國立交通大學

財務金融研究所

碩士論文

應用合適的損失函數於選擇權模型評價:避險績效

A relevant loss function in option valuation - Hedging performance

研究生：張智琦

指導教授：李昭勝 教授

中華民國九十四年六月

應用合適的損失函數於選擇權模型評價:避險績效

A relevant loss function in option valuation - Hedging performance

研究生：張智琪　　Student: Chih-Chi Chang

指導教授：李昭勝教授　　Advisor: Dr.Jack C. Lee

國立交通大學

財務金融研究所

碩士論文

A Thesis

Submitted to Institute of Finance

College of Management

National Chiao Tung University

in partial Fulfillment of the Requirements

for the Degree of

Master

of

Science in Finance

June 2005

Hsinchu, Taiwan, Republic of China

中華民國九十四年六月

應用合適的損失函數於選擇權模型評價:避險績效

研究生：張智琦　　　　　　　　　　　　　　　　　　指導教授：李昭勝 教授

國立交通大學財務金融研究所

摘要

在這篇論文裡，我們研究在評比選擇權訂價模型時，不同損失函數的適用性。我們的分析顯示用一個合理的評價標準去衡量不同模型間相對優點是相當重要的，基於避險者的角度，我們建議使用避險誤差作為損失函數來評定樣本外的績效。以台指選擇權為資料，用 Black-Scholes 模型描述三種不同形式的誤差並和 CEV 模型做比較，實證結果發現不適當的評比標準反而會使較好的模型被拒絕，因此選擇一個恰當的損失函數是必需的。

A relevant loss function in option valuation
- Hedging performance

Student: Chih-Chi Chang

Advisor: Dr. Jack C. Lee

Institute of Finance

National Chiao Tung University

June 2005

Abstract

In this thesis, we investigate the adequacy of different loss functions when evaluating option pricing models. Our analysis shows that it is important to have a reasonable yardstick to assess the relative merits of competing models. Based on the viewpoint of a hedger, we recommend the hedging error to be the loss function in judging out-of-sample performance. We illustrate the effect on the three types of error in an application of the Black-Scholes model to TAIEX index options and compare with the constant elasticity of variance model. Our empirical results show that a better model may be rejected when using an inappropriate criterion. Thus, it is vital to choose a relevant loss function.

Acknowledgements

這篇論文能夠完成，著實需要感謝很多人，首先要感謝我的指導教授李昭勝老師在這兩年來給予細心的指導及鼓勵，接著要感謝統計所博士班的牛維方學長，總是耐心的解決我的疑問並適時的給予意見，讓這篇論文能夠順利的進行，另外還要感謝我碩士班的同學，明俊、婉儀、信德、筱芳、兆輝、偉立、冠宇、禹丹、燕晴，謝謝你們這一路上的扶持，讓我能夠更有信心去面對困難。最後，家人的付出及支持對我而言是最有力的支柱，僅將這篇論文，獻給我最親愛的家人。

Contents

1. Introduction

Over the past few decades, the derivatives market has been expanding dramatically. Accompanied with this expansion is a delicate problem in pricing and risk management of derivatives. In particular, option pricing has played a central role in the theory of asset pricing. Since Black and Scholes (1973, henceforth BS) and Merton (1973), the theory of option pricing has advanced considerably. Because of the inconsistencies between the assumptions of the BS model and real world many researchers proposed alternative approaches to remedy. For instance, an important class of models specifies the volatility of the underlying asset as a deterministic function of time and the underlying asset (Derman and Kani, 1994; Dupire, 1994; Rubinstein, 1944). In addition, more new models that each relax some of the restrictive BS assumptions has developed, e.g. the constant elasticity of variance model by Cox (1975), and Schroder (1989), the stochastic interest rate option models of Merton (1973) and Amin and Jarow (1992), the stochastic volatility models of Scott (1987), Hull and White (1987), Heston (1993), Melino and Turnbull (1990), jump model of Bates(1996), and discrete-time GARCH models of Duan (1995), Heston and Nandi (2000) etc..

The adequacy of an option-pricing model is typically evaluated in an out-of-sample pricing exercise. Generally, we utilize the method that minimizes the price differences to the observed market prices. However, the choice of the particular loss function for the in-sample estimation and the out-of-sample influences the result of that pricing model. Christoffersen and Jacobs (2004) emphasize that consistency in the choice of loss functions is crucial. First, for any given model, the loss function used in parameter estimation and model evaluation should be the same, otherwise suboptimal parameter estimates may be obtained. Second, when comparing models, the estimation loss function should be identical across models, otherwise it will lead to inappropriate comparisons. In contrast, empirical researchers are inconsistent in their choice of the loss functions. They do not align the estimation and evaluation loss functions and therefore the results of these studies may be misleading. It is well known in the statistics literature that the choice of loss function is critical for model estimation and evaluation. Engle (1993) pointed that the choice of loss function implicitly defines the model under consideration. Therefore, it may happen that a misspecified model may outperform a "correctly specified" model, if different loss functions in estimation and evaluation are used. Renault (1997) argued that standard theoretical option valuation models imply a deterministic option price and thus do not tell the empirical researcher how to specify the error term. The choice of loss function is vital because it implicitly assumes a particular error structure.

However, the majority of option valuation studies focus on the theoretical models and the importance of the consistent loss function is ignored. They use different loss functions at the estimation and evaluation stages. In contrast, Dumas, Flemming, and Whaley (1998) compare the out-of-sample performance of the ad hoc BS model with the out-of-sample performance of deterministic volatility models implemented with identical in- and out-of-sample loss function. In this thesis, we focus only on evaluation of the option pricing models. The inconsistency problem can be avoided because we estimate the parameters with the stock data instead of the option data.

Although Christoffersen and Jacobs (2004) do not recommend any particular loss function, they mention that the choice of loss function is particularly important in option valuation. However, the particular loss function used in the empirical analysis characterizes the model specification under consideration. Therefore, it is still possible that a misspecified model will outperform a "correctly specified" model when the "inappropriate" loss function is used even if the loss functions are aligned. For different purposes, like hedging, speculating, or market making, a variety of loss functions can be applied. Generally, the participants of option markets trade to hedge their portfolio and option valuation models are also based on hedge related arguments. Bakshi, Cao and Chen (1997) mention that hedging errors measure how well a model captures the dynamic properties of option and underlying security prices. In other words, in-sample and out-of-sample pricing errors reflect a model's static performance, while hedging errors reflect the model's dynamic performance. In the later, we will discuss the loss function in more detail.

The purpose of this thesis is to provide evidence that hedging error is more appropriate in the evaluation of models than other conventional loss functions, especially pricing error.

In the next section, we discuss the drawbacks of the conventional loss functions and emphasize the adequacy of the hedging error with the BS model. A description of the TAIEX option data is provided in section 3. Section 4 presents the empirical results for the BS model and compare with the more complex CEV model in section 5. Finally, section 6 concludes.

2. Choice of loss function

The inappropriate loss function may lead to some faults in option valuation. We describe the deficiency of \$MSE and %MSE and then recommend the hedging error as the criterion.

