Chapter 1
Introduction

Quality measures can be used to evaluate a process's performance. In general; there are several issues to be discussed in assessing product quality. One important issue of them is the process capability index. Process capability is defined to be the range over which the measurements of a process vary when the process variation is due to random causes only. We can realize whether the product meets its specifications and the process is stable by using the process capability index. Indubitably, process capability and related analysis have been studied in recent years. Properties of the univariate processes have been investigated extensively, but are comparatively neglected for multivariate processes where multiple dependent characteristics are involved in quality measurement. Frequently, a manufactured product has multiple quality characteristics. That is, the process capability analysis involves more than one engineering specification. For this reason, multivariate methods for assessing process capability are proposed.

1.1 Motivation

Process capability index has been widely used to measure the capability of a process according to its manufacturing specifications in industry. In particular, customers always ask their suppliers to provide the capability indices for the product characteristics in the supply chain partnership. Also, process capability indices can be used as the benchmarking for quality improvement activities. The capability indices, C_p, C_{pk} and C_{pm}, are widely used to evaluate a process performance based on a single engineering specification. A large number of papers have dealt with the statistical properties and the estimation of these univariate indices. Kotz and Lovelace (1998) provided a review of these indices in their textbook. In addition, Kotz and Johnson (2002) provided a compact survey and comments on some 170 publications on process capability indices during the years 1992 to 2000.

Process capability is defined to be the range over which the measurements of a process vary when the process variation is due to random causes only. Process capability indices provide an effective measure of process capability. We can realize whether the product meets its specifications and the process is stable by using the process capability index. When a process is in a good state of statistical control, its process capability index will be large. On the contrary, it will be small when a process is out of control. Frequently, a manufactured product has multiple quality characteristics. That is, the process capability analysis involves more than one engineering specification. For this reason, multivariate methods for assessing process capability are proposed.

1.2 Literature Review

Most researches have been devoted to capability measures with single quality characteristic. However, it is quite common that the manufactured product involves more than one quality characteristic. That is, manufactured items require values of several different characteristics for adequate description of their quality. Each of those characteristics must satisfy certain specifications. The assessed quality of a product depends on the combined effects of those characteristics, rather than on their individual values. For example, automobile paint needs a range of light reflective abilities and a range of adhesion abilities. A paint that satisfies one criterion but not the other is undesirable. Those characteristics are related through the compositions of the paint. It is therefore natural to consider a bivariate characterization of this paint.

As for the tolerance region about multiple characteristics, we often take an ellipsoidal region or a rectangular region. For more complex engineering specifications, the tolerance region will be very complicated. For instance, a drawing of a connecting rod in a combustion engine consists of crank-bore inner diameter, pin-bore inner diameter, rod length, bore true-location and so on. In multivariate processes, we usually assumes that the observations X have a multivariate normal distribution $N_v(\mu,\Sigma)$, where v is the dimension of variables, μ is the mean vector and Σ represents the variance-covariance matrix of X. Also T is the target vector, $\overline{X}$ is the sample mean vector and S is the sample covariance matrix.

Table 1. Development of the multivariate capability index

Index	Reference
$C_{pM} = \sqrt{\dfrac{nr}{\sum_{i=1}^{n}(x_{ir}-T_1)^T A^{-1}(x_{ir}-T_v)}}$	Chan *et al.*, 1991
${}_vC_p = \dfrac{k}{k_v} = C_p^{1/v}$ ${}_vC_{pm} = \dfrac{{}_vC_p}{\left(1+\dfrac{(\mu-T)^T A^{-1}(\mu-T)}{v}\right)_v}$ where $C_p = \dfrac{volume\left(X^T A^{-1} X \le K^2\right)}{volume\left(X^T A^{-1} X \le \chi^2_{v,0.9973}\right)}$	Pearn *et al.*, 1992
$MC_R = \mid SA^{-1}\mid + (\mu-T)^T A^{-1}(\mu-T)$ $E[L(X)] = tr(A^{-1}S) + (\mu-T)^T A^{-1}(\mu-T)$ where $L(X) = (X-T)^T A^{-1}(X-T)$	Kotz and Johnson, 1993

Table 1. Development of the multivariate capability index (continued)

Index	Reference
$MC_{pk} = \frac{1}{3}\Phi^{-1}(p)$	Wierda, 1993
$MC_{pm} = \frac{C_p}{D}$ where $C_p = \frac{V_0(\text{modified specification region})}{\lvert\Sigma\rvert^{1/2}(\pi\chi^2_{v,\,0.9973})^{1/2}(\Gamma(v/2+1))^{-1}}$ $D = \left[1+(\mu-T)^T\Sigma^{-1}(\mu-T)\right]^{1/2}$	Taam *et al.*, 1993
$MC_p = \frac{r_0}{r}$ where $r = \min\{c : \Pr(h(X-\mu_0)) \geq 1-\alpha\}$	Chen, 1994
$\lfloor C_{pM}, PV, LI \rfloor$ where $C_{pM} = \left[\frac{\prod_{i=1}^{v}(USL_i - LSL_i)}{\prod_{i=1}^{v}(UPL_i - LPL_i)}\right]^{1/v}$ $PV = P\left(T^2 > \frac{v(n-1)}{n-v}F(v, n-v)\right)$ $LI = \begin{cases} 1 & \text{if process region within tolerance region} \\ 0 & \text{otherwise} \end{cases}$	Shahriari *et al.*, 1995
Multivariate Data Using Geometric Distance Approach	Wang and Hubele, 1999
Multivariate Data (Normal and Non-Normal Data) Using Principal Component Analysis	Wang and Du, 2000
$S^T_{pk} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}2\Phi(3S_{pkj})-1)+1\right]/2\right\}$	Chen *et al.*, 2003

In order to handle the issue for cases with multiple quality characteristics, multivariate methods for assessing process capability are proposed. The development of a multivariate capability index is shown in Table 1. According to the literature review, the multivariate process capability indices can be obtained from:

(1) the ratio of a tolerance region to a process region

(2) the function of the multivariate probability distribution which is used to compute the probability of the non-conforming product
(3) approaches that consist of loss functions and vector representation
(4) based on the geometric distance approach and principal component analysis

Here, we may conclude that four constraints exist in the conventional methods. First, the normality assumption on multivariate data is usually required. Second, the confidence intervals of the multivariate capability indices are difficult to derive. Third, the estimated probability of non-conforming for some conventional methods is not available. Finally, the computation for high dimensions is not obtainable except by the geometric distance approach and the principal analysis.

1.3 Research Objective

Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability according to manufacturing specifications. Properties of the univariate processes have been investigated extensively, but are comparatively neglected for multivariate processes where multiple dependent characteristics are involved in quality measurement. However, the relevant statistical properties for the multivariate capability indices are rarely discussed. In order to calculate the capability index value, sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments due to sampling errors. The approach by simply looking at the calculated values of the estimated index and then making a conclusion on whether the given process is capable, is highly unreliable since the sampling errors have been ignored. In our investigation, we focus on obtaining the statistical properties for the multivariate process capability indices. According to the relevant statistical properties for the multivariate process capability indices, the lower bounds of the multivariate capability indices can be obtained to judge whether the processes meet the present capability requirement.

1.4 Organization

The contents of this article divide five chapters. In Chapter 1, we address a systematic description for process capability indices, inclusive of motivation, literature review and research objective. In Chapter 2, we proposed the asymptotic distribution of yield index for processes with multiple independent characteristics or multiple correlated characteristics. In Chapter 3, we introduced how to obtain the relevant statistical properties of $\hat{MC}_p$ and $\hat{MC}_{pm}$. In Chapter 4, we considered the supplier selection problem based on manufacturing precision in which the processes involve multiple quality characteristics. We derived the distribution of the corresponding test statistic, and provided critical values required for the comparison purpose. In Chapter 5, we make the conclusion of the overall contents in this article.

Chapter 2
Process Yield Measure for Processes with Multiple Quality Characteristics

Process yield has been the most basic and common criterion used in the manufacturing industry for measuring process performance. It is closely related to the production cost as well as customers' satisfaction. Process yield is currently defined as the percentage of processed product unit passing inspection. That is, the product characteristic must fall within the manufacturing tolerance. For product units rejected (non-conformities), additional costs would be incurred to the factory for scrapping or repairing the product. All passed product units are equally accepted by the producer, which incurs the factory no additional cost. For processes with high yield, it produces few percentages of non-conforming products. That is, most of products produced in this process satisfy the requirement of specifications. In many cases, a benchmark of minimum 99.73% for assessing the process is suggested. Enterprises get more profit and cost down with high process yield, hence companies make their efforts to increase the process yield. For normally distributed processes with a single characteristic, the index S_{pk} (Boyles (1994)) is used to establish the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on the process yield, defined as

$$\begin{aligned} S_{pk} &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(\frac{USL-\mu}{\sigma})+\frac{1}{2}\Phi(\frac{\mu-LSL}{\sigma})\right\} \\ &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi(\frac{1-C_{dr}}{C_{dp}})+\frac{1}{2}\Phi(\frac{1+C_{dr}}{C_{dp}})\right\} \end{aligned}, \qquad (2.1)$$

where $C_{dr} = (\mu - m)/d$, $C_{dp} = \sigma/d$, $m = (USL + LSL)/2$, $d = (USL - LSL)/2$. It provides an exact measure of process yield. If $S_{pk} = c$, then the process yield can be expressed as yield=$2\Phi(3c)-1$. Obviously, there is a one-to-one correspondence between S_{pk} and the process yield.

The relationships between the process yield and the process capability indices have been discussed extensively for processes with single characteristics, but comparatively neglected for processes with multiple characteristics. Considering a production process in which, possibly dependent, quality characteristics determine the quality of the product. In other words, the product has multiple correlated characteristics. We are concerned with the probability of producing a good product satisfying all its specifications. Assume that the observations X have a multivariate normal distribution, $N_v(\mu,\Sigma)$, where v is the dimension of variables, μ is the mean vector and Σ represents the variance-covariance matrix of X. The components of the vectors LSL and USL are the v lower and upper specification limits, respectively. Under the assumptions mentioned, the probability that a production process produces a good product is

$$p = \int_{[LSL,USL]} N_v(X \mid \mu, \Sigma)\, dX .$$

It is also called the true process yield.

2.1 Process Yield for Multiple Independent Characteristics

The Yield Index S_{pk}^T

Considering processes with multiple characteristics (assuming characteristics are mutually independent), Chen *et al.* (2003) defined the yield index as

$$S_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3S_{pkj})-1\right)+1\right]/2\right\}, \qquad (2.2)$$

where S_{pkj} denote the S_{pk} value of the jth characteristic for $j = 1, 2, ..., v$, and v is the number of characteristics. This index provides an exact measure of the overall process yield when the characteristics are mutually independent. A one-to one correspondence relationship between the index S_{pk}^T and the overall process yield P can be established as

$$P = \prod_{j=1}^{v} P_j = \prod_{j=1}^{v}\left[2\Phi(3S_{pkj})-1\right] = 2\Phi(3S_{pk}^T)-1. \qquad (2.3)$$

Hence, the new index S_{pk}^T provides an exact measure of the overall process yield. For example, if S_{pk}^T=1, then the entire process yield would be exactly 99.73%. Table 2 display various commonly used capability requirements and the corresponding overall process yield.

Table 2. Some S_{pk}^T values and the corresponding nonconformities.

S_{pk}^T	Yield	PPM
1.00	0.9973002039	2699.796
1.33	0.9999339267	66.073
1.50	0.9999932047	6.795
1.67	0.9999994557	0.544
2.00	0.9999999980	0.002

For a process with v characteristics, if the requirement for the overall process capability is $S_{pk}^T \geq c_0$, a sufficient condition for the requirement to each single characteristic can be obtained by

$$S_{pkj} \geq \frac{1}{3}\Phi^{-1}\left(\frac{\sqrt[v]{2\Phi(3c_0)-1}+1}{2}\right),\ j=1,2,\ldots,v. \tag{2.4}$$

(see Chen *et al*. (2003)). For example, if c_0 is set to be 1 with v=5, i.e. the overall process yield is set to be no less than 0.9973. The overall capability requirement $S_{pk}^T \geq 1$ would be satisfied if each single characteristic yield is no less than $(0.997300204)^{1/5}$ =0.99945950, and the capability for all the five characteristics is

$$S_{pkj} \geq \frac{1}{3}\Phi^{-1}\left(\frac{\sqrt[5]{2\Phi(3)-1}+1}{2}\right)=1.153, \text{ for } j=1,2,\ldots,v.$$

An Estimator and its Asymptotic Distribution

To estimate the yield measurement index S_{pk}^T, we consider the following natural estimator $\hat{S}_{pk}^T$, expressed as

$$\hat{S}_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3\hat{S}_{pkj})-1\right)+1\right]/2\right\}, \tag{2.5}$$

where $\hat{S}_{pkj} \sim N(\ S_{pkj},(a_j^2+b_j^2)/36n(\phi(3S_{pkj}))^2\ ),\ j=1,2,\ldots,v.$ (see Lee *et al*.(2002)). Applying the first-order expansion of v-variate Taylor, the natural estimator $\hat{S}_{pk}^T$ can be expressed as

$$\hat{S}_{pk}^T = f(S_{pk1},S_{pk2},\ldots,S_{pkv}) + \frac{\partial f(S_{pk1},S_{pk2},\ldots,S_{pkv})}{\partial \hat{S}_{pk1}}(\hat{S}_{pk1}-S_{pk1}) + \ldots + \frac{\partial f(S_{pk1},S_{pk2},\ldots,S_{pkv})}{\partial \hat{S}_{pkv}}(\hat{S}_{pkv}-S_{pkv}).$$

Thus, we can derive

$$E(\hat{S}_{pk}^T) = S_{pk}^T \quad \text{and} \quad Var(\hat{S}_{pk}^T) = \frac{k(v)}{36n(\phi(3S_{pk}^T))^2}, \text{ where}$$

$$k(v) = \sum_{j=1}^{v}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}.$$

(See Corollary 1 in Appendix).

In addition, since $\hat{S}_{pk}^T$ is linear combination of $\hat{S}_{pk1}, \hat{S}_{pk2},\ldots$ and $\hat{S}_{pkv}$, we know $\hat{S}_{pk}^T$ is from normal distribution. So $\hat{S}_{pk}^T$ has the asymptotic normal distribution with the mean S_{pk}^T and variance $k(v)$. That is, the asymptotic distribution of $\hat{S}_{pk}^T$ is

$$\hat{S}_{pk}^T \sim N(S_{pk}^T, \frac{k(v)}{36n(\phi(3S_{pk}^T))^2}). \tag{2.6}$$

When $v=1$, the asymptotic distribution of $\hat{S}_{pk}^{T}$ can be expressed as

$$\hat{S}_{pk} \sim N(S_{pk}, \frac{1}{36n(\phi(3S_{pk}))^2}(a^2+b^2)),$$

which is equivalent to that derived from Lee *et al*. (QREI, 2002).

