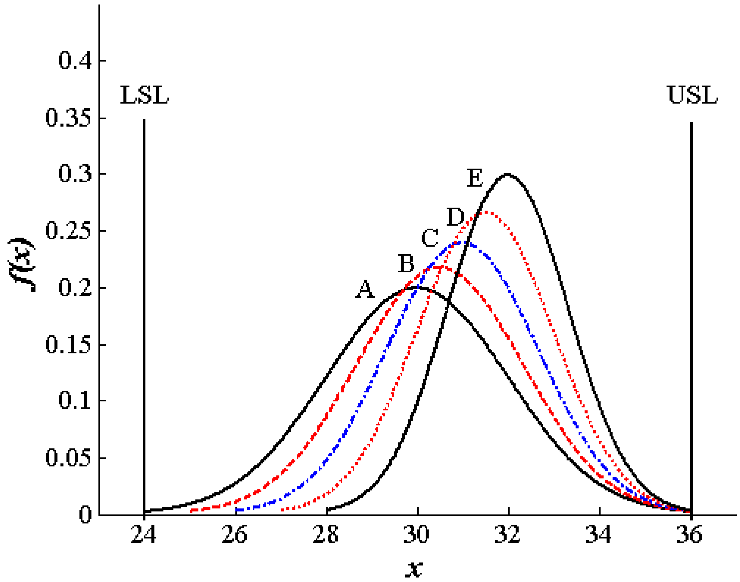


Figure 2-1. Distribution of five processes with USL = 36.0, LSL = 24.0.

Table 2-1(a). The C_{pk} value, calculated yield, and actual yield of five different processes in Figure 2-1.

Process	μ	σ	C_{pk}	Calculated Yield	Actual Yield
A	30.0	2.00	1.0	0.9973	0.9973
B	30.5	1.83	1.0	0.9973	0.9985
C	31.0	1.67	1.0	0.9973	0.9986
D	31.5	1.50	1.0	0.9973	0.9986
E	32.0	1.33	1.0	0.9973	0.9987

Table 2-1(b). The S_{pk} value , calculated yield, and actual yield of five different processes in Figure 2-1.

Process	μ	σ	S_{pk}	Calculated Yield	Actual Yield
A	30.0	2.00	1.000000	0.9973	0.9973
B	30.5	1.83	1.055311	0.9985	0.9985
C	31.0	1.67	1.067441	0.9986	0.9986
D	31.5	1.50	1.068365	0.9986	0.9986
E	32.0	1.33	1.068385	0.9987	0.9987

$\phi(\cdot)$ is the probability density function (PDF) of the standard normal variable, $C_{dr} = (\mu - \mathrm{m})/\mathrm{d}$, and $C_{dp} = \sigma/\mathrm{d}$. Therefore, $\hat{S}_{pk}$ is asymptotically normal-distributed with mean S_{pk} and variance $(a^2 + b^2)/36n\phi^2(3S_{pk})$. Furthermore, Pearn and Chuang [34] investigated the accuracy of the natural estimator of S_{pk}, using a simulation technique to find the relative bias and the relative mean square error for some commonly used quality requirement.

Most of the results obtained regarding the statistical properties of estimated capability indices are based on one single sample. However, to use estimators based on several small multiple samples and interpret the results as if they were based on a single sample may result in incorrect conclusions. In order to use past in-control data from multiple samples to make decisions regarding process capability, the distribution of the estimated capability index based on multiple samples should be taken into account. So, following we will investigate the sampling distribution of S_{pk} on multiple samples.

2.3. Estimating S_{pk} Under Multiple Samples

For the case when the studied characteristic of the process is normally distributed and we have m multiple samples where the sample size of the ith sample is n. Let x_{ij}, $i = 1,\dots, m$; $j = 1,\dots, n$, be the characteristic value of the $m\times n$ samples with mean μ and variance σ^2. Assume that the process is in statistical control during the time period that the multiple samples are taken. Consider the process is monitored using a $\bar{X}$-chart together with a S-chart. Then, for each multiple sample, let $\bar{x}_i$ and s_i^2 denote the sample mean and sample variance, respectively, of the ith sample and let N denote the total number of observations, i.e

$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}\ ,\quad s_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2 \quad \text{and} \quad N = \sum_{i=1}^{m} n = mn\,.$$

As an estimator of μ, we use the overall sample mean, i.e.

$$\hat{\mu} = \bar{\bar{x}} = \frac{1}{m}\sum_{i=1}^{m}\bar{x}_i = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\ .$$

We consider two ways to compute the variance estimator in estimating S_{pk} (Hubele and Vänman [18]). One estimator of σ^2 is the pooled variance estimator defined as

$$\hat{\sigma}^2 = s_p^2 = \frac{1}{mn}\sum_{i=1}^{m}(n-1)s_i^2 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2\ .$$

The other is an un-pooled variance estimator defined as

$$\hat{\sigma}^2 = s_u^2 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2\ .$$

The natural estimator of S_{pk} is

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\hat{\mu}}{\hat{\sigma}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\mu}-LSL}{\hat{\sigma}}\right)\right\}.$$

It is obviously that the sampling distribution of $\hat{S}_{pk}$ is a very complex function of $\hat{\mu}$ and $\hat{\sigma}$. However, a useful approximation could be obtained by the following expansion of S_{pk}. For deriving convenience, we use the notations in Lee's paper:

$$C_{dr} = \frac{\mu - \mathrm{m}}{\mathrm{d}},\ C_{dp} = \frac{\sigma}{\mathrm{d}},\ \hat{C}_{dr} = \frac{\hat{\mu}-\mathrm{m}}{\mathrm{d}},\ \hat{C}_{dp} = \frac{\hat{\sigma}}{\mathrm{d}},$$

and then the estimator of S_{pk} can be rewritten as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)\right\}.$$

Let

$$Z = \sqrt{mn}\left(\hat{\mu}-\mu\right) \text{ and } Y = \sqrt{mn}\left(\hat{\sigma}^2-\sigma^2\right).$$

We note that $\hat{\mu}$ is a complete sufficient statistic and $\hat{\sigma}^2$ (for either s_p^2 or s_u^2) is an ancillary statistic, so by Basu's theorem Z and Y are independent. Since the first two moments of $\hat{\mu}$ and $\hat{\sigma}^2$ exist, by the Central Limit Theorem Y converges to $\mathrm{N}(0, 2\sigma^4)$ under both estimators, s_p^2 and s_u^2, and Z converges to $\mathrm{N}(0,\sigma^2)$ as mn goes to infinity. Consequently, by the Taylor's expansion $\hat{S}_{pk}$ can be expressed as

$$\hat{S}_{pk} = S_{pk} + \frac{W}{6\sqrt{mn}\phi(3S_{pk})} + O_p\left(\frac{1}{mn}\right),$$

where

$$W = -\frac{1}{2\sigma^2}Y\left[\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right]-\frac{1}{\sigma}Z\left[\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right]$$

which is normally distributed with mean zero and variance a^2+b^2,

$$a = \frac{1}{\sqrt{2}}\left\{\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\},\ b = \phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)$$

and ϕ is the pdf of the standard normal distribution (See Appendix I for explicit derivation). We let

$$\hat{S}'_{pk} = \hat{S}_{pk} - O_p\left(\frac{1}{mn}\right)$$

Thus, our $\hat{S}'_{pk}$ is normally distributed, i.e.

$$\hat{S}'_{pk} \sim N\left(S_{pk},\ \frac{a^2+b^2}{36mn\phi^2(3S_{pk})}\right).$$

For testing process performance, we consider the following null and alternative hypotheses:

$H_0: S_{pk} \le c$, c is a specified value. (Process is incapable)

$H_1: S_{pk} > c$. (Process is capable)

The testing statistic is

$$T = \left(\hat{S}_{pk} - c\right)\frac{6\sqrt{mn}\phi(3\hat{S}_{pk})}{\sqrt{\hat{a}^2+\hat{b}^2}}$$

where $\hat{a}$ and $\hat{b}$ are estimates of a and b, with C_{dr} and C_{dp} replaced by $\hat{C}_{dr}$ and $\hat{C}_{dp}$ respectively. The null hypothesis H_0 is rejected at α level if $T > z_\alpha$, where z_α is the upper $100\alpha\%$ point of the standard normal distribution. An approximate $1-\alpha$ confidence interval for S_{pk} is

$$\left(\hat{S}_{pk} - z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{a}^2+\hat{b}^2}}{6\sqrt{mn}\phi(3\hat{S}_{pk})},\ \hat{S}_{pk} + z_{\frac{\alpha}{2}}\frac{\sqrt{\hat{a}^2+\hat{b}^2}}{6\sqrt{mn}\phi(3\hat{S}_{pk})}\right).$$

Table 2-2. Some different processes and corresponding $mnVar(\hat{S}'_{pk})$ with S_{pk} = 1.0.

μ	σ	C_{dr}	C_{dp}	a	b	$mnVar(\hat{S}'_{pk})$
8.0000	2.0000	0.0000000	0.3333333	0.0188027	0.0000000	0.499999935
8.0452	1.9995	0.0075315	0.3332483	0.0187932	0.0006001	0.499999793
8.0908	1.9979	0.0151253	0.3329907	0.0187643	0.0012004	0.499995833
8.1371	1.9953	0.0228474	0.3325531	0.0187158	0.0018014	0.499978579
8.1846	1.9915	0.0307724	0.3319220	0.0186472	0.0024033	0.499930778
8.2339	1.9865	0.0389895	0.3310765	0.0185575	0.0030066	0.499826688
8.2857	1.9799	0.0476115	0.3299852	0.0184455	0.0036117	0.499628976
8.3407	1.9716	0.0567897	0.3286013	0.0183095	0.0042190	0.499284463
8.4004	1.9611	0.0667404	0.3268532	0.0181470	0.0048292	0.498717347
8.4668	1.9478	0.0777965	0.3246260	0.0179547	0.0054431	0.497816042
8.5431	1.9303	0.0905222	0.3217203	0.0177273	0.0060618	0.496407586
8.6361	1.9065	0.1060173	0.3177420	0.0174563	0.0066871	0.494199792

2.4. Lower Confidence Bound of S_{pk}

We note that, for the same S_{pk}, the variance of $\hat{S}'_{pk}$ increases as the process mean closes to the center of the specification limits, and would be largest when the process mean is at the center of the specification limits. For two processes with the same S_{pk}, i.e. the same process yield, one with process mean away from the center of the specification limits must have smaller variance in order to have process yield equal to the other. Also, the process with smaller variance would have smaller variance of $\hat{S}'_{pk}$. Table 2-2 shows some different processes and corresponding $mnVar(\hat{S}'_{pk})$ with LSL = 2.0, USL = 14.0, and S_{pk} =1.0.

Table 2-3. Approximated LB for various m, $\hat{S}_{pk}$, n = 5(5)50, and α = 0.05, 0.025, 0.01.

$\hat{S}_{pk}$		1.0			1.33			1.5			1.67			2.0		
m	n	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
3	5	0.7690	0.7364	0.7018	1.0253	0.9819	0.9358	1.1535	1.1046	1.0528	1.2817	1.2274	1.1698	1.5380	1.4729	1.4037
	10	0.8248	0.7980	0.7690	1.0997	1.0640	1.0253	1.2372	1.1970	1.1535	1.3747	1.3301	1.2817	1.6496	1.5961	1.5380
	15	0.8522	0.8287	0.8030	1.1362	1.1050	1.0707	1.2783	1.2431	1.2046	1.4204	1.3813	1.3384	1.7044	1.6575	1.6061
	20	0.8694	0.8482	0.8248	1.1592	1.1309	1.0997	1.3041	1.2723	1.2372	1.4491	1.4137	1.3747	1.7388	1.6964	1.6496
	25	0.8815	0.8620	0.8403	1.1754	1.1493	1.1204	1.3223	1.2930	1.2605	1.4693	1.4367	1.4006	1.7631	1.7240	1.6807
	30	0.8907	0.8725	0.8522	1.1876	1.1633	1.1362	1.3361	1.3088	1.2783	1.4846	1.4542	1.4204	1.7815	1.7450	1.7044
	35	0.8980	0.8808	0.8616	1.1973	1.1744	1.1488	1.3470	1.3212	1.2925	1.4968	1.4681	1.4361	1.7961	1.7617	1.7233
	40	0.9040	0.8876	0.8694	1.2053	1.1835	1.1592	1.3560	1.3315	1.3041	1.5067	1.4795	1.4491	1.8080	1.7753	1.7388
	45	0.9090	0.8934	0.8759	1.2119	1.1912	1.1679	1.3635	1.3401	1.3139	1.5150	1.4890	1.4600	1.8180	1.7868	1.7519
	50	0.9132	0.8983	0.8815	1.2176	1.1977	1.1754	1.3699	1.3475	1.3223	1.5221	1.4972	1.4693	1.8265	1.7966	1.7631
6	5	0.8248	0.7980	0.7690	1.0997	1.0640	1.0253	1.2372	1.1970	1.1535	1.3747	1.3301	1.2817	1.6496	1.5961	1.5380
	10	0.8694	0.8482	0.8248	1.1592	1.1309	1.0997	1.3041	1.2723	1.2372	1.4491	1.4137	1.3747	1.7388	1.6964	1.6496
	15	0.8907	0.8725	0.8522	1.1876	1.1633	1.1362	1.3361	1.3088	1.2783	1.4846	1.4542	1.4204	1.7815	1.7450	1.7044
	20	0.9040	0.8876	0.8694	1.2053	1.1835	1.1592	1.3560	1.3315	1.3041	1.5067	1.4795	1.4491	1.8080	1.7753	1.7388
	25	0.9132	0.8983	0.8815	1.2176	1.1977	1.1754	1.3699	1.3475	1.3223	1.5221	1.4972	1.4693	1.8265	1.7966	1.7631
	30	0.9202	0.9063	0.8907	1.2269	1.2084	1.1876	1.3803	1.3595	1.3361	1.5337	1.5106	1.4846	1.8404	1.8127	1.7815
	35	0.9257	0.9127	0.8980	1.2342	1.2169	1.1973	1.3885	1.3690	1.3470	1.5428	1.5212	1.4967	1.8514	1.8254	1.7961
	40	0.9301	0.9178	0.9040	1.2401	1.2238	1.2053	1.3952	1.3768	1.3560	1.5503	1.5298	1.5067	1.8603	1.8357	1.8080
	45	0.9338	0.9222	0.9089	1.2451	1.2295	1.2119	1.4008	1.3833	1.3634	1.5565	1.5370	1.5150	1.8677	1.8444	1.8179
	50	0.9370	0.9259	0.9132	1.2493	1.2345	1.2176	1.4056	1.3888	1.3698	1.5618	1.5432	1.5221	1.8741	1.8518	1.8265

The lower bounds displayed in Table 2-3 are calculated under the condition that process mean is on the center of specification limits for assurance purpose. This approach ensures that the conclusions made based on the lower bounds have the smallest type I error α, the risk of wrongly concluding an incapable process as capable. When the practitioner wants to know what the least process yield (or say S_{pk}) is, necessary samples could be taken from the "stable" process to calculate the $\hat{S}_{pk}$ and check the lower bound. The lower bound represents the minimal S_{pk} of the process with $1-\alpha$ confidence level.

For the convenience of practitioners, we also develop a Matlab program to calculate the lower bounds (see Appendix II). Table 2-3 shows the lower bounds LB computed from the normal approximation for $\hat{S}_{pk}$ = 1.0, 1.33, 1.5, 1.67, 2.0, n = 5(5)50, m = 3, 6, and α = 0.05, 0.025, 0.01. For example, sampling with number of multiple samples m = 3 and each of sample size n = 50, resulting in sampling estimate $\hat{S}_{pk}$ = 1.67, we then conclude that the process has at least S_{pk} = 1.5221 with 95% confidence level.

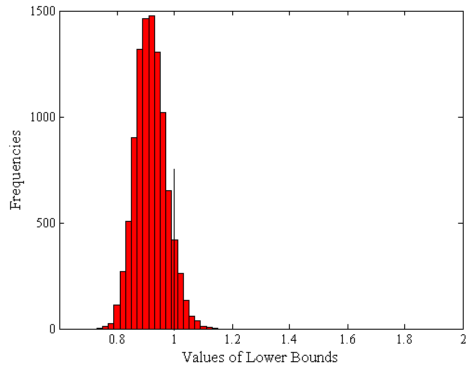


Figure 2-2(a). Histogram of 10000 lower bounds with S_{pk} = 1.00.

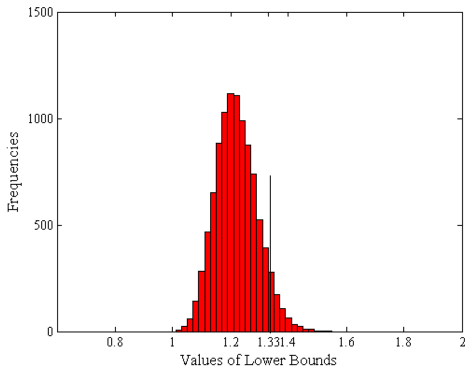


Figure 2-2(b). Histogram of 10000 lower bounds with S_{pk} = 1.33.

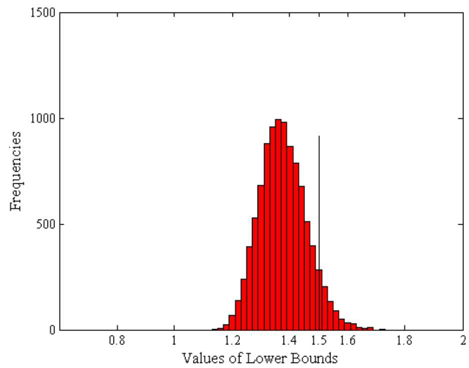


Figure 2-2(c). Histogram of 10000 lower bounds with S_{pk} = 1.50.

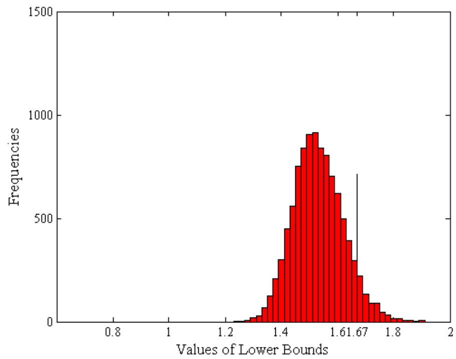


Figure 2-2(d). Histogram of 10000 lower bounds with S_{pk} = 1.67.