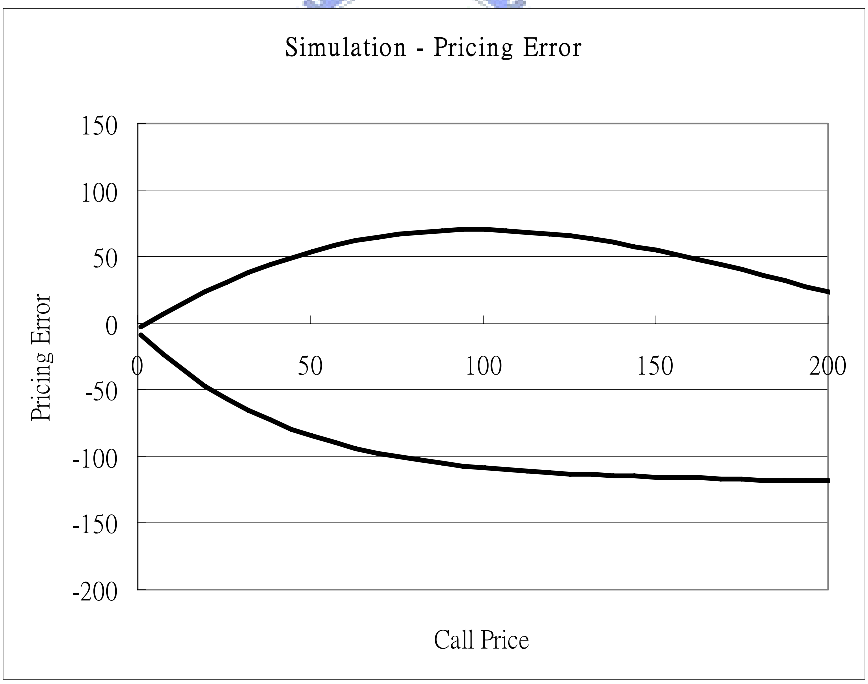
2.1. Discussion on \$MSE and %MSE

We first discuss the mean-squared dollar errors, which is the most frequently used criterion for evaluating the performance of different option valuation models. The form is

$$\$MSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(C_i - C_i(\theta)\right)^2, \tag{1}$$

where C_i and $C_i(\theta)$ are respectively the data and model option prices, respectively, and n is the number of option contracts used. The \$MSE loss function has the advantage that the errors are easily interpreted as \$-errors once the square root is taken of the mean-squared error. However, the relatively wide range of option prices across moneyness and maturity raises the problem of heteroskedasticity for \$MSE-based option valuation. To present the heteroskedasticity of \$MSE, we simulate the trend with different strike prices and maturity dates. Figure 1 indeed shows that the pricing error is not homogeneous as we describe previously.

Figure 1: Simulation the trend of pricing error



We set that volatility is 0.04, 0.26 respectively, and compute the pricing error with the closing prices of call options, which have different strike prices and maturity dates in May 18, 2005.

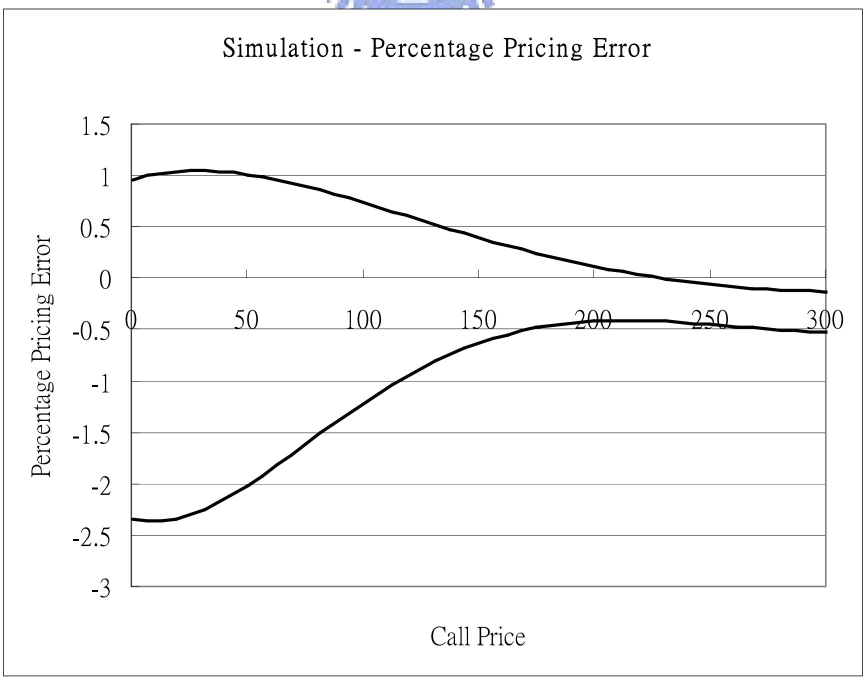
Because the \$MSE loss function implicitly assigns a lot of weight to options with high valuations (in-the-money and long time-to-maturity contracts) and therefore high \$-errors, some researchers instead favor the relative or percent mean-squared error loss function, defined as

$$\%MSE(\theta)=\frac{1}{n}\sum_{i=1}^{n}\left(\left(C_i-C_i(\theta)\right)/C_i\right)^2, \tag{2}$$

where that the %-sign is a convenient short-hand for relative loss. We do not in fact multiply the relative loss by 100, and thus the losses are not expressed in percent but rather decimals.

The %MSE loss function has the advantage that a \$1 error on a \$50 dollar option carries less weight than a \$1 error on a \$5 option, which is sensible from a rate of return perspective. The disadvantage is that short time-to-maturity out-of-the money options with valuations close to zero will implicitly be assigned a lot of weight and can thus create numerical instability as figure 2.

Figure 2: Simulation of the trend of percentage pricing error



Based on the above consideration of heteroskedasticity, \$MSE and %MSE seem inappropriate to be the criteria for option valuation.

We will give an example to verify the implied volatility or hedging ratio is more relevant than option price. Loss functions based on only prices and pricing errors can be generally inadequate. Consider, under Black and Scholes' framework, the following two option contracts.

	Contract 1	Contract 2
Stock price	100	100
Strike	100	95
time to maturity	0.5	0.09
Interest rate	0	0
Actual price	5	5
Implied volatility	17.74%	2.24%
δ	0.525	1
Price estimated with σ=20%	5.64	5.64
δ	0.528	0.812

The two contracts have the same prices but very different implied volatilities. Suppose that a trader obtains his volatility as 20% and then his fair prices for the two contracts will each be 5.64. Clearly, both MSE's and MAE's will be almost identical. However, pricing contract 2 with volatility 20% can lead to much larger losses (or gains) since the two δ values appear to be very different than those of contract 1.

This example illustrates that loss functions such as MSE or MAE can be very insensitive to the parameters of the model. In fact, it should be noted that the price of an option contract contains no information unless its specification and the price of the underlying asset are all taken into consideration.

2.2. Dynamic hedging performance

As discussed above, we abandon these criteria and recommend a meaningful and applicable loss function for model evaluation.

We first examine hedges in which only a single instrument (i.e., the underlying stock) can be employed. In addition, we first describe the assumptions of the Black-Scholes-Merton differential equation,

i. The risk-free interest rate r is constant.

ii. The underlying stock price dynamics are described in continuous time by the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dZ_t \qquad t \in [0,\infty) \tag{3}$$

where Z_t represents a Wiener process with respect to real-world probability P.

iii. The stock pays no dividends, and there are no stock splits or other corporate actions during the period [t,T].

iv. Finally, there are no transaction costs and no bid-ask spreads.