Inference Based on $\hat{S}_{pk}^{T}$

From equation (2.6), it shows that $\hat{S}_{pk}^{T}$ is an asymptotic unbiased estimator of S_{pk}^{T}. Also, according the asymptotic distribution of $\hat{S}_{pk}^{T}$ given in equation (2.6), hypothesis testing and a confidence interval for S_{pk}^{T} can be constructed. To test whether a given process is capable, we may consider the following statistical hypothesis testing:

$$H_0 : S_{pk}^{T} \le c_0 \quad \text{(process is not capable)}$$
$$H_1 : S_{pk}^{T} > c_0 \quad \text{(process is capable)}$$

where c_0 is the standard minimal criteria for S_{pk}^{T}. The test can be executed by using the testing statistic

$$T = \frac{6(\hat{S}_{pk}^{T} - c_0)\sqrt{n}\phi(3\hat{S}_{pk}^{T})}{\sqrt{k(v)}}. \tag{2.7}$$

The null hypothesis H_0 is rejected at α level if $T > Z_{\alpha}$, where Z_{α} is the upper 100α% point of the standard normal distribution. An approximate $100(1-\alpha)$% confidence interval for S_{pk}^{T} can be expressed as

$$\left[\hat{S}_{pk}^{T} - \frac{\sqrt{k(v)}}{6\sqrt{n}\phi(3\hat{S}_{pk}^{T})}Z_{\alpha/2},\ \hat{S}_{pk}^{T} + \frac{\sqrt{k(v)}}{6\sqrt{n}\phi(3\hat{S}_{pk}^{T})}Z_{\alpha/2}\right].$$

Also an approximate $100(1-\alpha)$% lower confidence bound for S_{pk}^{T} can be expressed as

$$\hat{S}_{pk}^{T} - \frac{\sqrt{k(v)}}{6\sqrt{n}\phi(3\hat{S}_{pk}^{T})}Z_{\alpha}. \tag{2.8}$$

2.2 Principal Component Analysis

PCA is a useful statistical technique that has been widely applied to face recognition and image compression, which is a common technique for finding patterns in high dimensional data. It is a way of identifying patterns in data, and expressing the data in such a way as to highlight their similarities and differences. In many cases the patterns in data can be difficult to find in high dimensional applications, particularly when graphical representation is not available, PCA is a powerful tool in such situations. The other main advantage of PCA is that after finding patterns in the data, one could compress the data by reducing the number of dimensions without losing much information. PCA is a multivariate technique in which a number of related variables are transformed to a set of uncorrelated linear functions of the original measurements. The first principal component linearly combines all of the original variables in which the maximum variation among the objects is displayed. The second, third, and further components are, similarly, the linear combinations representing the next largest variation, irrespective of those represented by previous ones. In most practical applications, analyzing the major components can retain most of the information regarding the variability of the process. In general, multivariate methods often assume the data satisfy multivariate normal distribution. But in applying the PCA technique one does not require such assumption.

Assume that X is a $v \times n$ sample data matrix, where v is the number of product quality characteristic from one part and n is the sample size of part measured. Also, $\overline{X}$ is the sample mean vector ($v \times 1$) of observations and S is a $v \times v$ symmetric matrix representing the covariance between observations. Engineering specifications are given for each quality characteristic, where LSL and USL are their v-vectors of the lower specification limits and upper specification limits, respectively. The vector T ($v \times 1$) represents the target values of the v quality characteristics. In addition, the spectral decomposition can be used to obtain $D = U^T SU$, where D is a diagonal matrix. The diagonal elements of D, $\lambda_1, \lambda_2, ..., \lambda_v$ are the eigenvalues of S and the columns of U, $u_1, u_2, ..., u_v$ are the eigenvectors of S. Consequently, the ith principal component (PCi) is expressed as

$$PC_i = u_i^T x, \ \forall i = 1, 2, \ldots, v, \tag{2.9}$$

where x is $v \times 1$ vectors on the original variables. The engineering specifications and target values of $PC_i s$ are as follows:

$$\begin{cases} LSL_{PC_i} = u_i^T LSL \\ USL_{PC_i} = u_i^T USL, \ \forall i = 1,2,\ldots,v. \\ T_{PC_i} = u_i^T T \end{cases} \tag{2.10}$$

Similarly, the relevant sample estimators, S^2 and $\overline{X}$ of $PC_i s$ can defined as

$$\begin{cases} S^2_{PC_i} = \lambda_i \\ \overline{X}_{PC_i} = u_i^T \overline{X} \end{cases}, \ \forall i = 1,2,\ldots,v. \tag{2.11}$$

The ratio of each eigenvalue to the summation of the eigenvalues is the proportion of variability associated with each principal component variable. That is,

$$\lambda_i / \sum_{i=1}^{v} \lambda_i, \ \forall i = 1, 2, \ldots, v.$$

However, only a few principal components can explain the most of the total variability (about 80%~90%). Anderson (1963) proposed a χ^2 test for identifying the significant components. It is

$$\chi^2 = -(n-1)\sum_{j=k+1}^{v} \ln \lambda_j + (n-1)(v-k)\ln \frac{\sum_{j=k+1}^{v} \lambda_j}{v-k}, \tag{2.12}$$

where χ^2 has $r = (1/2)(v-k)(v-k+1)-1$ degrees of freedom. Jackson (1980) further applied the test to the hypothesis $H_0 : \lambda_{k+1} = \ldots = \lambda_v$ against the alternatives with at least has one different eigenvalue. Referring to this method, we can choose the suitable number of $PC_i s$ rightly.

2.3 Process Yield for Multiple Correlated Characteristics

Assume that the multivariate processes data are from a multivariate normal distribution. In this case, the principal components can be applied to the capability study. Consequently, the new variables (principal components) are mutually independent and normal distributed (see Theorem 1 in the Appendix). Applying the equation (2.2), the combined yield index for the multivariate processes data can be determined by

$$TS_{pk,PC} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3S_{pkj;PC})-1\right)+1\right]/2\right\}, \tag{2.13}$$

where $S_{pkj;PC_i}$ represents the univariate measure of process yield index for the *ith* principal component. Also the estimator of $TS_{pk,PC}$ is defined as

$$\hat{TS}_{pk;PC} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3\hat{S}_{pkj;PC})-1\right)+1\right]/2\right\}. \tag{2.14}$$

Analogy to equation (2.6), we can verify that $\hat{TS}_{pk;PC}$ has the asymptotic normal distribution with the mean $TS_{pk;PC}$ and variance $k(v, PC) / 36n(\phi(3TS_{pk;PC}))^2$, where

$$k(v, PC) = \sum_{j=1}^{v}\left\{(a_{j;PC}^2 + b_{j;PC}^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki;PC})-1\right)^2}{\left(2\Phi(3S_{pkj;PC})-1\right)^2}\right]\right\}.$$

That is, the asymptotic distribution of $\hat{TS}_{pk;PC}$ is

$$\hat{TS}_{pk;PC} \sim N(TS_{pk;PC}, \frac{k(v, PC)}{36n(\phi(3TS_{pk;PC}))^2}). \qquad (2.15)$$

Thus, an approximate $100(1-\alpha)\%$ lower confidence bound for $TS_{pk;PC}$ can be obtained, and an approximate $100(1-\alpha)\%$ lower confidence bound for the true process yield can also be obtained by using the one-to-one correspondence between $TS_{pk;PC}$ and the process yield (Yield=$2\Phi(3TS_{pk;PC})-1$).

2.4 An Application Example

The previous study (Wang and Chen, 1998) presented a trivariate quality control involving the joint control of the depth (D), the length (L) and the width (W) of a plastic product. Fifty observations are collected from a plastic production line. The specified limits for D, L, and W are set at [2.1, 2.3], [304.5, 305.1] and [304.5, 305.1], respectively. The specification of the target value is $T^T = [2.2, 304.8, 304.8]$. The p-value for Mardia's SW statistic is 0.32. Thus, the assumption of multivariate normality can not be rejected at 95% confidence level. The sample mean vector and sample covariance matrix were

$$\bar{X}^T = [2.1616, 304.7182, 304.7678] \text{ and } S = \begin{bmatrix} 0.002051 & 0.000785 & 0.000656 \\ 0.000785 & 0.001717 & 0.001204 \\ 0.000656 & 0.001204 & 0.002034 \end{bmatrix}.$$

Figure 1 illustrate the process points and tolerance region.

By performing the principal components analysis, the eigenvectors and eigenvalues can be obtained. Table 3 shows the loading and eigenvalue of PCs using the principal component analysis. First, the test of the hypothesis $H_0 : \lambda_1 = \lambda_2 = \lambda_3$, produced a value of $\chi_5^2 = 36.47$, which is significant at the 95% confidence level. That is, the hypothesis is rejected. Then, testing the hypothesis $H_0 : \lambda_2 = \lambda_3$ produced a value of $\chi_2^2 = 8.19$, which is also significant at the 95% confidence level. Thus, we used the first two *PCs* to evaluate the capability at 89.04% total variability. The principal components are USL_{PC1} =-368.1421, LSL_{PC1} =-368.9698, $\bar{X}_{PC1}$ =-368.4682, T_{PC1} =-368.5560, S_{PC1} =0.0609, USL_{PC2} =216.8123, LSL_{PC2} =216.5499, $\bar{X}_{PC2}$ =216.6794, T_{PC2} =216.6811 and S_{PC2} =0.0382. $\hat{S}_{pk1;PC}$ and $\hat{S}_{pk2;PC2}$ can be calculated as 1.8262 and 1.1437, respectively. Applying the equation (2.15), the approximate 95% lower confidence bound for the combined index, $TS_{pk;PC}$, is 0.9556. Using the one-to-one correspondence between $TS_{pk;PC}$ and the process yield (Yield=$2\Phi(3TS_{pk;PC})-1$), the approximate 95% lower confidence bound for the true process yield is 0.995854. Notably, this process does not meet the process yield requirement.

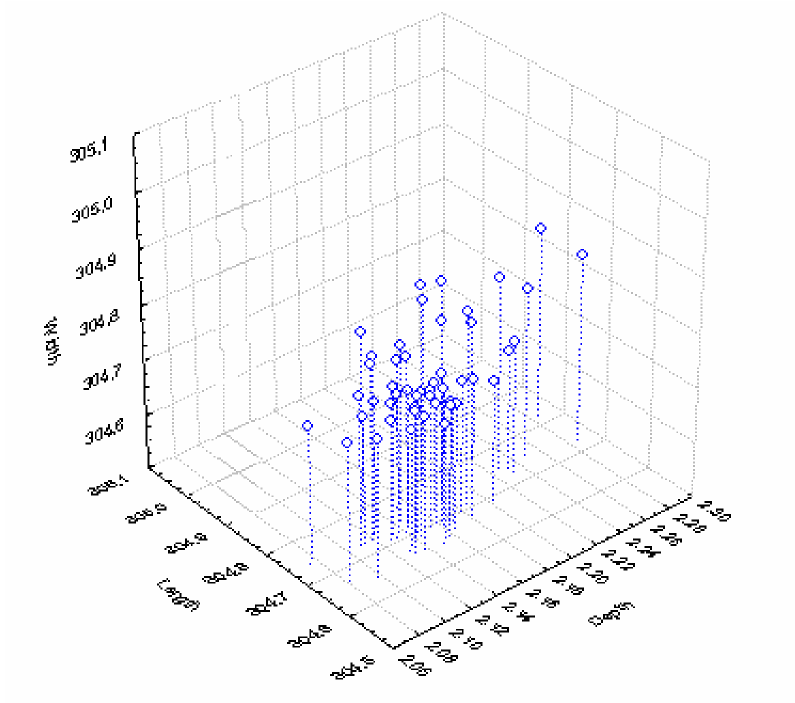

Figure 1. Process points and tolerance region for example

Table 3. The results of the first two PCs for example

		PC1 Loading	PC2 Loading
Variable	D	-0.522185	-0.838481
	L	-0.582407	0.217154
	W	-0.622997	0.499794
Eigenvalue		0.003709	0.001457
% Explained of total variability		63.9253	25.1129

Chapter 3
Distributional and Inferential Properties of Multivariate Capability Indices

Process capability indices have been widely used in the manufacturing industry for measuring process reproduction capability according to manufacturing specifications. Properties of the univariate processes have been investigated extensively, but are comparatively neglected for multivariate processes where multiple dependent characteristics are involved in quality measurement. Although some multivariate capability indices have been proposed, no papers discuss the statistical properties of these indices. It is needed to investigate the statistical properties for the multivariate process capability indices. In this chapter, the statistical properties for the multivariate process indices MC_p and MC_{pm} are investigated. The estimator of MC_p and its properties are presented in Section 3.1. A confidence interval, lower confidence bound and hypothesis testing for MC_p are presented in Section 3.2. In Section 3.3, an approximate confidence interval for MC_{pm} is derived. A simulation study is conducted to ascertain the accuracy of the approximation. Finally, an application example is chosen to illustrate the proposed methodology.

3.1 Estimation of MC_p

Here, we focus on the multivariate process indices MC_p and MC_{pm} (Taam *et al*, 1993). The multivariate capability index MC_p is defined as

$$MC_p = \frac{vol.(\text{modified tolerance region})}{vol.[(X-\mu)'\Sigma^{-1}(X-\mu) \le k(q)]} = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|\Sigma|^{1/2}[\Gamma(v/2+1)]^{-1}}, \tag{3.1}$$

where $k(q)$ is the 99.73th percentile of the χ^2 distribution with v degrees of freedom, $|\Sigma|$ is the determinant of Σ, and $\Gamma(\cdot)$ is the gamma function. An estimator of MC_p can be expressed as

$$\hat{MC}_p = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|S|^{1/2}[\Gamma(v/2+1)]^{-1}}, \tag{3.2}$$

where S is the sample variance-covariance matrix and $|S|$ is the determinant of S. According to the equation (3.1), $\hat{MC}_p$ can be expressed as $MC_p \times (|S|/|\Sigma|)^{-1/2}$. Let $X = (X_1, X_2, ..., X_n)'$ represents a *n*-dimensional vector of measurements from a multivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2, ..., \mu_v)'$, T is the vector of target values, Σ is the process variance-covariance matrix. Using the following theorem, we can derive the distribution of $\hat{MC}_p$.

Theorem 2: The distribution of the generalized variance $|S|$ of a sample $X_1, X_2, ..., X_n$ from $N_v(\mu, \Sigma)$ is the same as the distribution of $|\Sigma|/(n-1)^v$ times the product of v independent factors, the distribution of the ith factor being the χ^2 distribution with $n-i$ degrees of freedom.