2.5. Accuracy of the Normal Approximation

In order to assess the normally approximated distribution of $\hat{S}'_{pk}$, we simulate with 10,000 replications to generate 10,000 estimates of $\hat{S}_{pk}$, calculate their lower bounds, and compare with the real (preset) S_{pk} for various commonly used quality requirement. Figures 2-2(a) to 2-2(d) show histograms of lower bounds each of 10,000 replications with α = 0.05, m = 3, n = 50, S_{pk} = 1.00, 1.33, 1.50, and 1.67, respectively. Table 2-4 displays the actual type I errors for various m, n, S_{pk}, and each with 10,000 simulated lower bounds.

Table 2-4. Simulated type I errors α for various m, n, and S_{pk} with 10,000 lower bounds.

m	S_{pk}	n=10	n=20	n=30	n=50	n=100	n=150	n=200
1	1.00	0.1520	0.1156	0.1015	0.0812	0.0788	0.0680	0.0695
	1.33	0.1593	0.1147	0.0987	0.0839	0.0750	0.0680	0.0694
	1.50	0.1634	0.1161	0.1015	0.0869	0.0707	0.0677	0.0656
	1.67	0.1629	0.1132	0.1053	0.0842	0.0764	0.0673	0.0663
	2.00	0.1691	0.1188	0.1012	0.0845	0.0737	0.0701	0.0684
2	1.00	0.1126	0.0897	0.0817	0.0748	0.0655	0.0628	0.0602
	1.33	0.1124	0.0901	0.0823	0.0715	0.0677	0.0632	0.0614
	1.50	0.1148	0.0940	0.0842	0.0706	0.0684	0.0692	0.0663
	1.67	0.1158	0.0939	0.0823	0.0754	0.0702	0.0588	0.0584
	2.00	0.1223	0.0904	0.0844	0.0715	0.0639	0.0613	0.0641
3	1.00	0.1017	0.0838	0.0742	0.0692	0.0629	0.0579	0.0563
	1.33	0.1011	0.0825	0.0726	0.0651	0.0671	0.0560	0.0556
	1.50	0.1016	0.0791	0.0784	0.0638	0.0632	0.0600	0.0567
	1.67	0.1048	0.0854	0.0681	0.0681	0.0624	0.0586	0.0556
	2.00	0.1063	0.0805	0.0772	0.0693	0.0664	0.0644	0.0576
6	1.00	0.0816	0.0693	0.0705	0.0634	0.0619	0.0573	0.0555
	1.33	0.0831	0.0750	0.0669	0.0576	0.0624	0.0545	0.0578
	1.50	0.0832	0.0727	0.0677	0.0634	0.0599	0.0611	0.0555
	1.67	0.0861	0.0744	0.0673	0.0576	0.0609	0.0603	0.0575
	2.00	0.0763	0.0741	0.0699	0.0681	0.0626	0.0568	0.0558
9	1.00	0.0739	0.0675	0.0687	0.0573	0.0583	0.0561	0.0553
	1.33	0.0752	0.0655	0.0652	0.0546	0.0590	0.0546	0.0550
	1.50	0.0762	0.0671	0.0610	0.0628	0.0528	0.0561	0.0535
	1.67	0.0804	0.0688	0.0616	0.0623	0.0597	0.0577	0.0554
	2.00	0.0748	0.0713	0.0641	0.0636	0.0609	0.0533	0.0498
12	1.00	0.0707	0.0643	0.0593	0.0569	0.0553	0.0534	0.0557
	1.33	0.0751	0.0657	0.0559	0.0570	0.0625	0.0568	0.0521
	1.50	0.0671	0.0652	0.0641	0.0590	0.0559	0.0559	0.0545
	1.67	0.0728	0.0682	0.0625	0.0597	0.0587	0.0577	0.0554
	2.00	0.0705	0.0599	0.0645	0.0575	0.0542	0.0538	0.0520

Table 2-5. Ratios of the average of 10,000 lower bounds and the real S_{pk}, i.e. $\overline{LB}/S_{pk}$.

m	S_{pk}	n=10	n=20	n=30	n=50	n=100	n=150	n=200
1	1.00	0.8056	0.8286	0.8483	0.8726	0.9030	0.9178	0.9275
	1.33	0.8122	0.8287	0.8487	0.8729	0.9032	0.9179	0.9273
	1.50	0.8156	0.8302	0.8505	0.8730	0.9029	0.9178	0.9271
	1.67	0.8171	0.8312	0.8511	0.8732	0.9032	0.9186	0.9274
	2.00	0.8239	0.8341	0.8510	0.8739	0.9038	0.9177	0.9278
2	1.00	0.8283	0.8611	0.8810	0.9030	0.9274	0.9388	0.9460
	1.33	0.8314	0.8636	0.8802	0.9025	0.9276	0.9396	0.9472
	1.50	0.8319	0.8627	0.8817	0.9021	0.9285	0.9400	0.9469
	1.67	0.8304	0.8628	0.8815	0.9037	0.9277	0.9391	0.9466
	2.00	0.8358	0.8629	0.8816	0.9020	0.9277	0.9394	0.9471
3	1.00	0.8496	0.8820	0.8982	0.9179	0.9390	0.9493	0.9552
	1.33	0.8493	0.8806	0.8979	0.9170	0.9402	0.9497	0.9555
	1.50	0.8484	0.8810	0.8983	0.9179	0.9390	0.9494	0.9555
	1.67	0.8506	0.8815	0.8976	0.9176	0.9393	0.9493	0.9558
	2.00	0.8516	0.8812	0.8987	0.9185	0.9401	0.9498	0.9556
6	1.00	0.8807	0.9095	0.9241	0.9394	0.9563	0.9632	0.9682
	1.33	0.8803	0.9100	0.9239	0.9388	0.9559	0.9634	0.9684
	1.50	0.8810	0.9106	0.9241	0.9394	0.9561	0.9636	0.9683
	1.67	0.8825	0.9096	0.9241	0.9387	0.9556	0.9635	0.9685
	2.00	0.8819	0.9102	0.9240	0.9400	0.9563	0.9635	0.9679
9	1.00	0.8988	0.9245	0.9369	0.9499	0.9630	0.9697	0.9734
	1.33	0.8994	0.9236	0.9357	0.9495	0.9637	0.9700	0.9735
	1.50	0.8988	0.9248	0.9361	0.9499	0.9634	0.9699	0.9736
	1.67	0.8995	0.9244	0.9363	0.9500	0.9638	0.9702	0.9737
	2.00	0.8996	0.9247	0.9369	0.9499	0.9635	0.9697	0.9735
12	1.00	0.9104	0.9329	0.9442	0.9561	0.9681	0.9736	0.9770
	1.33	0.9100	0.9330	0.9434	0.9558	0.9683	0.9736	0.9769
	1.50	0.9093	0.9333	0.9441	0.9557	0.9682	0.9738	0.9771
	1.67	0.9095	0.9337	0.9441	0.9560	0.9682	0.9736	0.9773
	2.00	0.9106	0.9323	0.9446	0.9556	0.9677	0.9735	0.9772

The results in Table 2-4 show that when m = 12, n = 50, the confidence level of the normal approximation is almost equal to the preset 1-α (the confidence levels are all greater than 94%). As we know, the simulation results are in large variation, and by Central Limit Theorem the average is in small variation, so we calculate the average of the lower bounds and compare to the real S_{pk}. Table 2-5 shows the ratios of the average lower bounds relative to the real S_{pk}. It is noted that no matter what the real S_{pk} is, the ratios of $\overline{LB}/S_{pk}$ are almost equal with the same m and n. Thus, it is reasonable to estimate the true S_{pk} from the ratios. For example, when m = 3 and n = 200, practitioners can repeat the sampling procedure, obtain the average lower bound, and estimate the real S_{pk} by $\overline{LB}/0.9558$.

Table 2-6. Sample sizes required for the normal approximation to converge with α = 0.05.

m	S_{pk}	Designated Accuracy, ε									
		0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
1	1.00	193	238	301	392	534	769	1201	2135	4802	19208
	1.33	342	422	534	697	949	1366	2135	3795	8537	34147
	1.50	433	534	676	882	1201	1729	2702	4802	10805	43217
	1.67	534	659	834	1089	1483	2135	3335	5929	13339	53354
	2.00	769	949	1201	1568	2135	3074	4802	8537	19208	76830
2	1.00	97	119	151	196	267	385	601	1068	2401	9604
	1.33	171	211	267	349	475	683	1068	1898	4269	17074
	1.50	217	267	338	441	601	865	1351	2401	5403	21609
	1.67	267	330	417	545	742	1068	1668	2965	6670	26677
	2.00	385	475	601	784	1068	1537	2401	4269	9604	38415
3	1.00	65	80	101	131	178	257	401	712	1601	6403
	1.33	114	141	178	233	317	456	712	1265	2846	11383
	1.50	145	178	226	294	401	577	901	1601	3602	14406
	1.67	178	220	278	363	495	712	1112	1977	4447	17785
	2.00	257	317	401	523	712	1025	1601	2846	6403	25610
6	1.00	33	40	51	66	89	129	201	356	801	3202
	1.33	57	71	89	117	159	228	356	633	1423	5692
	1.50	73	89	113	147	201	289	451	801	1801	7203
	1.67	89	110	139	182	248	356	556	989	2224	8893
	2.00	129	159	201	262	356	513	801	1423	3202	12805
9	1.00	22	27	34	44	60	86	134	238	534	2135
	1.33	38	47	60	78	106	152	238	422	949	3795
	1.50	49	60	76	98	134	193	301	534	1201	4802
	1.67	60	74	93	121	165	238	371	659	1483	5929
	2.00	86	106	134	175	238	342	534	949	2135	8537
12	1.00	17	20	26	33	45	65	101	178	401	1601
	1.33	29	36	45	59	80	114	178	317	712	2846
	1.50	37	45	57	74	101	145	226	401	901	3602
	1.67	45	55	70	91	124	178	278	495	1112	4447
	2.00	65	80	101	131	178	257	401	712	1601	6403

We further consider how many sample size n should be taken to ensure that the sampling estimator is closed enough to the real S_{pk} within a designated accuracy ε (Pearn *et al.* [36]). Table 2-6 displays the sample sizes required for the normal approximation to converge to the real S_{pk} within a designated accuracy ε less than 0.10, 0.09, 0.08, 0.07, 0.06, 0.05, 0.04, 0.03, 0.02, 0.01, respectively, and the derivation is briefly done as follows:

$$\Pr\left\{\left|\hat{S}'_{pk} - S_{pk}\right| \le \varepsilon\right\} \ge 1-\alpha \Rightarrow \Pr\left\{\frac{\hat{S}'_{pk} - S_{pk}}{\sqrt{Var\left(\hat{S}'_{pk}\right)}} \le \frac{\varepsilon}{\sqrt{Var\left(\hat{S}'_{pk}\right)}}\right\} \ge 1-\frac{\alpha}{2} \Rightarrow \frac{\varepsilon}{\sqrt{Var\left(\hat{S}'_{pk}\right)}} \ge \Phi^{-1}\left(1-\alpha/2\right)$$

$$\Rightarrow \frac{a^2+b^2}{36mn\phi^2(3S_{pk})} \le \frac{\varepsilon^2}{\left[\Phi^{-1}(1-\alpha/2)\right]^2} \Rightarrow mn \ge \frac{\left(a^2+b^2\right)\left[\Phi^{-1}\left(1-\alpha/2\right)\right]^2}{36\phi^2(3S_{pk})\varepsilon^2}.$$

For example, for $m = 9$, $S_{pk} = 1.33$ with risk $\alpha = 0.05$, a sample size of $n \ge 3795$ ensures that the difference between the sampling $\hat{S}_{pk}$ and the real S_{pk} is smaller than 0.01. Thus, if the sampling $\hat{S}_{pk} = 1.33$, then we can conclude that the actual performance $S_{pk} > 1.32$ with 95% confidence level. This convergence investigated is not for practical purpose, but to illustrate the behavior and the rate of convergence for the normal approximation.

2.6. An Application Example

The integrated circuits (IC) industry has been the most popular industry for previous years. Products of integrated circuits are various types such as office automation equipment (copiers, facsimile machines, printers, etc.), vending machines, banking terminals, CD or DVD players, battery chargers, etc. We investigated a company in Taiwan manufacturing one-cell rechargeable Li-ion battery packs which have advantages of low current consumption, high withstand voltage, high accuracy voltage detection, over current and short circuit protection, and wide operating temperature range. Among the advanced features, the most important one is the high accuracy voltage detection. Once the voltage detector falls down, the lifetime or reliability of the Li-ion battery pack will be discounted. The preset upper and lower specification limits of the over charge detector are USL = 4.40 V, LSL = 4.30 V, and target value is set to T= 4.35 V.

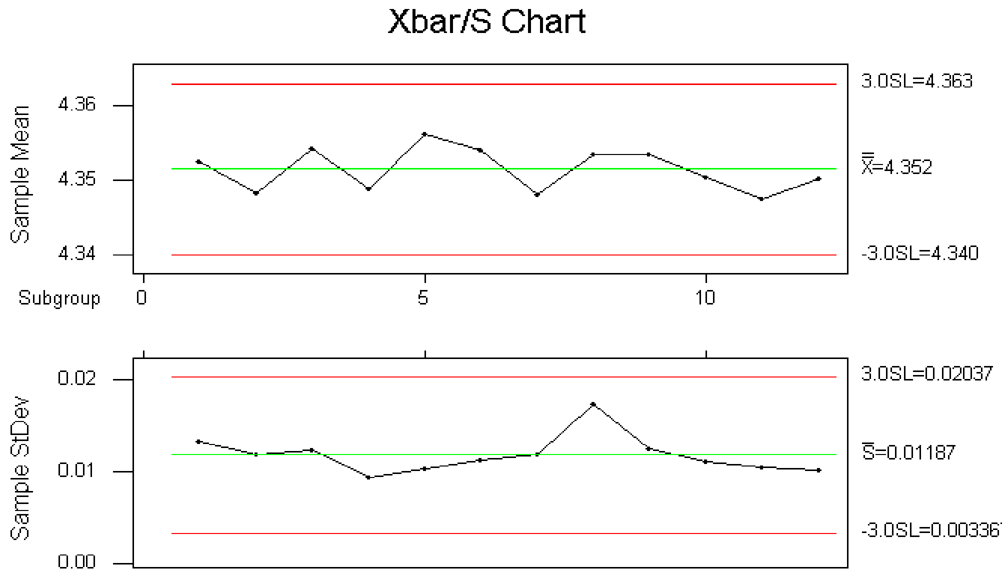


Figure 2-3. The $\bar{X}-S$ charts based on the collected 12 samples each of size 50.

Table 2-7. The collected electrical characteristic data of 12 samples each of size 50.

subsample	1	2	3	4	5	6	7	8	9	10	11	12
$\bar{x}_i$	4.3526	4.3483	4.3544	4.3490	4.3563	4.3542	4.3482	4.3537	4.3535	4.3505	4.3476	4.3502
s_i	0.0133	0.0120	0.0124	0.0093	0.0104	0.0114	0.0119	0.0174	0.0126	0.0112	0.0104	0.0102

Suppose the minimal precision requirement for this process is set to S_{pk} = 1.0. We calculate the overall sample mean $\bar{\bar{x}}_i$ = 4.35154, the pooled sample standard deviation $s_p = \sqrt{s_p^2}$ = 0.01192, the un-pooled sample standard deviation $s_u = \sqrt{s_u^2}$ = 0.01225, and

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\hat{\mu}}{\hat{\sigma}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\mu}-LSL}{\hat{\sigma}}\right)\right\}$$

$$=\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.40-4.35154}{0.01192}\right)+\frac{1}{2}\Phi\left(\frac{4.35154-4.30}{0.01192}\right)\right\}$$

$$\text{or } \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{4.40-4.35154}{0.01225}\right)+\frac{1}{2}\Phi\left(\frac{4.35154-4.30}{0.01225}\right)\right\}$$

= 1.3871 or 1.3503.

We run the program in Appendix II to find the lower bound as 1.3242 (or 1.2890). Thus, we conclude that the true value of the process capability S_{pk} would be no less than 1.3242 (or 1.2890) with 95% confidence level.

To estimate the real S_{pk}, the factory manager could implement a weekly based control system by repeating the sampling procedure for consecutive, say 10 weeks, then calculate the average lower bounds. For example, the ten weeks lower bounds result $\overline{LB} = 1.4527$. Refer to Table 2-5, the biggest ratio of $\overline{LB}/S_{pk}$ for $m = 12$ and $n = 50$ is 0.9561, then he can estimate the real $S_{pk} = 1.4527/0.9561 \doteq 1.52$. The corresponding process yield then could be estimated as 0.999994885, or equally, fraction of defectives is 5.115 ppm.

Chapter 3

Procedure of the Convolution Method for Estimating Production Yield With Sample Size Information

3.1. Introduction

As mentioned in chapter 1, numerous process capability indices (PCI) have been proposed to provide numerical measures on the production yield as well as process performance. Those indices, such as C_a, C_p, C_{pk}, C_{pm}, and C_{pmk}, are defined to emphasize the process centering, process precision or process loss, and only the index S_{pk}, provides an exact measure on the production yield. Note that the capability indices are designed to monitor the performance for stable normal or near-normal processes with symmetric tolerances. In practice, the process mean μ and the process variance σ^2 are unknown. To calculate the index value, sample data must be collected, and a great degree of uncertainty may be introduced into the assessments due to the sampling errors. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their distributions, learning that capability measures must be reported in confidence intervals or via capability testing. Statistical properties of the estimators of those indices under various process conditions have been investigated extensively, including Chan *et al.* [4], Pearn *et al.* [35], Kotz and Johnson [23], Vännman and Kotz [49], Vännman [47], Kotz and Lovelace [25], Chen [5], Zhang [54], Kotz and Johnson [24], Lee *et al.* [26], Xie *et al.* [53], Spiring *et al.* [42], Pearn *et al.* [38], Pearn *et al.* [36],Pearn *et al.* [39], Montgomery [31], Wu [52]. In this chapter, we propose the convolution method based on the S_{pk} index for estimating the production yield for single sample.

3.2. The Yield Index S_{pk}

Boyles [3] proposed a yield measurement index, referred to as S_{pk}, based on the production yield of normal processes. The yield index S_{pk}, as defined previously, also can be alternatively expressed as

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\},$$

where $C_{dr} = (\mu - \mathrm{m})/\mathrm{d}$, $C_{dp} = \sigma/\mathrm{d}$, m = $(USL + LSL)/2$ is the midpoint of the specification limits, and d = $(USL - LSL)/2$ is the half length of the specification interval.