To make the point precise, imagine a situation in which a financial institution intends to hedge a short position in a call option with $\tau = T - t$ periods to expiration and strike price K. As before, we use the BS model for the discussion of the basic idea. Consider a hedging portfolio consisting of a long position in Δ_t shares of the stock and a short position in one call option. We calculate the value of Δ_t so that the portfolio is riskless. A riskless portfolio therefore consists of

$$Long : \Delta_t \ shares$$

and

$$Short : 1 \ call \ option.$$

Define Π_t as the value of the portfolio at time t. By definition,

$$\Pi_t = \Delta_t S_t - C_t . \tag{4}$$

The change $d\Pi$ in the value of the portfolio in the time interval dt is given by

$$d\Pi = \Delta_t dS - dC \,. \tag{5}$$

The $It\hat{o}'s\ Lemma$ shows that the function of the call option price follows the process

$$dC = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S}\sigma S dZ \tag{6}$$

Substituting equations (3) and (6) into equation (4) yields

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt \tag{7}$$

Because this equation does not involve dZ (Winner process), the portfolio must be riskless during time dt. The assumptions listed in the preceding paragraph imply that the portfolio must instantaneously earn the risk-free interest rate. Otherwise, it will exist the arbitrage opportunity. Therefore, the change of the portfolio value satisfies

$$d\Pi = r\Pi dt \tag{8}$$

Theoretically, the constructed partial hedge requires continuous rebalancing to reflect the changing market conditions. In practice, only discrete rebalancing is possible. To derive a hedging effectiveness measure, suppose that portfolio rebalancing takes place at intervals of length δt. As described above, the value of the portfolio is Π_t at time t. Next, at time $t+\delta t$, the value of the risk-free portfolio becomes $e^{r\delta t} \cdot \Pi_t$. Thus, we can write the difference between the adjacent portfolios as

$$\Pi_{t+\delta t} - \Pi_t = e^{r\delta t}\Pi_t - \Pi_t = \Pi_t (e^{r\delta t} - 1) \approx 0 \tag{9}$$

The hedging error is

$$\begin{aligned} HE_l &= \Pi_{t+\delta t} - e^{r\delta t}\Pi_t \\ &= \left(\Delta_t S_{t+\delta t} - C_{t+\delta t}\right) - e^{r\delta t}\left(\Delta_t S_t - C_t\right) \end{aligned} \tag{10}$$

At the same time, reconstruct the portfolio, repeat the hedging error calculation at time $t+2\delta t$, and so on. Record the hedging errors HE_l, for l = 1, 2…, M, where M is the number of the corresponding contracts which traded at time t and time $t+\delta t$ respectively. Finally, compute the mean absolute hedging error (MAHE) as a function of rebalancing frequency δt:

$$\$MAHE = \frac{1}{M}\sum_{l=1}^{M}\left|HE_l\right| \tag{11}$$

Single-instrument hedging errors under other models can be determined with the same argument.

Besides, we examine the hedging error from a different perspective. It is from the standpoint of the seller of an option who hedges the position by holding a portfolio invested in the underlying asset and cash according to proportions consistent with the Black and Scholes formula. The option and the hedging portfolio is called global portfolio. Consider now holding a fixed portfolio of D shares of stock and Q dollars of risk-free security over the interval, δt. If volatility is constant, the option can be replicated by the following portfolio value at time t:

$$P_t = D_t S_t + Q_t \tag{12}$$

In the replicated portfolio, the quantities of stock and risk free bonds are constantly adjusted to achieve

$$\begin{aligned} D &= C_s \\ Q &= C - C_s S \end{aligned} \tag{13}$$

The difference between the call and the portfolio over the period δt is

$$\delta P - \delta C = C_s \delta S + (e^{r\delta t} - 1)(C - C_s S) - \delta C \tag{14}$$

which is equivalent to HE_l. Equation（14）measures the cash flow generated by a synthetic portfolio that balances a liability for the predicted option price with an offsetting position in the replicated portfolio. If the model is specified correctly, the synthetic portfolio will be self-financing, that is, it will generate neither positive nor negative cash flows. But if the model is misspecified, the synthetic portfolio will generate random cash flows. The size of the cash flows generated by the synthetic portfolio over the life of the option provides a metric with which to evaluate the option pricing model.

The residual risk incurred by such a discrete strategy is typically measured by the variance of the hedging error, denoted by HE_l, that is the variance of the difference between changes in the value of the option to be hedged, when rebalancing is performed every δt. Under standard assumptions (the underlying assets follows a geometric Brownian motion with constant volatility), Leland (1985) has derived theoretically

$$HE = \frac{1}{2} C_{ss} S^2 \left[\sigma^2 \delta t - \left(\frac{\delta S}{S} \right)^2 \right] + O\left(\delta t^{3/2}\right) \quad , \tag{15}$$

$$E(HE) = \frac{1}{2} C_{ss} S^2 \left[\sigma^2 \delta t - \left(\frac{\delta S}{S} \right)^2 \right] \quad , \tag{16}$$

and argued that $E\left(HE_l\right) = 0$ as $\delta t \to 0$. Henrotte (1993) and Martellini (2000) show the variance of HE_l

$$Var\left(HE_l\right) \sim \frac{1}{2} \Gamma_0^{\ 2} S_0^{\ 4} \sigma^4 \delta t^2 \tag{17}$$

Hence, the hedging error has a zero mean. It also has a vanishing variance as the time period, δt, approaches zero, that is as the time resolution approaches infinity, so that one recovers a BS type of perfect replication as a limiting case.

Hedging errors measure how well a model captures the dynamic properties of options and underlying security prices. In other words, out-of-sample pricing errors reflect a model's static performance, while hedging errors reflect the model's dynamic performance.

2.3. The difficulties in hedging and the causes of the hedging errors

Hedging performance is a meaningful method to judge which model is the best, but several irresistible factors could lead to biases in hedging effectiveness. First, frequent adjustments for portfolio can result in more efficient hedging performance if the transaction costs are not considered. Because the replicated portfolio is not rebalanced continuously, the cash flow in equation (10) may be non-zero even if the constant volatility model is true. An argument due to Leland (1985) shows that, in the absence of misspecification, the size of the cash flows in equation (10) can be made arbitrarily small by using a suitably chosen revision period. Second, when transaction costs become significant, hedging strategies can be infeasible. That is, hedging strategies without cost consideration are indeed inoperative. Third, our dynamic hedge only takes the delta value into account but in fact the option prices changes nonlinearly in the price of the asset, the hedging effects can be biased. Besides, the market illiquidity is also a special source of model risk in the hedging of derivatives. Finally, since delta is a function of the volatility of the underlying asset, estimation of volatilities plays a central role and affects hedging performances.

All of the factors above can result in poor performances in hedging. Many authors have been devoted to solving this problem. In this thesis, only adequacy of loss functions for option pricing is emphasized, so we will not discuss more about hedging strategies.

3. The Taiwan TAIEX Index Options

3.1. Market description

We begin with a brief description of the Taiwan Stock Exchange Corporation (TSEC). The Exchange maintains a total of 28 stock price indices, to allow investors to grab both overall market movement and different industrial sectors' performances conveniently. The indices may be grouped into market value indices and price average indices. The former are similar to the Standard & Poor's Index, weighted by the number of outstanding shares, and the latter are similar to the Dow Jones Industrial Average and the Nikkei Stock Average. The TSEC Capitalization Weighted Stock Index (TAIEX) is the most widely quoted of all TSEC indices.

TAIEX covers all of the listed stocks excluding preferred stocks, full-delivery stocks and newly listed stocks, which are listed for less than one calendar month. Up to December 2002, 593 issues were selected as component stocks from the 638 companies listed on the Exchange.