Proof: See Anderson (2003) on page 268.

From the above theorem, we can have that $|S|/|\Sigma|$ is the distribution of $\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}/(n-1)^v$. Let $y=\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}$. Then, we have

$$\frac{MC_p^2}{\hat{MC}_p^{\,2}}\sim\frac{y}{(n-1)^v}.$$

Using the transformation method, the probability density function of $\hat{MC}_p$ can be expressed as

$$f(x)=f_Y[g^{-1}(x)]\times\left|\frac{d}{dx}g^{-1}(x)\right|=f_Y[\frac{MC_p^2(n-1)^v}{x^2}]\times\frac{2(n-1)^v MC_p^2}{x^3},\ \text{for}\ x>0. \qquad (3.3)$$

When there is only single quality characteristic, that is, v=1, the probability density function of $\hat{MC}_p$ is equivalent to the pdf of $\hat{C}_p$. So, we have

$$\frac{C_p^2}{\hat{C}_p^2}\sim\frac{y}{(n-1)}\ \text{where}\ y=\chi^2_{n-1}.$$

From equation (3.3) and the pdf of y is

$$f_Y(y)=\frac{(\frac{1}{2})^{(n-1)/2}y^{(n-1)/2-1}e^{-y/2}}{\Gamma(\frac{n-1}{2})},$$

the pdf of $\hat{C}_p$ is given by

$$f(x)=\frac{(n-1)^{(n-1)/2}}{C_p\Gamma(\frac{n-1}{2})2^{(n-3)/2}}\times(\frac{C_p}{x})\times e^{-[(n-1)/2]\times(C_p/x)^2},\ \text{for}\ x>0. \qquad (3.4)$$

When there are two quality characteristics, that is, v=2, we have

$$\frac{MC_p^2}{\hat{MC}_p^{\,2}}\sim\frac{y}{(n-1)^2}\ \text{where}\ y=\chi^2_{n-1}\times\chi^2_{n-2}.$$

It can be shown that $\chi^2_{n-1}\times\chi^2_{n-2}\sim(\chi^2_{2n-4})/4$ (See Corollary 2 in Appendix). Thus, we have

$$\frac{MC_p^2}{\hat{MC}_p^{\,2}}\sim\frac{z}{4(n-1)^2},\ \text{where}\ z=(\chi^2_{2n-4})^2.$$

Using the transformation method, the pdf of z is

$$f(z)=\frac{(\frac{1}{2})^{(n-1)}z^{n/2-2}e^{-\sqrt{z}/2}}{\Gamma(n-2)},\ \text{for}\ z>0.$$

Similarly, from equation (3.3) and the pdf of z, the pdf of $\hat{MC}_p$ is given by

$$f(x)=\frac{(n-1)^{(n-2)}}{MC_p\Gamma(n-2)}\times(\frac{MC_p}{x})^{n-1}\times e^{-(n-1)\times(MC_p/x)},\ \text{for } x>0. \quad (3.5)$$

When there are three quality characteristics, that is, v =3, we have

$$\frac{MC_p^2}{\hat{MC}_p^2}\sim\frac{y}{(n-1)^3},\ \text{where } y=\chi^2_{n-1}\times\chi^2_{n-2}\times\chi^2_{n-3}.$$

Let $z_1=\chi^2_{n-1}\times\chi^2_{n-2}\sim(\chi^2_{2n-4})/4$ and $z_2=\chi^2_{n-3}$. Then, we have

$$\frac{MC_p^2}{\hat{MC}_p^2}\sim\frac{z_1 z_2}{(n-1)^3}.$$

Let $u=z_1\times z_2$ and $v=z_1$. Using the transformation method, the joint pdf of $u\times v$ is

$$f(u,v)=\frac{(\frac{1}{2})^{(n-1)/2}\times v^{-1/2}\times u^{(n-5)/2}\times e^{-(\sqrt{v}+u/2v)}}{\Gamma(n-2)\Gamma(\frac{n-3}{2})},\ \text{for } u,v>0.$$

Then, the pdf of u is given by

$$f_U(u)=\int_0^\infty\frac{(\frac{1}{2})^{(n-1)/2}\times v^{-1/2}\times u^{(n-5)/2}\times e^{-(\sqrt{v}+u/2v)}}{\Gamma(n-2)\Gamma(\frac{n-3}{2})}dv,\ \text{for } u>0.$$

Similarly, from equation (3.3) and the pdf of u, the pdf of $\hat{MC}_p$ is given by

$$f(x)=\frac{(n-1)^3\times(\frac{1}{2})^{n/2-3/2}}{MC_p\Gamma(n-2)\Gamma(\frac{n-3}{2})}\times(\frac{MC_p}{x})^{n-2}\times\int_0^\infty v^{-1/2}\times e^{-[\sqrt{v}+(n-1)^3(MC_p/x)^2/(2v)]}dv,\ \text{for } x>0. \quad (3.6)$$

Since the hth moment of a χ^2 distribution with v degrees of freedom is $2^h\Gamma(v/2+h)/\Gamma(v/2)$ and the moment of a product of independent variables is the product of the moments of the variables, the hth moment of $|S|/|\Sigma|$ can be obtained as

$$E(|S|/|\Sigma|)^h=\frac{2^{vh}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)+h]}{(n-1)^{vh}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)]}. \quad (3.7)$$

Now we can derive the rth moment of $\hat{MC}_p$ according to the equation (3.7) and $\hat{MC}_p=MC_p\times(|S|/|\Sigma|)^{-1/2}$. Thus, we have

$$E(\hat{MC}_p^{\,r})=E(MC_p^r\times(|S|/|\Sigma|)^{-r/2})=MC_p^r\times E(|S|/|\Sigma|)^{-r/2}. \quad (3.8)$$

Now, we can substitute r=1 and h=-1/2 into the equations (3.7) and (3.8), respectively. Then, we have

$$E(\hat{MC}_p)=MC_p\times\frac{2^{-v/2}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)-\frac{1}{2}]}{(n-1)^{-v/2}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)]}=MC_p\times(\frac{n-1}{2})^{v/2}\frac{\Gamma[\frac{1}{2}(n-v)-\frac{1}{2}]}{\Gamma[\frac{1}{2}(n-1)]}=\frac{1}{b_v}\times MC_p, \quad (3.9)$$

where $b_v=(\frac{2}{n-1})^{v/2}\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-v)-\frac{1}{2}]}$

is an unbiasing factor, so that $b_v\times\hat{MC}_p$ is an unbiased estimator of MC_p. Again, we can substitute r=2 and h=-1 into the equations (3.7) and (3.8), respectively. Then, we have

$$E(\hat{MC}_p^{\,2})=MC_p^2\times\frac{2^{-v}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)-1]}{(n-1)^{-v}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)]}. \quad (3.10)$$

From the equations (3.9) and (3.10), we have the variance of $\hat{MC}_p$ as

$$Var(\hat{MC}_p)=MC_p^2\times\frac{2^{-v}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)-1]}{(n-1)^{-v}\prod_{i=1}^{v}\Gamma[\frac{1}{2}(n-i)]}-\left[\frac{1}{b_v}\times MC_p\right]^2. \quad (3.11)$$

When there is only single quality characteristic, that is, v=1, according to the equations (3.9) and (3.11), we can find that the expectation and variance of $\hat{MC}_p$ are equal to those of $\hat{C}_p$ (See Kotz and Lovelace, 1998). That is, the expectation and variance of $\hat{C}_p$ can be obtained as

$$E(\hat{C}_p)=MC_p\times\sqrt{\frac{n-1}{2}}\frac{\Gamma[\frac{1}{2}(n-2)]}{\Gamma[\frac{1}{2}(n-1)]}=\frac{1}{b_1}\times C_p \text{ and } Var(\hat{C}_p)=[\frac{n-1}{n-3}-\frac{1}{b_1^2}]\times C_p{}^2,$$

where $b_1=\sqrt{\frac{2}{n-1}}\times\frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]}$

is a correction factor, so that $b_1\times\hat{C}_p$ is an unbiased estimator of C_p. When there are two quality characteristics, that is, v=2, according to the equations (3.9) and (3.11), the expectation and variance of $\hat{MC}_p$ can be obtained as

$$E(\hat{MC}_p)=MC_p\times\frac{n-1}{n-3}=\frac{1}{b_2}\times MC_p \text{ and}$$

$$Var(\hat{MC}_p)=MC_p^2\times\frac{(n-1)^2}{(n-3)^2(n-4)}=\frac{1}{n-4}\times\frac{1}{b_2^2}\times MC_p^2,$$

where $b_2 = \frac{n-3}{n-1}$

is a correction factor, so that $b_2 \times \hat{MC}_p$ is an unbiased estimator of MC_p.When there are three quality characteristics, that is, $\hat{v}$ =3, according to the equations (3.9) and (3.11), the expectation and variance of $\hat{MC}_p$ can be obtained as

$$E(\hat{MC}_p) = MC_p \times (\frac{n-1}{2})^{3/2} \times \frac{\Gamma[\frac{1}{2}(n-3)-\frac{1}{2}]}{\Gamma[\frac{1}{2}(n-1)]} = \frac{1}{b_3} \times MC_p \quad \text{and}$$

$$Var(\hat{MC}_p) = MC_p^2 \times [\frac{(n-1)^3}{(n-3)(n-4)(n-5)} - \frac{(n-1)^3}{2(n-3)^2} \times \frac{\Gamma^2(\frac{1}{2}n-2)}{\Gamma^2(\frac{1}{2}n-\frac{3}{2})}]$$

$$= [\frac{(n-1)^3}{(n-3)(n-4)(n-5)} - \frac{1}{b_3^2}] \times MC_p^2,$$

where $b_3 = \frac{n-3}{n-1} \times \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(\frac{1}{2}n-\frac{3}{2})}{\Gamma(\frac{1}{2}n-2)}$

is a correction factor, so that $b_3 \times \hat{MC}_p$ is an unbiased estimator of MC_p.

3.2 Confidence Intervals and Hypothesis Testing for MC_p

Since $\hat{MC}_p = MC_p \times (|S|/|\Sigma|)^{-1/2}$, like other statistics, is subject to the sampling variation, it is critical to compute an interval to provide a range which includes the true MC_p with high probability. Based on the definition, a 100($1-\alpha$) % confidence interval for MC_p can be established as:

$$1-\alpha = P\{L \le MC_p \le U\} = P\left\{L \le \hat{MC}_p \times (|S|/|\Sigma|)^{1/2} \le U\right\} = P\left\{\frac{L}{\hat{MC}_p} \le (|S|/|\Sigma|)^{1/2} \le \frac{U}{\hat{MC}_p}\right\}$$

$$= P\left\{\frac{L}{\hat{MC}_p} \le \sqrt{\frac{\chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}}{(n-1)^v}} \le \frac{U}{\hat{MC}_p}\right\}$$

$$= P\left\{\frac{L^2(n-1)^v}{\hat{MC}_p^2} \le \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v} \le \frac{U^2(n-1)^v}{\hat{MC}_p^2}\right\}.$$

Let $y = \chi^2_{n-1} \times \chi^2_{n-2} \times \cdots \times \chi^2_{n-v}$, then we have $\int_{L^2(n-1)^v/\hat{MC}_p^2}^{U^2(n-1)^v/\hat{MC}_p^2} f_Y(y)dy = 1-\alpha$

So, we have

$F_Y^{-1}(\alpha/2) = \dfrac{L^2(n-1)^v}{\hat{MC}_p^2}$ and $F_Y^{-1}(1-\alpha/2) = \dfrac{U^2(n-1)^v}{\hat{MC}_p^2}$, where $F_Y(z) = \int_0^z f(y)dy$.

Thus, a $100(1-\alpha)$ % confidence interval for MC_p can be obtained as:

$$\left[\hat{MC}_p\sqrt{\frac{F_Y^{-1}(\alpha/2)}{(n-1)^v}}, \hat{MC}_p\sqrt{\frac{F_Y^{-1}(1-\alpha/2)}{(n-1)^v}}\right]. \quad (3.12)$$

Furthermore, a $100(1-\alpha)$ % lower confidence bound for MC_p can be obtained as

$$\left[\hat{MC}_p\sqrt{\frac{F_Y^{-1}(\alpha)}{(n-1)^v}}\right]. \quad (3.13)$$

When there is only quality characteristic, that is, v =1, a $100(1-\alpha)$ % confidence interval and lower confidence bound for C_p are given by

$$\left[\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\alpha/2}}{(n-1)}}, \hat{C}_p\sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{(n-1)}}\right] \text{ and } \left[\hat{C}_p\sqrt{\frac{\chi^2_{n-1,\alpha}}{(n-1)}}\right]. \quad (3.14)$$

When there are two quality characteristics, that is, v =2, we can find that the distribution of $\chi^2_{n-1}\times\chi^2_{n-2}$ is equal to $(\chi^2_{2n-4})^2/4$. Thus, a $100(1-\alpha)$% confidence interval and lower confidence bound for MC_p are given by

$$\left[\hat{MC}_p\sqrt{\frac{(\chi^2_{2n-4,\alpha/2})^2}{4\times(n-1)^2}}, \hat{MC}_p\sqrt{\frac{(\chi^2_{2n-4,1-\alpha/2})^2}{4\times(n-1)^2}}\right] \text{ and } \left[\hat{MC}_p\sqrt{\frac{(\chi^2_{2n-4,\alpha})^2}{4\times(n-1)^2}}\right]. \quad (3.15)$$

When there are three quality characteristics, that is, v =3, we can find that the distribution of $\chi^2_{n-1}\times\chi^2_{n-2}\times\chi^2_{n-3}$ can be expressed as $y = [(\chi^2_{n-2})^2/4]\times\chi^2_{n-3}$.

Now, let $y_1 \sim (\chi^2_{2n-4})^2/4$ and $y_2 \sim \chi^2_{n-3}$. So, we have $y = y_1 y_2$. Let $w = y_1$. Using the transformation method, the probability density function of y is given by

$$f_Y(y) = \int_0^\infty f_{y_1,y_2}(y_1,y_2)dy_1 = \int_0^\infty \frac{(\frac{1}{2})^{(n-1)/2}\times y_1^{-1/2}\times y_2^{(n-5)/2}\times e^{-y_1^{1/2}-\frac{y_2}{2y_1}}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}dy_1,\ 0\le y<\infty.$$

Thus, a $100(1-\alpha)$ % confidence interval and lower confidence bound for MC_p are given by

$$\left[\hat{MC}_p\sqrt{\frac{F_Y^{-1}(\alpha/2)}{(n-1)^3}}, \hat{MC}_p\sqrt{\frac{F_Y^{-1}(1-\alpha/2)}{(n-1)^3}}\right] \text{ and } \left[\hat{MC}_p\sqrt{\frac{F_Y^{-1}(\alpha)}{(n-1)^3}}\right], \quad (3.16)$$

where $F_Y(y)=\int_0^y\int_0^\infty \frac{(\frac{1}{2})^{(n-1)/2}\times x^{-1/2}\times y^{(n-5)/2}\times e^{-x^{1/2}-\frac{y}{2x}}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}dxdy$. Efficient Mathematica programs are developed to obtain the equation (3.16).