As mentioned previously, the index C_{pk} can only provide interval estimation on the production yield. The indices C_{pm} and C_{pmk} are defined by being related to the customer's loss. Only the yield index S_{pk} can provide a one-to-one correspondence to the production yield, which can be expressed as

$$Yield = 2\Phi(3S_{pk}) - 1.$$

Table 3-1 summarizes the corresponding production yields as well as non-conformities in parts per million (PPM) for S_{pk} = 1.0(0.1)2.0, including the most commonly used performance requirements: 1.00, 1.33, 1.50, 1.67, and 2.00. For example, if a process has capability index value S_{pk} = 1.50, then the yield of the process is 0.999993205 and the corresponding non-conformities is roughly 7 parts per million.

Table 3-1. Various S_{pk} values and the corresponding production yields as well as non-conformities in PPM.

S_{pk}	Yield	PPM
1.00	0.997300204	2699.796
1.10	0.999033152	966.848
1.20	0.999681783	318.217
1.30	0.999903807	96.193
1.33	0.999933927	66.073
1.40	0.999973309	26.691
1.50	0.999993205	6.795
1.60	0.999998413	1.587
1.67	0.999999456	0.544
1.70	0.999999660	0.340
1.80	0.999999933	0.067
1.90	0.999999988	0.012
2.00	0.999999998	0.002

Assume that $X_1,\ldots,X_n$ be a random sample of the characteristic from a normal process. The natural estimator of S_{pk} is defined as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\bar{X}}{S}\right)+\frac{1}{2}\Phi\left(\frac{\bar{X}-LSL}{S}\right)\right\},$$

and can also be expressed as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)\right\},$$

where $\hat{C}_{dr} = (\bar{X}-\mathrm{m})/\mathrm{d}$ and $\hat{C}_{dp} = S/\mathrm{d}$ are natural estimators of C_{dr} and C_{dp}, respectively, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ is the sample variance. The distribution of the natural estimator of S_{pk} is mathematically intractable as it is a complex function of the statistics $\bar{X}$ and S^2 (or $\hat{C}_{dr}$ and $\hat{C}_{dp}$). However, we can profile the sampling distribution of S_{pk} by using a simulation technique. Figure 3-1 shows the histograms of $\hat{S}_{pk}$ with simulation parameters S_{pk} = 1.0, $\xi = (\mu-\mathrm{m})/\sigma$ = 0, and sample size n = 20, 30, 50, 80 each with 10,000 simulated $\hat{S}_{pk}$. The histograms reveal that the probability density function (PDF) of $\hat{S}_{pk}$ is nearly bell-shaped, symmetric to the real S_{pk} for large sample sizes, and slightly skewed to the right for small sample sizes.

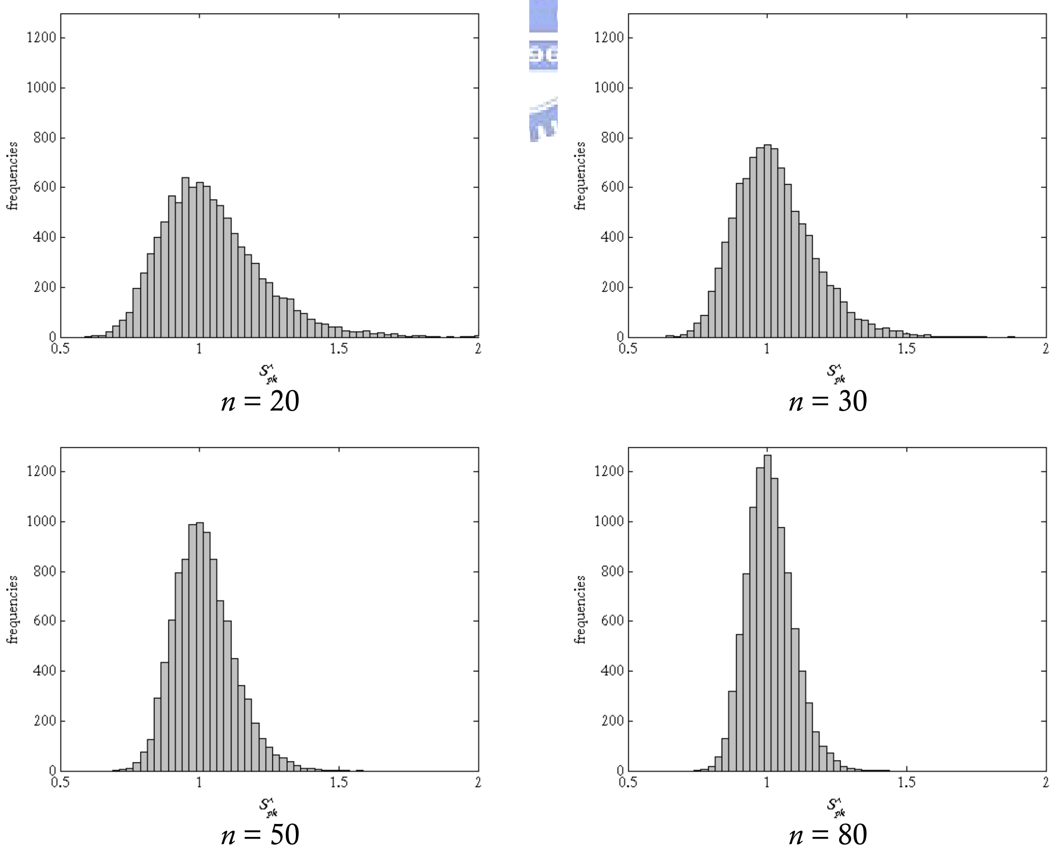


Figure 3-1. Histograms of $\hat{S}_{pk}$ with simulation parameters S_{pk} = 1.0 and ξ = 0.

Many researchers have focused on the sampling distribution of S_{pk}. Lee *et al.* [26] derived a normal approximated distribution of the estimated S_{pk}. Pearn *et al.* [34] investigated the accuracy of the normal approximation computationally, and suggested that a sample size greater than 150 is required for the normal approximation sufficiently accurate. Pearn and Cheng [33] further derived a normal approximated distribution of the estimated S_{pk} under multiple samples, and investigated the sample sizes required to converge to S_{pk} within a designated accuracy. Chen [6] considered that the formula of the normal approximation is messy and cumbersome to deal with. Chen [6] applied four bootstrap methods to find the lower confidence bounds on S_{pk}, and showed that the standard bootstrap (SB) method significantly outperforms the other three bootstrap methods in coverage fraction. We note, however, the bootstrap re-sampling method results in different solutions each time, while the theoretical sampling distribution approach provides a unique lower bound for the same sample estimates.

The distribution of $\hat{S}_{pk}$ is analytically intractable, but approximate distributions of $\hat{S}_{pk}$ can be obtained. In the following sections, two approximate distributions are considered and compared to the distribution of the estimated S_{pk} obtained via simulations.

3.3. Normal Approximation of $\hat{S}_{pk}$: $\hat{S}'_{pk}$

Lee *et al.* [26] considered a normal approximation of $\hat{S}_{pk}$, which is denoted $\hat{S}'_{pk}$ in this paper. The normal distribution of $\hat{S}'_{pk}$ is distributed with a mean S_{pk} and a variance $(a^2+b^2)/[36n\phi^2(3S_{pk})]$, i.e.

$$\hat{S}'_{pk} \sim N\left(S_{pk},\ \frac{a^2+b^2}{36n\phi^2(3S_{pk})}\right),$$

where

$$a=\frac{1}{\sqrt{2}}\left\{\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right\},\ \text{and}\ \ b=\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right).$$

The normal approximation is useful in statistical inferences for S_{pk}. Consider the following null versus alternative hypotheses:

H_0 : $S_{pk} \le$ C, a specified value;

H_1 : $S_{pk} >$ C.

The decision rule with $1-\alpha$ confidence level should be that to reject the null hypothesis H_0 if the sample statistic $\hat{S}_{pk}$ is equal to or larger than the critical value c_0, where c_0 satisfies the following equation

$$\Pr\left\{\hat{S}'_{pk} \geq c_0 \mid H_0 : S_{pk} \leq C\right\} \leq \alpha .$$

Lee *et al.* [26] suggested performing the hypothesis testing with the test statistic

$$T = \left(\hat{S}_{pk} - C\right)\frac{6\sqrt{n}\phi(3\hat{S}_{pk})}{\sqrt{\hat{a}^2 + \hat{b}^2}},$$

where $\hat{a}$ and $\hat{b}$ are the natural estimators of a and b, with C_{dr} and C_{dp} replaced by $\hat{C}_{dr}$ and $\hat{C}_{dp}$, respectively. Then, the decision rule becomes that the null hypothesis H_0 would be rejected if $T \geq z_\alpha$, where z_α is the upper $100\alpha\%$ point of the standard normal distribution.

This approach is intuitive and reasonable, but introduces additional sampling errors from estimating a and b (or C_{dr} and C_{dp}) with $\hat{a}$ and $\hat{b}$ (or $\hat{C}_{dr}$ and $\hat{C}_{dp}$). Thus, it would certainly become less reliable. For example, in Table 3-2 the sample estimate of S_{pk} in Process B is larger than the one in Process A, but contradictorily it turns out a smaller test statistic T in Process B. Table 3-2 shows a couple of examples for testing H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ in which the sample estimate of S_{pk} is larger (e.g. Processes B, D, F, and H), but on the contrary, the corresponding test statistic T is smaller.

Table 3-2. Contradiction between $\hat{S}_{pk}$ and test statistic T in Lee's method.

Process	$\bar{X}$	S	$\hat{C}_{dr}$	$\hat{C}_{dp}$	$\hat{S}_{pk}$	T
A	7.695115	1.365970	0.139023	0.273194	1.114490	0.807547
B	7.674245	1.372115	0.134849	0.274423	1.114555	0.807412
C	7.707630	1.335160	0.141526	0.267032	1.134942	1.207505
D	7.681125	1.342895	0.136225	0.268579	1.135032	1.207252
E	7.683340	1.314965	0.136668	0.262993	1.156439	1.063747
F	7.650165	1.324405	0.130033	0.264881	1.156573	1.063459
G	7.700125	1.219685	0.140025	0.243937	1.234395	1.929673
H	7.680760	1.224995	0.136152	0.244999	1.234452	1.929267

Pearn *et al.* [36] showed that for a specific S_{pk} (e.g. S_{pk} = C), the variance of $\hat{S}'_{pk}$ would be the largest with on-center processes, i.e. with $\xi = (\mu - m)/\sigma = 0$. Consequently, the critical value of testing H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$ would be the largest, and the test statistic T would be the smallest with $\xi = 0$. Hence, for practical purpose we would obtain the test statistic (or critical value) with $\xi = 0$ without having to further estimate the parameter ξ (or parameters a and b). The test statistic T obtained in this way is increasing in $\hat{S}_{pk}$, and there would be no contradiction. Pearn *et al.* [36] listed in the Table III of the published paper the critical values c_0 of the $\hat{S}'_{pk}$ approach which were obtained by the following probability

$$\Pr\left\{\hat{S}'_{pk} \geq c_0 \mid S_{pk} \leq \text{C and } \xi = 0\right\} \leq \alpha .$$

Lee *et al.* [26] showed that the normal distribution of $\hat{S}'_{pk}$ can produce an adequate approximation to the actual distribution of $\hat{S}_{pk}$ for a large enough sample size. However, Pearn *et al.* [36] noted that the normal approximation would significantly under-calculate the critical values for small sample sizes, and suggested that a sample of size greater than 150 is recommended in real applications, for which the magnitude of under-calculation would be as large as 0.02 at most. Since the critical value of the $\hat{S}'_{pk}$ approach is significantly under-calculated for small sample sizes, it is necessary to do some improvement.

3.4. Convolution Approximation of $\hat{S}_{pk}$: $\hat{S}''_{pk}$

The critical value obtained from the normal approximation is significantly under-calculated for small sample sizes. Thus, we go further to do some improvement by considering a convolution approximation of the estimated S_{pk}. First, we define the two random variables $Z = \sqrt{n}(\bar{X} - \mu)/\sigma$ and $Y = \sqrt{n}(S^2 - \sigma^2)/2\sigma^2$. The two random variables Z and Y are independent since $\bar{X}$ and S^2 are independent variables. It is well-known that the variable Z follows the standard normal distribution $N(0, 1)$ according to the famous Central Limit Theory, and Y can be expressed as a function of a chi-square random variable with $n-1$ degrees of freedom, i.e.

$$Z \sim N(0,1), \quad Y \sim \tfrac{\sqrt{n}}{2}\left(\tfrac{\chi^2_{n-1}}{(n-1)} - 1\right).$$

Then, we can rewrite the form of $\hat{S}_{pk}$ as the following analytical expansion:

$$\hat{S}_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 ZY + D_5 Y^2 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$D_1 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_0}{6\phi(3S_{pk})}\right),\ D_2 = \frac{1}{\sqrt{n}}\left(\frac{-\lambda_1}{6\phi(3S_{pk})}\right),\ D_3 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0^2}{8\left[\phi(3S_{pk})\right]^2} - \frac{\lambda_1}{12\phi(3S_{pk})}\right),$$

$$D_4 = \frac{1}{n}\left(\frac{S_{pk}\lambda_0\lambda_1}{4\left[\phi(3S_{pk})\right]^2} + \frac{\lambda_0 - \lambda_2}{6\phi(3S_{pk})}\right),\ D_5 = \frac{1}{n}\left(\frac{S_{pk}\lambda_1^2}{8\left[\phi(3S_{pk})\right]^2} + \frac{3\lambda_1 - \lambda_3}{12\phi(3S_{pk})}\right), \text{ and}$$

$$\lambda_k = \left(\frac{1-C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1-C_{dr}}{C_{dp}}\right) + (-1)^{k+1} \times \left(\frac{1+C_{dr}}{C_{dp}}\right)^k \phi\left(\frac{1+C_{dr}}{C_{dp}}\right),\ k = 0,\ 1,\ 2,\ 3.$$

Let

$$\hat{S}''_{pk} = S_{pk} + D_1 Z + D_2 Y + D_3 Z^2 + D_4 ZY + D_5 Y^2 .$$

The cumulative distribution function (CDF) of $\hat{S}''_{pk}$, $F_{\hat{S}''_{pk}}(x)$, then can be derived by the probability

$$F_{\hat{S}''_{pk}}(x) = \Pr\left\{\hat{S}''_{pk} - x \le 0\right\}$$

$$= \Pr\left\{ D_3\left(Z + \frac{D_1 + D_4 Y}{2D_3}\right)^2 - \frac{E_1(Y+E_3)^2}{4D_3} + \frac{\Delta_1(x)}{4D_3E_1} \le 0 \right\},$$

where

$$E_1 = D_4^2 - 4D_3D_5,\ \ E_2 = D_1D_4 - 2D_2D_3,\ \ E_3 = \tfrac{E_2}{E_1},\ \ E_4 = 4D_3S_{pk} - D_1^2, \text{ and}$$

$$\Delta_1(x) = E_2^2 - E_1(4D_3x - E_4).$$

The explicit form of the CDF of $\hat{S}''_{pk}$ is presented in the Appendix. The CDF of $\hat{S}''_{pk}$ consists of eight parts according to the signs of D_3, E_1 and y_0+E_3, where $y_0 = -\frac{\sqrt{n}}{2}$ is the minimal value of the variable Y. Applying the Leibniz's rule for derivatives, we can also obtain the probability density function (PDF) of $\hat{S}''_{pk}$.

Again, we consider the following hypothesis testing

$H_0 : S_{pk} \le C$, a specified value;

$H_1 : S_{pk} > C$.

It is inevitable to face the same problem or contradiction as in the normal approximation. Thus, we examine the behavior of the critical values c_0 against the parameter ξ before we do the hypothesis testing for S_{pk}. We perform extensive calculations to obtain the critical values c_0 for ξ = 0(0.05)3.0, n = 20(10)200, S_{pk}= 1.0(0.1)2.0, 1.33, 1.67, and confidence level $1-\alpha$ = 0.95. Figure 3-2 shows parts of the results for ξ versus the critical values. The parameter values we investigated, ξ = 0(0.05)3.0, cover a wide range of applications with process capability $S_{pk} \geq 1.0$. Note that for an on-center process the yield index $S_{pk} < 1.0$ indicates that six-sigma of the process is larger than the manufacturing specification tolerance, i.e. $6\sigma > USL - LSL$, and such a process is said to be inadequate. The results of our extensive calculations show the following features of the critical values obtained from the convolution approximation.

(i) The critical value obtains its maximum with ξ around 0.5, minimum with ξ = 0, and stays at the same value for $\xi \geq 1.0$ in all cases.

(ii) The critical value reaches its maximum with ξ slightly larger than 0.5 for $n \leq 50$, and with ξ slightly smaller than 0.5 for $n > 50$.

(iii) The larger the sample size n, the *smaller* the difference between the maximal and minimal critical values.

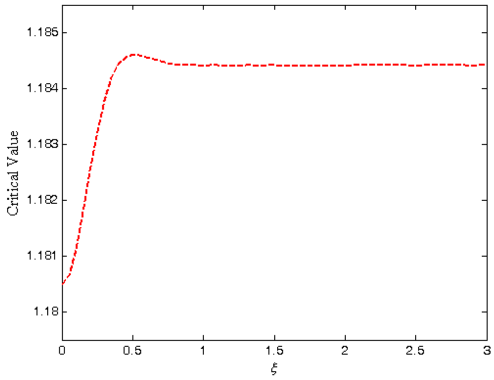
(iv) The larger the sample size n, the *larger* the difference between the maximal critical value (with ξ around 0.5) and the converged critical value (with $\xi \geq 1.0$).

(v) The larger the value of S_{pk}, the *larger* the difference between the maximal and minimal critical values.

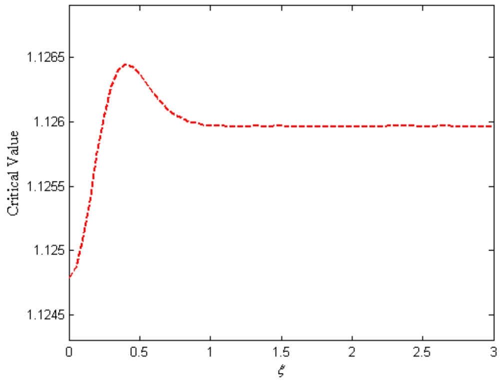
(vi) The larger the value of S_{pk}, the *smaller* the difference between the maximal critical value (with ξ around 0.5) and the converged one (with $\xi \geq 1.0$).