Trading on the Exchange starts at 09:00 and closes at 13:30. The orders can be entered half an hour before the trading session starts. Buy or sell orders are in standard unit or multiples of standard units. One standard trading unit is 1,000 shares, which is applicable to all listed stocks. Orders below 1,000 shares are considered odd-lot and that over 500,000 shares are block trading.

Under current trading rules, the closing price of a stock is simply the last traded price of the intra-day continuous auction of the trading day. Given that the closing price of securities is widely used by market participants as a benchmark for portfolio valuation as well as index calculation, the new system will accumulate orders for 5 minutes (from 1:25 p.m. to 1:30 p.m.) before the closing call auction.

The TSEC imposes a 7% price limits for all traded stocks. Within a trading day, the price for a single stock cannot move more than 7% from the previous price after adjusting for dividend and stock splits. Therefore, the maximum close to close 1-day return is 7% and the minimum return is -7%.

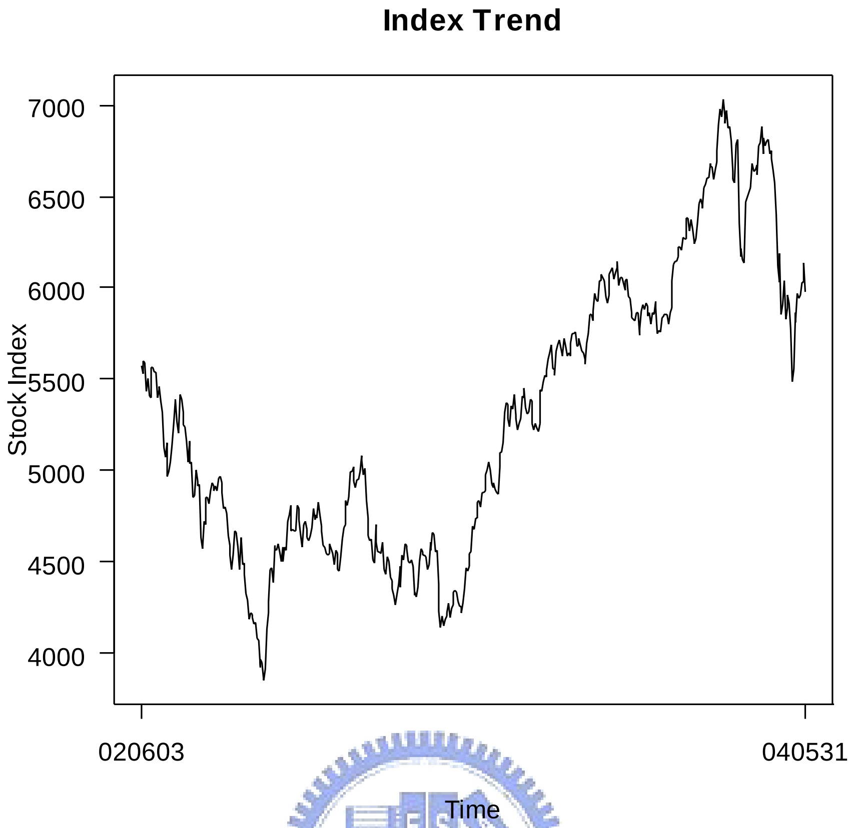
The official derivative market for risky assets, which is known as Taiwan Futures Exchange (TAIFEX), trades future contracts on TAIEX, the equivalent option contracts for calls and puts, and individual option contracts for blue-chip stocks. Trading in the derivative market started in 1998. The market has experienced tremendous growth from the very beginning. Launched on December 24th, 2001, the TAIEX index options achieved a total of 43,824,511 contracts by the end of 2004, accounting for 74.10% of the market total for the year.

The TAIEX option contract is a cash-settled European option with trading during the three nearest consecutive months and the other 2 months of the March quarterly cycle (March, June, September, and December). The last trading day is the third Wednesday of the delivery month and the expiration day is the first business day following the last trading day. Trading occurs from 08:45 to 13:45. During the sample period covered by this research, the contract size is 50 New Taiwan dollars times the TAIEX index, and prices are quoted in points, with a minimum price change of one-tenth point (NT$5). The exercise prices are given in 100 index point intervals in spot month, the next two calendar months and 200 index point intervals in the additional two months from the March quarterly cycle.

It is important to point out that liquidity is concentrated on the nearest expiration contract.

Thus, during 2002 and 2004 almost 90% of crossing transactions occurred in contracts of this type.

Figure 3: TAIEX index level each trading day during the period of June 2002 through May 2004.



3.2. The data

Based on the following consideration, we use TAIEX call option traded daily on TAIFEX during the period from July 1, 2002 through June 30, 2004 for our empirical work. To ease computational burden and avoid the non-simultaneous data, for each day in the sample, only the last reported quote (prior to 1:30 PM) of each option contract is employed in the empirical tests. Note that the recorded TAIEX index values are not the daily closing index levels. Rather, they are the corresponding index levels at the moment when the option quote is recorded. Thus, there is no nonsynchronous price issue here.

Several exclusion filters are applied to construct the option price data. First, option price quotes that are time-stamped later than 1:30 PM are eliminated. This ensures that the spot price is recorded synchronously with its option counterpart. Second, as options with less than six days and more than 100 days to expiration may induce liquidity-related biases, they are excluded from the sample.

Table 1: Sample Properties of TAIEX index options

	Moneyness S/K	Days-to-Expiration <30	30-60	>60	Subtotal
OTM	<0.94	15.15 (15.41) {973}	32.07 (44.05) {813}	72.25 (47.86) {175}	{1961}
	0.94 ~ 0.97	39.77 (28.61) {624}	88.56 (40.66) {566}	128.08 (57.29) {165}	{1355}
ATM	0.97~1.00	83.84 (40.52) {673}	144.86 (51.78) {572}	190.07 (66.76) {109}	{1354}
	1.00 ~ 1.03	164.19 (51.89) {610}	222.87 (62.06) {431}	289.95 (85.74) {60}	{1101}
ITM	1.03 ~ 1.06	280.38 (64.49) {427}	325.32 (79.26) {233}	404.24 (93.36) {29}	{689}
	≧1.06	523.79 (170.03) {340}	507.72 (158.47) {174}	687.54 (214.93) {41}	{555}
Subtotal		{3647}	{2789}	{579}	{7015}

The reported numbers are respectively the average quoted price, the standard deviation which are shown in parentheses, and the total number of observations (in braces), for each moneyness and maturity category.

These criteria yield a final daily sample of 7015 observations. Table 1 describes the sample properties of the call option prices employed in this work. Average prices, standard deviations, and the number of available calls are reported for each moneyness category. Moneyness is defined as the ratio of the spot price to the exercise price.

A call option is said to be deep out-of-the money if the ratio S/K belongs to the interval $(0, 0.94)$; out-of-the-money (OTM) if $0.94 \le S/K < 0.97$; at-the-money (ATM) when $0.97 \le S/K < 1.03$; in-the-money (ITM) when $1.03 \le S/K < 1.06$; and deep-in-the-money if $S/K > 1.06$.

To proxy for riskless interest rates, we use the daily series of annualized Taiwan deposit 1 month rates from the Bank of Taiwan.

4. Empirical results

We examine the homogeneity of pricing error and hedging error respectively with the BS model.