In hypothesis testing, we determine whether or not a hypothesized value of a parameter is true or not, based on the sample taken and the parameter estimate derived from it. That is, we are trying to find out where the estimated capability is relative to either true capability, hypothesized capability, or how different the estimated and true capabilities are. To do this, we estimate an index value, compare it to a lower bound c_0, and compute the so-called *p-value*. The quantity p refers to the actual risk of incorrectly concluding that the process is capable for a particular test. In general, we want *p-value* to be no greater than 0.05. To test whether a given process is capable, we may consider the following statistical hypothesis testing:

H_0: $MC_p \leqq c_0$ (process is not capable)
H_1: $MC_p > c_0$ (process is capable)

, where c_0 is the standard minimal criteria for MC_p . The critical value, c, can be determined as:

$$\alpha = P\left\{\hat{MC}_p > c \mid MC_p = c_0\right\} = P\left\{\frac{MC_p}{\sqrt{\frac{\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}}{(n-1)^v}}} > c \mid MC_p = c_0\right\}.$$

Let $y=\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}$, then we have

$$\alpha = P\left\{\frac{c_0}{\sqrt{\frac{y}{(n-1)^v}}} > c\right\} = P\left\{\frac{c_0^{\,2}}{\frac{y}{(n-1)^v}} > c^2\right\} = P\left\{y < \frac{(n-1)^v c_0^{\,2}}{c^2}\right\}, \text{ that is,}$$

$$\int_0^{(n-1)^v c_0^{\,2}/c^2} f_Y(y)dy = \alpha, \text{ so we have}$$

$$F_Y^{-1}(\alpha) = \frac{(n-1)^v c_0^{\,2}}{c^2}.$$

Thus, the critical value can be expressed as

$$c = c_0\sqrt{\frac{(n-1)^v}{F_Y^{-1}(\alpha)}} \tag{3.17}$$

When there is only single quality characteristic, that is, v=1, and $y=\chi^2_{n-1}$, the critical value is equal to

$$c_0\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}.$$

When there are two quality characteristics, that is, v=2, and $y=\chi^2_{n-1}\times\chi^2_{n-2}\sim(\chi^2_{2n-4})^2/4$, the critical value is equal to

$$c_0\sqrt{\frac{4(n-1)^2}{(\chi^2_{2n-4,\alpha})^2}}.$$

When there are three quality characteristics, that is, v=3, and $y=\chi^2_{n-1}\times\chi^2_{n-2}\times\chi^2_{n-3}$, the critical value is equal to

$$c_0\sqrt{\frac{(n-1)^3}{F_Y^{-1}(\alpha)}}, \text{ where } F_Y(y)=\int_0^y\int_0^\infty \frac{(\frac{1}{2})^{\frac{n-1}{2}}\times x^{\frac{-1}{2}}\times y^{\frac{n-5}{2}}\times e^{-x^{\frac{1}{2}}-\frac{y}{2x}}}{\Gamma(n-2)\times\Gamma((n-3)/2)}dxdy.$$

From the definition of c, it is obvious that the value of $\hat{MC}_p$ must be higher than the original target value for the true MC_p. The amount of difference required depends on the sample size, n. The power of the test, β, is given by

$$\beta(MC_p)=P\left\{\hat{MC}_p>c\mid MC_p\right\}$$

$$=P\left\{\frac{MC_p}{\sqrt{\frac{\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}}{(n-1)^v}}}>c\mid MC_p\right\}=P\left\{y<\frac{(n-1)^v MC_p^2}{c^2}\mid MC_p\right\}. \tag{3.18}$$

where y is the probability density function of $\chi^2_{n-1}\times\chi^2_{n-2}\times\cdots\times\chi^2_{n-v}$. From the above equation, we can obtain the operating characteristic (OC) curve for MC_p, which plots the true value of $1-\beta(MC_p)$ vs. MC_p for two situations: (1) n=30, c=1.33; (2) n=70, c=1.46. When v=1, 2 and 3, several operating characteristic (OC) curves for MC_p are shown in Figures 2-6. From these operating characteristic curves, we found that some interesting results are:

(i) From these graphs, it is obvious that when the c_0 is larger, the chance of incorrectly concluding the process is not capable is smaller.

(ii) When n and c of the two are the same, then their OC curves are similar regardless of v=1, v=2 or v=3.

(iii) When c is smaller, the chance of incorrectly concluding the process is not capable will be smaller.

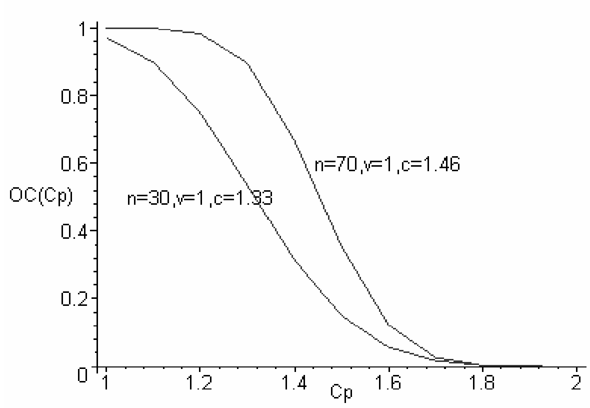

Figure 2. Operating Characteristic Curve for MC_p when v =1

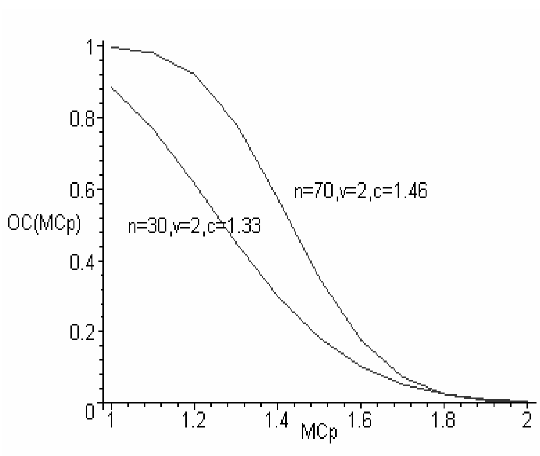

Figure 3. Operating Characteristic Curve for $M\ C_p$ when v =2

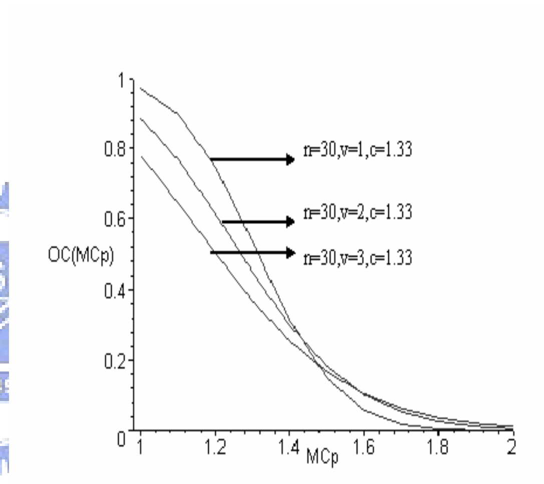

Figure 5. Operating Characteristic Curve for MC_p when n =30

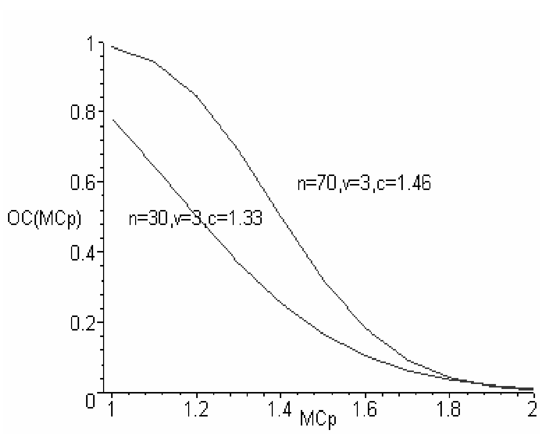

Figure 4. Operating Characteristic Curve for $M\ C_p$ when v =3

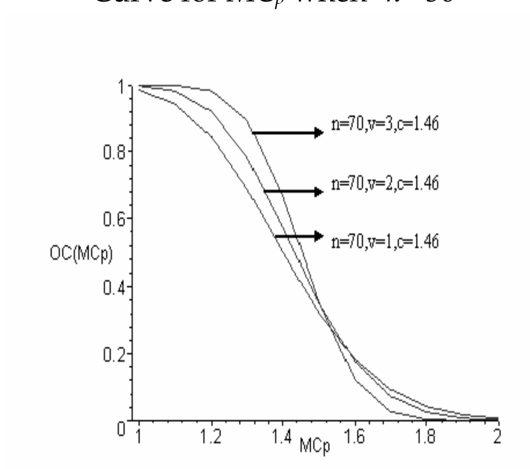

Figure 6. Operating Characteristic Curve for MC_p when n =70

3.3 Approximate Confidence Intervals for MC_{pm}

The multivariate capability index MC_{pm} is defined in the following, where $D=[1+(\mu-T)^{'}\Sigma^{-1}(\mu-T)]^{1/2}$,

$$MC_{pm}=\frac{MC_p}{D}. \tag{3.19}$$

Note that, MC_{pm} is less than 1, the process is not close to the specified tolerance region. On the other hand, MC_p is larger than 1 which only indicates that the process variation is smaller than the specified range of variation.

An estimator of MC_{pm} can be expressed as

$$\hat{MC}_{pm}=\frac{\hat{MC}_p}{\hat{D}}=\frac{MC_p\times(|S|/|\Sigma|)^{-1/2}}{\hat{D}}=\frac{MC_p\times D\times(|S|/|\Sigma|)^{-1/2}}{D\times\hat{D}}=MC_{pm}\times\frac{D}{\hat{D}}\times(|S|/|\Sigma|)^{-1/2},$$

where $\hat{D}=\left[1+\left(\frac{n}{n-1}\right)(\overline{X}-T)^{'}S^{-1}(\overline{X}-T)\right]^{1/2}$.

Since $\hat{MC}_{pm}=MC_{pm}\times\frac{D}{\hat{D}}\times(|S|/|\Sigma|)^{-1/2}$ and $|S|/|\Sigma|\ \sim\ \frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v}$,

we have

$$\left(MC_{pm}/\hat{MC}_{pm}\right)\times D=\hat{D}\times(|S|/|\Sigma|)^{1/2},\text{ then}$$

$$\left(MC_{pm}/\hat{MC}_{pm}\right)^2\times D^2=\hat{D}^2\times(|S|/|\Sigma|)=\hat{D}^2\times\frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v},$$

where $D^2=[1+(\mu-T)^{'}\Sigma^{-1}(\mu-T)]$, $\hat{D}^2=\left[1+\left(\frac{n}{n-1}\right)(\overline{X}-T)^{'}S^{-1}(\overline{X}-T)\right]$.

Based on the definition, a $100(1-\alpha)$ % confidence interval for MC_{pm} is given as follow:

$$1-\alpha=P\{L\le MC_{pm}\le U\}$$

$$=P\left\{\left(\frac{L}{\hat{MC}_{pm}}\right)^2D^2\le\left(\frac{MC_{pm}}{\hat{MC}_{pm}}\right)^2D^2\le\left(\frac{U}{\hat{MC}_{pm}}\right)^2D^2\right\}$$

$$=P\left\{\left(\frac{L}{\hat{MC}_{pm}}\right)^2D^2\le\hat{D}^2\times\left(\frac{\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v}}{(n-1)^v}\right)\le\left(\frac{U}{\hat{MC}_{pm}}\right)^2D^2\right\}$$

$$=P\left\{\left(\frac{L}{\hat{MC}_{pm}}\right)^2 D^2(n-1)^v \le \hat{D}^2\times\chi^2_{n-1}\chi^2_{n-2}\cdots\chi^2_{n-v} \le \left(\frac{U}{\hat{MC}_{pm}}\right)^2 D^2(n-1)^v\right\}.$$

Let $z=\hat{D}^2\times\chi^2_{n-1}\chi^2_{n-2}\ldots\chi^2_{n-v}$. The above equation can be rewritten as

$$\int_{L^2(n-1)^v D^2/\hat{MC}^2_{pm}}^{U^2(n-1)^v D^2/\hat{MC}^2_{pm}} f_Z(z)\,dz = 1-\alpha .$$

So, we have $F_Z^{-1}(\alpha/2)=\dfrac{L^2(n-1)^v D^2}{\hat{MC}^2_{pm}}$ and $F_Z^{-1}(1-\alpha/2)=\dfrac{U^2(n-1)^v D^2}{\hat{MC}^2_{pm}}$,

where $F_Z(z)=\int_0^z f_Z(z)dz$. Thus, a $100(1-\alpha)$ % confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{MC}_{pm}\sqrt{\frac{F_Z^{-1}(\alpha/2)}{(n-1)^v D^2}},\hat{MC}_{pm}\sqrt{\frac{F_Z^{-1}(1-\alpha/2)}{(n-1)^v D^2}}\right] \text{ and } \left[\hat{MC}_{pm}\sqrt{\frac{F_Z^{-1}(\alpha)}{(n-1)^v D^2}}\right]. \quad (3.20)$$

In fact, D and τ^2 are unkown values, we can use $\hat{D}$ and $\hat{\tau}^2$ to estimate the values of D and τ^2,

where $\hat{D}=\left[1+\left(\dfrac{n}{n-1}\right)(\overline{X}-T)'S^{-1}(\overline{X}-T)\right]^{1/2}$ and $\hat{\tau}^2=n(\overline{X}-T)'S^{-1}(\overline{X}-T)$.