(vii) The difference between the maximal critical value (with ξ around 0.5) and the converged critical value (with $\xi \geq 1.0$) is always less than 0.0006.

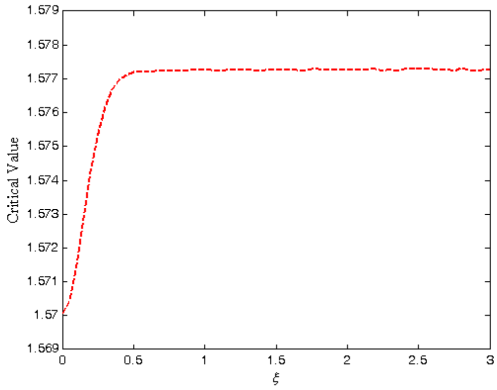
(viii) The critical value is increasing in S_{pk} (the testing parameter), and decreasing in sample size n, which is definite in the statistical inference.



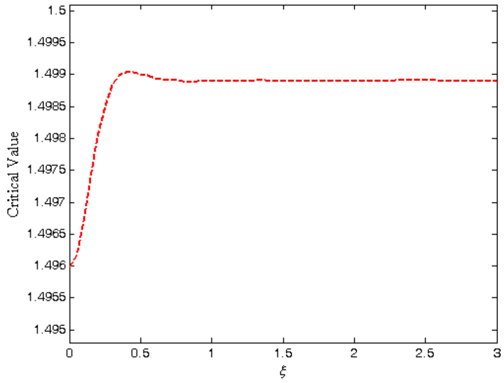
n = 50, S_{pk} = 1.00



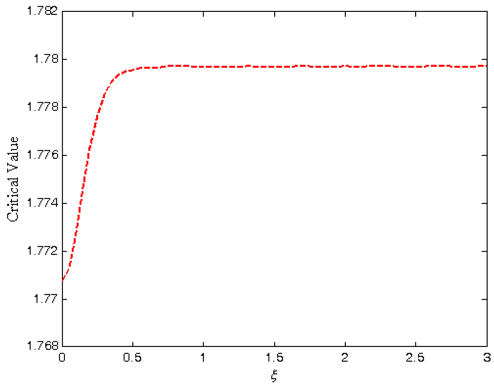
n = 100, S_{pk} = 1.00



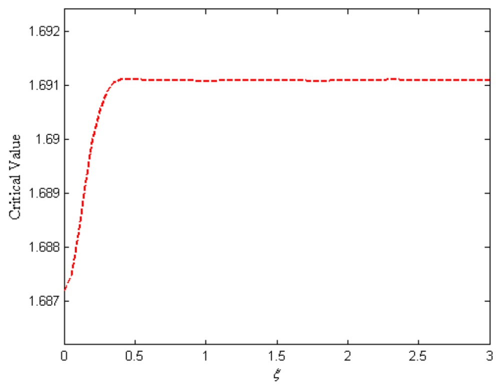
n = 50, S_{pk} = 1.33



n = 100, S_{pk} = 1.33



n = 50, S_{pk} = 1.50



n = 100, S_{pk} = 1.50

Figure 3-2. ξ versus critical values for various n and S_{pk}.

For assurance purpose, we calculate the critical value based on the convolution approximation with $\xi = 0.5$ to obtain the maximal critical value c_0 for testing the hypotheses H_0: $S_{pk} \leq$ C versus H_1: S_{pk} > C, since the critical value c_0 reaches its maximum with ξ around 0.5 in all cases.

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 \mid S_{pk} \leq \text{C and } \xi = 0.5\right\} \leq \alpha .$$

Thus, the level of confidence can be ensured, and the decisions made based on such an approach are indeed more reliable. We note that the above result is impossible to prove mathematically.

3.5. Comparisons of Both Approximations

3.5.1. Comparison of probability curves

We perform extensive calculations to draw the PDFs and CDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ as well as the density and distribution curves of the estimated S_{pk} via simulation for process parameters ξ = 0(0.25)1.0, S_{pk} = 1.0(0.25)2.0, and sample size n = 30, 50, 80, 100. Each of the density and distribution curves is obtained by 1,000,000 simulated $\hat{S}_{pk}$. Parts of the calculation results are presented in Figure 3-3. The calculation results also reveal the following general features.

(i) The density curve of $\hat{S}_{pk}$ is nearly bell-shaped, symmetric to the real S_{pk}, and so are the PDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$.

(ii) The tail probability of $\hat{S}''_{pk}$ is closer to the one of the simulated $\hat{S}_{pk}$ than that of $\hat{S}'_{pk}$.

(iii) The CDFs of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ are closer to the distribution curves of $\hat{S}_{pk}$ with a large ξ than those with a small ξ.

(iv) The larger the sample sizes n, the smaller the variance of $\hat{S}_{pk}$, $\hat{S}'_{pk}$, and $\hat{S}''_{pk}$, which is definite for all sample estimators.

The calculation results show that the cumulative distribution functions of $\hat{S}''_{pk}$ are closer to the distribution curves of the simulated $\hat{S}_{pk}$ than CDFs of $\hat{S}'_{pk}$. Though the distribution function of $\hat{S}''_{pk}$ (the convolution method) is more complicated than that of $\hat{S}'_{pk}$ (the normal approximation), it does produce a more accurate approximation to the sampling distribution of S_{pk} than the normal approximation. Besides, by the high development of computer technology, complex functions are no longer a problem for calculation. Making a decision accurately is relatively more important than easy calculation.

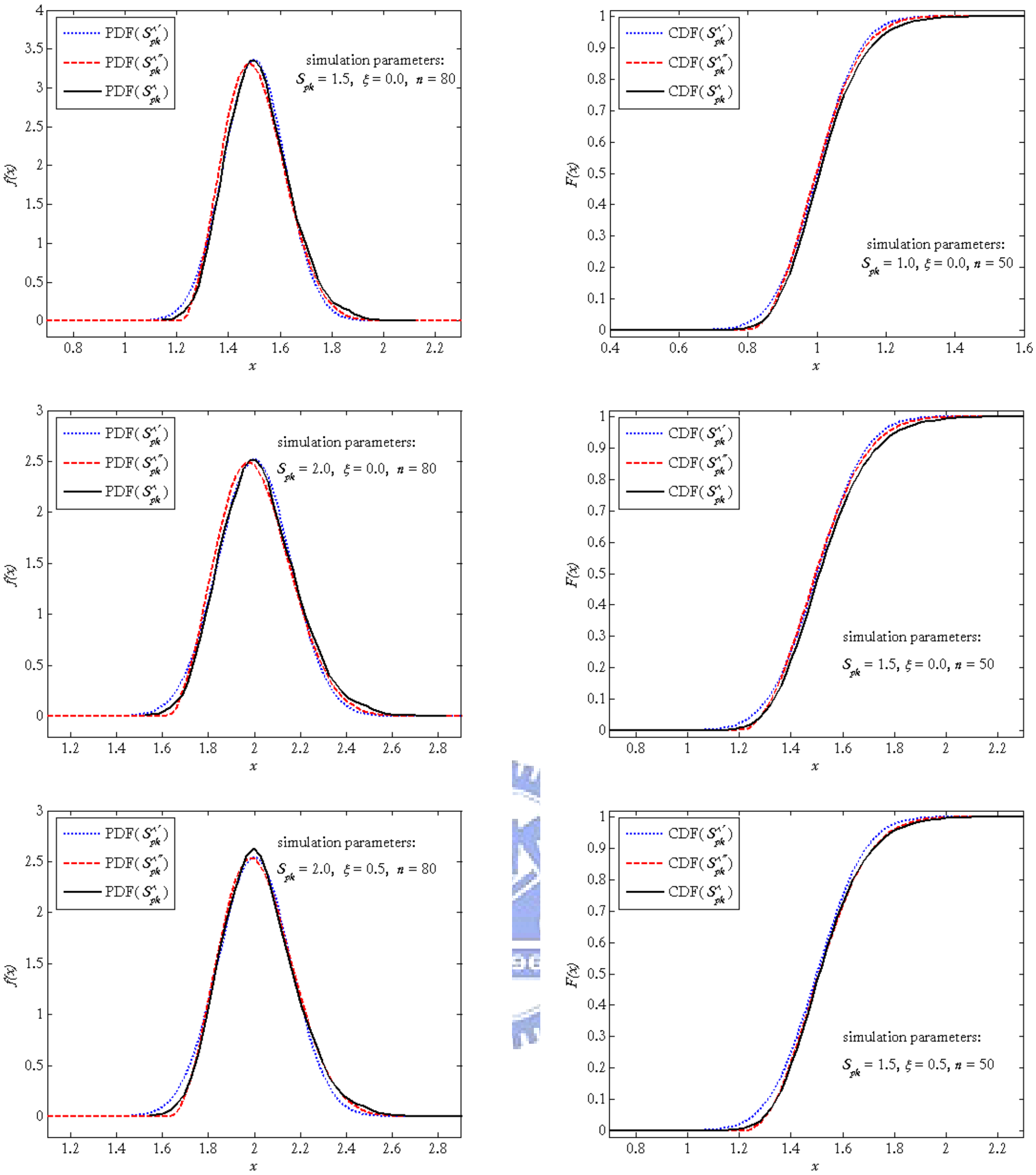


Figure 3-3. The PDF (l.h.s.) and CDF (r.h.s.) of $\hat{S}'_{pk}$ and $\hat{S}''_{pk}$ as well as the density and distribution curves of $\hat{S}_{pk}$ via simulation.

3.5.2. Comparison of critical values

Decision rule for testing hypotheses H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$ based on the normal and convolution approximations are conducted, respectively. The critical value which is closer to the critical value of the simulated $\hat{S}_{pk}$ is regarded as the more accurate and reliable one. We know that for the same S_{pk}, the maximal variance of $\hat{S}'_{pk}$ occurs at $\xi = 0$, i.e. process mean is on the center of the specification limits (Pearn *et al.* [36]). Thus, when testing the hypotheses based on the distribution of $\hat{S}'_{pk}$, we would set $\xi = 0$ to obtain the maximal critical value of the normal approximation. On the other hand, we would set $\xi = 0.5$ while testing the hypotheses based on the convolution method for the same reason.

To compute the critical value of the convolution method, we develop a Matlab program (available on request). The program reads the minimal capability requirement C, the significant level α, and the sample size n, and outputs with the critical value. Table 3-3 shows the critical values for the normal and convolution approximations as well as the one of $\hat{S}_{pk}$ via simulation, for testing hypotheses H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$ with significant level $\alpha = 0.05$. The critical values of $\hat{S}_{pk}$ (via simulation) and $\hat{S}'_{pk}$ (the normal approximation) are extracts from those presented in the paper of Pearn *et al.* [36]. We note that the critical values of the convolution approximation are always larger than those of the normal approximation, and are closer to the critical values of the simulated $\hat{S}_{pk}$, the $100(1-\alpha)$ percentile point of $\hat{S}_{pk}$ under H_0. That is the accuracy of the convolution method is greater than the normal approximation.

Note that the normal approximation significantly under-calculates the critical values for small sample sizes, particularly for $n \leq 40$, as the magnitude of the under-calculation exceeds 0.1. The large magnitude of under-calculation results in huge probability of wrongly rejecting H_0: $S_{pk} \leq C$ while actually the yield index S_{pk} is smaller than or equal to a specific value C, which incurring the risk of the customers by accepting products with less quality assurance. Therefore, for short run applications, one should avoid using the normal approximation. It is also noted that the magnitude of under-calculation of the normal approximation can be as large as 0.03 for n = 110, and 0.02 for n = 150. Thus, in real applications a sample of size greater than 150 is recommended for using the normal approximation (Pearn *et al.* [36]).

The convolution method also under-calculates the critical values, but it always provides a closer estimate to the critical value of $\hat{S}_{pk}$ than the normal approximation. The magnitude of the under-calculation of the convolution approximation is as large as 0.03 for n = 60, 0.02 for n = 70, and 0.01 for n = 90. As we know previously, the magnitude of under-calculation of the normal approximation is as large as 0.03 for n = 110, and 0.02 for n = 150. That is, if the allowable magnitude of under-calculation is 0.02, a sample size of 70 is enough for using the convolution method, while a sample size of 150 is enough for the normal approximation, which is more than twice sample sizes of the convolution method. Thus, the proposed convolution method does provide a better reliability assurance than the existing normal approximation.

Table 3-3. Critical values of the two approximations versus the simulated ones.

S_{pk}	1.00			1.33			1.50			1.67			2.00		
n	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$	$\hat{S}'_{pk}$	$\hat{S}''_{pk}$	$\hat{S}_{pk}$
20	1.26	1.31	1.37	1.68	1.74	1.82	1.89	1.97	2.05	2.11	2.19	2.30	2.52	2.63	2.74
25	1.23	1.27	1.31	1.64	1.69	1.75	1.85	1.91	1.98	2.06	2.13	2.20	2.47	2.56	2.63
30	1.21	1.25	1.28	1.61	1.66	1.70	1.82	1.87	1.93	2.03	2.09	2.14	2.43	2.50	2.57
35	1.20	1.23	1.25	1.59	1.63	1.67	1.80	1.84	1.89	2.00	2.05	2.10	2.39	2.46	2.51
40	1.18	1.21	1.23	1.58	1.61	1.64	1.78	1.82	1.85	1.98	2.02	2.06	2.37	2.42	2.47
45	1.17	1.20	1.22	1.56	1.59	1.61	1.76	1.80	1.82	1.96	2.00	2.02	2.35	2.40	2.43
50	1.16	1.18	1.20	1.55	1.58	1.60	1.75	1.78	1.80	1.95	1.98	2.01	2.33	2.38	2.40
55	1.16	1.18	1.19	1.54	1.56	1.58	1.74	1.77	1.79	1.93	1.97	1.99	2.31	2.36	2.38
60	1.15	1.17	1.18	1.53	1.55	1.57	1.73	1.75	1.77	1.92	1.95	1.98	2.30	2.34	2.36
65	1.14	1.16	1.17	1.52	1.54	1.56	1.72	1.74	1.76	1.91	1.94	1.96	2.29	2.32	2.34
70	1.14	1.15	1.16	1.52	1.54	1.55	1.71	1.73	1.77	1.90	1.93	1.95	2.28	2.31	2.33
75	1.13	1.15	1.15	1.51	1.53	1.54	1.70	1.72	1.74	1.89	1.92	1.94	2.27	2.30	2.31
80	1.13	1.14	1.15	1.50	1.52	1.53	1.70	1.72	1.73	1.89	1.91	1.93	2.26	2.29	2.31
85	1.13	1.14	1.14	1.50	1.51	1.53	1.69	1.71	1.72	1.88	1.90	1.92	2.25	2.28	2.30
90	1.12	1.13	1.14	1.49	1.51	1.52	1.68	1.70	1.71	1.88	1.90	1.91	2.25	2.27	2.28
95	1.12	1.13	1.14	1.49	1.50	1.51	1.68	1.70	1.71	1.87	1.89	1.90	2.24	2.26	2.27
100	1.12	1.13	1.13	1.49	1.50	1.50	1.67	1.69	1.70	1.86	1.88	1.89	2.23	2.26	2.27
105	1.11	1.12	1.13	1.48	1.49	1.50	1.67	1.69	1.70	1.86	1.88	1.89	2.23	2.25	2.26
110	1.11	1.12	1.13	1.48	1.49	1.50	1.67	1.68	1.69	1.86	1.87	1.89	2.22	2.24	2.25
115	1.11	1.12	1.12	1.47	1.49	1.49	1.66	1.68	1.69	1.85	1.87	1.88	2.22	2.24	2.25
120	1.11	1.11	1.12	1.47	1.48	1.49	1.66	1.67	1.68	1.85	1.86	1.87	2.21	2.23	2.24
125	1.10	1.11	1.12	1.47	1.48	1.49	1.66	1.67	1.68	1.84	1.86	1.86	2.21	2.23	2.24
130	1.10	1.11	1.12	1.47	1.48	1.48	1.65	1.67	1.68	1.84	1.86	1.86	2.20	2.22	2.23
135	1.10	1.11	1.11	1.46	1.47	1.48	1.65	1.66	1.67	1.84	1.85	1.86	2.20	2.22	2.23
140	1.10	1.11	1.11	1.46	1.47	1.48	1.65	1.66	1.67	1.83	1.85	1.86	2.20	2.21	2.22
145	1.10	1.10	1.11	1.46	1.47	1.48	1.65	1.67	1.66	1.83	1.84	1.85	2.19	2.21	2.22
150	1.10	1.10	1.11	1.46	1.47	1.47	1.64	1.65	1.66	1.83	1.84	1.85	2.19	2.21	2.21
155	1.09	1.10	1.10	1.45	1.46	1.47	1.64	1.65	1.66	1.83	1.84	1.84	2.19	2.20	2.21
160	1.09	1.10	1.10	1.45	1.46	1.47	1.64	1.65	1.65	1.82	1.84	1.84	2.19	2.20	2.21
165	1.09	1.10	1.10	1.45	1.46	1.46	1.64	1.65	1.65	1.82	1.83	1.84	2.18	2.20	2.20
170	1.09	1.10	1.10	1.45	1.46	1.46	1.63	1.64	1.65	1.82	1.83	1.84	2.18	2.19	2.20
175	1.09	1.09	1.10	1.45	1.46	1.46	1.63	1.64	1.65	1.82	1.83	1.83	2.18	2.19	2.20
180	1.09	1.09	1.10	1.45	1.45	1.46	1.63	1.64	1.65	1.82	1.83	1.83	2.17	2.19	2.19
185	1.09	1.09	1.09	1.44	1.45	1.46	1.63	1.64	1.64	1.81	1.82	1.83	2.17	2.18	2.19
190	1.08	1.09	1.09	1.44	1.45	1.45	1.63	1.64	1.64	1.81	1.82	1.83	2.17	2.18	2.19
195	1.08	1.09	1.09	1.44	1.45	1.45	1.63	1.63	1.64	1.81	1.82	1.82	2.17	2.18	2.18
200	1.08	1.09	1.09	1.44	1.45	1.45	1.62	1.63	1.64	1.81	1.82	1.82	2.16	2.18	2.18