4.1. Black-Scholes (BS) Option Pricing Model

Black and Scholes (1973) proposed the option pricing model. The model of stock price behavior developed is

$$dS = \mu S dt + \sigma S dZ, \tag{18}$$

where μ is the expected rate of return of the stock price and σ is its volatility; dZ is a Weiner process. Due to the model implies S_T has a lognormal distribution,

$$\ln S_T \sim n\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right] \quad , \tag{19}$$

we estimate the volatility from historical data. Define:

$n+1$: Number of observations

S_i : Stock price at end of ith ($i = 0,1,...,n$) interval

τ : Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \qquad \textit{for } i = 1,2,...,n.$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

where $\bar{u}$ is the mean of the u_i.

It follows that σ itself can be estimated by $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}.$$

4.2. BS Option Pricing Formula

The Black-Scholes formulas for the prices at time zero of a European call option on a non-dividend-paying stock is

$$\begin{aligned} C &= e^{-r(T-t)}E_Q[\max(S_T - K, 0)] \\ &= S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \end{aligned} \tag{20}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{21}$$

The function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution, S_t is the stock price at time t, K is the strike price, r is the risk-free rate, and $(T-t)$ is the time to maturity.

4.3. Hedge ratio with Delta (Δ) and Gamma (Γ) in the BS model

Consider a portfolio consisting of a long position in Δ (or Δ^*) shares of the stock and a short position in one call option. We calculate the value of Δ (or Δ^*) that makes the portfolio riskless. In other words, the delta of an option, Δ (or Δ^*), is defined as the rate of change of the option price with respect to the price of the stock price. In general,

$$\Delta = \frac{\partial C}{\partial S} \tag{22}$$

where C is the price of the call option and S is the stock price.

Further, Gamma represents the rate of change of the delta as the underlying risk S_t changes. Changes in delta were seen to play a fundamental role in determining the price of a vanilla option. Hence, gamma is another important Greek.

The delta and gamma in the BS model are

$$\begin{aligned} \Delta^{BS} &= N(d_1), \\ \Gamma^{BS} &= \frac{N'(d_1)}{S\sqrt{T-t}}. \end{aligned} \tag{23}$$

4.4. Comparison of the homogeneity among different loss functions

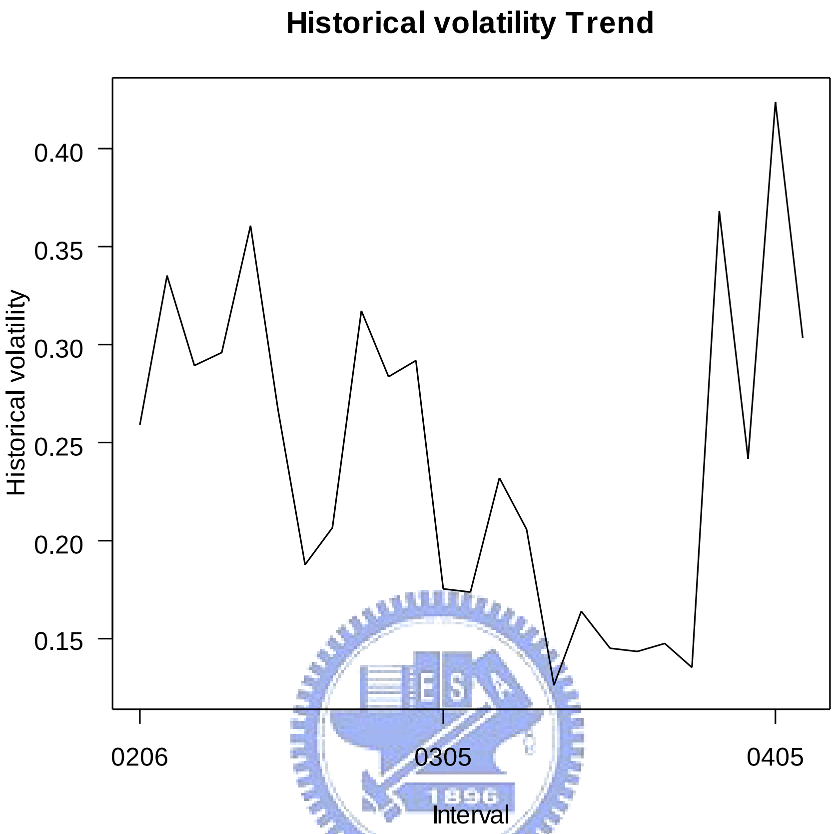
We define

$\text{Pricing Error (PE)} = C - C(\theta)$

$\text{Percentage Pricing Error (PPE)} = \dfrac{C - C(\theta)}{C}$

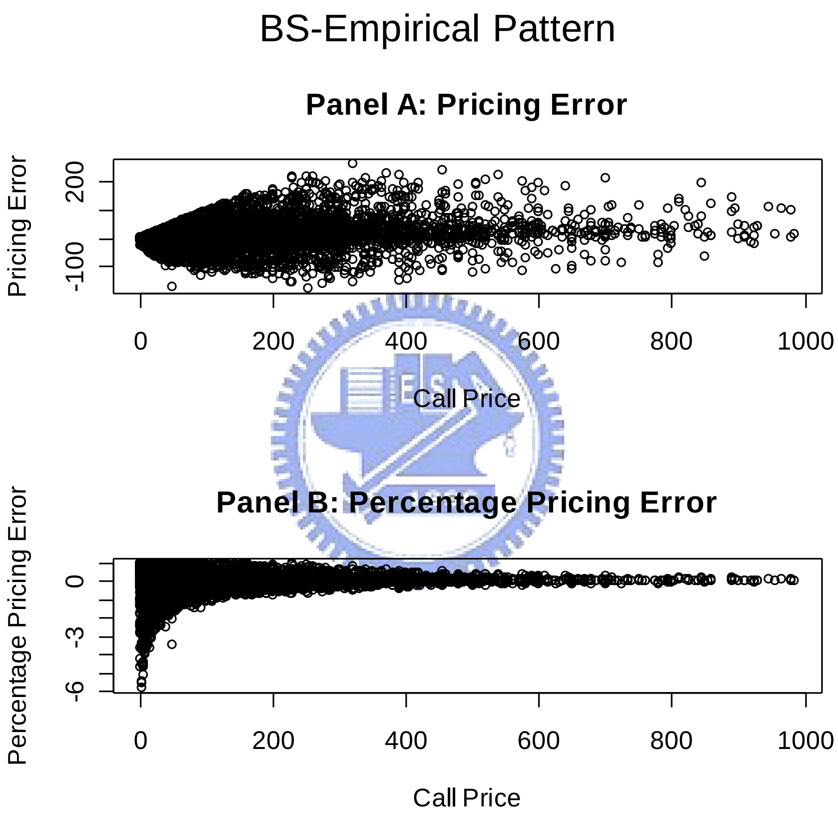
and HE has been shown in the section 2.2. To obtain the hedging results, we follow the three steps below. First, estimate the set of parameter values implied by all call options. Next, on day *t*, use these parameter estimates and the current day's stock index and interest rates to compute the hedge ratio and construct the hedging portfolio. Finally, calculate the hedging error as of day *t*+1 if the portfolio is rebalanced daily. These steps are repeated for each option and every trading day in the sample. We now estimate the historical volatility monthly with the data of TAIEX index from June, 2002 to May,2004 and Figure 4 shows the trend of the historical volatilities.

Figure 4: Historical volatility trend each month during the period of June 2002 through May 2004.