Thus, an approximate $100(1-\alpha)$ % confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^v\hat{D}^2}},\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^v\hat{D}^2}}\right] \text{ and } \left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^v\hat{D}^2}}\right]. \quad (3.21)$$

From Corollary 3 in the Appendix, the pdf of z can be found. When there are two quality characteristics, that is, v=2, an approximate $100(1-\alpha)$ % confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^2\hat{D}^2}},\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^2\hat{D}^2}}\right] \text{ and } \left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^2\hat{D}^2}}\right], \quad (3.22)$$

where

$$\hat{F}_Z(z)=\int_0^z\int_1^\infty \frac{\frac{1}{2}e^{-\frac{1}{2}\hat{\tau}^2}}{x\Gamma(n-2)\Gamma[(n-2)/2]}(z/x)^{(n-4)/2}e^{-\sqrt{z/x}}\sum_{i=0}^{\infty}\frac{(\hat{\tau}^2/2)^i(x-1)^i\Gamma(n/2+i)}{i!\,\Gamma(i+1)x^{\frac{1}{2}n+i}}dxdz, \text{ for } z\ge 0.$$

When there are three quality characteristics, that is, v=3, an approximate $100(1-\alpha)$ % confidence interval and lower confidence bound for MC_{pm} are given by

$$\left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha/2)}{(n-1)^3\hat{D}^2}},\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(1-\alpha/2)}{(n-1)^3\hat{D}^2}}\right] \text{ and } \left[\hat{MC}_{pm}\sqrt{\frac{\hat{F}_z^{-1}(\alpha)}{(n-1)^3\hat{D}^2}}\right], \quad (3.23)$$

where

$$\hat{F}_Z(z)=\int_0^z\int_1^\infty\int_0^\infty \frac{(1/2)^{(n-1)/2}x^{-1/2}e^{-\sqrt{x}-z/(2wx)}}{\Gamma(n-2)\Gamma[(n-3)/2]}\frac{e^{-\frac{1}{2}\hat{\tau}^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]}\sum_{i=0}^{\infty}\frac{(\hat{\tau}^2/2)^i(w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{n/2+i}}dxdwdz \quad .$$

Table 4. Observed Coverage of 95% Confidence Limits

			n=25		n=45		n=65	
μ	Σ	MC_{pm}	CI	LCB	CI	LCB	CI	LCB
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{0.75\chi^2_{2,0.9973}} & \frac{4}{0.75\chi^2_{2,0.9973}}\\ \frac{4}{0.75\chi^2_{2,0.9973}} & \frac{5}{0.75\chi^2_{2,0.9973}}\end{bmatrix}$	0.75	0.968	0.959	0.965	0.969	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{\chi^2_{2,0.9973}} & \frac{4}{\chi^2_{2,0.9973}}\\ \frac{4}{\chi^2_{2,0.9973}} & \frac{5}{\chi^2_{2,0.9973}}\end{bmatrix}$	1	0.968	0.959	0.966	0.970	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{1.25\chi^2_{2,0.9973}} & \frac{4}{1.25\chi^2_{2,0.9973}}\\ \frac{4}{1.25\chi^2_{2,0.9973}} & \frac{5}{1.25\chi^2_{2,0.9973}}\end{bmatrix}$	1.25	0.968	0.959	0.966	0.970	0.952	0.952
$\begin{bmatrix}13\\13\end{bmatrix}$	$\begin{bmatrix}\frac{5}{1.5\chi^2_{2,0.9973}} & \frac{4}{1.5\chi^2_{2,0.9973}}\\ \frac{4}{1.5\chi^2_{2,0.9973}} & \frac{5}{1.5\chi^2_{2,0.9973}}\end{bmatrix}$	1.5	0.968	0.959	0.966	0.970	0.952	0.952

Note: CI=confidence interval; LCB=lower confidence bound

In order to ascertain the performance of the confidence interval and the lower confidence bound in (3.22), a simulation study was conducted. In this study, random samples of size 25, 45 and 65 were generated from the multivariate normal distribution with a plethora of combinations of μ, Σ and MC_{pm}. The specification limits were assumed to be, without loss of generality, LSL_1=10, T_1=13, USL_1=16, LSL_2=12, T_2=13, USL_2=14. For each combination, 1000 random samples were generated and, for each of these samples, the corresponding confidence intervals and lower confidence bound were assessed. The proportion of times that each of these limits contains the actual value of the index was recorded. The frequency of coverage for the limit is a binomial random variable with p=0.95 and N=1000. Thus, a 99% confidence interval for the coverage proportion is $0.95\pm 2.576\sqrt{0.95\times 0.05/1000}$. Hence, the limits from 0.932 to 0.968 are the critical values for a statistical test at the 99% confidence level of the hypothesis that the p is 0.95.

The obtained results are summarized in Table 4. More specifically, Table 4 presents that the observed coverage of 95% confidence interval as well as the observed coverage of the lower confidence bound are within the nominal interval at 99% confidence level. Thus, we can ascertain the performance of the confidence interval and the lower confidence bound in (3.22).

3.4 An Application Example

Taam *et al*. (1993) and Karl *et al*. (1994) discussed a geometric dimensioning and tolerancing (GD&T) drawing that specifies a target value for pin diameter corresponding to the midpoint of allowable pin sizes and allowable perpendicularity of the pin depending on its size. The specifications require a pin diameter between 9 and 11 tenths of an inch (all units in tenths of an inch) and the center line of the pin to be within a cylinder of diameter 0.5 at maximum material condition (MMC, i.e., maximum pin diameter), increasing to a cylinder 2.5 diameter at least material condition (LMC, i.e., minimum pin diameter). Therefore, the tolerance of perpendicularity depends on the pin diameter: for a pin with a diameter of 9, the allowable perpendicular tolerance zone is a 2.5 diameter cylinder, whereas for a pin diameter of 11, the allowable perpendicular tolerance is a 0.5 diameter cylinder. A pin meeting this specification will fit a gage with an 11.5 diameter hole. These GD&T specifications result in a 3-dimensional tolerance region in the shape of a frustum, as illustrated in Figure 7. The center of the specification is $T^T = [0,0,10]$. The sample mean vector and sample covariance matrix for 70 observations were

$$\overline{X}^T = [-0.0124,\ -0.0062,\ 10.0586] \text{ and } S = \begin{bmatrix} 0.01313 & -0.00371 & 0.00884 \\ -0.00371 & 0.01618 & -0.01031 \\ 0.00884 & -0.01031 & 0.06473 \end{bmatrix}.$$

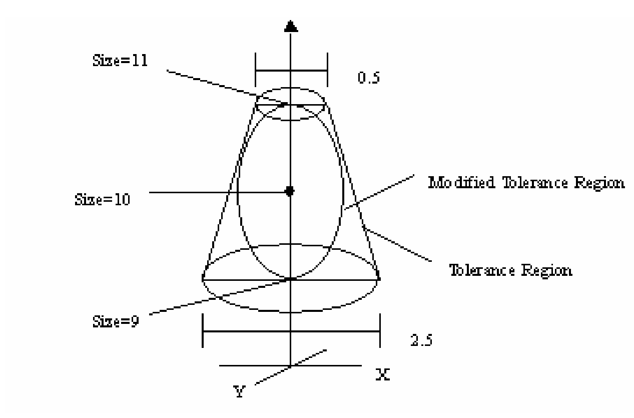

Figure 7. Tolerance Region and Modified Tolerance Region for Example

From the data, we have $\chi^2_{3,0.9973} = 14.1563$ and $|S| = 0.00001092$. Then we have

$$\hat{MC}_p = \frac{\frac{4}{3}\pi\times[(11-9)/2]\times(2.5/2)\times(0.5/2)}{|S|^{1/2}(\pi\times\chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 1.7752.$$

Now, v=3 and n=70 , the expectation and variance of $\hat{MC}_p$ can be calculated as 1.0570× MC_p and 0.025685× MC_p^2, respectively. According to the equation (3.16), a 95% confidence interval for MC_p is calculated as

$$\left[1.7752\sqrt{\frac{164939}{69^3}}, 1.7752\sqrt{\frac{533052}{69^3}}\right] = [1.2579, 2.2613].$$

Also, a 95% lower confidence bound is

$$1.7752\sqrt{\frac{182304}{69^3}} = 1.3224.$$

To judge whether this process meets the present capability requirement, we consider a statistical hypothesis testing for MC_p: H_0: $MC_p \leqq 1$ vs H_1: $MC_p > 1$. According to the equation (3.17), the critical value

$$c = 1\times\sqrt{\frac{69^3}{182304}} = 1.3423.$$

Since $\hat{MC}_p$=1.7752>1.3423, we can conclude that the MC_p is larger than 1 at 95% confidence level. It implies that this process variation is smaller than the specified range of variation. From the data, we have $\hat{D}$=1.0437 and $\hat{\tau}^2$=6.1608. Thus, the $\hat{MC}_{pm}$ be calculated as 1.7752/1.0437=1.7009. According to the equation (3.23), an approximate 95% confidence interval for MC_{pm} is calculated as

$$\left[1.7009\sqrt{\frac{184666}{(69)^3\times1.0437^2}}, 1.7009\sqrt{\frac{619698}{(69)^3\times1.0437^2}}\right] = [1.2219, 2.2383].$$

Also, according to the equation (3.23), an approximate 95% lower confidence bound for MC_{pm} is

$$1.7009\sqrt{\frac{204521}{(69)^3\times1.0437^2}} = 1.2859.$$

Therefore, we can conclude that the MC_{pm} is larger than 1 at 95% confidence level. It implies that this process is close to the specified target.

Chapter 4
Select Better Suppliers Based on Manufacturing Precision for Processes with Multivariate Data

In this chapter, we considered the supplier selection problem based on manufacturing precision in which the processes involve multiple quality characteristics. The multivariate process index MC_p (Taam, *et al.*, 1993) is used to judge whether the capability of one process is superior to another process. We derived the distribution of the corresponding test statistic, and provided critical values required for the comparison purpose. We developed an effective test procedure for practitioners to make reliable decisions in their in-plant applications involving supplier selections.

4.1 Comparing Two Manufacturing Precision Using MC_P

An estimator of MC_p can be expressed as

$$\hat{MC}_p = \frac{vol.(\text{modified tolerance region})}{vol.(\textit{estimated } 99.73\% \textit{ process region})} = \frac{vol.(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}|S|^{1/2}[\Gamma(v/2+1)]^{-1}},$$

where S is the sample variance-covariance matrix, and $|S|$ is the determinant of S. From Equation (3.1), $\hat{MC}_p$ can be rewritten as $MC_p \times (|S|/|\Sigma|)^{-1/2}$. Let $X = (X_1, X_2, ..., X_n)'$ be an n-dimensional vector of measurements taken from a multivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2, ..., \mu_v)'$, target vector *T*, process variance-covariance matrix Σ. Using the above theorem (Theorem 2), we may obtain the distribution of $\hat{MC}_p$.Chou (1994) developed a procedure using univariate C_p to determine whether or not two processes are equally capable, which allows one to select the supplier with better quality. However, for processes with multiple characteristics (multivariate data) no methods are available for comparing two processes with multivariate data. For this purpose, we consider the problem of comparing two multivariate processes using MC_p. The hypothesis testing would be as follows: $H_0: MC_{p1} \le MC_{p2}$ (Process I is not better than Process II) versus $H_1: MC_{p1} > MC_{p2}$ (process I is better than process II). The critical value c can be determined as:

$$\alpha = P\left\{\frac{\hat{MC}_{p1}}{\hat{MC}_{p2}} > c \mid MC_{p1} = MC_{p2}\right\} = P\left\{\frac{MC_{p1}(|S_1|/|\Sigma_1|)^{-1/2}}{MC_{p2}(|S_2|/|\Sigma_2|)^{-1/2}} > c \mid MC_{p1} = MC_{p2}\right\}$$

$$= P\left\{\frac{(|S_1|/|\Sigma_1|)^{-1/2}}{(|S_2|/|\Sigma_2|)^{-1/2}} > c\right\} = P\left\{\frac{(|S_2|/|\Sigma_2|)}{(|S_1|/|\Sigma_1|)} > c^2\right\} \tag{4.1}$$

From the above theorem, $|S|/|\Sigma|$ is distributed as $\chi_{n-1}^2 \times \chi_{n-2}^2 \times \cdots \times \chi_{n-v}^2 / (n-1)^v$. When there are two quality characteristics, that is, $v = 2$, Equation (4.1) becomes

$$\alpha = P\left\{ \frac{\chi_{n_2-1}^2 \times \chi_{n_2-2}^2}{(n_2-1)^2} \frac{(n_1-1)^2}{\chi_{n_1-1}^2 \times \chi_{n_1-2}^2} > c^2 \right\} = P\left\{ \frac{(\chi_{2n_2-4}^2)^2}{(n_2-1)^2} \frac{(n_1-1)^2}{(\chi_{2n_1-4}^2)^2} > c^2 \right\}$$

($\because \chi_{n-1}^2 \times \chi_{n-2}^2 \sim (\chi_{2n-4}^2)^2 / 4$ (See Corollary 2 in Appendix)).

$$= P\left\{ \frac{(\frac{\chi_{2n_2-4}^2}{2n_2-4})^2}{(\frac{\chi_{2n_1-4}^2}{2n_1-4})^2} \frac{(n_1-1)^2}{(n_2-1)^2} \frac{(2n_2-4)^2}{(2n_1-4)^2} > c^2 \right\} = P\left\{ (F_{2n_2-4,\, 2n_1-4})^2 > c^2 \frac{(n_2-1)^2}{(n_1-1)^2} \frac{(2n_1-4)^2}{(2n_2-4)^2} \right\},$$

so we have

$$F_{2n_2-4,\, 2n_1-4,\, 1-\alpha} = c \frac{(n_2-1)}{(n_1-1)} \frac{(2n_1-4)}{(2n_2-4)}.$$

Thus, the critical value can be expressed as

$$c = F_{2n_2-4,\, 2n_1-4,\, 1-\alpha} \frac{(n_1-1)}{(n_2-1)} \frac{(2n_2-4)}{(2n_1-4)}. \tag{4.2}$$

When there are three quality characteristics, that is, $v = 3$, equation (4.1) can be expressed as

$$\alpha = P\left\{ \frac{\chi_{n_2-1}^2 \times \chi_{n_2-2}^2 \times \chi_{n_2-3}^2}{(n_2-1)^3} \frac{(n_1-1)^3}{\chi_{n_1-1}^2 \times \chi_{n_1-2}^2 \times \chi_{n_1-3}^2} > c^2 \right\}$$

$$= P\left\{ \left(\frac{\chi_{2n_2-4}^2}{\chi_{2n_1-4}^2} \right)^2 \frac{\chi_{n_2-3}^2}{\chi_{n_1-3}^2} \frac{(n_1-1)^3}{(n_2-1)^3} > c^2 \right\}$$

$$= P\left\{ \left(\frac{\frac{\chi_{2n_2-4}^2}{2n_2-4}}{\frac{\chi_{2n_1-4}^2}{2n_1-4}} \right)^2 \frac{\frac{\chi_{n_2-3}^2}{n_2-3}}{\frac{\chi_{n_1-3}^2}{n_1-3}} \frac{(2n_2-4)^2}{(2n_1-4)^2} \frac{(n_2-3)}{(n_1-3)} \frac{(n_1-1)^3}{(n_2-1)^3} > c^2 \right\}$$

$$= P\left\{ \left(F_{2n_2-4,\, 2n_1-4}\right)^2 * F_{n_2-3,\, n_1-3} > c^2 \frac{(2n_1-4)^2}{(2n_2-4)^2} \frac{(n_1-3)}{(n_2-3)} \frac{(n_2-1)^3}{(n_1-1)^3} \right\}. \tag{4.3}$$

Let $z = xy$, where $x \sim (F_{2n_2-4,\, 2n_1-4})^2$, $y \sim F_{n_2-3,\, n_1-3}$, then equation (4.3) can be expressed as

$$\int_0^{\frac{c^2(2n_1-4)^2(n_1-3)(n_2-1)^3}{(2n_2-4)^2(n_2-3)(n_1-1)^3}} f_z(Z)dz = 1-\alpha .$$

Thus, the critical value can be expressed as

$$c=\sqrt{F_z^{-1}(1-\alpha)\frac{(2n_2-4)^2}{(2n_1-4)^2}\frac{(n_2-3)}{(n_1-3)}\frac{(n_1-1)^3}{(n_2-1)^3}}\ , \qquad (4.4)$$

$$f_Z(z) \;= \int_0^\infty k(n_1,n_2)\frac{x^{\frac{(2n_2-4)}{4}-2}\left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1+\frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2}\left(1+\frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}}, \text{ for x, } z\ge 0,$$

$$k(n_1,n_2)=\frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma((n_2+n_1-6)/2)\left(\frac{2n_2-4}{2n_1-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$$

(See Corollary 4 in Appendix).