3.5.3. Comparison of powers

The power of test calculates the probability of correctly rejecting the null hypothesis H_0: $S_{pk} \le C$ while actually $S_{pk} > C$. It is well known that the power of test is the larger the better. As we know previously, the critical value of the normal approximation is highly under-calculated for small sample sizes. Consequently, the power (probability of rejecting H_0) would be highly over-calculated if the under-calculated critical value is used as the testing rule. To compare both approximations on the same basis, we define and calculate the power of both approximations as the probability that the sample estimator is larger than the critical value c_0 of simulated $\hat{S}_{pk}$ as follows:

$$\text{power}(\hat{S}'_{pk}) = \Pr(\hat{S}'_{pk} > c_0 \mid S_{pk} \text{ under } H_1,\ \xi = 0)\,,$$

$$\text{power}(\hat{S}''_{pk}) = \Pr(\hat{S}''_{pk} > c_0 \mid S_{pk} \text{ under } H_1,\ \xi = 0.5)\,.$$

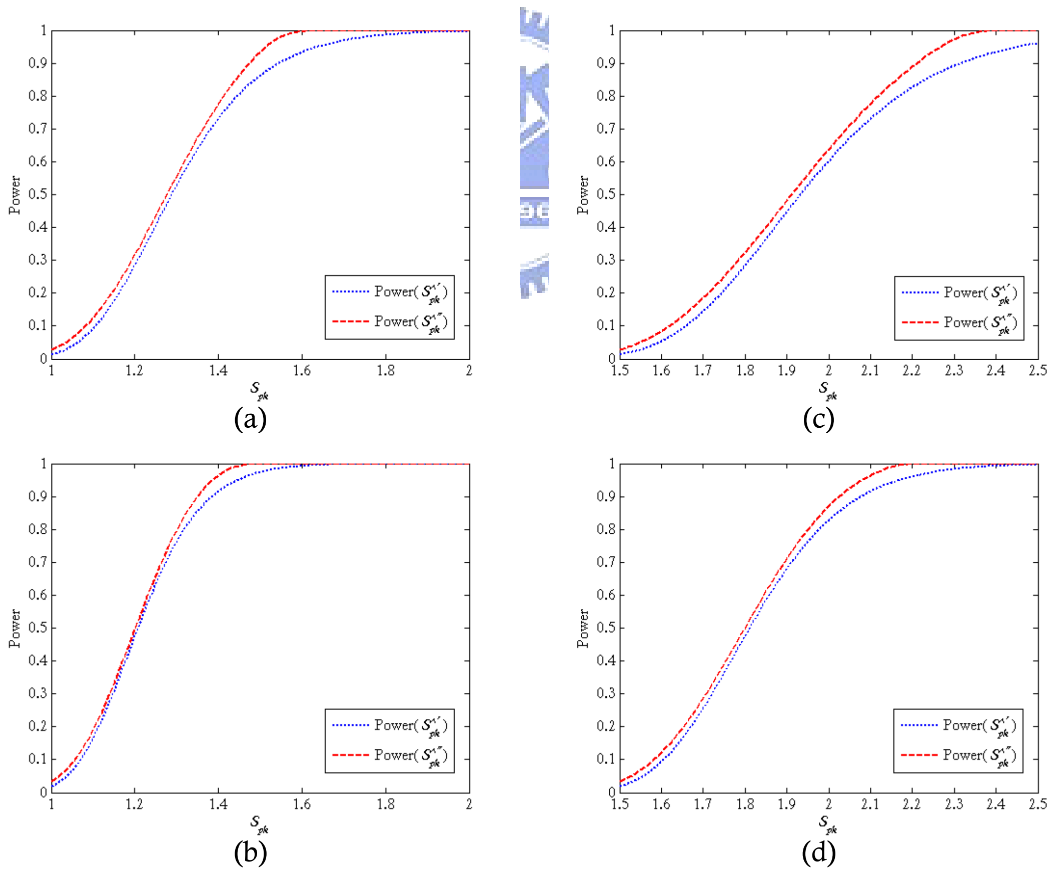


Figure 3-4. Power curves for testing (a) H_0: $S_{pk} \le 1.0$ vs H_1: $S_{pk} > 1.0$, n = 30; (b) H_0: $S_{pk} \le 1.0$ vs H_1: $S_{pk} > 1.0$, n = 50; (c) H_0: $S_{pk} \le 1.5$ vs H_1: $S_{pk} > 1.5$, n = 30; (d) H_0: $S_{pk} \le 1.5$ vs H_1: $S_{pk} > 1.5$, n = 50.

Figure 3-4 shows the powers of the normal and convolution approximations for testing (a) H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ with sample size $n = 30$, (b) H_0: $S_{pk} \leq 1.0$ versus H_1: $S_{pk} > 1.0$ with sample size $n = 50$, (c) H_0: $S_{pk} \leq 1.5$ versus H_1: $S_{pk} > 1.5$ with sample size $n = 30$, and (d) H_0: $S_{pk} \leq 1.5$ versus H_1: $S_{pk} > 1.5$ with sample size $n = 50$ and $\alpha = 0.05$. Obviously, the power of the convolution method is always greater than the normal approximation, which means the capability of correctly rejecting the bad product lots of the convolution method is stronger than that of the normal approximation.

So far we know that the tail probability and critical value of the convolution method are closer to those of the simulated $\hat{S}_{pk}$ than those of the normal approximation, and the power of the convolution method is always greater than that of the normal approximation. All of above indicate that the convolution method does make a more accurate and reliable approximation to the sampling behavior of S_{pk} than the normal approximation.

Following, we develop an efficient step-by-step procedure based on the convolution method for testing hypotheses H_0: $S_{pk} \leq C$ versus H_1: $S_{pk} > C$, where C is the minimal capability requirement defined by the customer or product designer. Engineers or practitioners can easily apply the procedure to their in-plant applications to obtain reliable decisions.

Procedure for using the convolution method

Step 1. Decide the minimal capability requirement C of S_{pk} (normally set to 1.00, 1.33, 1.50, 1.67 or 2.0), and the significant level α (normally set to 0.10, 0.05, or 0.025).

Step 2. Randomly sample n samples from the products.

Step 3. Calculate the sample estimate of $\hat{S}_{pk}$.

Step 4. Check out Table 3-3 or run the Matlab program (available on request) for the critical value based on the corresponding capability requirement of S_{pk}, significant level and sample size n.

Step 5. Conclude that the product capability S_{pk} is larger than the minimal capability requirement C, and production yield is larger than $2\Phi(3\times \text{minimal requirement})-1$ with $100(1-\alpha)\%$ confidence level, if the sample estimate $\hat{S}_{pk}$ is larger than or equal to the critical value c_0 of the convolution method. Otherwise, we do not have sufficient information to make such a conclusion.

3.6. Accuracy Analysis

The information of required sample size is important for in-plant applications, as it directly relates to the cost of the data collection plan. Following, we investigate the accuracy of the convolution method which provides useful information about the sample size required for designated power levels and for convergence.

3.6.1. Sample size required for designated power

The decision rule of hypothesis testing depends solely on the significant level α, the maximal probability of Type I error, and ignores the probability of Type II error β. Once the sample size n and α risk are chosen for testing a hypothesis, the power of test $1-\beta$, the probability of correctly rejecting H_0 while H_1 is true, will be fixed. To decrease the β risk and at the meantime maintain the α risk in a small level, the sample size should be increased.

The required sample size of the convolution method can be obtained by a recursive search with the following two constraints:

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 \mid S_{pk} \leq \mathrm{C} \text{ and } \xi = 0.5\right\} \leq \alpha \text{, and}$$

$$\Pr\left\{\hat{S}''_{pk} \geq c_0 \mid S_{pk} > \mathrm{C} \text{ and } \xi = 0.5\right\} \geq 1-\beta \text{,}$$

where c_0 is the critical value of the convolution method. Table 3-4 shows the minimal sample size required for testing H_0: $S_{pk} \leq$ C, C = 1.0, 1.33, 1.50, 1.67, while actually S_{pk} = C + h, h = 0.15(0.05)0.35, with designated α levels = 0.1, 0.05, 0.025, and power levels = 0.7, 0.8, 0.9, 0.95.

Note that the sample size required is a function of the α and power levels, the minimal capability requirement C of S_{pk}, and the difference between the actual value of S_{pk} and the minimal requirement C. Table 3-4 shows that the larger the difference, the smaller the sample size required for fixed α and power levels. For fixed α, minimal requirement C, and actual value of S_{pk}, the sample size increases as the designated power level increases. This phenomenon can be explained easily, since the smaller the difference and the greater the desired power level, the more sample size should be collected to account for the smaller uncertainty in the estimation.

Table 3-4. Sample size required for designated power levels of the convolution method.

(a) H_0: $S_{pk} \le 1.0$ vs H_1: $S_{pk} > 1.0$

α	S_{pk}	power			
		0.7	0.8	0.9	0.95
0.1	1.15	83	113	161	207
	1.20	49	66	212	265
	1.25	33	44	62	78
	1.30	24	32	44	55
	1.35	18	24	33	40
0.05	1.15	120	156	212	265
	1.20	71	91	124	154
	1.25	47	61	82	101
	1.30	34	44	59	72
	1.35	26	33	44	53
0.025	1.15	158	199	262	321
	1.20	93	116	153	187
	1.25	62	77	101	125
	1.30	44	56	73	88
	1.35	33	42	55	66

(b) H_0: $S_{pk} \le 1.33$ vs H_1: $S_{pk} > 1.33$

α	S_{pk}	power			
		0.7	0.8	0.9	0.95
0.1	1.48	142	194	278	360
	1.53	83	113	161	207
	1.58	55	75	106	135
	1.63	40	53	75	95
	1.68	30	40	56	71
0.05	1.48	205	267	366	458
	1.53	120	156	212	264
	1.58	79	103	140	173
	1.63	57	74	99	123
	1.68	43	56	75	92
0.025	1.48	270	340	451	554
	1.53	157	198	262	321
	1.58	104	131	172	210
	1.63	74	94	123	149
	1.68	56	71	93	112

(c) H_0: $S_{pk} \le 1.5$ vs H_1: $S_{pk} > 1.5$

α	S_{pk}	power			
		0.7	0.8	0.9	0.95
0.1	1.65	179	244	352	454
	1.70	104	142	203	261
	1.75	69	94	133	170
	1.80	49	67	94	120
	1.85	37	50	71	90
0.05	1.65	259	337	462	579
	1.70	150	195	267	333
	1.75	99	129	175	218
	1.80	71	92	125	155
	1.85	54	69	94	116
0.025	1.65	340	429	569	700
	1.70	197	249	329	403
	1.75	130	164	216	264
	1.80	93	117	154	187
	1.85	70	88	116	141

(d) H_0: $S_{pk} \le 1.67$ vs H_1: $S_{pk} > 1.67$

α	S_{pk}	power			
		0.7	0.8	0.9	0.95
0.1	1.82	220	301	433	560
	1.87	128	174	250	321
	1.92	84	113	163	210
	1.97	60	82	116	148
	2.02	46	62	87	110
0.05	1.82	318	415	569	712
	1.87	185	240	328	407
	1.92	122	158	215	268
	1.97	87	113	153	190
	2.02	65	85	115	142
0.025	1.82	418	528	700	860
	1.87	242	305	404	497
	1.92	159	201	265	325
	1.97	114	143	189	230
	2.02	85	108	142	172

3.6.2. Sample size required for convergence

Table 3-5 displays the sample sizes required for the convolution approximation to converge to S_{pk} within a designated accuracy $\varepsilon = 0.12(0.01)0.03$.

$$\Pr\left\{\left|\hat{S}''_{pk} - S_{pk}\right| \le \varepsilon\right\} \ge 1-\alpha$$

For example, for S_{pk} = 1.33 with risk α = 0.025, a sample size of $n \ge 3831$ ensures that the difference between sample estimate and actual parameter would be no greater than 0.03 with 97.5% confidence. Thus, if $\hat{S}_{pk}$ = 1.33, then we may conclude that the actual S_{pk} is greater than 1.3, actually in the interval of (1.30, 1.36), with 97.5% confidence. Note that the investigation is not for practical purpose. But, the computations illustrate the rate of convergence for the convolution approximation to converge to actual S_{pk}.

Table 3-5. Sample size required for the convolution approximation to converge.

S_{pk}	α	Designated Accuracy, ε									
		0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03
1.00	0.1	88	106	129	159	203	266	363	523	819	1459
	0.05	127	151	184	228	289	378	518	744	1164	2072
	0.025	167	200	242	299	380	496	676	975	1524	2288
1.33	0.1	159	189	230	284	361	472	644	929	1210	1789
	0.05	226	270	327	405	513	672	875	1319	1543	2706
	0.025	298	355	430	531	673	880	1199	1727	2183	**3831**
1.50	0.1	203	242	294	364	461	603	822	1184	1852	3296
	0.05	289	345	418	517	655	840	1168	1683	2631	4680
	0.025	380	453	549	678	859	1122	1529	2203	3443	5225
1.67	0.1	253	302	366	453	574	750	1022	1473	2303	4097
	0.05	361	430	521	644	811	1066	1452	2092	3271	5818
	0.025	473	564	683	767	1068	1396	1901	2738	4280	7455
2.00	0.1	366	437	529	654	828	1082	1474	2124	3321	5444
	0.05	521	621	752	929	1177	1538	2094	3017	4716	8387
	0.025	683	731	985	1217	1540	2013	2741	3946	6170	10970

Chapter 4

Product Acceptance Determination Based on the S_{pk} Index

4.1. Introduction

Acceptance sampling plans are practical tools for quality assurance applications, which provide the supplier and the customer a general decision rule for lot sentencing that meets both of their requirements for product quality. Acceptance sampling plans, however, cannot avoid the risk of accepting undesired poor product lots, nor can it avoid the risk of rejecting good product lots unless 100% inspection is implemented. A well-designed sampling plan can effectively reduce the difference between the required and the actual supplied product quantity. An acceptance sampling plan is a statement regarding the required sample size for product inspection and the associated acceptance criteria for sentencing each individual product lot. The criteria used for measuring the performance of an acceptance sampling plan is based the operating characteristic (OC) curve which quantifies the risks for vendors and buyers. The OC curve plots the probability of accepting an individual lot versus the actual product fraction of defectives, which displays the discriminatory power of the sampling plan.

For product quality protection and company's profit, both the vendor and the buyer would focus on certain points on the OC curve to reflect their benchmark risk. The vendor (supplier) would usually focus on a specific product quality level, called *acceptable quality level* (*AQL*), which would yield a high probability, $1-\alpha$, of accepting a lot. On the other hand, the buyer (consumer) would focus on a point at the other end of the OC curve, called *lot tolerance percent defective* (*LTPD*), which would result in a low probability, β, of accepting a lot. The *LTPD* is a quality level specified by the consumer, setting a specified low probability of accepting a lot for product with defective level as high as *LTPD*, and is the poorest quality level that the consumer is willing to accept. The α probability, also called the *producer's risk*, occurs when an acceptable product lot is rejected. The β probability, also called the *consumer's risk*, occurs when the product with unacceptable quality is accepted. Thus, a well-designed sampling plan must provide a probability of at least $1-\alpha$ of accepting a lot if the product quality level is at the contracted *AQL* level and a probability of no more than β if the level of the product quality is at the *LTPD* level, the designated undesired quality level preset by the customer. That is, the OC curve of the acceptance sampling plan must pass through the two designated points (*AQL*, $1-\alpha$) and (*LTPD*, β).

There are a number of different ways of classifying the acceptance sampling plans. One major classification is by attributes and variables. The primary advantage of variables sampling plans is that the same operating characteristic curve can be obtained with a smaller sample size than that required by an attributes sampling plan. The precise measurements required by a variables plan would probably cost more than the simple classification of items required by an attributes plan, but the reduction in sample size may more than offset this exact expense. Such saving may be especially marked if inspection is destructive and the item is expensive (see Schilling [41], Duncan [9] and Montgomery [31]). The basic concepts and models of statistically based on variables sampling plans were introduced by Jennett and Welch [19]. Lieberman and Resnikoff [28] developed extensive tables and OC curves for various *AQLs* for MIL-STD-414 sampling plan. Owen [32] considered variables sampling plans based on the normal distribution, and developed sampling plans for various levels of probabilities of Type I error when the standard deviation is unknown. Das and Mitra [8] have investigated the effect of non-normality on the performance of the sampling plans. Guenther [11] developed a systematic search procedure, which can be used with published tables of binomial, hyper-geometric, and Poisson distributions to obtain the desired acceptance sampling plans. Stephens [43] provided a closed form solution for single sample acceptance sampling plans using a normal approximation to the binomial distribution. Hailey [12] presented a computer program to obtain single sampling plans with minimum sample size based on either the Poisson or binomial distribution. Hald [13] gave a systematic exposition of the existing statistical theory of lot-by-lot sampling inspection by attributes and provided some tables for the sampling plans. Comparisons between variables sampling plans and attributes sampling plans were investigated by Hamaker [14], who concluded that the expected sample size required by variable sampling is smaller than those for comparable attributes sampling plans. Govindaraju and Soundararajan [10] developed variables sampling plans that match the OC curves of MIL-STD-105D. Suresh and Ramanathan [46] developed a sampling plan based on a more general symmetric family of distributions.

Due to the sampling cannot guarantee that every defective item in a lot will be inspected, the sampling plan involves risks of not adequately reflecting the quality conditions of the lot. Such risk is even more significant as the rapid advancement of the manufacturing technology and stringent customers demand is enforced. Particularly, when the required product fraction of defectives is very low, often measured in parts per million (PPM). The required number of inspection items must be enormously large in order to adequately reflecting the actual product fraction of defectives or process yield. The yield index S_{pk} establishes the relationship between the manufacturing specifications and the actual process performance, and provides an exact yield measure on the normal processes. In this paper, we consider a variables sampling plan based on the S_{pk} index as a quality benchmark to deal with lot sentencing problem for processes with very low fraction of defectives.

4.2. Process Capability Indices

4.2.1. Process capability indices C_a, C_p, C_{pk}, C_{pm}, C_{pmk}

Process capability indices have been proposed to the manufacturing industry, to provide numerical measures on process performance. Those indices establish the relationship between the actual process performance and the manufacturing specifications, which have been the focus of the recent research in statistical and quality assurance literatures. The explicit forms of the indices are defined as follows:

$$C_a = 1 - \frac{|\mu - \mathrm{m}|}{\mathrm{d}}, \quad C_p = \frac{USL - LSL}{6\sigma}, \quad C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \text{ and } C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\},$$

where USL and LSL are the upper and lower specification limits, respectively, m = (USL + LSL)/2 is the midpoint of the specification limits, d = (USL − LSL)/2 is the half length of the specification interval, T is the target value, μ is the process mean, and σ is the process standard deviation.