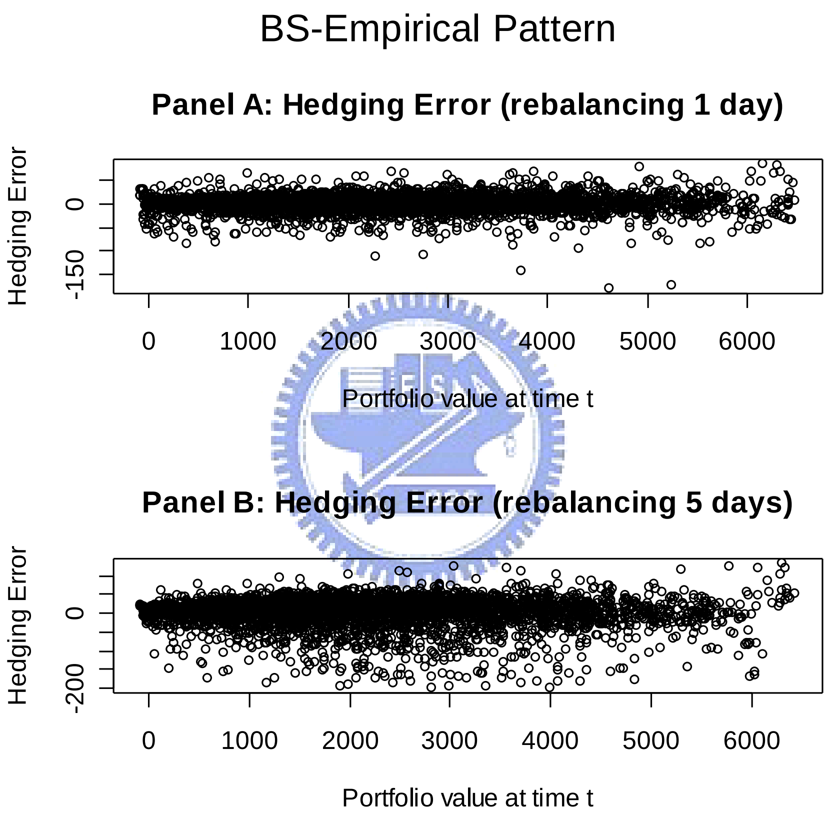
Using the TAIEX index option data described in previous section and the historical volatilities, we can compute the three errors with the BS model. Figure 5 shows patterns of the two error forms. It is apparent the both patterns of PE and PPE display the heteroskedasticity and the HE almost supplies a homogeneity pattern. Our discussion about the three errors is verified. The PE and PPE are inappropriate to judge what model is better and HE works better. On pricing error, the maximum error from the cheap call option almost equals to the price and the maximum error from the expensive call is also the same. The consequence of one dollar error over one dollar market price is more serious than over one hundred dollars. The phenomenon is particularly apparent at the interval (0,200) of panel A. In addition, pricing errors $= C - \hat{C} > 0$ imply price underrated by the BS model and price must be more than zero so that the errors cannot exceed the upper line with the slope being one. The reason for some errors to exceed the lower line with the slope being one is that the option pricing model overrate more than 100% of the price. Panel A also shows that the BS model is not sensitive to pricing for the cheap calls and easy to overrate.

Figure 5: PE and PPE patterns of BS model



However, percentage pricing error proposed to solve the drawback from the pricing error represents the error in one dollar call on average. But it may cause the overcorrected result, e.g., with an error of five dollars, the error rate would be 500% if the call price is one dollar while the error rate would be only 5%, if the call price is one hundred dollars. It leads to the model which is the best determined entirely by fraction of calls overvalued significantly so that many calls priced perfectly are ignored. Thus, we desire to find a criterion with which error pattern scatter equally. Figure 6 results the hedging error satisfies the property better than the others. The empirical with the BS model shows that the hedging error is more appropriate for model valuation.

Figure 6: HE patterns of BS model



5. Comparison with an alternative model

So far, the empirical analysis has focused on testing the impact of different loss functions using the BS model as the underlying option pricing model. We now consider the CEV (constant elasticity of variance) option pricing model and investigate whether the previous results are true for more structural models.

5.1. Constant Elasticity of Variance (CEV) Option Pricing Model

Cox (1975) has derived the renowned constant elasticity of variance (CEV) option pricing model and Schroder (1989) has subsequently extended the model by expressing the CEV option pricing formula in terms of the noncentral Chi-square distribution. The diffusion process of stock price S in a CEV model model can be defined as

$$dS = \mu S dt + \delta S^{\beta/2} dZ, \tag{26}$$

where μ is known as the expected rate of return; δ and β are constants; dZ is a Weiner process. The CEV model assumes the following relationship between stock price S and volatility:

$$\sigma(S,t) = \delta \cdot S^{(\beta-2)/2} \tag{27}$$

The elasticity of return variance with respect to price equals $\beta - 2$, and, if $\beta < 2$, volatility and price are inversely related. In the limiting case $\beta = 2$, the CEV model returns to the conventional Black-Scholes model in which the variance rate is independent of stock price. In option pricing theory, the risk-neutrality assumption allows us to replace the expected rate of return by risk-free rate of interest, r, and hence the only unobservable value is the volatility. The parameters δ and β can be estimated from the transition probability density function $f(S_T, T; S_t, t)$ derived by Cox (1975).

$$f(S_T, T; S_t, t) = (2-\beta) k^{1/(2-\beta)} \left(x w^{1-2\beta}\right)^{1/(4-2\beta)} e^{-x-w} I_{1/(2-\beta)}\left(2\sqrt{xw}\right), \tag{28}$$

where

$$k = \frac{2(r-a)}{\delta^2 (2-\beta)\left[e^{(r-a)(2-\beta)\tau} - 1\right]},$$
$$x = k S_t^{2-\beta} e^{(r-a)(2-\beta)\tau},$$
$$w = k S_T^{2-\beta},$$

and $\tau = T - t$; $I_q(\cdot)$ is the modified Bessel function of the first kind of order q; r denotes the riskless interest rate; and a denotes the continuous proportional dividend rate.

5.2. CEV Option Pricing Formula

Following Schroder (1989), the European call option pricing formula for $\beta < 2$ under the CEV model is

$$
\begin{aligned}
C &= e^{-r\tau}\int_E^{\infty} f(S_T, T; S_t, t)(S_T - E)\,dS_T \\
&= S_t e^{-a\tau}\int_y^{\infty} e^{-w-x}(w/x)^{1/(4-2\beta)} I_{1/(2-\beta)}\left(2\sqrt{xw}\right)dw \\
&\quad + Ee^{-r\tau}\int_y^{\infty} e^{-w-x}(x/w)^{1/(4-2\beta)} I_{1/(2-\beta)}\left(2\sqrt{xw}\right)dw \\
&= S_t e^{-a\tau} Q[2y; 2+2/(2-\beta), 2x] \\
&\quad - Ee^{-r\tau} Q[2y; 2-2/(2-\beta), 2x] \\
&= S_t e^{-a\tau} Q[2y; 2+2/(2-\beta), 2x] \\
&\quad - Ee^{-r\tau}\left(1 - Q[2x; 2/(2-\beta), 2y]\right)
\end{aligned}
\tag{29}
$$

where E is the exercise price, $y = kE^{2-\beta}$, and $Q[x; v, \lambda]$ is the noncentral Chi-square complementary distribution function with v degrees of freedom and noncentral parameter λ.