Tables 5-7 display the critical values for various sample sizes in the case with $v=2$ under confidence levels $\alpha=0.05$, $\alpha=0.025$ and $\alpha=0.01$. Tables 8-10 display the critical values for various sample sizes in the case of $v=3$ under $\alpha=0.05$, $\alpha=0.025$ and $\alpha=0.01$. For practical and convenient purpose, a step-by-step procedure is provided below: Step 1. Determine the sample size n_i for each supplier and the α-risk (the chance of incorrectly rejecting a better supplier). Step 2. Take a random sample from each process and calculate the sample covariance matrix. Step 3. Calculate the test statistic $\hat{MC}_{p1}/\hat{MC}_{p2}$ and the critical value c. Step 4. If $\hat{MC}_{p1}/\hat{MC}_{p2} > c$, then we reject H_0 and conclude that $MC_{p1} > MC_{p2}$. From the definition of c, it is clear that the value of $\hat{MC}_{p1}/\hat{MC}_{p2}$ must be higher than the original target value for the true MC_{p1}/MC_{p2}. The power of the test can be also computed below: The power of the test, β, is given by

$$\beta(MC_{p1}/MC_{p2}) = P\left\{\hat{MC}_{p1}/\hat{MC}_{p2} > c \mid MC_{p1}/MC_{p2}=k\right\}. \qquad (4.5)$$

Take $v=2$, and 3, equation (4.5) can be expressed as

$$P\left\{F_{2n2-4,\,2n1-4} > c\frac{(n_2-1)}{(n_1-1)}\frac{(2n_1-4)}{(2n_2-4)}\frac{(MC_{p2})}{(MC_{p1})}\right\},$$

$$P\left\{\left(F_{2n_2-4,\,2n_1-4}\right)^2 * F_{n_2-3,\,n_1-3} > c^2\frac{(2n_1-4)^2}{(2n_2-4)^2}\frac{(n_1-3)}{(n_2-3)}\frac{(n_2-1)^3}{(n_1-1)^3}\frac{(MC_{p2})^2}{(MC_{p1})^2}\right\}.$$

Table 5. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.05$ $(v = 2)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	2.33	2.31	2.29	2.29	2.28	2.28	2.28	2.27	2.27	2.27
20	1.81	1.74	1.71	1.69	1.68	1.68	1.67	1.67	1.66	1.66
30	1.68	1.60	1.56	1.54	1.52	1.51	1.51	1.50	1.50	1.49
40	1.62	1.53	1.49	1.46	1.45	1.44	1.43	1.42	1.42	1.41
50	1.59	1.49	1.44	1.42	1.40	1.39	1.38	1.37	1.37	1.36
60	1.57	1.46	1.42	1.39	1.37	1.36	1.35	1.34	1.34	1.33
70	1.55	1.45	1.40	1.37	1.35	1.34	1.33	1.32	1.31	1.31
80	1.54	1.43	1.38	1.35	1.33	1.32	1.31	1.30	1.30	1.29
90	1.53	1.42	1.37	1.34	1.32	1.31	1.30	1.29	1.28	1.28
100	1.52	1.41	1.36	1.33	1.31	1.30	1.29	1.28	1.27	1.27

Table 6. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.025$ $(v = 2)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	2.76	2.69	2.67	2.65	2.64	2.64	2.63	2.63	2.63	2.63
20	2.06	1.94	1.89	1.87	1.85	1.84	1.83	1.83	1.82	1.82
30	1.89	1.75	1.70	1.67	1.65	1.63	1.62	1.62	1.61	1.61
40	1.81	1.67	1.61	1.57	1.55	1.54	1.53	1.52	1.51	1.50
50	1.76	1.62	1.55	1.52	1.50	1.48	1.47	1.46	1.45	1.44
60	1.74	1.58	1.52	1.48	1.46	1.44	1.43	1.42	1.41	1.40
70	1.72	1.56	1.50	1.46	1.43	1.41	1.40	1.39	1.38	1.38
80	1.70	1.55	1.48	1.44	1.41	1.39	1.38	1.37	1.36	1.35
90	1.69	1.53	1.46	1.42	1.40	1.38	1.36	1.35	1.35	1.34
100	1.68	1.52	1.45	1.41	1.39	1.37	1.35	1.34	1.33	1.32

Table 7. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.01$ $(v = 2)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	3.37	3.25	3.20	3.17	3.16	3.15	3.14	3.14	3.13	3.13
20	2.39	2.21	2.14	2.10	2.08	2.06	2.05	2.04	2.03	2.03
30	2.15	1.96	1.88	1.83	1.81	1.79	1.78	1.77	1.76	1.75
40	2.05	1.84	1.76	1.71	1.68	1.66	1.65	1.64	1.63	1.62
50	1.99	1.78	1.69	1.64	1.61	1.59	1.58	1.56	1.55	1.55
60	1.95	1.74	1.65	1.60	1.57	1.54	1.53	1.51	1.50	1.50
70	1.93	1.71	1.62	1.57	1.53	1.51	1.49	1.48	1.47	1.46
80	1.91	1.69	1.59	1.54	1.51	1.49	1.47	1.45	1.44	1.43
90	1.89	1.67	1.58	1.52	1.49	1.47	1.45	1.43	1.42	1.41
100	1.88	1.66	1.56	1.51	1.47	1.45	1.43	1.42	1.41	1.40

Table 8. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.05$ $(v = 3)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	2.94	3.14	3.19	3.21	3.22	3.23	3.24	3.24	3.24	3.25
20	1.95	2.00	1.99	1.99	1.99	1.98	1.98	1.98	1.98	1.97
30	1.73	1.75	1.73	1.72	1.71	1.70	1.70	1.69	1.69	1.69
40	1.64	1.64	1.61	1.60	1.59	1.58	1.57	1.56	1.56	1.56
50	1.59	1.58	1.55	1.53	1.52	1.50	1.50	1.49	1.49	1.48
60	1.56	1.54	1.51	1.48	1.47	1.46	1.45	1.44	1.44	1.43
70	1.53	1.51	1.48	1.45	1.44	1.43	1.42	1.41	1.40	1.40
80	1.51	1.49	1.45	1.43	1.41	1.40	1.39	1.38	1.38	1.37
90	1.50	1.47	1.44	1.41	1.40	1.38	1.37	1.36	1.36	1.35
100	1.49	1.46	1.42	1.40	1.38	1.37	1.36	1.35	1.34	1.34

Table 9. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.025$ $(v = 3)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	3.64	3.81	3.84	3.86	3.87	3.87	3.88	3.88	3.88	3.88
20	2.29	2.28	2.26	2.24	2.23	2.22	2.22	2.21	2.21	2.21
30	2.01	1.96	1.92	1.90	1.88	1.87	1.86	1.85	1.85	1.84
40	1.89	1.82	1.78	1.75	1.73	1.71	1.70	1.70	1.69	1.68
50	1.82	1.75	1.70	1.66	1.64	1.63	1.61	1.60	1.60	1.59
60	1.78	1.70	1.64	1.61	1.59	1.57	1.56	1.55	1.54	1.53
70	1.75	1.66	1.61	1.57	1.55	1.53	1.51	1.50	1.49	1.49
80	1.73	1.64	1.58	1.54	1.52	1.50	1.48	1.47	1.46	1.46
90	1.71	1.62	1.56	1.52	1.49	1.48	1.46	1.45	1.44	1.43
100	1.70	1.60	1.54	1.50	1.48	1.46	1.44	1.43	1.42	1.41

Table 10. Critical value for testing $H_1 : MC_{p1} > MC_{p2}$ under $\alpha = 0.01$ $(v = 3)$

n1	n2 10	20	30	40	50	60	70	80	90	100
10	4.68	4.79	4.81	4.81	4.82	4.82	4.82	4.82	4.82	4.82
20	2.76	2.67	2.62	2.58	2.56	2.55	2.54	2.53	2.52	2.52
30	2.38	2.25	2.18	2.14	2.11	2.09	2.08	2.06	2.06	2.05
40	2.22	2.07	1.99	1.94	1.91	1.89	1.87	1.86	1.85	1.84
50	2.13	1.97	1.88	1.83	1.80	1.78	1.76	1.75	1.74	1.73
60	2.07	1.90	1.82	1.77	1.73	1.71	1.69	1.67	1.66	1.65
70	2.03	1.86	1.77	1.72	1.68	1.66	1.64	1.62	1.61	1.60
80	2.00	1.83	1.74	1.68	1.65	1.62	1.60	1.58	1.57	1.56
90	1.98	1.80	1.71	1.65	1.62	1.59	1.57	1.55	1.54	1.53
100	1.96	1.78	1.69	1.63	1.60	1.57	1.55	1.53	1.52	1.51

4.2 An Application Example

To illustrate how the proposed method can be applied to the actual data collected from the factories, we present a real-world example of an electronic component manufacturer making ceramic multilayer capacitors applicable to consumer electronics, telecommunications, automotive parts, and data processing devices. The capacitor consists of a rectangular block of ceramic in which a number of interleaved electrodes are contained. A cross section of the ceramic multi-layer capacitor structure is depicted in Figure 8. For a particular model of the ceramic multilayer capacitor investigated, the electrical characteristics are displayed in Table 11.

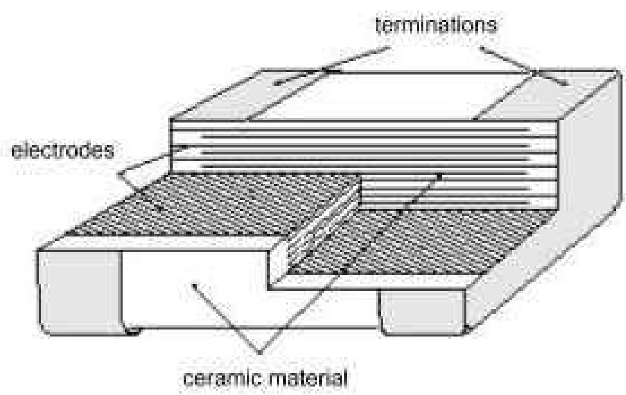

Figure 8. Structure of a ceramic multi-layer capacitor

Consider a production process with multiple characteristics following a multivariate normal distribution, which is taken from a multilayer capacitor factory in Taiwan adapting the six-sigma quality improvement program. To compare product quality between a supplier versus another, 50 random samples are taken from the two processes. The quality control of the process involves the simultaneous control of the layer thickness, the layer length, and the layer width. The lower and upper specification limits for layer thickness, layer length, and layer width have been set to $[1.45, 1.75]$, $[3.0, 3.4]$ and $[1.45, 1.75]$, respectively. The sample covariance matrices are summarized below, where S_1 represents the data from supplier I, and S_2 represents the data from supplier II.

$$S_1 = \begin{bmatrix} 0.00193 & 0.00046 & 0.00086 \\ 0.00046 & 0.00097 & 0.00075 \\ 0.00086 & 0.00075 & 0.00167 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.00236 & 0.00029 & 0.00003 \\ 0.00029 & 0.00176 & 0.00097 \\ 0.00003 & 0.00097 & 0.00161 \end{bmatrix}.$$

Table 11. Specification of Y5V/BME/1206/22uF/6.3V.

Capacitance range	22uF, Size 1206
Tolerance on capacitance after 1000 hours	-20% to +80%
Rated voltage U_R (DC)	6.3V
Test voltage (DC) for 1 minute	2.5 X U_R
Tan D (Note 1)	<=12.50 %
Insulation resistance after 1 minute at U_R (DC)	$R_{ins.}$ x C > = 500 s
Maximum capacitance change as a function of temperature	+30% to -80%
Ageing	Typically, 12.5% per time decade
Terminations	NiSn plated
Resistance to soldering heat	260° C, 10 sec

To compare the two processes, we consider a test with null hypothesis $H_0: MC_{p1} \le MC_{p2}$ against the alternative hypothesis $H_1: MC_{p1} > MC_{p2}$, where MC_{p1} and MC_{p2} represent the process capability indices of the two suppliers respectively. The test procedure can be described as: Step 1. For the two suppliers with sample sizes $n_1 = n_2 = 50$, set the confidence level as 0.05. Step 2. Calculate the sample covariance. From the above result, we obtain

$$S_1 = \begin{bmatrix} 0.00193 & 0.00046 & 0.00086 \\ 0.00046 & 0.00097 & 0.00075 \\ 0.00086 & 0.00075 & 0.00167 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.00236 & 0.00029 & 0.00003 \\ 0.00029 & 0.00176 & 0.00097 \\ 0.00003 & 0.00097 & 0.00161 \end{bmatrix}.$$

Step 3. Calculate the test statistic $\hat{MC}_{p1}/\hat{MC}_{p2}$ and critical value c:

$$\hat{MC}_{p1} = \frac{\frac{4}{3}\pi\times(0.3/2)\times(0.4/2)\times(0.3/2)}{|S_1|^{1/2}(\pi\times\chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 2.13239, \quad \hat{MC}_{p2} = \frac{\frac{4}{3}\pi\times(0.3/2)\times(0.4/2)\times(0.3/2)}{|S_2|^{1/2}(\pi\times\chi^2_{3,0.9973})^{3/2}[\Gamma(2.5)]^{-1}} = 1.28415.$$

Therefore, $\hat{MC}_{p1}/\hat{MC}_{p2}$=2.13239/1.28415=1.6605, and c=1.52 (refer to Table 8). Step 4. Because 1.6605>1.52, we conclude that with 95% confidence Process (Supplier) I is superior to Process (Supplier) II. In order to show the sensitivity of the test procedure, the power curve of the test is depicted in Figure 9 under the ratio value of MC_{p1}/MC_{p2}=0.8 to 2.3.