The index C_a measures only the process centering (process accuracy), and ignores the process variation (process precision). The index C_p measures the overall process variation relative to the specification tolerance, therefore can not reflect the tendency of process centering (see Juran [20], Sullivan [44], [45] and Kane [21]). In order to reflect the deviations of process mean from the target value, several indices similar in nature to C_p, such as C_{pk}, C_{pm}, C_{pmk}, have been proposed. Those indices take into consideration the magnitude of process variance as well as process location (see Hsiang and Taguchi [16], Chan *et al* [4], Boyles [2], Ruczinski [40] and Pearn *et al* [35]). The index C_{pk} have a relationship to the actual process yield, which can be expressed as

$$2\Phi(3C_{pk}) - 1 \le Yield \le \Phi(3C_{pk}),$$

(Boyles [2]). To emphasize the loss in a product's worth when one of its characteristics departs from the target value T, the two indices C_{pm} and C_{pmk} are defined by being related to the idea of squared error loss $loss(X) = (X - T)^2$.

4.2.2. The yield index S_{pk}

For a long time, process yield has been a standard criterion used in the manufacturing industry as a common measure on process performance, and defined as the percentage of processed product unit that falls within the manufacturing specification limits. For product units falling out of the manufacturing tolerance, additional cost would be incurred to the factory for scrapping or repairing the product. All passed product units, which incur no additional cost to the factory, are equally accepted by the producer. The above indices C_{pk}, C_{pm}, C_{pmk} can provide only a lower bound estimation on the process yield $Yield \ge 2\Phi(3 \times index\ value) - 1$. Note that the highest value that a process yield might be would not be concerned. For example, if the index value is C, then the yield of the process would be equal to or greater than $2\Phi(3C) - 1$.

On the other hand, the yield index S_{pk}, proposed by Boyles [3], can provide an exact measure on the process yield, which can be expressed as

$$Yield = 2\Phi(3S_{pk}) - 1.$$

The S_{pk} index is defined as

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu}{\sigma}\right)+\frac{1}{2}\Phi\left(\frac{\mu-LSL}{\sigma}\right)\right\}.$$

The S_{pk} index outperforms other indices in providing a one-to one relationship to the process yield.

We remark that the indices presented above are designed to monitor the performance for stable normal or near-normal processes with symmetric tolerances. In practice, the process mean μ and the process variance σ^2 are unknown. To calculate the index value, sample data must be collected, and a great degree of uncertainty may be introduced into the assessments due to sampling errors. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their distributions, learning that capability measures must be reported in confidence intervals or via capability testing. Statistical properties of the estimators of those indices under various process conditions have been investigated extensively, including Chan *et al.* [4], Pearn *et al.* [35], Kotz and Johnson [23], Vännman and Kotz [49], Ruczinski [40], Vännman [47], Kotz and Lovelace [25], Borges and Ho [1], Hoffman [15], Zimmer *et al.* [55], Kotz and Johnson [24], Lee *et al.* [26], Spiring *et al.* [42], Pearn *et al.* [36], Montgomery [31], Hubele *et al.* [17], Lin and Sheen [29], Wang [50], Wu [52], Mathew *et al.* [30].

4.2.3. Sampling distribution of the estimated S_{pk}

To estimate the yield measurement index S_{pk}, we consider the following natural estimator $\hat{S}_{pk}$, expressed as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\bar{X}}{S}\right)+\frac{1}{2}\Phi\left(\frac{\bar{X}-LSL}{S}\right)\right\}$$

$$= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)\right\},$$

where $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ is the sample mean, $S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ is the sample variance, $\hat{C}_{dr}=(\bar{X}-\mathrm{m})/\mathrm{d}$, and $\hat{C}_{dp}=S/\mathrm{d}$.

The exact distribution of $\hat{S}_{pk}$ is analytically intractable. However, a useful approximate distribution of $\hat{S}_{pk}$ can be furnished by considering an expansion of $\hat{S}_{pk}$. Lee *et al* [26] considered a normal approximation to the distribution of $\hat{S}_{pk}$ by using an expansion technique. The expansion of $\hat{S}_{pk}$, which is denoted $\hat{S}'_{pk}$ in this paper, is normally distributed, and can be expressed as follows

$$\hat{S}'_{pk} \sim N\left(S_{pk}, \frac{a^2+b^2}{36n\phi^2(3S_{pk})} \right),$$

where

$$a = \frac{1}{\sqrt{2}} \left\{ \frac{USL-\mu}{\sigma} \phi\left(\frac{USL-\mu}{\sigma} \right) + \frac{\mu - LSL}{\sigma} \phi\left(\frac{\mu - LSL}{\sigma} \right) \right\}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{\mathrm{d}-(\mu-\mathrm{m})}{\sigma} \phi\left(\frac{\mathrm{d}-(\mu-\mathrm{m})}{\sigma} \right) + \frac{\mathrm{d}+(\mu-\mathrm{m})}{\sigma} \phi\left(\frac{\mathrm{d}+(\mu-\mathrm{m})}{\sigma} \right) \right\}, \text{ and}$$

$$b = \phi\left(\frac{USL-\mu}{\sigma} \right) - \phi\left(\frac{\mu - LSL}{\sigma} \right) = \phi\left(\frac{d-(\mu-m)}{\sigma} \right) - \phi\left(\frac{d+(\mu-m}{\sigma} \right).$$

Thus, the approximate probability density function of the approximate distribution can be expressed as:

$$f(x) = \sqrt{\frac{18n\phi^2(3S_{pk})}{\pi(a^2+b^2)}} \exp\left[-\frac{18n\phi^2(3S_{pk})}{a^2+b^2}(x-S_{pk})^2 \right], \quad -\infty < x < \infty .$$

4.3. Designing S_{pk} Variables Sampling Plan

Consider a variables sampling plan for controlling the lot fraction of defectives (nonconformities). Since the quality characteristic is a variables, there will exist either a *USL* or an *LSL*, or both, that defined the acceptable values of this parameter. A well-designed sampling plan must provide a probability of at least $1-\alpha$ of accepting a lot if the lot fraction of defectives is at the contracted *AQL*. The sampling plan must also provide a probability of acceptance no more than β if the lot fraction of defectives is at the *LTPD* level, the designated undesired level preset by the buyer. Thus, the acceptance sampling plan must have its OC curve passing through those two designated points (*AQL*, $1-\alpha$) and (*LTPD*, β). To determine whether a given process is capable, we can first consider the following testing hypothesis:

H_0: $p = AQL$ (process is capable),

H_1: $p = LTPD$ (process is incapable).

For normally distributed processes with single characteristic, the index S_{pk} is used to establish the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on the process yield. That is, the null hypothesis with proportion defective, H_0: $p = AQL$ is equivalent to test process capability index with H_0: $S_{pk} \geq S_{AQL}$, where S_{AQL} is the level of acceptable quality for S_{pk} index. For instance, if the fraction of defectives $p = AQL$ of vendor's product is about 66 PPM, then the probability of consumer accept the lots will larger than $100(1-\alpha)\%$. On the other hand, if the fraction of defectives of vendor's product, $p = LTPD$, is about 2700 PPM, then the probability of consumer would accept no more than $100\beta\%$. Then, from the relationship between the index value and fraction of defectives, we could obtain the equivalent $S_{AQL} = 1.33$ and $S_{LTPD} = 1.00$ based on the process index S_{pk}.

Therefore, the required inspection sample size n and critical acceptance value c_0 for the sampling plans are the solution to the following two nonlinear simultaneous equations.

$$\Pr\{\text{Accepting the lot} \mid p = AQL\} \geq 1-\alpha ,$$

$$\Pr\{\text{Accepting the lot} \mid p = LTPD\} \leq \beta .$$

As described earlier, the asymptotic sampling distribution of $\hat{S}_{pk}$ is normally distributed with a mean S_{pk} and a variance $(a^2+b^2)/36n\phi^2(3S_{pk})$. The probability of accepting the lot then can be expressed as:

$$\pi_A(S_{pk}) = \Pr(\hat{S}_{pk} \geq c_0 \mid S_{pk} = c)$$

$$= \int_{c_0}^{\infty} \sqrt{\frac{18\text{n}}{\pi}} \frac{\phi(3c)}{\sqrt{a^2+b^2}} \exp\left[-\frac{18n\left(\phi(3c)\right)^2}{a^2+b^2}(x-c)^2\right] dx ,$$

Therefore, the required inspection sample size n and critical acceptance value c_0 of $\hat{S}_{pk}$ for the sampling plans can be obtained by solving the following two nonlinear simultaneous equations.

$$1-\alpha \leq \int_{c_0}^{\infty} \sqrt{\frac{18\text{n}}{\pi}} \frac{\phi(3S_{AQL})}{\sqrt{a^2+b^2}} \exp\left[-\frac{18n\left(\phi(3S_{AQL})\right)^2}{a^2+b^2}(x-S_{AQL})^2\right] dx ,$$

$$\beta \geq \int_{c_0}^{\infty} \sqrt{\frac{18\mathrm{n}}{\pi}} \frac{\phi(3S_{LTPD})}{\sqrt{a^2+b^2}} \exp\left[-\frac{18n(\phi(3S_{LTPD}))^2}{a^2+b^2}(x-S_{LTPD})^2 \right] dx \, ,$$

where $S_{AQL} > S_{LTPD}$. We note that the required sample size n is the smallest possible value of n satisfying the above two equations, and determining the $[n]$ as sample size, where $[n]$ means the least integer greater than or equal to n. Since the calculation of critical values involve two parameters a and b, functions of the two indices C_p and C_a, we have to consider the effect of a^2+b^2 from various (C_p, C_a) for given a fixed performance requirement S_{pk}.

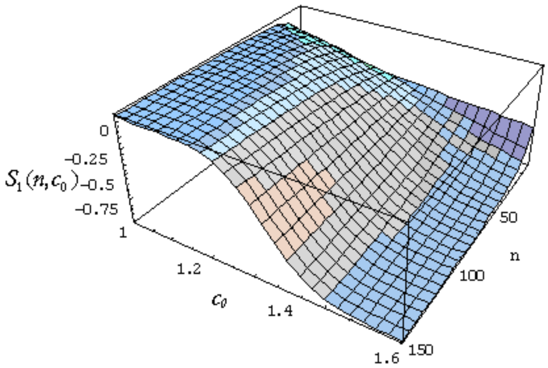


Figure 4-1(a). Surface plot of S_1.

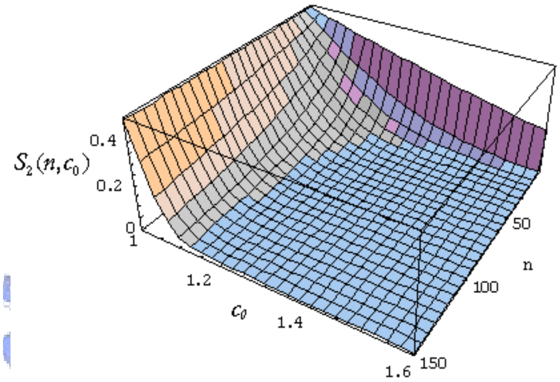


Figure 4-1(b). Surface plot of S_2.

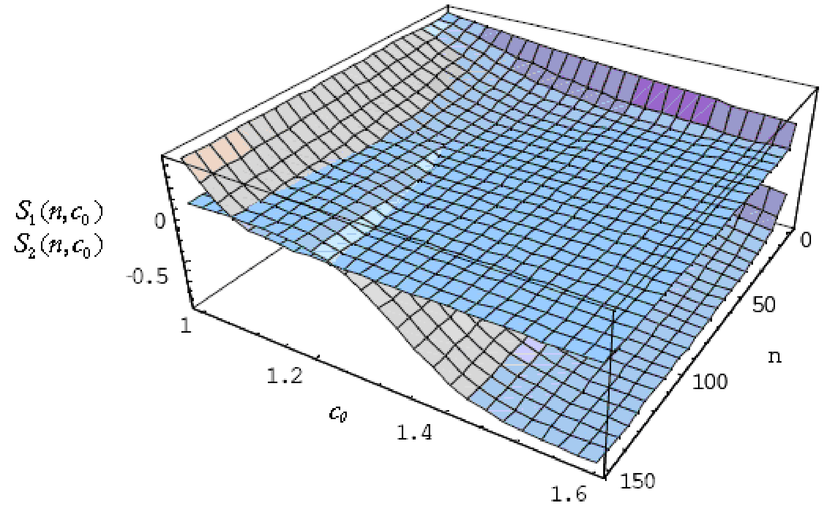


Figure 4-2. Surface plot of S_1 and S_2.

Pearn *et al* [36] discussed that for given a fixed performance requirement S_{pk}, the differences among those calculated critical values corresponding to various values of C_p and C_a, are sufficiently small and can be neglected. Also, they showed that the factor $(a^2+b^2)/36\phi^2(3S_{pk})$

is insensitive to the value changes of C_p and C_a in all cases, except for C_a = 1. Consequently, the critical values c_0 may be considered as a constant, which is independent of the process characteristics C_p and C_a for a fixed performance requirement S_{pk}. n

In order to illustrate how we solve the above two nonlinear simultaneous equations, we let

$$S_1(n,c_0)=\int_{c_0}^{\infty}\sqrt{\frac{18\text{n}}{\pi}}\frac{\phi(3S_{AQL})}{\sqrt{a^2+b^2}}\exp\left[-\frac{18n\left(\phi(3S_{AQL})\right)^2}{a^2+b^2}(x-S_{AQL})^2\right]dx-(1-\alpha)\,,$$

$$S_2(n,c_0)=\int_{c_0}^{\infty}\sqrt{\frac{18\text{n}}{\pi}}\frac{\phi(3S_{LTPD})}{\sqrt{a^2+b^2}}\exp\left[-\frac{18n\left(\phi(3S_{LTPD})\right)^2}{a^2+b^2}(x-S_{LTPD})^2\right]dx-\beta\,.$$

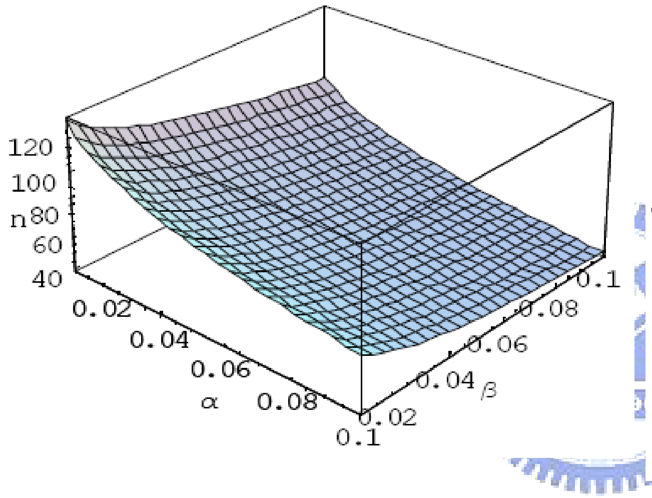


Figure 4-3(a). The required sample size n as surface plot with α = 0.01 (0.01) 0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.33, 1.00).

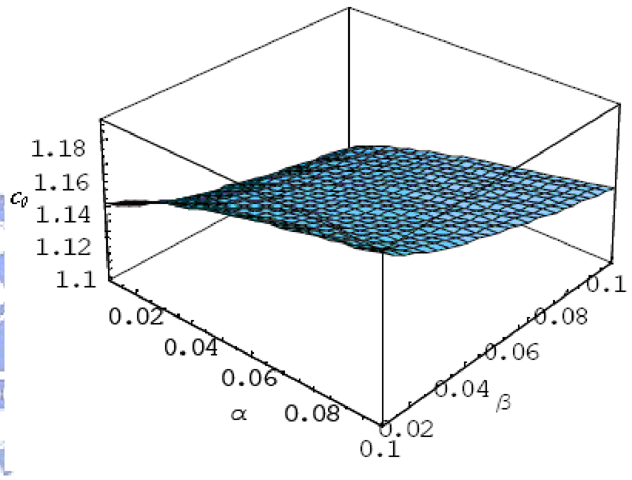


Figure 4-3(b). The critical acceptance value c_0 as surface plot with α = 0.01 (0.01) 0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.33, 1.00).

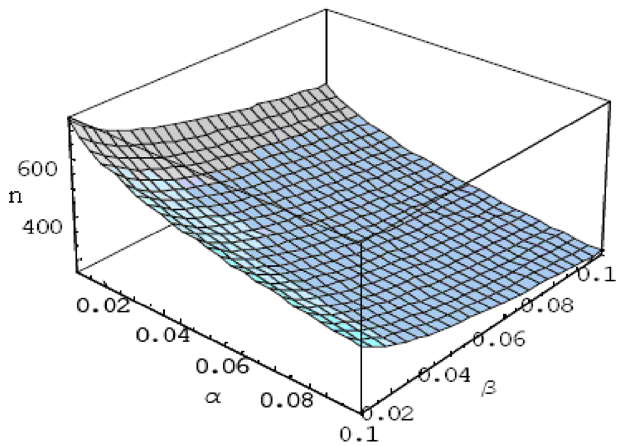


Figure 4-3(c). The required sample size n as surface plot with α = 0.01 (0.01) 0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.50, 1.33).

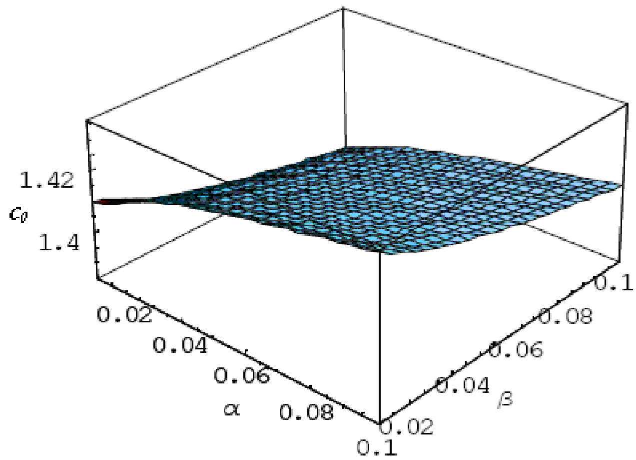


Figure 4-3(d). The critical acceptance value c_0 as surface plot with α = 0.01 (0.01) 0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.50, 1.33).

For S_{AQL} = 1.33 and S_{LTPD} = 1.00, Figures 4-1(a) and 4-1(b) display the surface plots of equations $S_1(n, c_0)$ and $S_2(n, c_0)$ with α-*risk* = 0.05 and β-*risk* = 0.05, respectively. Figure 4-2 displays the surface plots of equations $S_1(n, c_0)$ and $S_2(n, c_0)$ simultaneously with α-*risk* = 0.05 and β-*risk* = 0.05 under S_{AQL} = 1.33 and S_{LTPD} = 1.00, respectively. In Figure 4-2, interaction of $S_1(n, c_0)$ and $S_2(n, c_0)$ is (n, c_0) = (66, 1.1412), which is the solution to the two nonlinear simultaneous equations. That is, in this case, the minimum required sample size n = 66 and critical acceptance value c_0 = 1.1412 of the sampling plan based on the capability index S_{pk}.