For $\beta > 2$, Emanuel and MacBeth (1982) derive a pricing formula, which can be shown to be equivalent to

$$
\begin{aligned}
C &= S_t e^{-a\tau} Q[2x; 2+2/(\beta-2), 2y] \\
&\quad - Ee^{-r\tau}\left(1 - Q[2y; 2/(\beta-2), 2x]\right)
\end{aligned}
\tag{30}
$$

5.3. Hedge ratio with Delta (Δ) and Gamma (Γ) in the CEV model

The components of hedge ratio, $\Delta^{CEV} = \dfrac{\partial C}{\partial S}$ and $\Gamma^{CEV} = \dfrac{\partial^2 C}{\partial S^2}$, in the CEV model are tough to derive in closed-forms. Instead, we utilize numerical method to obtain the differentiation.

Lemma 1.

Suppose a function $f \in C^2[a,b]$. *For any* $x_0 \in (a,b)$, *consider some* $h \neq 0$ *such that* $x_0 + h \in (a,b)$. *The approximation of first derivative of the function* f *at* x_0 *is*

$$f'(x_0) = \frac{1}{12h}\left[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right] + \frac{h^4}{30} f^{(5)}(\xi),$$

where

$$\xi \in (x_0 - h, x_0 + h), \text{ the error term is of the form } O(h^4).$$

The approximation of second derivative is

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right] - \frac{h^2}{12} f^{(4)}(\xi).$$

Also, $\xi \in (x_0 - h, x_0 + h)$, *the error term is of the form* $O(h^2)$.

Proof. Please see Burden and Faires (2001).

The numerical result applying Theorem 1 can aid portfolio hedging and then we can calculate the hedging error easily.

5.4. CEV model parameter estimation

The parameters, μ, δ and β, shown in Table 3 are estimated by maximizing the likelihood function of the transition probability density function $f(S_T, T; S_t, t)$. Table 2 contains the descriptive statistics for all parameters during the sample period.

Table 2: Descriptive statistics for TAIEX

Parameters	μ	δ	β
Mean	0.0256	2.2735	1.6842
STD	0.8285	3.2527	0.2996
kurtosis	-1.1196	1.9977	-1.2436
skewness	-0.0061	1.7562	-0.5297
Min.	-1.4254	0.1204	1.1817
Max.	1.4231	10.7540	1.9962

Table 3: TAIEX parameter estimates

Sample period	μ	δ	β
Jun. 2002	-1.0252	2.6792	1.4506
Jul.	-0.0444	0.8323	1.7713
Aug.	-0.3411	9.2376	1.1817
Sep.	-1.3187	0.2968	1.9919
Oct.	1.1817	9.6418	1.2088
Nov.	0.4246	0.7104	1.7611
Dec.	-0.5900	0.2032	1.9777
Jan. 2003	1.4231	3.3128	1.3420
Feb.	-1.4254	0.8228	1.7545
Mar.	-0.5638	0.7268	1.7654
Apr.	-0.4933	0.3028	1.9904
May	1.0724	2.4023	1.3732
Jun.	0.5071	0.1574	1.9782
Jul.	0.6886	0.2357	1.9779
Aug.	0.6101	0.5145	1.7830
Sep.	-0.1721	0.1782	1.9143
Oct.	0.9364	5.1989	1.1997
Nov.	-0.6960	0.1458	1.9916
Dec.	0.0495	0.4602	1.7118
Jan. 2004	0.9760	0.1204	1.9962
Feb.	0.8826	0.1332	1.9908
Mar.	-0.7026	10.7540	1.2255
Apr.	-0.6876	1.3122	1.6136
May	-0.0783	4.1835	1.4694

The period of parameter estimates is June 2002 - May 2004. We maximize the likelihood function of asset price to estimate monthly.

5.5. Comparison of the homogeneity

Figure 7 and Figure 8 also show that the HE is more appropriate to evaluate model. Besides, Figure 8 indicates that more frequent hedge results in little hedging error without transaction cost consideration and it corresponds to the description in subsection 2.2. Both BS and CEV models points out that PE and PPE can be misleading in evaluating model and reject the more appropriate one.

Figure 7: PE and PPE patterns of CEV model

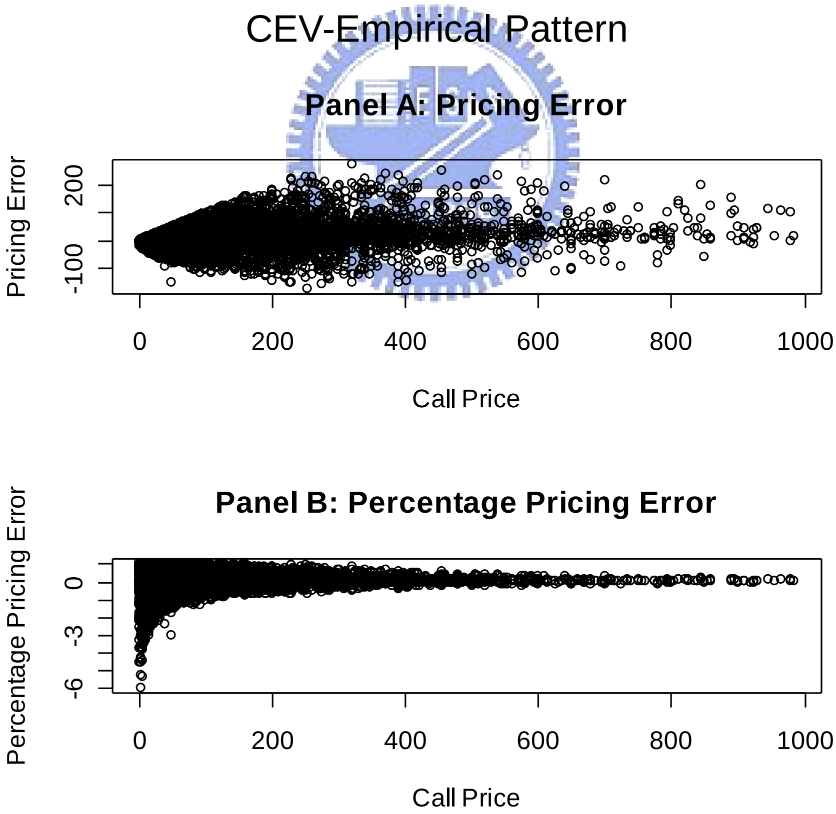
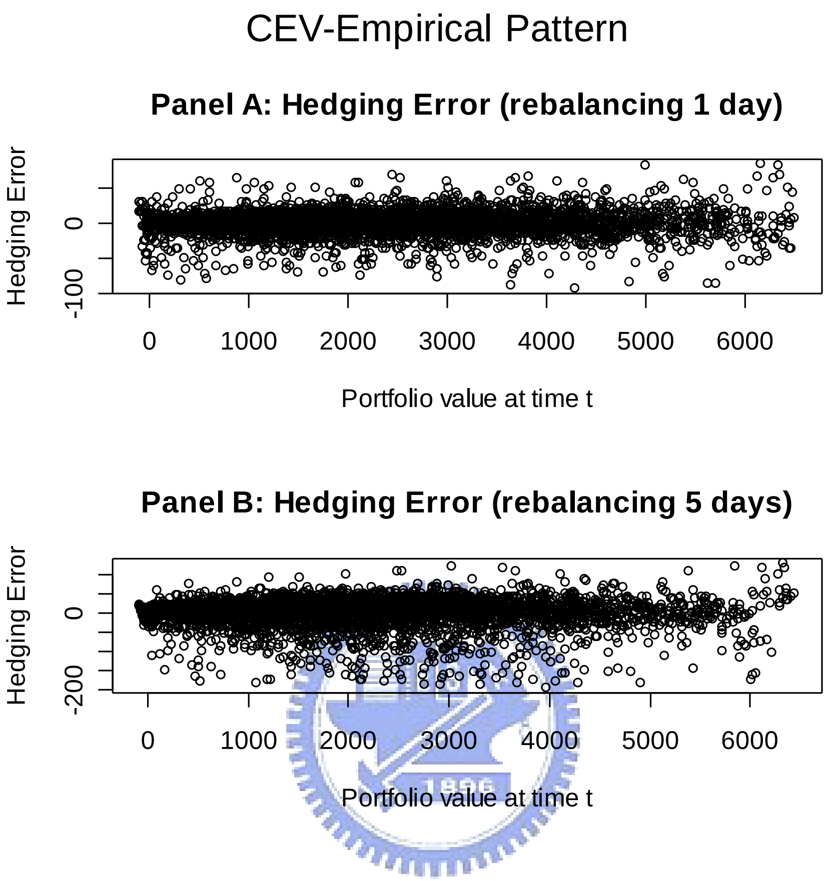