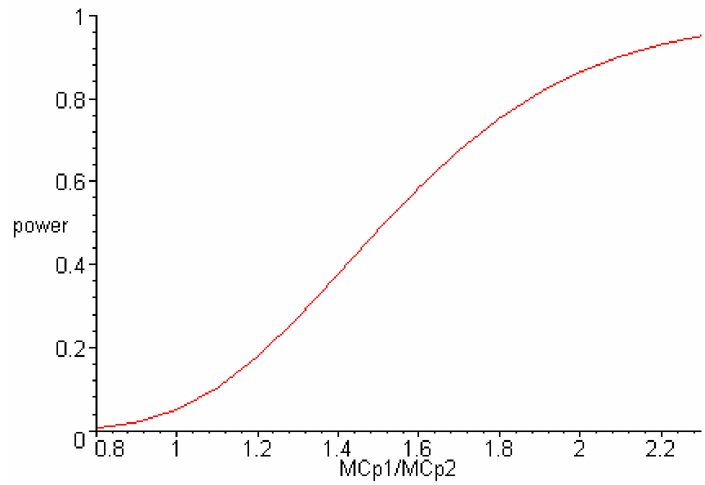

Figure 9. Power curve of testing

Chapter 5
Conclusions

Process capability indices, which establish the relationship between the actual process performance and the manufacturing specifications, have been the focus in quality assurance and capability analysis for the past fifteen years. Those capability indices quantifying process performance are essential to any successful quality improvement activities and quality program implementation. Most researches have been devoted to capability measures with single quality characteristic. However, it is quite common that the manufactured product involves more than one quality characteristic. That is, it requires several different characteristics for adequate product description. Each of those characteristics must satisfy certain specifications. The assessed quality of a product depends on the combined effects of those characteristics rather than on their individual values.

In Chapter 2, we proposed the asymptotic distribution of yield index called $\hat{S}_{pk}^{T}$. Applying the asymptotic distribution of $\hat{S}_{pk}^{T}$, hypothesis testing and lower confidence bound for S_{pk}^{T} can be executed. This methodology can be used to handle process yield assurance for processes with multiple independent characteristics. For multivariate data, we proposed a generalized yield index, called $TS_{pk;PC}$, based on the yield index S_{pk}^{T} proposed by Chen *et al*. (2003), by using the PCA method. This methodology can be used to handle process yield assurance for processes with multiple correlated characteristics. The proposed procedures can be used to determine whether their production meets the present yield requirement, and make a reliable decision.

In Chapter 3, we introduced how to obtain the probability density function of $\hat{MC}_p$ and its expectation and variance. Also, we showed how to construct the confidence intervals and make a hypothesis testing with MC_p. In addition, we also derive the approximate confidence interval and lower confidence bound for MC_{pm} for v=2, 3. A simulation study was conducted to ascertain the accuracy of the approximation. Finally, a application example is used to illustrate these results. The practitioners can use the proposed procedure to determine whether their process meets the present capability requirement, and make a reliable decision.

In Chapter 4, we considered the supplier selection problem based on manufacturing precision in which the processes involve multiple quality characteristics. We derived the distribution of the corresponding test statistic, and provided critical values required for the comparison purpose. We developed an effective test procedure for practitioners to make reliable decisions in their in-plant applications involving supplier selections.

In the multivariate case, the specification limits and the actual process spread will be more difficult to define than their univariate counterparts. In general, the specification is assumed to be ellipsoidal in most multivariate process capability indices due to the assumption of the multivariate normal distribution. Up to the present day, the research of multivariate process capability indices is still very few in comparison to the research of the univariate process capability indices relatively. A current problem in multivariate capability indices is that there is no consistency regarding a methodology for evaluating the capability. In addition, it is difficult to obtain the relevant statistical properties to make a further inference. Obvious, further investigation in this field are needed.

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Appendix

Corollary 1: $\hat{S}_{pk}^{T}$ is defined as $\hat{S}_{pk}^{T} = \frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{v}\left(2\Phi(3\hat{S}_{pkj})-1\right)+1\right]/2\right\}$,

where $\hat{S}_{pkj}$ denotes the estimator of S_{pkj}, and the asymptotic distribution of $\hat{S}_{pkj}$ is $N(S_{pkj},(a_j^2+b_j^2)/36n(\phi(3S_{pkj}))^2)$, $j=1, 2,..., v$, then the asymptotic distribution of $\hat{S}_{pk}^{T}$ can expressed as

$$\hat{S}_{pk}^{T} \sim N(S_{pk}^{T}, \frac{1}{36n(\phi(3S_{pk}^{T}))^2}(\sum_{j=1}^{v}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\})).$$

Proof:

The first-order expansion of v-variate Taylor can be expressed as

$$f(X) = f(X_0) + \sum_{j=1}^{v}\frac{\partial f(X_0)}{\partial x_i}(x_i - x_{i0}), \text{ where } X = (x_1, x_2, ..., x_v).$$

We take $v=2$ for example to derive the asymptotic distribution of $\hat{S}_{pk}^{T}$. Given the asymptotic distribution of $\hat{S}_{pkj}$ is

$$\hat{S}_{pkj} \sim N(S_{pkj}, \frac{a_j^2+b_j^2}{36n(\phi(3S_{pkj}))^2}), \ \forall j=1,2,$$

where $\hat{S}_{pk1}$ and $\hat{S}_{pk2}$ are mutually independent. From the definition, let

$$\hat{S}_{pk}^{T} = f(\hat{S}_{pk1}, \hat{S}_{pk2}) = \frac{1}{3}\Phi^{-1}\left\{\left[(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1\right]/2\right\}.$$

By the first-order expansion of v-variate Taylor for $\hat{S}_{pk}^{T}$, $\hat{S}_{pk}^{T}$ can be expressed as

$$f(\hat{S}_{pk1}, \hat{S}_{pk2}) = f(S_{pk1}, S_{pk2}) + \frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk1}}(\hat{S}_{pk1} - S_{pk1}) + \frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk2}}(\hat{S}_{pk2} - S_{pk2}).$$

Thus, $E(\hat{S}_{pk}^{T}) = E(f(S_{pk1}, S_{pk2})) + E(\frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk1}}(\hat{S}_{pk1} - S_{pk1})) + E(\frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk2}}(\hat{S}_{pk2} - S_{pk2}))$

$$= f(S_{pk1}, S_{pk2}) + \frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk1}}E(\hat{S}_{pk1} - S_{pk1}) + \frac{\partial f(S_{pk1}, S_{pk2})}{\partial \hat{S}_{pk2}}E(\hat{S}_{pk2} - S_{pk2})$$

$= f(S_{pk1}, S_{pk2})$ ($\because \hat{S}_{pkj}$ is an asymptotic unbiased estimator of S_{pkj})

$$=\frac{1}{3}\Phi^{-1}\left\{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]/2\right\}$$

$$=\frac{1}{3}\Phi^{-1}\left\{\left[\prod_{j=1}^{2}\left(2\Phi(3\hat{S}_{pkj})-1\right)+1\right]/2\right\}$$

$$=S_{pk}^{T}.$$

Also, $Var(\hat{S}_{pk}^{T})=(\frac{\partial f(S_{pk1,}S_{pk2})}{\partial \hat{S}_{pk1}})^{2}Var(\hat{S}_{pk1})+(\frac{\partial f(S_{pk1,}S_{pk2})}{\partial \hat{S}_{pk2}})^{2}Var(\hat{S}_{pk2})$.

Since $f(\hat{S}_{pk1},\hat{S}_{pk2})=\frac{1}{3}\Phi^{-1}\left\{\left[(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1\right]/2\right\}$,

$$\frac{\partial f(\hat{S}_{pk1,}\hat{S}_{pk2})}{\partial \hat{S}_{pk1}}$$

$$=\frac{1}{3}\Phi^{-1'}\left\{\frac{\left[(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1\right]}{2}\right\}\frac{\partial}{\partial \hat{S}_{pk1}}\left[\frac{(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1}{2}\right]$$

$$=\frac{1}{3}\frac{1}{\Phi'\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1\right]}{2}\right\}\right\}}\left(\frac{1}{2}\times(2\Phi(3\hat{S}_{pk2})-1)\times 2\times\phi(3\hat{S}_{pk1})\times 3\right)$$

$$=\frac{(2\Phi(3\hat{S}_{pk2})-1)\phi(3\hat{S}_{pk1})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3\hat{S}_{pk1})-1)(2\Phi(3\hat{S}_{pk2})-1)+1\right]}{2}\right\}\right\}}.$$

Thus, we have $\frac{\partial f(S_{pk1},S_{pk2})}{\partial \hat{S}_{pk1}}=\frac{(2\Phi(3S_{pk2})-1)\phi(3S_{pk1})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}$.

Similarly, we have $\frac{\partial f(S_{pk1},S_{pk2})}{\partial \hat{S}_{pk2}}=\frac{(2\Phi(3S_{pk1})-1)\phi(3S_{pk2})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}$.

So $Var(\hat{S}_{pk}^{T})$

$$= \left(\frac{(2\Phi(3S_{pk2})-1)\phi(3S_{pk1})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}\right)^2 Var(\hat{S}_{pk1}) +$$

$$\left(\frac{(2\Phi(3S_{pk1})-1)\phi(3S_{pk2})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}\right)^2 Var(\hat{S}_{pk2})$$

$$= \left(\frac{(2\Phi(3S_{pk2})-1)\phi(3S_{pk1})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}\right)^2 \frac{a_1^2+b_1^2}{36n(\phi(3S_{pk1}))^2} +$$

$$\left(\frac{(2\Phi(3S_{pk1})-1)\phi(3S_{pk2})}{\phi\left\{\Phi^{-1}\left\{\frac{\left[(2\Phi(3S_{pk1})-1)(2\Phi(3S_{pk2})-1)+1\right]}{2}\right\}\right\}}\right)^2 \frac{a_2^2+b_2^2}{36n(\phi(3S_{pk2}))^2}$$

$$= \frac{1}{36n(\phi(3S_{pk}^T))^2}\left[(a_1^2+b_1^2)(2\Phi(3S_{pk2})-1)^2 + (a_2^2+b_2^2)(2\Phi(3S_{pk1})-1)^2\right]$$

$$= \frac{1}{36n(\phi(3S_{pk}^T))^2}\left(\sum_{j=1}^{2}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{2}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right).$$

In addition, since $\hat{S}_{pk}^T$ is linear combination of $\hat{S}_{pk1}$ and $\hat{S}_{pk2}$, we know $\hat{S}_{pk}^T$ is from normal distribution. So the asymptotic distribution of $\hat{S}_{pk}^T$ can expressed as

$$\hat{S}_{pk}^T \sim N\left(S_{pk}^T, \frac{1}{36n(\phi(3S_{pk}^T))^2}\left(\sum_{j=1}^{2}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{2}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right).$$

Consider v variables, the asymptotic distribution of $\hat{S}_{pk}^{T*}$ can be derived as

$$\hat{S}_{pk}^T \sim N\left(S_{pk}^T, \frac{1}{36n(\phi(3S_{pk}^T))^2}\left(\sum_{j=1}^{v}\left\{(a_j^2+b_j^2)\left[\frac{\prod_{i=1}^{v}\left(2\Phi(3S_{pki})-1\right)^2}{\left(2\Phi(3S_{pkj})-1\right)^2}\right]\right\}\right)\right).$$

Theorem 1: Let Σ be the covariance matrix associated with the random vector $X = \begin{bmatrix} x_1 \\ \vdots \\ x_v \end{bmatrix}$.

Let Σ have the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_v$ & eigenvectors $e_1, \ldots, e_v$. Then the ith principal component variable is given by $y_i = e_i' X = e_{i1} x_1 + \cdots + e_{iv} x_v$, $i = 1,2,\ldots,v$. With these choices, $\text{var}(y_i) = e_i' \Sigma e_i = \lambda_i$, $i = 1,2,\ldots,v$ and $\text{cov}(y_i, y_k) = e_i' \Sigma e_k = 0$, $i \neq k$.

Proof:

The proof can be found in Johnson and Wichern (2002) on page 428.

Corollary 2: If χ^2_{n-1} and χ^2_{n-2} are independently distributed, then $\chi^2_{n-1} \times \chi^2_{n-2}$ is distributed as $(\chi^2_{2n-4})^2 / 4$.

Proof : Let $x_1 \sim \chi^2_{n-1}$ and $x_2 \sim \chi^2_{n-2}$. The joint pdf of x_1 and x_2 is given by

$$f_{x_1,x_2}(x_1, x_2) = \frac{(\frac{1}{2})^{(n-1)/2} x_1^{n/2-3/2} e^{-x_1/2}}{\Gamma[(n-1)/2]} \times \frac{(\frac{1}{2})^{(n-2)/2} x_2^{n/2-2} e^{-x_2/2}}{\Gamma[(n-2)/2]}.$$

Let $z_1 = x_1$ and $z_2 = 2\sqrt{x_1 x_2}$. Using the transformation method, the solution is $x_1 = z_1$ and $x_2 = \frac{z_2^2}{4z_1}$, and the Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{z_2^2}{4\sqrt{z_1}} & \frac{z_2}{2z_1} \end{vmatrix} = \frac{z_2}{2z_1}.$$

So, we find that the joint pdf of $z_1 z_2$ is

$$f_{z_1,z_2}(z_1, z_2) = \frac{(\frac{1}{2})^{(n-1)/2} z_1^{n/2-3/2} e^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(\frac{1}{2})^{(n-2)/2} (\frac{z_2^2}{4z_1})^{n/2-2} e^{-(\frac{z_2^2}{4z_1})/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1}, 0 \leq z_1, z_2 \leq \infty.$$

Then, the marginal density fucntion of z_2 is obtained as follow:

$$f_{z_2}(z_2) = \int_0^\infty \frac{(\frac{1}{2})^{(n-1)/2} z_1^{n/2-3/2} e^{-z_1/2}}{\Gamma[(n-1)/2]} \times \frac{(\frac{1}{2})^{(n-2)/2} (\frac{z_2^2}{4z_1})^{n/2-2} e^{-(\frac{z_2^2}{4z_1})/2}}{\Gamma[(n-2)/2]} \times \frac{z_2}{2z_1} dz_1$$

$$= C_1 \times z_2^{n-3} \times \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(\frac{z_2^2}{4z_1})/2} dz_1, 0 \leq z_2 \leq \infty$$

, where $C_1 = \frac{(\frac{1}{2})^{2n-9/2}}{\Gamma[(n-1)/2]\times\Gamma[(n-2)/2]}$. Let $h(z_2) = \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(\frac{z_2^2}{4z_1})/2} dz_1$.