To investigate the behavior of the critical acceptance values and required sample sizes, we perform extensive calculations to obtain the solution of the two nonlinear equations with various parameters. Figure 4-3(a) displays the required sample size n as surface plot with the probabilities α = 0.01(0.01)0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.33, 1.00). Figure 4-3(b) displays the critical acceptance value c_0 as surface plot with the probabilities α = 0.01(0.01)0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.33, 1.00). Figures 4-3(c) and 4-3(d) show the required sample size n and the critical acceptance value c_0 as surface plot with α = 0.01(0.01)0.10 and β = 0.01(0.01)0.10 under (S_{AQL}, S_{LTPD}) = (1.50, 1.33), respectively.

From Figures 4-3(a) and 4-3(d), we observe that the larger of the risk (α or β) which producer or customer could suffer, the smaller is the required sample size n. This phenomenon can be explained intuitively, as if we hope that the chance of wrongly concluding a bad process as good or good lots as bad ones is small, then more sample information is needed to judge the lot quality. Further, for fixed S_{AQL}, S_{LTPD} and α-*risk*, the corresponding critical acceptance values become smaller when the β-*risk* becomes larger. On the other hand, for fixed S_{AQL}, S_{LTPD}, and β-*risk*, the corresponding critical acceptance values become larger when the α-*risk* becomes larger. This can also be explained by the same reasoning as above. Consequently, the required sample size is smaller when the difference between S_{AQL} and S_{LTPD} is significant since the judgment will then be relatively easier to reach correct decision.

For practical applications purpose, we calculate and tabulate the critical acceptance values and required sample sizes for the sampling plans, with commonly used α-*risk*, β-*risk*, S_{AQL} and S_{LTPD}. Table 4-1 display (n, c_0) values for α-*risk* = 0.01, 0.025(0.025)0.10 and β-*risk* = 0.01, 0.025(0.025)0.10, with various benchmarking quality levels, (S_{AQL}, S_{LTPD}) = (1.33, 1.00), (1.50, 1.33), (1.67, 1.50), (2.00, 1.67). Based on the designed sampling plan, the practitioners can determine the number of production items to be sampled for inspection and the corresponding critical acceptance value. For example, if the benchmarking quality level (S_{AQL}, S_{LTPD}) is set to (1.33, 1.00) with producer's α-*risk* = 0.01 and customer's β-*risk* = 0.05, then the corresponding sample size and critical acceptance value can be obtained as (n, c_0) = (101, 1.1142). The lot will be accepted if the 101 inspected production items yield measurements with $\hat{S}_{pk} \geq 1.1142$.

Table 4-1. (n, c_0) values for α-*risk* = 0.01, 0.025(0.025)0.10, β-*risk* = 0.01,0.025(0.025)0.10 with various (S_{AQL}, S_{LTPD}).

α-*risk*	β-*risk*	S_{AQL} = 1.33		S_{AQL} = 1.50		S_{AQL} = 1.67		S_{AQL} = 2.00	
		S_{LTPD} = 1.00		S_{LTPD} = 1.33		S_{LTPD} =1.50		S_{LTPD} = 1.67	
		n	c_0	n	c_0	n	c_0	n	c_0
0.010	0.010	132	1.1412	735	1.4100	929	1.5801	331	1.8206
	0.025	115	1.1276	630	1.4028	796	1.5729	286	1.8067
	0.050	101	1.1142	546	1.3956	690	1.5657	249	1.7929
	0.075	93	1.1044	495	1.3903	625	1.5604	226	1.7828
	0.100	87	1.0963	457	1.3859	577	1.5560	210	1.7744
0.025	0.010	110	1.1552	618	1.4173	780	1.5874	277	1.8347
	0.025	94	1.1412	522	1.4100	659	1.5801	235	1.8206
	0.050	82	1.1273	446	1.4026	563	1.5727	202	1.8064
	0.075	74	1.1170	400	1.3971	505	1.5672	182	1.7959
	0.100	68	1.1084	366	1.3925	462	1.5626	167	1.7870
0.050	0.010	92	1.1696	525	1.4247	663	1.5948	234	1.8491
	0.025	78	1.1555	437	1.4174	552	1.5875	196	1.8350
	0.050	66	1.1412	368	1.4100	464	1.5801	166	1.8206
	0.075	60	1.1306	326	1.4044	411	1.5745	148	1.8098
	0.100	55	1.1215	295	1.3996	373	1.5696	134	1.8005
0.075	0.010	81	1.1806	468	1.4302	592	1.6003	208	1.8600
	0.025	68	1.1665	386	1.4231	487	1.5932	172	1.8460
	0.050	57	1.1521	321	1.4157	405	1.5857	144	1.8316
	0.075	51	1.1412	282	1.4100	356	1.5801	127	1.8206
	0.100	46	1.1319	253	1.4051	320	1.5752	115	1.8112
0.100	0.010	73	1.1900	427	1.4350	540	1.6050	190	1.8693
	0.025	61	1.1761	348	1.4279	440	1.5980	155	1.8555
	0.050	51	1.1616	287	1.4206	362	1.5907	129	1.8412
	0.075	45	1.1507	250	1.4149	316	1.5850	113	1.8302
	0.100	41	1.1412	223	1.4100	282	1.5801	101	1.8206

For the proposed sampling plan to be practical and convenience to use, a step-by-step procedure is provided below.

Step 1: Decide the process capability requirements (i.e. set the values of S_{AQL} and S_{LTPD}), set the α-*risk*, the chance of wrongly concluding a capable process as incapable, and the β-*risk*, the chance of wrongly concluding a bad lot as good one.

Step 2: Check the Table 4-1 to find the critical value c_0 and the required sample size n for inspection based on given values of α-*risk*, β-*risk*, S_{AQL} and S_{LTPD}.

Step 3: Calculate the value of $\hat{S}_{pk}$ from these n inspected samples.

Step 4: Make decisions that accept the entire lot if the estimated $\hat{S}_{pk}$ value is greater than the critical value c_0. Otherwise, we reject the entire lot.

4.4. Accuracy of S_{pk} Variables Sampling Plans

To assess the accuracy of the S_{pk} sampling plan, we simulate N = 10000 lots in each combination shown in Table 4-1 to calculate the probability of accepting lots to compared with the probability of accepting the lot from the corresponding (α, β) and (S_{AQL}, S_{LTPD}). Table 4-2 displays the probability of accepting the lot from simulation according to the rule in Table 4-1.

For the case of (S_{AQL}, S_{LTPD}) = (1.33, 1.00) , the simulation results indicate for each combination of α-*risk* (the producer's risk) = 0.01, 0.025(0.025)0.10 and β-*risk* (the customer's risk) = 0.01, 0.025(0.025)0.10, the probabilities of accepting the lots are all greater than the corresponding $1-\alpha$ under S_{AQL} = 1.33, and the probabilities of accepting the lots are all slightly greater than the corresponding β under S_{LTPD} = 1.00, but the magnitude of $LP-\beta$ are no greater than 0.0205.

For the case of (S_{AQL}, S_{LTPD}) = (1.50, 1.33) , the simulation results indicate for each combination of α-*risk* = 0.01, 0.025(0.025)0.10 and β-*risk* = 0.01, 0.025(0.025)0.10, the probabilities of accepting the lots are almost greater than the corresponding $1-\alpha$ under S_{AQL} = 1.50 (except for α = 0.025 and β = 0.01, the probability of accepting the lot is 0.9749), and the probabilities of accepting the lots are all slightly greater than the corresponding β under S_{LTPD} = 1.33, but with $LP-\beta$ no greater than 0.0206.

Table 4-2. Probabilities of accepting the lot for α-*risk* = 0.01, 0.025(0.025)0.10, β-*risk* = 0.01,0.025(0.025)0.10 with various (S_{AQL}, S_{LTPD}) by simulation with N=10000.

α-*risk*	β-*risk*	S_{AQL} = 1.33		S_{AQL} = 1.50		S_{AQL} = 1.67		S_{AQL} = 2.00	
		S_{LTPD} = 1.00		S_{LTPD} = 1.33		S_{LTPD} =1.50		S_{LTPD} = 1.67	
		AP	*LP*	*AP*	*LP*	*AP*	*LP*	*AP*	*LP*
0.010	0.010	0.9937	0.0164	0.9906	0.0169	0.9874	0.0148	0.9922	0.0116
	0.025	0.9947	0.0338	0.9910	0.0370	0.9886	0.0278	0.9924	0.0293
	0.050	0.9943	0.0613	0.9919	0.0660	0.9902	0.0572	0.9931	0.0550
	0.075	0.9953	0.0872	0.9914	0.0943	0.9910	0.0818	0.9922	0.0775
	0.100	0.9948	0.1100	0.9911	0.1181	0.9899	0.1090	0.9926	0.1001
0.025	0.010	0.9821	0.0153	0.9749	0.0161	0.9733	0.0147	0.9808	0.0132
	0.025	0.9843	0.0353	0.9755	0.0337	0.9729	0.0327	0.9802	0.0283
	0.050	0.9868	0.0631	0.9753	0.0631	0.9729	0.0586	0.9802	0.0539
	0.075	0.9863	0.0881	0.9765	0.0910	0.9743	0.0820	0.9804	0.0799
	0.100	0.9870	0.1135	0.9757	0.1203	0.9743	0.1051	0.9816	0.1042
0.050	0.010	0.9661	0.0176	0.9524	0.0140	0.9439	0.0141	0.9564	0.0109
	0.025	0.9663	0.0393	0.9543	0.0331	0.9454	0.0308	0.9563	0.0287
	0.050	0.9643	0.0611	0.9532	0.0657	0.9478	0.0578	0.9584	0.0495
	0.075	0.9682	0.0894	0.9544	0.0905	0.9478	0.0787	0.9585	0.0774
	0.100	0.9665	0.1156	0.9531	0.1130	0.9519	0.1055	0.9599	0.0977
0.075	0.010	0.9397	0.0172	0.9308	0.0144	0.9199	0.0128	0.9286	0.0149
	0.025	0.9410	0.0363	0.9284	0.0375	0.9184	0.0288	0.9328	0.0304
	0.050	0.9450	0.0656	0.9259	0.0646	0.9236	0.0543	0.9330	0.0536
	0.075	0.9437	0.0878	0.9256	0.0893	0.9253	0.0804	0.9351	0.0771
	0.100	0.9455	0.1161	0.9287	0.1185	0.9252	0.1089	0.9352	0.1015
0.100	0.010	0.9144	0.0195	0.9030	0.0154	0.8906	0.0111	0.9053	0.0125
	0.025	0.9145	0.0347	0.9015	0.0367	0.8931	0.0291	0.9060	0.0301
	0.050	0.9174	0.0638	0.9073	0.0623	0.8944	0.0574	0.9070	0.0530
	0.075	0.9202	0.0956	0.9036	0.0906	0.8945	0.0783	0.9108	0.0778
	0.100	0.9236	0.1205	0.9015	0.1206	0.8947	0.1053	0.9084	0.1023

AP: the probability of accepting the lot under S_{AQL}
LP: the probability of accepting the lot under S_{LTPD}

For the case of (S_{AQL}, S_{LTPD}) = (1.67, 1.50) , the simulation results indicate for each combination of α-*risk* and β-*risk* = 0.01, 0.025(0.025)0.10, the probabilities of accepting the lots are all close to the corresponding $1-\alpha$ under S_{AQL} = 1.67 with $|AP-(1-\alpha)|$ no greater than 0.0094, and the probabilities of accepting the lots are all slightly greater than the corresponding β-*risk* under S_{LTPD} = 1.50, but with $LP-\beta$ no greater than 0.0089.

For the case of (S_{AQL}, S_{LTPD}) = (2.00, 1.67) , the simulation results indicate for each combination of $(\alpha, \beta) =$ 0.01, 0.025(0.025)0.10, the probabilities of accepting the lots are all greater than the corresponding $1-\alpha$ under S_{AQL} = 2.00, and the probabilities of accepting the lots are mostly greater than the corresponding β-*risk* under S_{LTPD} = 1.67 (except for $(\alpha, \beta) =$ (0.05, 0.05) and (0.05, 0.1), the probability of accepting the lot is 0.0495 and 0.0977, respectively), but with $LP-\beta$ no greater than 0.0054. The simulation results clearly indicate that the probabilities of accepting lots are all very close to the preset value of $1-\alpha$ or β in all the cases we investigated. Thus, the sampling plan based on the normal approximation to the distribution of the estimated S_{pk} is adequately reliable to the engineers for their in-plant applications.

4.5. An Application Example

To illustrate how the sampling plan can be established and applied to the actual data collected from the factories, we present a case study on an electronic component manufacturer, which developing ceramic multilayer capacitors for the consumer electronics, automotive parts, telecommunications, data processing. The capacitor consists of a rectangular block of ceramic in which a number of interleaved electrodes are contained. A cross section of the structure is shown in Figure 4-4. For a particular model of the ceramic multilayer capacitor investigated, the electrical characteristics are shown in Table 4-3.

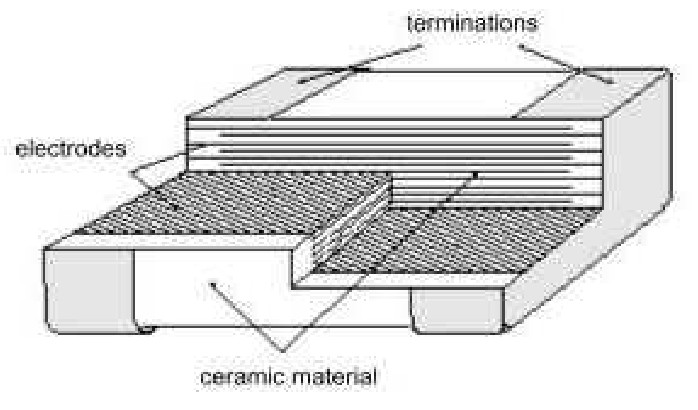


Figure 4-4. Construction of multi-layer ceramic capacitor.

The lower and upper specification limit for thickness of layer are set to, LSL = 1.45 mm, USL = 1.75 mm, respectively. If the data characteristic does not fall within the tolerance (LSL, USL), the lifetime or reliability of the capacitors will be discounted. In the contract, the values of

S_{AQL} and S_{LTPD} are set to 1.33 and 1.00 with the α-*risk* = 0.05 and β-*risk* = 0.10, respectively. That is, the sampling plans must provide a probability of at least 0.95 of accepting the lot if the lot proportion defective is at the S_{AQL} = 1.33 (which is equivalent to about 6.80 PPM fraction of defectives), and also provide a probability of no more than 0.10 of accepting the lot if the lot proportion defective is at the S_{LTPD} = 1.00 (which is equivalent to 2700 PPM fraction of defectives).

Table 4-3. Specification of Y5V/BME/1206/22uF/6.3V.

Capacitance range	22uF (Size 1206)
Tolerance on capacitance after 1000 hours	-20% to +80%
Rated voltage U_R (DC)	6.3V
Test voltage (DC) for 1 minute	2.5 X U_R
Tan D	<=12.50 %
Insulation resistance after 1 minute at U_R (DC)	$R_{ins.}$ x C > = 500 s
Maximum capacitance change as a function of temperature	+30% to -80%
Aging	Typically, 12.5% per time decade
Terminations	NiSn plated
Resistance to soldering heat	260° C, 10 sec

Table 4-4. The sample data with 55 observations (unit: mm).

1.554	1.571	1.531	1.644	1.610	1.747	1.655	1.621	1.687	1.463	1.592
1.632	1.679	1.627	1.546	1.543	1.633	1.520	1.725	1.507	1.619	1.690
1.375	1.630	1.506	1.724	1.591	1.540	1.667	1.642	1.607	1.576	1.449
1.500	1.690	1.562	1.612	1.524	1.693	1.618	1.638	1.545	1.501	1.625
1.619	1.560	1.496	1.652	1.623	1.468	1.623	1.572	1.576	1.653	1.642

Based on the above specified values in the contract, we could find the critical acceptance value and inspected sample size of the sampling plan (n, c_0) = (55, 1.1215) from the Table 4-1. Hence, the inspected samples are taken from the lot randomly and the observed measurements are displayed in Table 4-4. Based on these inspections, we obtain that

$$\bar{X} = 1.594,\ S = 0.076, \text{ and } \hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\bar{X}}{S}\right)+\frac{1}{2}\Phi\left(\frac{\bar{X}-LSL}{S}\right)\right\} = 0.6559.$$

Therefore, in this case, the lot will be rejected by the consumer, since the sample estimator from the inspections, 0.6559, is smaller than the critical acceptance value 1.1215 of the sampling plan significantly. We note that if existing sampling plans are applied here, it is almost certain that any sample of 55 capacitors taken from the process will contain zero defective items. All the products therefore will be accepted, which obviously provides no protection to the buyer at all.

Chapter 5

Conclusions and Future Works

How to measure process performance in the manufacturing industries is a major concern for the factory managers, and process yield is the most common and standard criteria. The capability index S_{pk} provides an exact yield measure (rather than yield range) for normal processes. Many researches supporting the use of capability indices have focused on process with single large representative sample. However, in practice the process performance is monitored by collecting multiple samples periodically. This dissertation considered this type of realistic data structure and investigated the sampling distribution of S_{pk} based on multiple samples.

We note that for processes with the same specification limits and process yield, the variance of $\hat{S}_{pk}$ would be largest while process mean is on the center of the specification limits, so we compute the lower bounds of S_{pk} under such condition for assurance purpose. It is noted that the lower bounds calculated by normal approximation distribution have larger risk of α especially for smaller total sample size $m \times n$. As mentioned previously, sampling with number of multiple samples m = 12, and number of sample size n = 50 is suggested to use the lower bounds from normal approximation to have the almost 1-α confidence level. Furthermore, if the sampling replications are allowed, practitioners can even estimate the real process capability from the average of lower bounds dividing by the ratio (i.e. $S_{pk} \doteq \overline{LB}$ /ratio).