Figure 8: HE patterns of CEV models



5.6. Performance of both models

To assess the quality of both models, three measurements, \$MSE, %MSE and \$MAHE, introduced in previous section are made. Based on the \$MAHE and %MSE in Table 4, the CEV model is better than the BS model. Nevertheless, according to the \$MSE, the BS model is much better. However, it may be unbelievable since we discuss previously. Table 5 indicates that a minor proportion of contracts that are obviously overrated become a critical factor in model valuation. Thus, %MSE cannot be a good criterion.

Table 4: Out-of-sample performance

Criteria	Model	
	BS	CEV
\$MSE	2178.1844	2289.5788
%MSE	0.5672	0.4606
\$MAHE (rebalancing 1 day)	10.0129	9.9998
\$MAHE (rebalancing 5 days)	26.0007	25.9253

Table 5: More analysis about %MSE

Percentage pricing error ≤ 1	Model	
	BS	CEV
%MSE	0.3961	0.2733
Percentage (%)	69.8478	59.3353
number of call options	384/7015	339/7015

6. Conclusion

The thesis investigates an important empirical issue concerning criterion selection for an option model valuation. So far, the empirical literatures have mainly focused on the relative performance of various option valuation models. The role and the importance of the loss functions at the evaluation stage have been overlooked frequently. However, different criteria lead to different results so how to choose a loss function is important.

We abandon the conventional criteria, like \$MSE and %MSE, for their characteristics of heteroskedasticity. Since delta hedging is the most common hedging strategy used by traders to protect against risk, so hedging error evaluates properly the hedging performance. Thus the \$MAHE is recommended here as a substitute for them.

To demonstrate these implications, we compare the empirical performance of the BS model to the performance of the CEV model. We find that when using an inappropriate loss function, the BS model performs better than the CEV model. However, the BS model performs somewhat worse than the CEV model when the appropriate loss function is used.

Finally, we stress that when one compares different models, the relevant loss function for option model valuation should be used carefully otherwise improper comparisons will be made.

References

Amin, K. I., and R. A. Jarrow, 1992. Pricing options on risky assets in a stochastic interest rates. Mathematical Finance 2, 217-238.

Bakshi, G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. Journal of Finance 52, 2003–2049.

Bakshi, C., Cao, C., Chen, Z., 2000. Pricing and hedging long-term options. Journal of Econometrics 94, 277-318.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637-659.

Bams, D., Lehnert, T., Wolff, C.C.P., 2004. Loss functions in option valuation: A framework for model selection. Working paper.

Bates, D., 1996. Jumps and stochastic volatility: exchange rate processes implicit in Deutschemark options. Review of Financial Studies 9, 69-107.

Burden, Richard L. and J. Douglas Faires, 2001, Numerical Analysis - 7th Edition, Brooks/Cole Publishing Company.

Christoffersen, P. and Jacobs, K., 2004. The importance of the loss function in option valuation. Journal of Financial Economics 72, 291-318.

C. F. Lee, Jack C. Lee, Y.L. Hsu and T.I Lin, 2004. Constant elasticity of variance option pricing model: Integration and Detailed Derivations. Working paper.

C. F. Lo and P.H. Yuen, 2000. Constant elasticity of variance option pricing model with time-dependent parameters. International Journal of Theoretical and Applied Finance 3, 661-674.

Cox, J., 1975. Notes on pricing I: Constant Elasticity of Variance Diffusions. Unpublished note, Stanford University, Graduate School of Business.

Derman, E., Kani, I., 1994. Riding on the smile. Risk 7, 32-39.

Duan, J., 1995. The GARCH option pricing model. Mathematical Finance 5, 13-32.

Dumas, B., Flemming, F., Whaley, R., 1998. Implied volatility functions: empirical tests. Journal of Finance 53, 2059-2106.

Dupire, B., 1994. Pricing with a smile. Risk 7,18-20.

Engle, R., 1993. A comment on Hendry and Clements on the limitations of comparing mean square forecast errors. Journal of Forecasting 12, 642–644.

Hull, J., Suo, W., 2002. A methodology for assessing model risk and its application to the implied volatility function model. Journal of Financial and Quantitative Analysis 37, 297–318.

Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatilities. Journal of Finance 42, 281-300.

Henrotte, P., 1993. Transaction costs and duplication strategies. Working paper, Stanford University and Groupe HEC.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6, 327-343.

Heston, S., Nandi, S., 2000. A closed-form GARCH option pricing model. Review of Financial Studies 13, 585-626.

K.L. Chu, H. Yang and K.C. Yuen. Estimation in the constant elasticity of variance model. University of Hong Kong.

Leland, H.E., 1985. Option pricing and replication with transaction costs. Journal of Finance 40, 1283-1301.

Martellini, L., 2000. Efficient option replication in the presence of transaction costs. Review of Derivatives Research 4-2, 107-131.

Martellini, L., Priaulet, P., 2001. Optimal dynamic hedging in the presence of transaction costs: An empirical investigation. Working paper.

Melino, A., 1995. Misspecification and the pricing and hedging of long-term foreign currency options. Journal of International Money and Finance 14-3, 373-393.

Melino, A., Turnbull, S., 1990. Pricing foreign currency options with stochastic volatility. Journal of Econometrics 45, 239-265.

Merton, R., 1973. Theory of rational option pricing. Bell Journal of Economics 4, 141-183.

Renault, E., 1997. Econometric models of option pricing errors. In: Kreps, D., Wallis, K. (Eds.), Advances in Economics and Econometrics. Seventh World Congress. Cambridge University Press, New York and Melbourne, pp. 223–278.

Rubinstein, M., 1994. Implied binomial trees. Journal of Finance 49, 771-818.

Schroder, M., 1989. Computing the constant elasticity of variance option pricing formula. Journal of Finance 44, 211-219.

Scott, L., 1987. Option pricing when the variance changes randomly: theory, estimators, and applications. Journal of Financial and Quantitative Analysis 22, 419-438.

Toft, K. B., 1996. On the mean-variance tradeoff in option replication with transactions. Journal of Financial and Quantitative Analysis 31-2, 233-263.