Hence, $h'(z_2) = (-\frac{z_2}{4z_1}) \times \int_0^\infty z_1^{-1/2} \times e^{-z_1/2-(\frac{z_2^2}{4z_1})/2} dz_1$.

Now, let $\frac{z_2^2}{4z_1} = w$. Using the transformation method, we find that

$$h'(z_2) = (-\frac{1}{2}) \times \int_0^\infty w^{-1/2} \times e^{-w/2-(\frac{z_2^2}{4w})/2} dw = (-\frac{1}{2}) \times h(z_2).$$

The above equation gives $h(z_2) = e^{(-z_2/2+C_2)}$, where C_2 is a constant. Thus, the pdf of z_2 is given as the following, where $C_3 = C_1 \times e^{-C_2}$. Therefore, we have $z_2 \sim \chi^2_{2n-4}$.

$$f_{z_2}(z_2) = C_1 \times e^{-C_2} \times z_2^{n-3} \times e^{-z_2/2} = C_3 \times z_2^{(2n-4)/2-1} \times e^{-z_2/2}, 0 \le z_2 \le \infty.$$

Theorem 3: Let $T^2 = n(\overline{X}-\mu_0)' S^{-1}(\overline{X}-\mu_0)$, where $X = (x_1, x_2, \ldots, x_n)'$ be a sample from $N(\mu, \Sigma)$ with mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_v)'$ and covariance matrix $\Sigma_{v\times v}$, and μ_0 is the vector of target values. The distribution of

$$\frac{T^2}{n-1} \times \frac{n-v}{v}$$

is a non-central *F* with v and $n-v$ degrees of freedom and non-centrality parameter $\tau^2 = n(\overline{X}-\mu_0)' \Sigma^{-1}(\overline{X}-\mu_0)$.

Proof: See Anderson (2003) on pages 174-176.

It can be shown that the pdf of T^2 is given by

$$\frac{e^{-\frac{1}{2}\tau^2}}{(n-1)\Gamma[\frac{1}{2}(n-v)]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i \left[t^2/(n-1)\right]^{\frac{1}{2}v+i-1} \Gamma(\frac{1}{2}n+i)}{i!\Gamma(\frac{1}{2}v+i)\left[1+t^2/(n-1)\right]^{\frac{1}{2}n+i}} . \qquad \text{(A1)}$$

(See Anderson (2003) on page 186)

Corollary 3:

1) For v=2, if $z = \chi^2_{n-1} \times \chi^2_{n-2} \times \hat{D}^2$, where $\hat{D}^2 = [1 + \frac{n}{n-1}(\overline{X}-\mu_0)' S^{-1}(\overline{X}-\mu_0)]$, then the pdf of z is

$$f_Z(z) = \int_1^\infty \frac{\frac{1}{2}e^{-\frac{1}{2}\tau^2}}{w\Gamma(n-2)\Gamma[(n-2)/2]}(z/w)^{(n-4)/2}e^{-\sqrt{z/w}}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(w-1)^i\Gamma(n/2+i)}{i!\Gamma(i+1)w^{\frac{1}{2}n+i}}dw, \text{for } z \geq 0.$$

2) For v=3, if $z = \chi^2_{n-1}\times\chi^2_{n-2}\times\chi^2_{n-3}\times\hat{D}^2$, where

$\hat{D}^2 = [1+\frac{n}{n-1}(\overline{X}-\mu_0)^{'}S^{-1}(\overline{X}-\mu_0)]$, then the pdf of z is

$$f_Z(z) = \int_1^\infty\int_0^\infty \frac{(1/2)^{(n-1)/2}x_1^{-1/2}e^{-\sqrt{x_1}-z/(2wx_1)}}{\Gamma(n-2)\Gamma[(n-3)/2]}\frac{e^{-\frac{1}{2}\tau^2}(z/w)^{(n-5)/2}}{\Gamma[(n-3)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(w-1)^i\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{\frac{1}{2}n+i}}dx_1dw, \text{ for } z \geq 0.$$

Proof :

1) From $\hat{D}^2 = [1+\frac{n}{n-1}(\overline{X}-\mu_0)^{'}S^{-1}(\overline{X}-\mu_0)]$ and Theorem 3, we find that

$$\hat{D}^2 = 1+\frac{1}{n-1}T^2.$$

Let $y = 1+\frac{1}{n-1}T^2$. Using the transformation method and the equation (A1), the pdf of y is obtained as follow:

$$f_Y(y) = f_{T^2}((n-1)(y-1))\times|(n-1)|$$

$$= \frac{e^{-\frac{1}{2}\tau^2}}{(n-1)\Gamma[(n-2)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(y-1)^i\Gamma(n/2+i)}{i!\Gamma(i+1)y^{\frac{1}{2}n+i}}\times(n-1)$$

$$= \frac{e^{-\frac{1}{2}\tau^2}}{\Gamma[(n-2)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(y-1)^i\Gamma(n/2+i)}{i!\Gamma(i+1)y^{\frac{1}{2}n+i}}, \text{ for } y \geq 1.$$

Let $x = \chi^2_{n-1}\times\chi^2_{n-2}$. From Corollary 2, we find that $x = (\chi^2_{2n-4})^2/4$. That is, the pdf of x is

$$f_X(x) = \frac{\frac{1}{2}x^{(n-4)/2}e^{-\sqrt{x}}}{\Gamma(n-2)}, \text{ for } x > 0.$$

Now, let $z = xy$ and w=y. Using the transformation method, the solution is

$x = z/w$ and y=w, and the Jacobian of the transformation is $J = \begin{vmatrix} \frac{1}{w} & \frac{-z}{w^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$. So,

we find that the joint pdf of zw is

$f_{ZW}(z,w) = f_{XY}(z/w, w)\times(1/w)$

$$= \frac{\frac{1}{2}(z/w)^{(n-4)/2} e^{-\sqrt{z/w}}}{w\Gamma(n-2)} \times \frac{e^{-\frac{1}{2}\tau^2}}{\Gamma[(n-2)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1) w^{\frac{1}{2}n+i}}, \text{ for } z \geq 0, w > 1.$$

Then, the marginal density function of z is obtained as follow:

$$f_Z(z) = \int_1^\infty \frac{\frac{1}{2}(z/w)^{(n-4)/2} e^{-\sqrt{z/w}}}{w\Gamma(n-2)} \times \frac{e^{-\frac{1}{2}\tau^2}}{\Gamma[(n-2)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1) w^{\frac{1}{2}n+i}} dw$$

$$= \int_1^\infty \frac{\frac{1}{2} e^{-\frac{1}{2}\tau^2}}{w\Gamma(n-2)\Gamma[(n-2)/2]} (z/w)^{(n-4)/2} e^{-\sqrt{z/w}} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (w-1)^i \Gamma(n/2+i)}{i!\Gamma(i+1) w^{\frac{1}{2}n+i}} dw, \text{ for } z \geq 0.$$

2) From $\hat{D}^2 = [1 + \frac{n}{n-1}(\overline{X}-\mu_0)' S^{-1} (\overline{X}-\mu_0)]$ and Theorem 3, we find that

$$\hat{D}^2 = 1 + \frac{1}{n-1} T^2 .$$

Let $y = 1 + \frac{1}{n-1} T^2$. Using the transformation method and the equation (A1), the pdf of y is obtained as follow:

$$f_Y(y) = f_{T^2}\big((n-1)(y-1)\big) \times |(n-1)|$$

$$= \frac{e^{-\frac{1}{2}\tau^2}}{(n-1)\Gamma[(n-3)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^{i+1/2} \Gamma(n/2+i)}{i!\Gamma(i+3/2) y^{\frac{1}{2}n+i}} \times (n-1)$$

$$= \frac{e^{-\frac{1}{2}\tau^2}}{\Gamma[(n-3)/2]} \sum_{i=0}^{\infty} \frac{(\tau^2/2)^i (y-1)^{i+1/2} \Gamma(n/2+i)}{i!\Gamma(i+3/2) y^{\frac{1}{2}n+i}}, \text{ for } y \geq 1.$$

Let $x_1 = \chi^2_{n-1} \times \chi^2_{n-2}$ and $x_2 = \chi^2_{n-3}$. From Corollary 2, we find that $x_1 = (\chi^2_{2n-4})^2/4$. That is, the pdf of x_1 is

$$f_{X_1}(x_1) = \frac{\frac{1}{2} x_1^{(n-4)/2} e^{-\sqrt{x_1}}}{\Gamma(n-2)}, \text{ for } x_1 > 0 .$$

Now, let $x = x_1 x_2$ and $u = x_1$. Using the transformation method, the solution is $x_1 = u$ and $x_2 = x/u$, and the Jacobian of the transformation is $J = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & \frac{-x}{u^2} \end{vmatrix} = \frac{1}{u}$. So, we find that the joint pdf of *xu* is

$$f_{XU}(x,u)=f_{X_1X_2}(u,x/u)\times\frac{1}{u}=\frac{(\frac{1}{2})^{(n-1)/2}\times u^{-1/2}\times x^{(n-5)/2}\times e^{-\sqrt{u}-\frac{x}{2u}}}{\Gamma(n-2)\times\Gamma[(n-3)/2]},\ 0\le x,u<\infty .$$

Then, the marginal density function of *x* is

$$f_X(x)=\int_0^\infty\frac{(\frac{1}{2})^{(n-1)/2}\times u^{-1/2}\times x^{(n-5)/2}\times e^{-\sqrt{u}-\frac{x}{2u}}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}du,\ x\ge 0 .$$

Again, let $z=xy$ and w=y. Using the transformation method, the solution is $x=z/w$ and y=w, and the Jacobian of the transformation is the $J=\begin{vmatrix}\frac{1}{w} & \frac{-z}{w^2}\\ 0 & 1\end{vmatrix}=\frac{1}{w}$.

So, the joint pdf of *zw* is obtained as follow:

$$f_{ZW}(z,w)=f_{XY}(z/w,w)\times(1/w)$$

$$=\int_0^\infty\frac{(1/2)^{(n-1)/2}\times u^{-1/2}\times(z/w)^{(n-5)/2}\times e^{-\sqrt{u}-\frac{z}{2wu}}}{\Gamma(n-2)\times\Gamma[(n-3)/2]}du\times\frac{e^{-\frac{1}{2}\tau^2}}{\Gamma[(n-3)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{\frac{1}{2}n+i}}\times\frac{1}{w}$$

$$=\int_0^\infty\frac{(1/2)^{(n-1)/2}u^{-1/2}e^{-\sqrt{u}-z/(2wu)}}{\Gamma(n-2)\Gamma[(n-3)/2]}du\,\frac{e^{-\frac{1}{2}\tau^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{\frac{1}{2}n+i}},\text{ for } z\ge 0, w>1.$$

Then, the marginal density function of *z* is

$$f_Z(z)=\int_1^\infty\int_0^\infty\frac{(1/2)^{(n-1)/2}u^{-1/2}e^{-\sqrt{u}-z/(2wu)}}{\Gamma(n-2)\Gamma[(n-3)/2]}\frac{e^{-\frac{1}{2}\tau^2}(z/w)^{(n-5)/2}}{w\Gamma[(n-3)/2]}\sum_{i=0}^{\infty}\frac{(\tau^2/2)^i(w-1)^{i+1/2}\Gamma(n/2+i)}{i!\Gamma(i+3/2)w^{\frac{1}{2}n+i}}dudw,\text{ for } z\ge 0.$$

Corollary 4: $x\sim(F_{2n_2-4,\,2n_1-4})^2$, $y\sim F_{n_2-3,\,n_1-3}$, if $z=xy$, then the p.d.f. of z is

$$f_Z(z)=\int_0^\infty k(n_1,n_2)\frac{\frac{1}{2}x^{(2n_2-4)/4-2}\left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1+\frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2}\left(1+\frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}}dz,\text{ for x, } z\ge 0,$$

where $k(n_1,n_2)=\dfrac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}$.

Proof : Let $t\sim(F_{2n_2-4,\,2n_1-4})$, $x=t^2$, where

$$f_t(t)=\frac{\Gamma[(2n_2+2n_1-8)/2]}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]}\times\frac{t^{(2n_2-4)/2-1}}{\left(1+(\frac{2n_2-4}{2n_1-4})t\right)^{(2n_2+2n_1-8)/2}}\text{, for } t\ge 0.$$

Using the transformation method, $t=\sqrt{x}$, and $J=\frac{1}{2\sqrt{x}}$ (Jacobian), so the p.d.f. of x is

$$f_x(x)=\frac{\Gamma[(2n_2+2n_1-8)/2]}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]}\times\frac{\frac{1}{2}x^{(2n_2-4)/4-1}}{\left(1+(\frac{2n_2-4}{2n_1-4})\sqrt{x}\right)^{(2n_2+2n_1-8)/2}}\text{, for } x\ge 0.$$

Now, let $z=xy$ given $x\sim(F_{2n_2-4,\,2n_1-4})^2$, $y\sim F_{n_2-3,\,n_1-3}$. Using the transformation method, $x=x$, $y=z/x$, and $J=1/x$ (Jaobian), so the joint pdf of x and z is

$$f_{x,z}(x,z)\quad=k(n_1,n_2)\frac{\frac{1}{2}x^{(2n_2-4)/4-2}\left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1+\frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2}\left(1+\frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}}\text{, for x, } z\ge 0,$$

$$k(n_1,n_2)=\frac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$$

Then the marginal density function of z is obtained as follows:

$$f_z(Z)=\int_0^\infty k(n_1,n_2)\frac{\frac{1}{2}x^{(2n_2-4)/4-2}\left(\frac{z}{x}\right)^{(n_2-3)/2-1}}{\left(1+\frac{(2n_2-4)}{(2n_1-4)}\sqrt{x}\right)^{(2n_2+2n_1-8)/2}\left(1+\frac{(n_2-3)}{(n_1-3)}\frac{z}{x}\right)^{(n_2+n_1-6)/2}}dz,\quad\text{for x, } z\ge 0,$$

where $k(n_1,n_2)=\dfrac{\Gamma[(2n_2+2n_1-8)/2]\Gamma[(n_2+n_1-6)/2]\left(\frac{2n_2-4}{2n_2-4}\right)^{(2n_2-4)/2}\left(\frac{n_2-3}{n_1-3}\right)^{(n_2-3)/2}}{\Gamma[(2n_2-4)/2]\Gamma[(2n_1-4)/2]\Gamma[(n_2-3)/2]\Gamma[(2n_1-3)/2]}.$