Production yield is the most common and standard criteria used in the manufacturing industry for measuring process performance. The yield index S_{pk} provides a one-to-one measure on the yield of normal processes, while no other indices can. The statistical properties of the natural estimator of S_{pk} are mathematically intractable, and the existing approach (the normal approximation) does not provide adequate accuracy, particularly, for small sample sizes. In this dissertation, we considered the convolution approximation. The proposed approach, indeed, outperforms the existing method in providing more accurate and reliable estimation for S_{pk} as well as production yield. An efficient step-by-step procedure is developed for using the convolution method to estimate the production yield. The accuracy of the convolution method is also investigated, which provides useful information about the sample size required for designated power levels, and for convergence. The sample size information and the efficient step-by-step procedure are useful to the practitioners for making reliable decisions regarding process performance based on production yield.

Process capability indices have been widely used in the manufacturing industry to determine whether a process is capable of reproducing items within a specified tolerance, which provides common quantitative measures on production quality. The index S_{pk} is used to establish the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on the process yield. In this paper, we developed a variables sampling plan based on process yield index S_{pk}, to deal with lot sentencing problem for situations with very low fraction of defectives. The proposed sampling plan provides a feasible inspection policy, which can be applied to products requiring low fraction of defectives where classical sampling plans cannot be applied. We developed an analytical method to obtain the critical acceptance values and the corresponding sample size required for inspection, providing the desired levels of protection to both producers and consumers. We also tabulated the required sample size n and the critical acceptance value c_0 for various α-*risks*, β-*risks*, and the fraction of defectives of process that correspond to acceptable quality levels. The results of a simulation in Table 4-2 is used to ascertain the accuracy of the sampling plan based on the normal approximation to the distribution of the estimated S_{pk}. Practitioners can determine the number of required inspection units and the critical acceptance value, and make reliable decisions. For illustrative purpose, we demonstrated the use of the derived results by presenting a case study on capacitors manufacturing process.

In this dissertation, we investigated the sampling distribution of S_{pk} under multiple samples. In the future, one may propose the distribution for processes with multiple characteristics. In addition, since the convolution method is more accurate than the normal approximation, we may design the S_{pk} variables sampling plan based on the convolution approach.

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Appendix

Appendix I: Taylor expansion of $\hat{S}_{pk}$ for multiple samples

Before our derivation, we need to define some notations:

$$C_{dr}=\frac{\mu-\mathrm{m}}{d},\ C_{dp}=\frac{\sigma}{d},\ \hat{C}_{dr}=\frac{\hat{\mu}-\mathrm{m}}{d},\ \hat{C}_{dp}=\frac{\hat{\sigma}}{d},$$

$Z=\sqrt{mn}\left(\hat{\mu}-\mu\right)$ and $Y=\sqrt{mn}\left(\hat{\sigma}^2-\sigma^2\right)$

By the Central Limit Theorem, Y converges to $\mathrm{N}(0,2\sigma^4)$ under both estimators, s_p^2 and s_u^2, and Z converges to $\mathrm{N}(0,\sigma^2)$ as mn goes to infinity.

$$\hat{C}_{dr}=C_{dr}+\frac{Z}{\sigma\sqrt{mn}}(C_{dp}),\ 1-\hat{C}_{dr}=(1-C_{dr})-\frac{Z}{\sigma\sqrt{mn}}(C_{dp}),\ 1+\hat{C}_{dr}=(1+C_{dr})+\frac{Z}{\sigma\sqrt{mn}}(C_{dp}),$$

$$\hat{C}_{dp}=C_{dp}\sqrt{1+\frac{Y}{\sigma^2\sqrt{mn}}},\ \frac{1}{\hat{C}_{dp}}=\frac{1}{C_{dp}}\left[1-\frac{Y}{2\sigma^2\sqrt{mn}}+O_p\left(\frac{1}{mn}\right)\right],$$

$$\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}=\frac{1-C_{dr}}{C_{dp}}+\frac{1}{\sqrt{mn}}\left[\frac{-Z}{\sigma}-\left(\frac{1-C_{dr}}{C_{dp}}\right)\frac{Y}{2\sigma^2}\right]+O_p\left(\frac{1}{mn}\right),$$

$$\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}=\frac{1+C_{dr}}{C_{dp}}+\frac{1}{\sqrt{mn}}\left[\frac{Z}{\sigma}-\left(\frac{1+C_{dr}}{C_{dp}}\right)\frac{Y}{2\sigma^2}\right]+O_p\left(\frac{1}{mn}\right),$$

$$\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)=\Phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1}{\sqrt{mn}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)\left[\frac{-Z}{\sigma}-\left(\frac{1-C_{dr}}{C_{dp}}\right)\frac{Y}{2\sigma^2}\right]+O_p\left(\frac{1}{mn}\right)$$

$$\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)=\Phi\left(\frac{1+C_{dr}}{C_{dp}}\right)+\frac{1}{\sqrt{mn}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\left[\frac{Z}{\sigma}-\left(\frac{1+C_{dr}}{C_{dp}}\right)\frac{Y}{2\sigma^2}\right]+O_p\left(\frac{1}{mn}\right)$$

$$\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)+\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)=\Phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\Phi\left(\frac{1+C_{dr}}{C_{dp}}\right)+\frac{1}{\sqrt{mn}}\frac{-Z}{\sigma}\left[\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right]+$$

$$\frac{1}{\sqrt{mn}}\frac{-Y}{2\sigma^2}\left[\left(\frac{1-C_{dr}}{C_{dp}}\right)\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\left(\frac{1+C_{dr}}{C_{dp}}\right)\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right]+O_p\left(\frac{1}{mn}\right)$$

The estimator of S_{pk} can be rewritten as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1-\hat{C}_{dr}}{\hat{C}_{dp}}\right)+\frac{1}{2}\Phi\left(\frac{1+\hat{C}_{dr}}{\hat{C}_{dp}}\right)\right\}.$$

Applying the Taylor's expansion $\Phi^{-1}(x+y)=\Phi^{-1}(x)+\frac{y}{\phi\left(\Phi^{-1}(x)\right)}+O_p(y^2)$, we can obtain

$$3\hat{S}_{pk} = 3S_{pk} + \frac{W}{2\sqrt{mn}\phi(3S_{pk})} + O_p\left(\frac{1}{mn}\right),$$

where $W=-\frac{Y}{2\sigma^2}\left[\frac{1-C_{dr}}{C_{dp}}\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+\frac{1+C_{dr}}{C_{dp}}\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right]-\frac{Z}{\sigma}\left[\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)-\phi\left(\frac{1+C_{dr}}{C_{dp}}\right)\right].$

Thus

$$\hat{S}_{pk} = S_{pk} + \frac{W}{6\sqrt{mn}\phi(3S_{pk})} + O_p\left(\frac{1}{mn}\right).$$

Appendix II: Calculation of lower bounds for multiple samples

```
alpha=0.05;
m=12;          % number of sub-samples
n=50;          % number of sample size
S=1.0;         % value of Ŝpk
Spk=S;
for i=1:1:100000
Spk=Spk-0.0001;
Cdp=1/(3*Spk);
a=sqrt(2)*3*Spk*normpdf(3*Spk);
p=normcdf((S-Spk)/sqrt(a*a/(36*m*n*normpdf(3*Spk)*normpdf(3*Spk))));
if p>(1.-alpha) break;
end; end
fprintf('The true value of the process capability Spk is no less than %g',Spk)
```

Appendix III: CDF and PDF of $\hat{S}''_{pk}$

A. Notations

To simplify the derivation, we define the following notations. Let

$$Z=\sqrt{n}(\bar{X}-\mu)/\sigma \text{, and } Y=\sqrt{n}(S^2-\sigma^2)/2\sigma^2 .$$

Consider the following analytical expansion of $\hat{S}_{pk}$

$$\hat{S}_{pk}=S_{pk}+D_1Z+D_2Y+D_3Z^2+D_4ZY+D_5Y^2+O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$D_1=\frac{1}{\sqrt{n}}\left(\frac{-\lambda_0}{6\phi(3S_{pk})}\right),\ D_2=\frac{1}{\sqrt{n}}\left(\frac{-\lambda_1}{6\phi(3S_{pk})}\right),\ D_3=\frac{1}{n}\left(\frac{S_{pk}\lambda_0^2}{8\left[\phi(3S_{pk})\right]^2}-\frac{\lambda_1}{12\phi(3S_{pk})}\right),$$

$$D_4=\frac{1}{n}\left(\frac{S_{pk}\lambda_0\lambda_1}{4\left[\phi(3S_{pk})\right]^2}+\frac{\lambda_0-\lambda_2}{6\phi(3S_{pk})}\right),\ D_5=\frac{1}{n}\left(\frac{S_{pk}\lambda_1^2}{8\left[\phi(3S_{pk})\right]^2}+\frac{3\lambda_1-\lambda_3}{12\phi(3S_{pk})}\right) \text{, and}$$

$$\lambda_k=\left(\frac{1-C_{dr}}{C_{dp}}\right)^k\phi\left(\frac{1-C_{dr}}{C_{dp}}\right)+(-1)^{k+1}\times\left(\frac{1+C_{dr}}{C_{dp}}\right)^k\phi\left(\frac{1+C_{dr}}{C_{dp}}\right),\ k=0,\ 1,\ 2,\ 3.$$

Let

$$\hat{S}''_{pk}=S_{pk}+D_1Z+D_2Y+D_3Z^2+D_4ZY+D_5Y^2 .$$

The CDF of $\hat{S}''_{pk}$ then can be derived by the probability

$$F_{\hat{S}''_{pk}}(x)=\Pr\left\{\hat{S}''_{pk}-x\le 0\right\}$$

$$=\Pr\left\{D_3\left(Z+\frac{D_1+D_4Y}{2D_3}\right)^2-\frac{E_1(Y+E_3)^2}{4D_3}+\frac{\Delta_1(x)}{4D_3E_1}\le 0\right\},$$

where

$$E_1=D_4^2-4D_3D_5\ ,E_2=D_1D_4-2D_2D_3\ ,\ \ E_3=\tfrac{E_2}{E_1}\ ,\ \ E_4=4D_3S_{pk}-D_1^2\ ,$$

$$\Delta_1(x)=E_2^2-E_1(4D_3x-E_4)\text{ , and }\ \Delta_2(Y;x)=E_1(Y+E_3)^2-\tfrac{\Delta_1(x)}{E_1}\ .$$

Moreover, we define the following notations through the derivation:

$$u_1(x)=-E_3-\sqrt{\frac{\Delta_1(x)}{E_1^2}}\ ,\ v_1(x)=-E_3+\sqrt{\frac{\Delta_1(x)}{E_1^2}}\ ,$$

$$u_2(y;x) = -\left(\frac{D_1 + D_4 y}{2D_3}\right) - \sqrt{\frac{\Delta_2(y;x)}{4D_3^2}}, \quad v_2(y;x) = -\left(\frac{D_1 + D_4 y}{2D_3}\right) + \sqrt{\frac{\Delta_2(y;x)}{4D_3^2}},$$

$y_0 = -\frac{\sqrt{n}}{2}$ is the minimum value of the random variable Y, and $\psi(\cdot)$ is the PDF of the random variable Y.

B. CDF and PDF of $\hat{S}''_{pk}$

Case 1: For $D_3 < 0$, $E_1 < 0$, $y_0 > -E_3$.

$$x < \frac{E_4 + E_2 E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{v_1(x)}^{\infty} \psi(y)dy + \int_{y_0}^{v_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy;$$

$$x \geq \frac{E_4 + E_2 E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = 1.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2 E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = -\frac{2D_3\psi[v_1(x)]}{\sqrt{\Delta_1(x)}} + \int_{y_0}^{v_1(x)} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy + \frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x \geq \frac{E_4 + E_2 E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = 0;$$

Case 2: For $D_3 < 0$, $E_1 < 0$, $y_0 < -E_3$.

$$x < \frac{E_4 + E_2 E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{v_1(x)}^{\infty} \psi(y)dy + \int_{y_0}^{v_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)\,dy;$$

$$\frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3} \le x < \frac{E_4 + E_2E_3}{4D_3},$$

$$F(x) = \int_{y_0}^{u_1(x)} \psi(y)dy + \int_{v_1(x)}^{\infty} \psi(y)dy + \int_{u_1(x)}^{v_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy;$$

$$x \ge \frac{E_4 + E_2E_3}{4D_3},$$

$$F(x) = 1.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = -\frac{(2D_3)\psi[v_1(x)]}{\sqrt{\Delta_1(x)}} + \int_{y_0}^{v_1(x)} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy +$$

$$\frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$\frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3} \le x < \frac{E_4 + E_2E_3}{4D_3},$$

$$f(x) = -\frac{(2D_3)\{\psi[u_1(x)] + \psi[v_1(x)]\}}{\sqrt{\Delta_1(x)}} + \int_{u_1(x)}^{v_1(x)} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy +$$

$$\frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}} +$$

$$\frac{(2D_3)\{\Phi[u_2(u_1(x);x)] + \Phi[-v_2(u_1(x);x)]\}\psi[u_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x \ge \frac{E_4 + E_2E_3}{4D_3},$$

$$f(x) = 0.$$

Case 3: For $D_3 < 0$, $E_1 > 0$, $y_0 > -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{y_0}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy;$$

$$x \geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{y_0}^{v_1(x)} \psi(y)dy + \int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = \int_{y_0}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy;$$

$$x \geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = -\frac{(2D_3)\psi[v_1(x)]}{\sqrt{\Delta_1(x)}} + \int_{v_1(x)}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy +$$

$$\frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}.$$

<u>Case 4</u>: For $D_3 < 0$, $E_1 > 0$, $y_0 < -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3},$$

$$F(x) = \int_{y_0}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy;$$

$$\frac{E_4 + E_2E_3}{4D_3} \leq x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{u_1(x)}^{v_1(x)} \psi(y)dy + \int_{y_0}^{u_1(x)} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy +$$

$$\int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy;$$

$$x \geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{y_0}^{v_1(x)} \psi(y)dy + \int_{v_1(x)}^{\infty} \{\Phi[u_2(y;x)] + \Phi[-v_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2E_3}{4D_3},$$

$$f(x) = \int_{y_0}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy;$$

$$\frac{E_4 + E_2E_3}{4D_3} \le x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = \int_{y_0}^{u_1(x)} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy +$$

$$\frac{(2D_3)\{\Phi[u_2(u_1(x);x)] + \Phi[-v_2(u_1(x);x)]\}\psi[u_1(x)]}{\sqrt{\Delta_1(x)}} +$$

$$\int_{v_1(x)}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy - \frac{(2D_3)\{\psi[v_1(x)] + \psi[u_1(x)]\}}{\sqrt{\Delta_1(x)}} +$$

$$\frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x \ge \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = -\frac{(2D_3)\psi[v_1(x)]}{\sqrt{\Delta_1(x)}} + \int_{v_1(x)}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy +$$

$$\frac{(2D_3)\{\Phi[u_2(v_1(x);x)] + \Phi[-v_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}.$$

Case 5: For $D_3 > 0$, $E_1 > 0$, $y_0 > -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{v_1(x)}^{\infty} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy;$$

$$x \ge \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{y_0}^{\infty} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = \int_{v_1(x)}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy +$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)] - \Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x \geq \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = \int_{y_0}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy.$$

<u>Case 6</u>: For $D_3 > 0$, $E_1 > 0$, $y_0 < -E_3$.

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$F(x) = \int_{v_1(x)}^{\infty} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy;$$

$$\frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3} \leq x < \frac{E_4 + E_2E_3}{4D_3},$$

$$F(x) = \int_{y_0}^{u_1(x)} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy + \int_{v_1(x)}^{\infty} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy;$$

$$x \geq \frac{E_4 + E_2E_3}{4D_3},$$

$$F(x) = \int_{y_0}^{\infty} \{\Phi[v_2(y;x)] - \Phi[u_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x < \frac{E_4 + E_2E_3}{4D_3} - \frac{E_1(y_0 + E_3)^2}{4D_3},$$

$$f(x) = \int_{v_1(x)}^{\infty} \frac{\{\phi[u_2(y;x)] + \phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}} dy +$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)]-\Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3}\leq x<\frac{E_4+E_2E_3}{4D_3},$$

$$f(x)=\int_{v_1(x)}^{\infty}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy+\int_{y_0}^{u_1(x)}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy+$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)]-\Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}+$$

$$\frac{(2D_3)\{\Phi[v_2(u_1(x);x)]-\Phi[u_2(u_1(x);x)]\}\psi[u_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x\geq\frac{E_4+E_2E_3}{4D_3},$$

$$f(x)=\int_{y_0}^{\infty}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy.$$

Case 7: For $D_3 > 0$, $E_1 < 0$, $y_0 > -E_3$.

$$x<\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$F(x) = 0$;

$$x\geq\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$F(x)=\int_{y_0}^{v_1(x)}\{\Phi[v_2(y;x)]-\Phi[u_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x<\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$f(x) = 0$;

$$x\geq\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$f(x)=\int_{y_0}^{v_1(x)}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy+$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)]-\Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}.$$

Case 8: For $D_3 > 0$, $E_1 < 0$, $y_0 < -E_3$.

$$x<\frac{E_4+E_2E_3}{4D_3},$$

$F(x) = 0;$

$$\frac{E_4+E_2E_3}{4D_3}\le x<\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$F(x)=\int_{u_1(x)}^{v_1(x)}\{\Phi[v_2(y;x)]-\Phi[u_2(y;x)]\}\psi(y)dy;$$

$$x\ge\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$F(x)=\int_{y_0}^{v_1(x)}\{\Phi[v_2(y;x)]-\Phi[u_2(y;x)]\}\psi(y)dy.$$

Using the Leibniz's rule for derivatives, we get the PDF as follows:

$$x<\frac{E_4+E_2E_3}{4D_3},$$

$f(x) = 0;$

$$\frac{E_4+E_2E_3}{4D_3}\le x<\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$f(x)=\int_{u_1(x)}^{v_1(x)}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy+$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)]-\Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}+$$

$$\frac{(2D_3)\{\Phi[v_2(u_1(x);x)]-\Phi[u_2(u_1(x);x)]\}\psi[u_1(x)]}{\sqrt{\Delta_1(x)}};$$

$$x\ge\frac{E_4+E_2E_3}{4D_3}-\frac{E_1(y_0+E_3)^2}{4D_3},$$

$$f(x)=\int_{y_0}^{v_1(x)}\frac{\{\phi[u_2(y;x)]+\phi[v_2(y;x)]\}\psi(y)}{\sqrt{\Delta_2(y;x)}}dy+$$

$$\frac{(2D_3)\{\Phi[v_2(v_1(x);x)]-\Phi[u_2(v_1(x);x)]\}\psi[v_1(x)]}{\sqrt{\Delta_1(x)}}.$$